SETS OF DISJOINT LINES IN PG(3,q)

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Using combinatorial methods, this paper discusses sets of pairwise skew lines in a finite projective 3-spaces, particularly the case in which there are points disjoint from all lines of the set but no lines skew to all lines of the set.

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1. Spreads and partial spreads. Let Σ be a projective space PG(3,q) of dimension 3 and finite order q. Then Σ contains $(q+1)(q^2+1)$ points and an equal number of planes, and $(q^2+1)(q^2+q+1)$ lines. It will be convenient to consider lines and planes as sets of points and to treat the incidence relation as set inclusion. Each plane contains q^2+q+1 points and an equal number of lines. Each line contains q+1 points and is contained in an equal number of planes. Each point is contained in q^2+q+1 planes and an equal number of lines.

A spread of lines of Σ is a set of q^2+1 lines of Σ which are pairwise disjoint, or skew; it can also be defined as a set of lines such that each point (or each plane) is incident with exactly one of the lines.

A packing of lines in Σ is a set of q^2+q+1 spreads such that every line is in exactly one spread of the set. Spreads of lines exist in every PG(3,q) and packings are known to exist in some cases.

A more general concept is that of a spread or packing of disjoint PG(m-1,q)'s, subspaces of dimension m-1, in PG(n-1,q), where m is a divisor of n. Generalizations to infinite geometries can also be formulated. All spreads and packings mentioned in this paper are to be taken as spreads and packings of lines in a finite 3-space. The terminology and the recent theory of spreads are due to R. H. Bruck [1,2], although related ideas have been considered earlier. A linear congruence [3] in Σ is a special case of a spread. A finite geometry which admits a packing is an instance of a balanced incomplete block design which is resolvable.

Example. The 15 points of PG(3,2) may be represented as follows by coordinate vectors over GF(2).

0001	0100	0111	1010	1101
0010	0101	1000	1011	1110
0011	0110	1001	1100	1111

Each of the 35 lines of this geometry contains 3 points and is displayed in one row of Table I. Each section of the table contains a spread, a set of 5 disjoint lines whose union is the set of all 15 points. The 7 spreads listed contain each line exactly once and therefore comprise a packing. Table II is for later use.

TABLE I.

Points of PG(3,2) arranged by lines, lines arranged into spreads.				
Spread 1,	line 1	0001	0010	0011
	line 2	0100	1000	1100
	line 3	0101	1010	1111
	line 4	0110	1011	1101
	line 5	0111	1001	1110
Spread 2,	line 1	0001	0100	0101
	line 2	0010	1000	1010
	line 3	0011	1101	1110
	line 4	0110	1001	1111
	line 5	0111	1011	1100
Spread 3,	line 1	0001	0110	0111
	line 2	0010	1001	1011
	line 3	0011	1100	1111
	line 4	0100	1010	1110
	line 5	0101	1000	1101
Spread 4,	line 1	0001	1000	1001
	line 2	0010	1101	1111
	line 3	0011	0100	0111
	line 4	0101	1011	1110
	line 5	0110	1010	1100
Spread 5,	line 1	0001	1010	1011
	line 2	0010	1100	1110
	line 3	0011	0101	0110
	line 4	0100	1001	1101
	line 5	0111	1000	1111
Spread 6,	line 1	0001	1100	1101
	line 2	0010	0101	0111
	line 3	0011	1001	1010
	line 4	0100	1011	1111
	line 5	0110	1000	1110
Spread 7,	line 1	0001	1110	1111
	line 2	0010	0100	0110
	line 3	0011	1000	1011
	line 4	0101	1001	1100
	line 5	0111	1010	1101

TABLE II.

Points of PG(3,4).	
001a	001b
1a00	1b00
1a1a	1b1b
1ab1	1ba1
1aab	1bba
010a	010b
10b0	10a0
11ba	11ab
1bb1	1aa1
1abb	1baa
011a	011b
10a1	10b1
11bb	11aa
1b10	1a10
1a0a	1b0b
100a	100b
11b1	11a1
01bb	01aa
1a1b	1b1a
1ba0	1ab0
101a	101b
11a0	11b0
01ba	01ab
1a01	1b01
1bbb	1aaa
110a	110b
01a1	01b1
10ab	10ba
1a11	1b11
1bb0	1aa0
111a	111b
01b0	01a0
10aa	10bb
1b0a	1a0b
1aba	1bab

Any set $\mathcal{S}$ of mutually disjoint lines of Σ will be called a partial spread. If $\mathcal{S}$ contains q^2+1-d lines, d will be called the deficiency of $\mathcal{S}$. A spread is thus a partial spread of deficiency zero. Exactly $(q+1)(q^2+1-d)$ points and an equal number of planes are incident with lines of $\mathcal{S}$, leaving residual sets of points and planes defined as follows.

(1.1) $\mathcal{A}$: the set of $d(q + 1)$ points of Σ not on lines of $\mathcal{S}$.

(1.2) $\mathcal{B}$: the set of $d(q + 1)$ planes of Σ not containing lines of $\mathcal{S}$.

A line ℓ is disjoint from all lines of $\mathcal{S}$ if and only if the $q+1$ points contained in ℓ are all points of $\mathcal{A}$. Equivalently, the $q+1$ planes containing ℓ are all planes of $\mathcal{B}$. If no such line exists, $\mathcal{S}$ will be called a complete partial spread.

The procedure of choosing lines of Σ one at a time, each disjoint from those previously chosen, will terminate in a complete partial spread and possibly in a spread. The conjecture and question which are taken up in this paper are motivated by the following observations, accumulated by the author (and his children) in empirical trials of this procedure with the 35 lines of PG(3,2) and the 130 lines of PG(3,3). Every partial spread in PG(3,2) can be completed to a spread of 5 lines, but not all partial spreads in PG(3,3) can be completed to spreads of 10 lines. The choice procedure is sometimes blocked at 7 lines. On the other hand, once we get as far as 8 or 9 lines, it always seems to be possible to find the remaining lines required for a spread.

Conjecture: There exists a critical size $d_0 = d_0(q)$ such that for $d < d_0$, every partial spread of deficiency d can be completed to a spread.

Question: If a partial spread of positive deficiency d is complete, what can be said about its structure, or about the structure of the residual sets $\mathcal{A}$ and $\mathcal{B}$ of points and planes?

This paper shows that the conjecture is true for $d_0 = 1 + \sqrt{q}$, gives an answer to the question in the case $d = 1 + \sqrt{q}$, and shows by an example with $q = 4$ that this case actually arises.

2. A lower bound for positive deficiency of a complete partial spread.

LEMMA 1. Let $\mathcal{S}$ be a partial spread in $\Sigma = PG(3,q)$ with residual sets $\mathcal{A}$ and $\mathcal{B}$ of points and planes. Then any line ℓ in Σ is incident with the same number, say $\lambda = \lambda(\ell)$, of points of $\mathcal{A}$ as planes of $\mathcal{B}$.

PROOF. The statement is trivial, with $\lambda = 0$, for $\ell \in \mathcal{S}$. To see it for $\ell \notin \mathcal{S}$, let ν be the number of lines of $\mathcal{S}$ which intersect ℓ. Since the lines of $\mathcal{S}$ are mutually skew, these lines meet ℓ in distinct points and lie in distinct planes on ℓ. The remaining $q+1-\nu$ points on ℓ are the points of $\mathcal{A}$ on ℓ, while remaining $q+1-\nu$ planes on ℓ are the planes of $\mathcal{B}$ on ℓ. Therefore the statement is true for $\ell \notin \mathcal{S}$, with $\lambda = q+1-\nu$.

THEOREM 1. If a partial spread $\mathcal{S}$ in $\Sigma = PG(3,q)$ has positive deficiency d and is complete, then

(2.1) $$d \geq 1 + \sqrt{q}.$$

PROOF. $\mathcal{S}$ contains $q^2 + 1 - d$ lines. The residual sets $\mathcal{A}$ and $\mathcal{B}$ of points and planes are not incident with lines of $\mathcal{S}$ have been defined and enumerated in (1.1) and (1.2). A plane π which contains ν lines of $\mathcal{S}$, $\nu = 0$ or 1, meets the remaining $q^2 + 1 - d - \nu$ lines of $\mathcal{S}$ in distinct points, accounting for

$$\nu(q+1) + q^2 + 1 - d - \nu = q^2 + \nu q + 1 - d$$

points of π on lines of $\mathcal{S}$. By subtraction plane π contains,

(2.2) $$q + d \text{ points of } \mathcal{A},\ \pi \in \mathcal{B},$$

(2.3) $$d \text{ points of } \mathcal{A},\ \pi \notin \mathcal{B}.$$

By a dual argument, a point P is contained in

(2.4) $\quad q + d$ planes of $\mathcal{B}$, $P \in \mathcal{A}$,

(2.5) $\quad d$ planes of $\mathcal{B}$, $P \notin \mathcal{A}$.

Let ℓ be incident with λ points of $\mathcal{A}$ and, by Lemma 1, with λ planes of $\mathcal{B}$. Using (1.1) and (2.2), there are $d(q+1)-\lambda$ points of $\mathcal{A}$ not on ℓ, of which $q+d-\lambda$ lie on each of the λ planes of $\mathcal{B}$ on ℓ. This gives the inequality

(2.6) $$d(q + 1) - \lambda \geq \lambda(q + d - \lambda) \quad ,$$

which reduces to

(2.6) $$(\lambda - d)(\lambda - q - 1) \geq 0 \quad .$$

Now take $\ell \notin \mathcal{S}$, and let $\mathcal{S}$ be complete. Then ℓ contains fewer than q+1 points of $\mathcal{A}$; otherwise it could be adjoined to $\mathcal{S}$. That is,

$$\lambda - q - 1 < 0 \quad ,$$

which with (2.7) implies

(2.8) $$\lambda - d \leq 0 \quad .$$

Let λ_{ij} be the number of points of $\mathcal{A}$ which are on the line of intersection of the i-th and j-th planes of $\mathcal{B}$, and let $\bar{\lambda}$ be the mean of λ_{ij} over the $(dq+d)(dq+d-1)$ ordered pairs i, j. There are $dq+d$ points of $\mathcal{A}$ in all, each of which is on $q+d$ planes of $\mathcal{B}$ and therefore on the lines of intersection of $(q+d)(q+d-1)$ pairs of planes. Counting points two ways, we have

(2.9) $$\bar{\lambda}(dq + d)(dq + d - 1) = (dq + d)(q + d)(q + d - 1) \quad .$$

Taking $\mathcal{S}$ complete, (2.8) implies

(2.10) $$\bar{\lambda} \leq d \quad .$$

Taking $\mathcal{S}$ with positive deficiency, $d \geq 1$, statements (2.9) and (2.10) imply

$$d(dq + d - 1) \geq (q + d)(q + d - 1).$$

Solving for d, and disregarding negative solutions, we obtain (2.1), noting for later use that equality holds only if equality holds in (2.10).

The next theorem is obvious but useful.

THEOREM 2. A partial spread $\mathcal{S}$ in $\Sigma = PG(3,q)$ with deficiency $d \leq q$ can be completed to a spread in at most one way.

PROOF. Suppose the residual set $\mathcal{A}$ contains d disjoint lines with which the spread can be completed. Then any other line can contain at most one point of each of these lines and must have points not in $\mathcal{A}$. Thus $\mathcal{A}$ does not contain any other sets of d lines, disjoint or otherwise.

Theorems 1 and 2 give the following.

COROLLARY 1. A partial spread in PG(3,q) with more than $q^2 - \sqrt{q}$ lines can be completed uniquely to a spread.

PROOF. The partial spread has deficiency less than $\sqrt{q} + 1$. It follows from Theorem 1 that if it is not a spread, it is not complete and at least one more line can be adjoined. Continuing, we obtain a spread. Theorem 2 implies uniqueness.

It is easy to enumerate partial spreads in PG(3,q) with up to four or five lines, while partial results can be obtained for the number of spreads. It can be proved that if ℓ, m, n and ℓ', m', n' are two sets of three mutually skew lines in Σ, there exists a collineation mapping ℓ onto ℓ', m onto m' and n onto n'. In this sense, all partial spreads of three or fewer lines are equivalent.

In PG(3,2), a partial spread with 3 lines has more than $q^2 - \sqrt{q}$ lines and it follows from Corollary 1 that any partial spread in this geometry can be completed to a spread.

In PG(3,3), Corollary 1 implies that a partial spread with more than $3^2 - \sqrt{3} \approx 7.27$ lines can be completed to a spread, confirming the empirical findings mentioned for partial spreads of 8 or 9 lines. On the other hand, the existence of complete partial spreads with 7 lines shows that the inequality of Theorem 1 is essentially the best possible for $q = 3$.

With $q = 4$, the possibility arises that (2.1) may hold with equality. This situation is discussed in the next theorem, and an example with $q = 4$ follows.

3. Complete partial spreads in $PG(3,s^2)$ with deficiency $1 + s$

DEFINITION. With Σ, $\mathcal{S}$, $\mathcal{A}$, and $\mathcal{B}$ as defined before, the residual geometry of $\mathcal{S}$ is a system Γ whose elements are certain points, lines, and planes of Σ specified as follows:

the points of $\mathcal{A}$,

the planes of $\mathcal{B}$,

the lines of intersection of planes of $\mathcal{B}$.

An incidence relation for elements of Γ is defined to coincide with the incidence relation for the same elements, regarded as elements of Σ.

We are now ready to investigate a complete partial spread whose deficiency d is equal to the lower bound $1 + \sqrt{q}$ of Theorem 1. Without loss of generality we may assume that q is a square, say

(3.1) $$q = s^2 ,$$

(3.2) $$d = 1 + s .$$

THEOREM 3. Let $\mathcal{S}$ be a complete partial spread of deficiency $d = s + 1$ in $\Sigma = PG(3,s^2)$. Then the residual geometry Γ of $\mathcal{S}$ is a $PG(3,s)$.

PROOF. For easy reference, we specialize (1.1), (1.2), (2.2), (2.3), (2.4), (2.5) to the present case as follows.

(3.3) $$\Gamma \text{ contains } (s + 1)(s^2 + 1) \text{ points} ,$$

(3.4) $$\Gamma \text{ contains } (s + 1)(s^2 + 1) \text{ planes} .$$

A plane π of Σ contains

(3.5) $s^2 + s + 1$ points of Γ, $\pi \in \Gamma$,

(3.6) $s + 1$ points of Γ, $\pi \notin \Gamma$.

A point P of Σ is contained in

(3.7) $s^2 + s + 1$ planes of Γ, $P \in \Gamma$,

(3.8) $s + 1$ planes of Γ, $P \notin \Gamma$.

By definition of lines of Γ and the geometric properties of Σ,

(D) any two planes of Γ intersect in exactly one line of Γ .

Let the line of intersection of the i-th and j-th planes of Γ contain λ_{ij} points of Γ. Since (2.1) holds with equality, the same is true for (2.10), implying

(3.9) $$\sum_{i,j} (\lambda_{ij} - d) = 0 \quad .$$

But $\mathscr{d}$ is complete and from (2.8) we have

(3.10) $$\lambda_{ij} \leq d \text{ , all } i,j \quad .$$

(3.9) and (3.10) imply $\lambda_{ij} = d$, all i,j, proving that

(3.11) every line of Γ contains exactly $s + 1$ points of Γ .

By Lemma 1,

(3.12) every line of Γ is contained in exactly $s + 1$ planes of Γ .

Two points of Γ are on a unique line of Σ and hence on at most one line of Γ. We must show that it is a line of Γ. From (3.4), Γ contains

(3.13) $(s + 1)(s^2 + s + 1)(s^3 + s^2 + s)$ ordered pairs of planes.

If these are enumerated by the lines in which they intersect. By (3.12), each line accounts for $(s+1)s$ ordered pairs, showing that the number of distinct lines of Γ is

(3.14) $$(s+1)(s^2+s+1)s/[(s+1)s] = (s^2 + 1)(s^2 + s + 1) \quad .$$

By (3.11), each of these lines contains $s+1$ points of Γ and accounts for

$(s+1)s$ ordered pairs of points. Taken jointly, the lines of Γ account for $(s+1)(s^2+1)(s^2+s+1)s$ pairs of points, which from (3.3) is the totality of such pairs. Therefore,

(A) any two points of Γ are contained in exactly one line of Γ.

From properties of Σ , if two points of Γ lie in a plane of Γ , which is also a plane of Σ , the line joining them lies entirely in the same plane. Thus

(C) if two points of Γ are in a plane of Γ , the line of Γ containing the two points is also in that plane.

Consider 3 points of Γ which are not all on a line of Γ. Then they are not collinear in Σ and hence determine a unique plane, say π , of Σ . From (C) and (3.11), π contains at least 3 lines and at least $3s$ points of Γ. Comparison with (3.5) and (3.6) shows that π must be a plane of Γ. Therefore,

(B) three points of Γ which are not collinear lie on a unique plane of Γ .

Three planes of Γ meet either in a common line of Γ or by pairs in 3 lines of Γ . In the latter case their intersection is a point of Σ which is on at least 3 lines of Γ and by (3.12) is on at least $3s$ planes of Γ . Comparison with (3.7) and (3.8) shows that it must be a point of Γ . Therefore,

(E) three planes of Γ which do not meet in a common line must intersect in a unique point of Γ .

Statements (A), (B), (C), (D), (E), taken as postulates, are sufficient to show that Γ is a projective space. We need existence postulates to the effect that Γ does not reduce to a degenerate form such as a single plane and that Γ has order s. (3.4) and (3.11), among others, will do, and the proof is complete.

Suppose that $\Sigma = PG(3,s^2)$ contains a complete partial spread $\mathcal{S}$ of $s^4 - s$ lines. A coordinate system can be set up for $\Sigma = PG(3,s^2)$ so that points are represented by non-null vectors $\underline{x} = (x_0, x_1, x_2, x_3)$ over the field $GF(s^2)$ with s^2 elements, where $\underline{x}$ and $\alpha\underline{x}$ represent the same point if α is a nonzero field element. If $\underline{x}$ and $\underline{y}$ are two points, then the points of the line joining $\underline{x}$ and $\underline{y}$ are those which can be expressed

(3.15) $\alpha \underline{x} + \beta \underline{y}$,

where α and β are field elements not both zero.

Without going into details, we remark that this field has a subfield $GF(s)$ with s elements, and the coordinate system may be set up so that the points of a specified subgeometry $\Gamma = PG(3,s)$ are precisely those which can be represented by coordinate vectors over the subfield. Let us make such a choice of coordinate system, taking Γ as the residual geometry of the partial spread $\mathcal{S}$. Two points $\underline{x}$ and $\underline{y}$ in Γ are joined by a line of Γ whose $s+1$ points may be represented by (3.15), where α and β are taken from $GF(s^2)$.

Taking $s = 2$, suppose that $\Sigma = PG(3,4)$ contains a complete partial spread $\mathcal{S}$ with 14 lines and deficiency 3, and let $\Gamma = PG(3,2)$ be the residual geometry of $\mathcal{S}$. Representing the elements of the subfield $GF(2)$ by 0 and 1 and the remaining elements of $GF(4)$ by $\underline{a}$ and $\underline{b}$, there is no loss of generality in representing the 15 points and 35 lines of Γ in the form displayed in Table I.

Each line of Γ contains 3 points and each line of Σ contains 5 points. Table II augments Table I by listing coordinate vectors for the two remaining points of Σ on each of the 35 lines of Γ. The 70 points of Σ thus obtained are seen to be distinct, exhausting the points of Σ which are not in Γ. These are the 70 points which must occur on the lines of $\mathcal{S}$.

Each section of Table I lists a spread of 5 lines. Each spread determines a set of 10 points, listed in the corresponding section of Table II. In each case this set may be observed to be the union of two disjoint lines; each of the two columns of each section of Table II displays one of these lines. In all, the 7 sections of Table II contain 14 such lines, mutually disjoint, exhausting the points of Σ which are not in Γ, and comprising a complete partial spread $\mathcal{S}$ with deficiency 3.

Thus we have an example, for $q = 4$, of a complete partial spread with deficiency equal to the lower bound $1 + \sqrt{q}$ of Theorem 1.

It may be fairly criticized that this example has been produced like a rabbit from a hat. The author's defense is that it is only a small rabbit (the assertions made for this example being easily verfied by computation), and that the hat is now empty (since the analogous construction using spaces Γ and Σ for $s > 2$ fails to furnish enough lines in $PG(3,s^2)$ for a partial spread of deficiency $1 + s$). The author does not know whether the inequality of Theorem 1 is the best possible for $q > 4$.

It would be of interest to have a lower bound for the number of lines (upper bound for the deficiency) in a complete partial spread.

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