ASYMPTOTIC METHODS OF EVALUATING \( \int_a^\infty f(x)dx \)

by

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NOTATION

The following notation will be used in this dissertation. Let $f$ be a real function of a real variable, $x$. $f'$ and $f''$ refer to the first and second derivatives of $f$ with respect to $x$. $v$ stands for $-f/f'$. $\int f$ is a shortened form of $\int_{x}^{\infty} f(t)dt$ or $\int_{c}^{x} f(t)dt$, depending on whether the first integral converges or diverges. It will also be written as $F(x)$ or simply, $F$. Leaving out explanatory symbols such as $(x)$ and integration limits in a context where they are understood clarifies complicated formulas.

Theorems, formulas, and examples will be numbered in a single ordering. They will be referred to by numbers in parentheses. Such references at the end of a line of proof justify the preceding statement. Numbers in square brackets refer to the bibliography listed at the end.

Parentheses will also be used to designate points of the $(x, f(x))$ plane, open intervals on the real line, and the extended order notation defined in Chapter II. Square brackets will also be used for closed intervals, and a combination of a round and a square bracket refers to an interval open on one end and closed on the other. Curly brackets refer to sets. $\{x; a < x < b\}$ is the set of all numbers, $x$, that are $> a$ and $< b$.

The symbols, $\implies$ and $\iff$ stand respectively for "implies" and "if and only if." These symbols are less powerful than the phrases, so that the sentence,

If $A$, then $B \implies C$

should be parenthesized

If $A$, then $(B \implies C)$. 
The symbol, \( \implies \), directly following a comma, should be read, "which implies that," and thus it introduces a new line of proof. An equal sign directly following a comma has the meaning, "which equals." The symbol, \( // \), designates the end of a proof.

The letters \( \varepsilon \) and \( \delta \) are always considered small, positive numbers, and \( M, N, \) and \( R, \) large positive numbers. \( R \) is also used briefly as the multiple correlation coefficient. \( m \) and \( n \) are always considered positive integers. \( i \) and \( j \) are integers that are often used as summation indices.

A single arrow, \( \rightarrow \), means "approaches a limit of." The tense is occasionally changed to "approaching" or "would approach" where appropriate. Infinite limits are allowed unless specifically prohibited. Thus, the statement, "\( f(x) \to \text{a limit} \)" means that one of the following situations holds.

1) There exists a real number, \( L, \) with the property that for every \( \varepsilon \) there exists \( N \) such that \( x > N \implies |f(x) - L| < \varepsilon; \)
   i.e., \( f(x) \to L. \)

2) For every \( R \) there exists \( N \) such that \( x > N \implies f(x) > R; \)
   i.e., \( f(x) \to \infty. \)

3) For every \( R \) there exists \( N \) such that \( x > N \implies f(x) < -R; \)
   i.e., \( f(x) \to -\infty. \)

The notation, \( f < g, \) stands for the statement, "there exists \( N \) such that \( x > N \implies f(x) < g(x).""

The word, constant, used in such phrases as \( f(x) = \text{constant} \cdot f(x), \)
and \( g(x) \to \) a constant, refers to a finite, real number \( \neq 0. \)
The order of \( f \) is a real number or \( +\infty \) or \( -\infty \). It is symbolized by \( \theta(f) \) or \( r \). When finite, it means that for every \( \epsilon \),
\[
x^{r-\epsilon} < f < x^{r+\epsilon}
\]
or the same is true of \( -f \). \( r = +\infty \) means that for every \( N \), \( x^N < f \) or the same is true of \( -f \). \( r = -\infty \) means that for every \( N \), \( |f| < x^{-N} \). The letter, \( \gamma \), stands for \( r/(r+1) \).
CHAPTER I
INTRODUCTION AND SUMMARY

When one is trying to solve a problem in statistical inference, and wishes to give an answer that is an exact probability statement, an essential step is often the evaluation of the integral, from some point, $a$, to $\infty$, of a known function, $f$. In statistics, this integral almost never has some simple, closed form, and the tables that have been constructed are incomplete, both with respect to probability levels and degrees of freedom.

There is a need for handy methods of approximating these integrals, and, indeed, many methods have been considered by statisticians. Nearly all of these methods, however, have been good near the center of the distribution and relatively poor on the tails.

The methods considered in this dissertation lead to approximations that get increasingly accurate as $a \to \infty$, so that they are especially applicable to the tail of a distribution. This is usually the place where most accuracy is needed, since research workers have found the .05, .01, and .001 levels most useful. There are many cases where even more extreme levels are needed, such as when lives are at stake, or in combining the results of tests. The methods considered here can thus be applied to the usual levels of untabulated distributions and to extreme levels of the common distributions.

The approximations depend only upon knowing the value of $f$ and a few of its derivatives at $a$, and that $f$ has desirable monotonic
properties. Sequences of increasingly accurate approximations are derived along with bounds for the errors, so that the application is both general and practical.

The methods considered apply to the lower tail by using the transformation, \( g(x) = f(-x) \). Proper integrals can be approximated by subtracting an approximation to \( F(b) \) from the approximation to \( F(a) \). (We are writing \( F(a) \) for \( \int_a^\infty f(x)dx \).) The methods are extended by analogy to finite tails and to sums of series.

A wide class of approximations previously studied have been the methods of mechanical quadrature, such as are described by Irwin, [20]. The value of \( f \) is obtained for a finite number of points and approximated elsewhere by fitting polynomials through these points. The approximation to \( f \) is then integrated and the whole procedure summed up by a handy formula.

This method applies only to finite intervals, since if one tries to extrapolate the polynomials, they approach \( \infty \). So no matter how far out one goes on the tail, there always remains an error tail that must be approximated and bounded. For the latter calculation, the methods of this dissertation are well suited. Indeed, by combining quadrature with a formula asymptotic as \( a \) approaches \( \infty \), such as is considered here, one can make the relative error as small as one pleases.

If one knew the median, or a lower bound for the distribution, (e.g. the gamma distribution has a lower bound of 0,) one could approximate the tail area by subtraction. In this case, the error, though numerically small, grows increasingly large relative to the integral as \( a \to \infty \).

Another class of well-known procedures consists of special cases of the method of orthogonal polynomials, which is thoroughly described by
Szego, [48]. In this method, we are given an interval, \((a, b)\), finite or infinite, and a positive, increasing function, \(\alpha\). Two functions, \(f\) and \(g\), are said to be orthogonal under these conditions if \((f \cdot g)\), which is short for \(\int_a^b f(x)g(x)\,d\alpha(x)\), is equal to 0. One can build up a sequence of polynomials, \(\{\phi_i\}\), of degree \(i = 0, 1, 2, \ldots\), every pair of which is orthogonal, and these are called orthogonal polynomials. For different values of \(a\) and \(b\) and different functions, \(\alpha\), one gets different sequences of orthogonal polynomials.

Now this orthogonal sequence has the following approximation property: the minimum distance between \(f\) and a linear combination of \(\phi_0, \ldots, \phi_n\) is always given by

\[
\frac{(\phi_0 \cdot f)}{(\phi_0 \cdot \phi_0)} \phi_0 + \ldots + \frac{(\phi_n \cdot f)}{(\phi_n \cdot \phi_n)} \phi_n,
\]

where the distance between \(f\) and \(g\) is defined as

\[
\int_a^b [f(x) - g(x)]^2 \,d\alpha(x).
\]

Thus, the integral of \(f\) can be approximated by integrating a finite number of polynomials.

This procedure has its advantages and disadvantages. It again fails for the tail, since the approximation is a polynomial and \(\to \infty\) with \(x\). And further, we have to compute \(n\) integrals, \((\phi_0 \cdot f), \ldots, (\phi_n \cdot f)\), in order to get the original integral. However, there are special cases when we can compute the coefficients by short cuts, and then we don't even have to know \(f\) itself. An example is the Gram-Charlier A series, which is the expansion of \(f(x)/(\text{normal density})\) in Hermite polynomials. \((a = -\infty, b = \infty, \alpha(x) = \int_{-\infty}^x t^2/2 \,dt\). The coefficients turn out to be linear functions of the moments of \(f\), which in many cases are more readily available than an exact knowledge of \(f\).
In the Gram-Charlier series, and also in the Edgeworth series, which is a rearrangement of terms of the former, we have \( f(x) \) equal to \( \phi(x) \) multiplied by a series of polynomials in \( x \). (\( \phi(x) \) is the normal density.) This can be integrated term by term to obtain \( F(x) = \xi(x) + (\phi \times \text{a series of polynomials in } x) \). This is not an approximation that approaches \( \infty \) with \( x \), but because \( F \) is very unlikely to have order \(-1, 2\), in the terminology of Chapter II, the relative error approaches \( \infty \) or 0.

Another method of approximation is called Laplace's method, [26]. The integral of a function with a sharply peaked maximum is approximated by means of the value of the function and its second derivative at the maximum. When this method is sharpened up by taking a path of integration in the complex plane to give \( f \) a steepest maximum, it is called the method of steepest descent. A one-sided form of Laplace's method is called the method of the stationary phase. This class of approximation methods, which is described by Erdelyi, [9], clearly applies to the center of the distribution rather than the tail.

The method of contour integration has simplified the calculation of many integrals. In its simplest form, one obtains the integral of \( f(x) \) from \(-\infty\) to \( \infty \) as the limit of the integral along a sequence of closed paths in the complex plane. One side of the path is a bigger and bigger interval on the real axis approaching the whole axis, and the other segment is chosen so that the integral over it approaches 0. Our integral has a finite left end point, so that in only very special cases does the integral over the complex segment approach 0. The situation is not improved if one applies the transformation, \( z = \log(x-a) \), carrying \( a \) into \( -\infty \). To get the integral of the transformed function over any complex path from \(-R\) to \( R \), say, we could just as well have taken the
integral over \( f \) along a path from \( a + e^{-R} \) to \( a + e^{R} \). And that, as we have observed, is most unlikely to approach 0.

In preparing the preceding abbreviated summary of known methods, I am indebted to Harold Hotelling's course on integral approximations at the University of North Carolina, and to D. L. Wallace's expository article, [54], on the same subject, which describe the foregoing and many other methods. The principal difference between the methods described by Wallace and those of this dissertation is that the former are asymptotic as a parameter, \( n \), approaches \( \infty \), while the latter are asymptotic as \( x \) approaches \( \infty \).

There must be hundreds or thousands of different methods of integration scattered through the literature. I cannot pretend to have searched them all out. What I have done, in addition to following up the aforementioned sources of information, is to check through Mathematical Reviews and the Fortschritte der Mathematik for general methods of integration that are good on the tails, and for all methods that bear resemblance to those of this dissertation. In Mathematical Reviews, I checked the sections on "Analysis, general," "Theory of functions of a real variable," (which changes to "Theory of measure and integration,") "Theory of series," "Numerical and graphical methods," and "Probability and statistics," from Volume 1, (1940), up to the present. In the Fortschritte I have checked for all volumes, (1868-1941), the sections on analysis in general, series, integration, differentiation, real variables, approximations, practical analysis, and probability. I have also checked through relevant sections of the Encyclopaedia der Mathematischen Wissenschaften, and all volumes of the journal, Mathematical Tables and Aids to Computation.
I was fortunate to be able to consult Hans J. Rohrbach, an expert on numerical analysis, about possible references in the literature. He was kind enough to furnish me with a short bibliography, which I have checked most carefully. A reference given both by him and by H. Hotelling, *Orders of Infinity*, by G. H. Hardy, [15], turned out to be most relevant to my work, and I read it with great care and checked its references. I have also read carefully *Asymptotic Expansions* by A. Erdelyi, [9], and checked its references.

The search showed that the subject of this dissertation has rarely been considered. Hotelling, [17], by a geometrical argument, has given an approximation for the distribution of \( t \) for the normal and non-normal cases that improves as \( t \) approaches \( \infty \). Wallace, [54, p. 647], relates a special procedure devised by D. Teichrow for the normal \( t \) distribution that is asymptotic as both \( t \) and \( n \) approach \( \infty \). Hardy, [15, p. 37], gives two integral approximation formulas that are asymptotic as \( a \) approaches \( \infty \). These are discussed in Chapter II, Formula 33. Ostrowski, [31], gives one of them more general conditions. (See Theorem 35.)

An expansion of a definite integral by Laplace in 1814, [26, p. 90], forms the subject of Chapter V of this dissertation. The expansion was given a simpler derivation and a remainder term by Winckler, [58], in 1817, and then, apparently forgotten. Although the emphasis of these authors was on the evaluation of a proper integral, the series is especially applicable to tail integrals. In Chapter V, it is shown that the expansion is a special case of a general iterative procedure. This procedure has desirable asymptotic properties when \( f \) has infinite order, and a modification is introduced that is suitable when \( f \) has finite order, to use the terminology of Chapter II.
Two other procedures are studied in this dissertation. Since it is natural to think of the Taylor expansion in connection with approximation procedures, its usefulness for approximating tail integrals is considered in Chapter IV. It cannot be applied directly, since all the terms approach \( \infty \), so we apply a transformation, \( y = \psi(x) \), carrying \( \infty \) into 0. The integral is now expanded around \( \psi(a) \) or 0. The expansion of \( F(x) \) in powers of \( 1/x \) is a special case of this method.

The best \( \psi \) one can choose is the transformation \( y = F(x) \), which reduces the Taylor series to one term. If we don't know \( F \), but have a good guess, \( F_1 \), the latter function would seem to be a good choice of \( \psi \). In that case, we expect the Taylor series to be well approximated by the first term, which is \( F_1(a)f(a)/f_1(a) \), (where \( f_1(a) \) is \( -F_1'(a) \), to be analogous to \( F \) and \( f \).) We can call this expression \( F_2(a) \), and we have an iterative procedure for approximating \( F \). The properties of this procedure are studied in Chapter VI.

In order to write the formulas for these procedures, and to study their errors and asymptotic properties, it is necessary to introduce the concept of order. In this connection, the Cauchy definition has been most useful, [3, p. 281]; namely, the order of \( f \) is \( r \) means that given any \( \epsilon \), from a certain point on, \( x^{r-\epsilon} < f < x^{r+\epsilon} \). Chapter II is devoted to order theorems based on this definition. Some of the theorems cover ground previously explored by Cauchy, [3], Borel, [2], and Hardy, [15]. These theorems are included for the convenience of having a collection of order theorems based on the Cauchy definition, exactly stated in a uniform notation.

In Chapter III, some general theorems about errors and bounds for integral approximations are proved. Chapters VII and VIII extend the procedures by analogy to finite tails and infinite series.
Chapter IX applies the methods of this dissertation to integrals useful in statistics. New and useful approximations are obtained for the $t$, gamma, beta, bivariate normal, and Pearson Type IV distributions.

The approximation formulae discussed in this dissertation are listed in an appendix for easy reference.
CHAPTER II
ORDER THEOREMS

In this chapter we will classify certain well-behaved functions according to their rate of increase as \( x \to \infty \). Various ways of doing this have been proposed. The most common notation is due to the influence of Landau, [25, p. 61], and Hardy, [15]. They write, \( f = o(g) \), for \( \frac{f(x)}{g(x)} \to 0 \) as \( x \to \infty \), \( f = O(g) \), for \( \frac{f(x)}{g(x)} \) is bounded as \( x \to \infty \), and \( f \sim g \) for \( \frac{f(x)}{g(x)} \to 1 \) as \( x \to \infty \).

For the purposes of this dissertation we will need to be more explicit. The trouble with the Landau concept of order is that if \( f = o(x^2) \), say, then \( f \) also \( = o(x^3) \). We want a notation that distinguishes between these two cases.

A notation proposed by Cauchy, [3, p. 281], and elaborated by Borel, [2], serves this purpose. They say that the order of \( f \) is \( r \) when \( x^{r-\varepsilon} < f < x^{r+\varepsilon} \) from a certain point onward. This distinguishes between the functions, \( x^2 \) and \( x^3 \), but fails to distinguish between \( x^2 \) and \( x^2 \log x \). Borel, [2], proposed a notation that discriminates among all types of regular increase, so that if \( f = o(g) \), then \( f \) and \( g \) have a different order notation. This notation is too precise, being too cumbersome to be useful.

Since Cauchy did not carry his concept beyond one or two theorems, and since no other writer (that I know of) has made a collection or order theorems based on the Cauchy definition and precisely stated and proved, a number of such theorems will be written down here. Of course, they
will, in many cases, cover ground that has been essentially explored before.

Definition. Let \( r \) be a real number. \( f \) has order \( r \), also written, \( \theta(f) = r \), means that for any \( \varepsilon > 0 \), \( x^{r-\varepsilon} < f < x^{r+\varepsilon} \), or the same thing is true of \( -f \). \( f \) has order \( \infty \) means that for any large number, \( N \), \( x^N < f \), or the same thing is true of \( -f \). \( f \) has order \( -\infty \) means that for any \( N \), \( |f| < x^{-N} \). We will say that \( f \) has order, if there exists \( r \), finite or infinite, such that \( \theta(f) = r \).

Example 1. There are, of course, many functions that do not have order. The following monotonic function does not have order: \( f \) is a function whose graph is a polygonal curve joining the points, \((1,1), (2,1/4), (5,1/5), (6,1/36), (37,1/37)\), and so on. Although \( f \) steadily decreases, it alternates between the functions, \( x^{-1} \) and \( x^{-2} \), and so cannot have order.

Theorem 2. \( \theta(f) < \theta(g) \iff |f| < |g| \), and \( f = o(g) \). If \( |f| \geq |g| \) and both have order, then \( \theta(f) \geq \theta(g) \).

Theorem 3. \( \theta(cf) = \theta(f) \).

Theorem 4. \( f \) has order \( \neq \infty \iff f > 0 \) or \( f < 0 \).

The proofs of these theorems are immediate. \( e^{-x} \sin x \) is an example of a function having order \( \infty \) and alternating in sign indefinitely.

Example 5. A function having order \( 0 \) does not have to \( \rightarrow \) a limit. The function \( \sin x + 2 \) is an example. All functions of positive order \( \rightarrow \infty \) and all functions of negative order \( \rightarrow 0 \).

Theorem 6. \( \theta(f) = r \Rightarrow \theta|f| = r \). If \( \theta|f| = r \), and if \( r = \infty \) or \( f \) is continuous, then \( \theta(f) = r \).

Proof of the last statement. If \( f \) is continuous and alternates in sign indefinitely, then \( f = 0 \) on a sequence of points approaching
\[ \infty, \implies |f| \text{ has order } -\infty, \implies \theta(f) = \theta|f| = -\infty. // \]

In the next theorem, we prove the equivalence of the Cauchy definition of order and one given by Borel, [2, p. 34].

**Theorem 7.** If \( f \) is continuous, or if \( f > 0 \) or \( f < 0 \), or if \( r = \infty \), then \( \theta(f) = r \), finite or infinite \( \iff \log|f|/\log x \to r \).

Proof. Under the assumptions of this theorem, \( \theta(f) = r \iff \theta|f| = r \), (6). Case 1: \( r \) is finite. \( \theta|f| = r \iff x^{r-\epsilon} < |f| < x^{r+\epsilon} \), \( \iff (r-\epsilon) \log x < \log |f| < (r+\epsilon) \log x, \iff \log |f|/\log x \to r \). A similar proof applies to the cases \( r = \pm \infty \). //

**Theorem 8.** 1) If \( f(x) \to 1 \) then \( f - 1 \sim \log f \), and hence \( \theta(f(x) - 1) = \theta(\log f) \), provided that one of the two has order.

2) If \( \theta(f) = 0 \) and \( f' \) has order, then \( \theta(1/f') = \theta(f') \).

3) If \( f \to a \) constant, \( c \), then \( 1/f - 1/c \sim \text{constant} \cdot f' \).

Proof. Part 1 is a consequence of the fact that \( \log y \sim y - 1 \) as \( y \to 1 \). Part 2: \( \theta(1/f') = \theta(f'/f^2) = \theta(f') \). Part 3: \( 1/f - 1/c = (c - f)/fc \sim (f - c)/-c^2 \). (\( 1/f' = f'/f^2 \sim f'/c^2 \)). //

**Theorem 9.** If \( f \) and \( g \) are positive, and \( \theta(f) \geq \theta(g) \), then \( \theta(f+g) = \theta(f) \). If \( \theta(f) > \theta(g) \), then \( \theta(f-g) = \theta(f) \). If \( \theta(f) = \theta(g) \), and \( f-g \) has order, then \( \theta(f-g) \leq \theta(f) \). If, in addition, \( f \) is not \( \sim g \), then \( \theta(f-g) = \theta(f) \).

Proof. \( \theta(f) > \theta(g) \implies x^{r-\epsilon} < f/2 < f+g < 2f < x^{r+\epsilon} \), where \( r = \theta(f) \), \( \implies \theta(f+g) = \theta(f) \). Now let us suppose that \( \theta(f) = \theta(g) = r \), \( x^{r-\epsilon} < f < x^{r+\epsilon} \) and \( x^{r-\epsilon} < g < x^{r+\epsilon} \), \( \implies x^{r-\epsilon} < \frac{f+g}{2} < x^{r+\epsilon} \), \( \implies \theta(f+g) = r \). Let us further suppose that \( f-g \) has order. Then \( \theta(f-g) \leq r \) because \( |f-g| < |f| + |g| \), (2). But if \( \theta(f-g) < r \), then \( f-g = o(g) \), \( \implies (f/g) - 1 \to 0 \), \( \implies f \sim g \). //

**Theorem 10.** If \( f \to L \), and \( g \to K \), constants, and the expressions in the theorem have order, then both \( \theta(fg - LK) \) and \( \theta(f/g - L/K) \)
= \max \{\theta(f-L), \theta(g-K)\}, \text{ unless, possibly, } \theta(f-L) = \theta(g-K) \text{ and } f-L \sim_{\frac{L}{K}} \frac{L}{K}, [\frac{L}{K}], \text{ in which case the symbol, } =, \text{ should be replaced by } \preceq.

Proof. Let \( f-L = \epsilon \) and \( g-K = \delta \). \( f \cdot g = L \cdot K + \epsilon \cdot K + \delta (L+\epsilon) \).

Case 1: \( \theta(\epsilon) > \theta(\delta) \). Then \( \theta(\epsilon \cdot K) > \theta(\delta (L+\epsilon)) \), \( \Rightarrow \theta(\epsilon \cdot K + \delta (L+\epsilon)) \)

= \( \theta(\epsilon K) \) by (9). Case 2: \( \theta(\epsilon) = \theta(\delta) \). Then \( \theta(\epsilon \cdot K + \delta (L+\epsilon)) = \theta(\epsilon) \)

unless \( \epsilon \cdot K \sim \delta (L+\epsilon), \text{ i.e., } \epsilon/\delta \sim -L/K \), since \( \epsilon \to 0 \), (9).

The division part of the theorem is obtained from the first part by substituting \( 1/g \) and \( 1/K \) for \( g \) and \( K \). \( \theta(1/g - 1/K) = \theta(g-K), \)

(8.3), proving part of what we want. The rest follows from the relation

\( \frac{1}{g} - \frac{1}{K} = \frac{g-K}{g \cdot K} \sim \frac{g-K}{g \cdot K^2} \), so that \( \frac{f-L}{g-K} \sim \frac{L}{K} \)

may be substituted for the relation

\( (f-L)/(1-g - 1/K) \sim -KL \), that one would obtain from the first part of the theorem. //

**Theorem 11.** 1) \( \theta(f) = r > 0 \Rightarrow \theta(f+c) = r \). If \( \theta(f) = 0 \), and

either \( f \to +\infty \), or \( f \) and \( c \) are both \( \geq 0 \), or \( f+c \) has order and \( f \not\to -c \), then \( \theta(f+c) = 0. \)

2) If \( f \) has order and \( a > 0 \), then \( \theta[f(ax + b)] = \theta(f) \).

Proof. 1) is an immediate corollary of (9). Proof of 2).

\( \lim \left( \frac{ax + b}{x} \right)^\alpha = \left[ \lim \left( \frac{ax + b}{x} \right)^\alpha \right] = a^\alpha \). Hence \( (ax + b)^\alpha \sim a^\alpha \cdot x^\alpha \). Case 1:

\( \theta(f) = r, \text{ finite}. \ x^{r-2\epsilon} < (ax + b)^{r-\epsilon} < f(ax + b)^{r+\epsilon} < x^{r+2\epsilon}, \)

\( \Rightarrow \theta[f(ax + b)] = r \). The other cases are similar. //

**Theorem 12.** If \( f \) and \( g \) have order, and their orders are not opposite infinities, then \( \theta(fg) = \theta(f) + \theta(g) \).

Proof. \( \frac{\log|fg|}{\log x} = \frac{\log|f|}{\log x} + \frac{\log|g|}{\log x} = \log|f| + \theta|g|, (7), \)

\( = \theta(f) + \theta(g), (6). \) Hence \( \theta|fg| = \theta(f) + \theta(g), (6) \). Now if either \( \theta(f) \) or \( \theta(g) = -\infty \), so is \( \theta(f) + \theta(g) \) and \( \theta|fg| = \theta|fg|, (6) \). Otherwise, \( f \) and \( g \), and hence, \( fg \), are of constant sign, so that the same result holds, (6). //
Theorem 13. If \( g \to \infty \) and \( f \) and \( g \) have order \( \neq 0 \) and \( \infty \), or \( \infty \) and \( 0 \), then \( \theta[f(g)] = \theta(f) \cdot \theta(g) \).

Proof. \[
\frac{\log|f(g)|}{\log x} = \frac{\log|f(g)|}{\log|g|} \cdot \frac{\log|g|}{\log x} \to \theta[f] \cdot \theta[g], (7),
\]
\[= \theta(f) \cdot \theta(g), (6). \] Hence \( \theta[f(g)] = \theta(f) \cdot \theta(g), (7). \) The proof continues as in (12). //

A more general version of this theorem will be discussed in Chapter VII.

Theorem 14. \( \theta(f) = r \iff \theta(f^\alpha) = \alpha r \), for all \( \alpha \).

Proof. \[
\frac{\log|f^\alpha|}{\log x} = \alpha \frac{\log|f|}{\log x} \to \alpha \theta[f] = \alpha \theta(f). \] The proof continues as in (12). //

Theorems 9, 12, and 13 embody the three principles that Hardy states as essential for any order notation, [15, p. 26]. (12) and (13) are proved by Cauchy, [3, p. 281], for finite \( r > 0 \).

Theorem 15. \( \theta(1/f) = -\theta(f) \), if \( f \) has order, (14). If \( f \) and \( g \) have order, not both \( \infty \) nor both \( -\infty \), then \( \theta(g/f) = \theta(g) - \theta(f), (12). \)

In order to talk about the integral of a function and its order, we adopt a definition given by Hardy, [15, p. 37]. We take the integral of \( f(x) \), written \( F(x) \), to be \( \int_{x}^{\infty} f(t)dt \), if that integral exists, and \( \int_{c}^{x} f(t)dt \) otherwise. \( f(x) = -F'(x) \) in the first case and \( F'(x) \) in the second. In the second case, \( F(x) \) is determined only up to an additive constant. By (11), this ambiguity has no effect on the order of \( F \).

Hardy has proved the following theorem, [15, p. 33].

Theorem 16. If \( f \) and \( g \) are \( \geq 0 \), then \( f = o(g) \implies F = o(G) \), and \( f \sim g \implies F \sim G. \)

Proof. Case 1: \( G(x) = \int_{x}^{\infty} g(t) \), convergent. Assume \( f = o(g) \).

For every \( \varepsilon \), there exists \( R \) such that for \( x > R \), \( f < \varepsilon g \). Hence for \( x > R \), \( F < \varepsilon G \), \( \implies F = o(G) \). If \( f \sim g \), \( (1-\varepsilon)g < f < (1+\varepsilon)g \), and
(1-\varepsilon)G < F < (1+\varepsilon)G, for x > R. Case 2: G(x) = \int_C^x g(t), divergent.

If \int_C^x f(x) is convergent, then f = o(g) and F = o(G). So suppose F(x) diverges and f = o(g). f < \varepsilon g for x > R, \Rightarrow \int_R^x f < \varepsilon \int_R^x g for x > R. Since \int_R^x g \to \infty with x, there exists x large enough so that \int_C^x f < \varepsilon \int_R^x g, \Rightarrow \int_C^x f < 2\varepsilon \int_R^x g < 2\varepsilon \int_C^x g. Now suppose f \sim g.

g(x) < (1+\varepsilon) f(x) for x > R, \Rightarrow \int_R^x g < (1+\varepsilon) \int_R^x f. For x large enough, \int_C^x g < \varepsilon \int_R^x f, \Rightarrow \int_C^x g < (1+2\varepsilon) \int_R^x f < (1+2\varepsilon) \int_C^x f. By alphabetic variation, \int_C^x f < (1+2\varepsilon) \int_C^x g for x large enough. // (Although this proof is basically similar to Hardy's proof, it is included for easy reference and compatibility of style.)

This theorem can be rewritten as follows:

Theorem 17. If f \geq 0 and if g/f \to a limit, (finite or infinite,) then \( G/F \to \lim g/f. \)

Proof. The theorem clearly follows from (16) if g > 0 or g < 0. Otherwise, let \( G_0 = \int |g| \). g/f must, in this case, \to 0, which implies that |g|/f \to 0, \Rightarrow G_0/F \to 0, by the first part. |G| < G_0, \Rightarrow G/f \to 0. // The theorem in this form was proved by Lettenmeyer, [27], for the case when f and g \to \infty.

Theorem 18. If \( \theta(f) = r \), finite or infinite, then \( \theta(F) = r+1 \). If, in addition, f' has order and f \to 0 or \infty, (which will automatically happen if r \neq 0,) then \( \theta(f') = r-1 \).

Proof. Case 1: r is finite. \( x^{r-\varepsilon} = o(f) \Rightarrow x^{r+1-\varepsilon} = o(F), (16) \).

Similarly, F = o(x^{r+1+\varepsilon}), so that \( \theta(F) = r+1 \). Case 2: r = \infty.
\( x^n = o(f), \Rightarrow x^{N+1} = o(F), (16), \Rightarrow \theta(F) = \infty. \) Case 3: r = -\infty.
\( f/x^{-\varepsilon} \to 0, \Rightarrow F/x^{-N+1} \to 0, (17), \Rightarrow \theta(F) = -\infty. \) The second statement of the theorem is an obvious consequence of the first. //

We require that f \to 0 or \infty so that it can fit the role of F in
the first statement. When (16) or (17) are applied, we must recall that 
F and G have been defined in such a way that they cannot either ap-
proach a constant.

The following theorems show some other ways in which the order con-
cept we have been using can be defined.

**Theorem 19.** \( \Theta(f) = r \implies \text{lub } \{\rho; \int_{a}^{\infty} x^{\rho} f(x)dx \text{ converges} \} = -(r+1) \).

**Proof.** Let \( g(x, \rho) = x^{\rho} f(x) \). Case 1: \( r \) is finite. Let \( \rho \) be
< -(r+1). \( \Theta(g) = r + \rho < -1, (12), \implies \Theta(G) < 0 \), and the integral con-
verges. \( \Theta(G) > 0 \) if \( \rho > -(r+1) \), \implies the integral diverges. Case 2:
\( r = \pm \infty \). Then \( \Theta(g) = \pm \infty \), \implies \Theta(G) = \pm \infty \).

**Example 20.** We have shown that the integral converges for \( \rho < -(r+1) \).

If \( \rho = -(r+1) \), the integral can either converge or diverge. For
example, if \( f(x) = x^{2} \), then \( \int_{a}^{\infty} x^{-(r+1)} f = \int_{a}^{\infty} x^{-3} x^{2} = \infty \). But if
\( f(x) = x^{2} / (\log x)^{2} \), then \( G = \int_{a}^{\infty} 1 / x (\log x)^{2} = -1 / \log x \mid_{a}^{\infty} = 1 / \log a \).

**Example 21.** The converse of Theorem 19 does not hold. If it did, any
positive function would have order, since \( \int_{a}^{\infty} x^{\rho} f \) is an increasing
function of \( \rho \), and there must therefore be a l.u.b., finite or infinite,
for \( \{\rho; \int_{a}^{\infty} x^{\rho} f \text{ converges.} \} \). The following is an example of a function,
f, for which \( \int_{a}^{\infty} x^{\rho} f \text{ converges for } \rho < -(r+1) \), diverges for
\( \rho > -(r+1) \), and yet does not have order. Let \( f(x) = x^{-2} \) with little
symmetrical triangles of height 1 and area \( 2^{-n} \) added at the points,
x = an integer, \( n \). \( \int_{x}^{\infty} x^{\rho} f(x)dx = \int x^{\rho-2} + \sum_{n=1}^{\infty} 2^{-n} \). The second term con-
verges for all \( \rho \). Hence, \( \int x^{\rho} f \text{ converges for } \rho < 1 \text{ and diverges for }
\rho > 1 \). Yet the function does not have order.

**Example 22.** From the preceding example can be derived an example of a
monotonic function, \( g \), which has order while its derivative, \( g' \), does
not. Let \( g(x) = \int_{x}^{\infty} f(t)dt \), where \( f \) is defined in Example 21.
g'(x) = f(x) and does not have order. \( g(x) = \int_{x}^{\infty} t^{-2} dt + \sum_{n=1}^{\infty} 2^{-n} \),
\[ x^{-1} + 2^{-n}[x]. \quad x^{-1-\epsilon} < g(x) < x^{-1+\epsilon}, \] so \( g \) has order. //

**Theorem 23.** \( \theta(f) = r > 0 \implies \text{lub} \{\rho; \int_{a}^{\infty} f^\rho \text{ converges} \} = -1/r. \quad r < 0 \implies \text{glb} \{\rho; \int_{a}^{\infty} f^\rho \text{ converges} \} = -1/r. \)

Proof. Let \( r > 0 \) and \( \rho > -1/r. \) Let \( g(x,\rho) = [f(x)]^\rho. \) \( \theta(r^\rho) = \rho r > -1, (14), \implies \theta(g) > 0 \) and diverges. If \( \rho < -1/r, \) then \( \theta(g) < 0 \) and converges. Similarly for \( r < 0. // \)

The following theorem is similar to one proved by Hardy for \( L\)-functions, namely Formula 33 later in this chapter.

**Theorem 24.** If \( f' \) is continuous, and \( xf'(x)/f(x) \to \) a limit, \( r, \) then \( \theta(f) = r. \) If, in addition, \( r \neq 0 \) or \( -\infty, \) then \( \theta(f') = r-1. \)

Proof. Case 1: \( f \geq 0 \) [or \( f \leq 0 \)]. \( \frac{f'}{f} \to r \implies \log f/\log x \to r, (17), \implies \theta(f) = r, (7). \) The only case where (17) doesn't work is where \( \log f \to \) a constant, \( c. \) Then \( \frac{\log f - c}{\log x} \to r \) by (17), \( \implies r = 0 = \theta(f), \) because \( f \to e^c. \) Case 2: \( f \) alternates in sign. Let \( g = |f|. \) At all points, \( x, \) where \( f(x) \neq 0, \) \( xg'(x)/g(x) = xf'(x)/f(x). \) At every point, \( x_0, \) where \( f(x_0) = 0, \) the definition of \( x_0 f'(x_0)/f(x_0) \) depends on the value of \( x_0 f'(x_0)/f(x_0) \) at points in a neighborhood of \( x_0 \) where \( f(x) \neq 0. \) At such points, \( xg'(x)/g(x) = xf'(x)/f(x). \) Hence, \( x_0 f'(x_0)/f(x_0) = xg'(x_0)/g(x_0). \) Furthermore \( f'(x_0) \) must be \( 0. \) (If it were \( > 0, \) say, there would exist a neighborhood of \( x_0 \) where \( f'(x) > 0, \) so that \( f(x) > 0 \) and \( < 0 \) for points in every sub-neighborhood. Hence, \( xf'(x)/f(x) \to \infty \) and \( -\infty \) as \( x \to x_0, \) a contradiction.) Thus, \( g'(x) \) is continuous at \( x_0, \implies \theta(g) = r, \) (Case 1), \( \implies \theta(f) = r, (6). \)

If \( r \) is finite and \( \neq 0, \) then \( xf' \sim rf, \) so that \( \theta(f') = r-1. \)

If \( r = \infty, \) then \( f = o(xf'), \) and \( xf' \) must be of constant sign, so that \( \theta(xf') = \theta(f') = \infty. // \)

**Example 25.** If \( r = -\infty, \) however, \( f \) could be \( e^{-x} \) with little \( 45^\circ \) slides tucked in at points, \( n, \) of length \( f(n)/2, \) so that \( f(n) \)
\[ e^{-x}/x^2. \] The corners can be rounded to make \( f' \) continuous, but \( f'(x) = -1 \) on the slides and \( e^{-x} \) elsewhere, and so does not have order. Thus \( xf'/f \to -\infty \), but \( f' \) does not have order. //

**Example 26.** Let \( f' \) vary between \( x^{-5} \) and \( x^{-6} \) like Example 1. Let \( f(x) = -\int_0^x f'(t)dt + 1 \). Then \( \theta(f) = 0 \) and \( xf'/f \to 0 \), yet \( f' \) doesn't have order. //

**Theorem 27.** If \( f'(xf') \to \) a limit, \( s \), and \( f' \) is continuous, then \( xf'/f \to 1/s \). (1/0 is interpreted to mean \( \infty \) or \( -\infty \).)

Proof. \( f'(xf') \to \) a limit, \( s \), \( s \neq 0 \) \( \implies \) \( xf'/f \to 1/s \). To complete the proof, we suppose that \( f'(xf') \to 0 \). There exists \( R \) such that for \( x > R \), \( f(x)/[xf'(x)] \) is defined and \( < 1 \) in absolute value. Suppose there exist two points, \( x_1 \) and \( x_2 > R \), where \( f(x) = 0 \). Then, by Rolle's theorem, either \( f(x) \equiv 0 \) between \( x_1 \) and \( x_2 \), which is impossible because \( xf(x)/f'(x) \) was assumed to be defined there, or there exists a point, \( x_3 \), between them where \( f'(x_3) = 0 \) but \( f(x_3) \neq 0 \). Then \( f(x)/[xf'(x)] \) becomes unbounded at \( x_3 \), a contradiction. So there exists at most one point \( > R \) where \( f(x) = 0 \). Now suppose there exist points, \( x_4 \), and \( x_5 \), greater than that point and with \( f'(x_4) > 0 \) and \( f'(x_5) < 0 \). Then, by continuity, there exists \( x_6 \) between them where \( f'(x_6) = 0 \). But here \( f(x_6) \neq 0 \), so \( f(x)/[xf'(x)] \) becomes unbounded again in a neighborhood of \( x_6 \), a contradiction. Thus, we have \( f(x) \) and \( f'(x) \) of constant sign for \( x \) large enough. Hence, if \( xf'/f \to 0 \), then \( f'(xf') \to \infty \) or \( -\infty \). //

**Example 28.** Let \( f(x) = 1 + e^{-x} \sin x \). \( f'(x) = e^{-x}(\cos x - \sin x) \). \( xf'/f \to 0 \), but \( f'(xf') \) doesn't \( \to \) anything. In fact, every real number and \( \pm \infty \) are limit points. // This is also a counter-example for the next theorem.

**Theorem 29.** If \( f' \) is continuous and \( xf'/f \to \) a non-0 limit, then \( f \) is monotonic and \( f'>0 \) or \( < 0 \).
Proof. \( xf'/f \to r \), finite, \( \Rightarrow \theta(f) = r \) and \( \theta(f') = r-1 \), (24), \( \Rightarrow \) the theorem. \( xf'/f \to +\infty \Rightarrow f/(xf') \to 0 \), \( \Rightarrow \) the theorem by the proof of (27). //

The following theorem in the case when \( t = l \) is similar to one proved by Hardy for L-functions, namely Formula 33 later in this chapter.

**Theorem 30.** 1) If \( f'' \) is continuous, and \( ff''/f'^2 \to \) a limit, \( t \), then \( f/(xf') \to 1-t \), \( xf'/f \to 1/(1-t) \), and \( \theta(f) = 1/(1-t) \).

We can simplify the notation here by writing \( v \) for \(-f/f'\). \( v' = ff''/f'^2 - 1 \). The theorem says that if \( v' \to \) a limit, then \( f \) has order \( r \), and \( v' \to -\frac{1}{r} \).

2) If \( f \) and \( f' \) have order and are of constant sign, and if \( \log|f'|/\log|f| \to \) a limit, then that limit is \( (r-1)/r \). This is considered to be \( 1 \) if \( r = +\infty \), and \( \infty \) if \( \theta(f) = 0 \).

**Proof of 1.** Let \( v' \to s \), finite or infinite, then \( v/x \to s \) by (17), \( \Rightarrow \theta(f) = -1/s \) by (27) and (24). The only case when one can't use (17) is when \( v \to a \) constant, \( c \). Then by (17), \( (v-c)/x \to s \). But in this case, both \( (v-c)/x \) and \( v/x \to 0 \), so that the theorem holds in this case, too. //

**Proof of 2.** Let \( r \) be finite.

Let \( L = \lim \frac{\log|f'|}{\log|f|} = \lim \frac{\log|f'|}{\log x} \frac{\log|f|}{\log x} = \frac{r-1}{r} \), for \( r \neq 0 \), and \( \to +\infty \) for \( r = 0 \), (7). Now let \( r = \infty \). Case 1: \( L \) is finite.

Suppose \( L \neq 1 \). \( \log f'/\log f \to L \), supposing for simplicity that \( f > 0 \), \( \Rightarrow f^{L-\epsilon} = o(f') \) and \( f' = o(f^{L+\epsilon}) \), where \( \epsilon < |1-L| \),

\( \Rightarrow f^{-L-\epsilon} = o(1/f') \) and \( L/f' = o(f^{-L+\epsilon}) \),

\( \Rightarrow f^{-L-\epsilon} f' = o(1) \) and \( L = o(f^{-L+\epsilon} f') \),

\( \Rightarrow f^{(1-L)-\epsilon} < x < f^{(1-L)+\epsilon} \) by (16), a contradiction, because both sides have order \( \infty \) or both \( \infty \).
Case 2: $L = \infty$. $\log f'/\log f \to \infty$. \( f^R = o(f') \), $\Rightarrow 1/f' = o(f^{-R})$, $\Rightarrow 1 = o(f^{-R}f')$, $\Rightarrow x = o(f^{-R+1})$ by (16), a contradiction. The proof for the case $\theta(f) = -\infty$ is precisely analogous. //

Theorem 31. If \( f'' \) is continuous and if $ff''/f'^2 \to$ a non-0 limit, \( t \), then \( f \) and \( f' \) are monotonic and \( f'' > 0 \) or \( < 0 \). If \( t = 0 \), then \( f \) is monotonic and \( f' > 0 \) or \( < 0 \). If \( t \) is finite, (including 0), then \( xf''/f' \to t/(1-t) = \theta(f') \). If \( t \not= 0, 1, \) or \( \pm \infty \), then $\theta(f'') = 2t-1$. \[ \frac{xf''}{f'} = \frac{t}{1-t} = \theta(f') \]

Proof. Case 1: \( t \not= 0, 1, \) or \( \pm \infty \). \( xf''/f' = (ff''/f'^2) (xf'/f) \).

The first factor $\to t$, the second, $1/(1-t)$, $\Rightarrow xf''/f' \to t/(1-t)$, so that $f$, $f'$, and $f''$ have finite orders, \( r, r-1, \) and \( r-2 \), (24).

Case 2: \( t = 0 \) or \( 1 \). If \( t = 0 \), then \( xf''/f' \to 0 \). If \( t = 1 \), \( xf'/f \to 0 \) or $\infty$, (27, 30), $\Rightarrow xf''/f' \to$ the same limit. In either case, and in the previous case, $\theta(f') = t/(1-t)$. If, in addition, \( t \not= 0 \), then \( f \) and \( f' \) are monotonic and \( f'' > 0 \) or \( < 0 \), (29).

When \( t = 0 \), \( \theta(f') = 0 \), so that \( f' > 0 \) or \( < 0 \), and \( f \) is therefore monotonic. Case 3: \( t = \pm \infty \). Suppose there exists an infinite sequence of points, where \( f' = 0 \). For every \( R \), there exists \( x_1 > R \) such that \( f'(x_1) \not= 0 \), for if not, \( f'(x) \equiv 0 \) for \( x > R \), and so $ff''/f'^2$ is not defined there. The set, \( \{ x; f'(x) = 0 \} \), is closed by continuity.

Let \( x_2 \) be the greatest point of that set < \( x_1 \), and \( x_3 \) be the least point > \( x_2 \). By Rolle's theorem, there exists \( x_4 \) between \( x_2 \) and \( x_3 \) where \( f''(x_4) = 0 \), and we have seen to it that \( f'(x_4) \not= 0 \). So $ff''/f'^2 = 0$ at \( x_4 \). This contradicts the assumption that $ff''/f'^2 \to \pm \infty$. Thus, \( f' > 0 \) or \( < 0 \). Hence, for \( x \) large enough, $ff''/f'^2 = 0$ where \( f'' \) does. Hence, there cannot be a sequence of points where \( f'' = 0 \) that $\to \infty$, or else $ff''/f'^2$ would not $\to \pm \infty$. So by the continuity of \( f'' \), \( f' > 0 \) or \( < 0 \). //
\( f' \) in Example 28, one gets an example where \( ff''/f''^2 \to 0 \), \( f'' \) alternates in sign, and \( f' \) is not monotonic.

**Example 32.** We can construct a function for which \( ff''/f''^2 \to \infty \) and for which \( f' \) does not have order. This is also an example of a function whose derivative is monotonic and which doesn't have order. The general idea is to let \( g(x) \) vary between \( x^{-5} \) and \( x^{-6} \) in such a way that \( |g'(x)| > x^{-8} \) and is monotonic. Then we can let \( f'(x) = -g(x) \), \( f''(x) = -g'(x) \), and \( f(x) = 1 - \int_x^\infty f'(x) \, dx \). \( ff'' \) is then \( > x^{-8} \) while \( f''^2 < x^{-10} \), so that \( ff''/f''^2 \to \infty \), but \( f' \) does not have order.

Let the graph of \( g \) be a tangent to \( x^{-5} \) at \((1,1)\) until that line hits \( x^{-6} \) at \( x_2 \). Then \( g(x) \) becomes \( x^{-7} + (x_1^{-6} - x_1^{-7}) \) until the tangent to that curve becomes tangent to \( x^{-5} \). Then \( g(x) \) follows the tangent back to \( x^{-6} \) again at \( x_2 \), and the process is repeated. \( x^{-7} + (x_1^{-6} - x_1^{-7}) \) crosses \( x^{-5} \), so the tangent must hit \( x^{-5} \) tangentially sooner or later. It may be that the tangent at \( x_1 \) crosses \( x^{-5} \). In that case, \( g(x) \) follows a straight line from \( x_1 \) tangent to \( x^{-5} \) and back to \( x^{-6} \) again. Thus, \( g'(x) \) is always as steep as or steeper than the tangent to \( x^{-7} + (x_1^{-6} - x_1^{-7}) \) which = \(-7x^{-8}\). To make \( g' \) continuous, we may round off the corners at \( x_1, x_2, \ldots \) with very small circular arcs. //

The definitions of order embodied in (19), the Cauchy definition, (24), and (30) consecutively narrow down the class of functions possessing order to those having more and more desirable monotonic properties. Hardy, [15, Ch. 3], defines a very restrictive class of very well-behaved functions that he calls L-functions. These are functions that can be formed from constants and the variable \( x \) by a finite number of the following operations: +, -, \( x \), \( x^\frac{1}{n} \), exp, and log. He has proved that all L-functions are ultimately monotonic, and that the class has the
pleasing property of being closed under differentiation. It follows that any algebraic combination of L-functions and their derivatives is an L-function, and \( \to \) some limit, as \( x \to \infty \).

Hardy has proved a theorem, [15, Th. 25], about L-functions, similar to (24) and (30) and put in the form of an integral approximation.

**Formula 33.** (Hardy) If \( f \) is an L-function, then \( \Theta(f) = r \), finite and \( \not \infty \Rightarrow F \sim xf/|r+1| \), and \( \Theta(f) = \pm \infty \Rightarrow F \sim f^2/f' \). (We can see by (24) and (30) that this theorem also holds if \( xf/F \) and \( ff'/f^2 \to \) limits.)

The second formula has the virtue of being independent of location. That is, if \( g(x) = f(x+c) \), we get the same approximation for \( G(x) \) as we did for \( F(x+c) \). This approximation, is also the first term of the Laplace-Winckler expansion, to be considered in Chapter V.

We may use Theorem 30 to extend the formula to the case where \( r \) is finite. We suppose that \( Ff'/f^2 \to \) a limit, \( t \), and that \( F \to \infty \), so that \( f = F' \). Then \( \Theta(F) = \frac{1}{\infty-t} = r + 1 \). Thus, \( t = l - \frac{1}{r+1} = \frac{r}{r+1} \). If \( F \to 0 \), the same derivation holds for \(-F\). Thus, we have

**Formula 34.** If \( \Theta(f) = r \) and \( Ff'/f^2 \to \) a limit, then \( F \sim \left| \frac{r}{r+1} f^2/f' \right| \).

(\( \frac{r}{r+1} \) is considered to be 1 if \( r = \pm \infty \).)

If we wish to apply Formulas 33 and 34 to functions that are not L-functions, we need some easy tests of whether \( xf/F \) and \( Ff'/f^2 \to \) limits. Such tests are provided by the next two theorems.

The first was proposed by Ostrowski, [31], as a problem which was solved by Doetsch, [8].

**Theorem 35.** If \( xf'/f \to r \), finite or infinite, then \( F \sim xf/|r+1| \).

Proof. \( xf'/f \to r \Rightarrow \frac{xf'+f}{f} \to r+1 \), \( \Rightarrow xf/F \to r+1 \), (17), unless \( xf \to \) a constant, \( c \neq 0 \). In that case, \( r = -1 \), and \( f \sim c/x \), \( \Rightarrow F \sim c \log x \), (16), \( \Rightarrow xf/F \to 0 \).
Theorem 36. If \( f f''/f'2 \to t \), finite or infinite, then \( F \sim |(f^2/f')/(2-t)| \).

Proof. \( f \) and \( f' \) are of constant sign and \( \neq 0 \) for large \( x \), (31).
\[
2 - \frac{f f''}{f'2} = \frac{f'2 f'' - f^2 f''}{f'2} f = \frac{f^2}{f'} f' \to 2 - t, \quad \Rightarrow f^2/(f'F)
\]
\( \to 2 - t \), (17), \( \Rightarrow F f'/f^2 \to 1/(2 - t) \). The constant signs take care of the case \( t = 2 \). The only case when (17) doesn't work is when \( f^2/f' \to c \neq 0 \). Let \( \theta(f) = r \), (30). \( \theta(f^2) = 2r \), \( \Rightarrow \theta(f') = 2r \), since
\( f^2 \sim cf' \), \( \Rightarrow r \neq 0 \), \( \Rightarrow \theta(f') = r-1 \), (18), \( \Rightarrow 2r = r-1 \), \( \Rightarrow r = + \infty \) or \( -1 \). If \( F(x) \to \infty \) or \( 0 \), then \( F f'/f^2 \to \infty \) or \( 0 \), because
\( f'/f^2 \to 1/c \), \( \Rightarrow \theta(F) = 0 \) or \( 1 \), respectively, (30). But \( \theta(F) = r+1 \), (18), \( = + \infty \) or \( 0 \). Thus, \( \theta(F) = 0 \), \( \theta(f) = -1 \), \( t = 2 \), and
\( F f'/f^2 \to \infty \) in accordance with the theorem. //

The previous classification of functions by the Cauchy definition of order is sufficient for many purposes of this dissertation. For some applications, however, it is desirable to subdivide the class of functions having infinite order.

We will use as a scale, functions of the type,
\[
\ldots e^{e^{e^x}}, e^{e^x}, e^x, x, e^{-x}, e^{-e^x}, e^{-e^{e^x}}, \ldots,
\]
defining the orders of these functions to be
\( (3,r), (2,r), (1,r), (0,r), -(1,r), -(2,r), \) and \( -(3,r) \).

A scale of functions is a notion introduced by Hardy. [15, Ch. 2], and refers to a set of functions ordered by the relation, \( f = o(g) \). This is closely connected to the notion of order, since if \( f_\alpha \) is a member of the scale, we can say that a function, \( g \), has order \( \alpha \) if
\( \theta < \) any member of the scale \( \nabla \) and \( \theta \) any member of the scale \( \nabla \).

(The symbols, \( > \) and \( < \), here refer to the ordering of the scale.) Hardy has suggested that two useful scales would be \( \{ \alpha \} \), which underlies the first part of this chapter, and the one referred to above, which underlies the remaining part.

**Definition.** Let \( r \) be a finite real number > 0, and let \( n \) be an integer > 0. \( \theta(f) = (n,r) \) means that \( \theta(n^{\text{th}} \log \text{ of } f) = r \).

\( \theta(f) = (-n,-r) \), also written, \( -(n,r) \), means that \( \theta((n-1)^{\text{st}} \log \text{ of } -\log f) = r \). \( \theta(f) = (0,r) \) means that for some finite \( r \), including possibly negative \( r \), \( \theta(f) = r \). \( \theta(f) \), which coincides with \( \theta(f) \) in this case, is considered finite. \( \theta(f) = (+1,0) \) means that \( \theta(f) = +\infty \) and \( \theta(\log f) = 0 \). \( \theta(f) = (n,0) \) means that \( \theta((n-1)^{\text{st}} \log f) = (1,0) \). \( \theta(f) = (-n,0) \) means that \( \theta(-\log f) = (n-1,0) \).

Example 36 shows a function with \( \theta(f) = -(1,0) \). If \( \theta(f) = (n,r) \), we will say that \( f \) has extended order.

Hardy, [15, p. 8], quotes a theorem of DuBois-Reymond showing that given any countable scale of functions, there exist functions whose increase is greater than any function on the scale. Hence, there are functions whose extended order is \( (+\infty, r) \), so to speak. Hardy constructs such a function by joining the points \( (n, \exp^n(n)) \) together by straight lines. We will not discuss such functions, since they have no apparent application to statistics, but will confine ourselves to functions having extended order.

If a function has extended order, it has order. The converse only holds if the function has finite order.

We have defined extended order in such a way that when \( \theta(f) = (n,r) \), \( n \) and \( r \) have the same sign.

---

1. \( \theta \) is a typewriter version of capital theta.
Definition. \( \theta(f) = \theta(g) \) means that \( \theta(f) = (n,r) = \theta(g) \). \( \theta(f) > \theta(g) \) means that \( \theta(f) = (n,r) \), \( \theta(g) = (m,s) \), and either \( n > m \), or \( n = m \) and \( r > s \).

**Theorem 37.** \( \theta(f) < \theta(g) \implies f < g \) and \( f = o(g) \). If \( f < g \) and they have extended order, then \( \theta(f) \leq \theta(g) \).

The proof is direct. 

We have defined extended order for \( f > 0 \). We now define \( \theta(-f) \) to be \( \theta(f) \). If \( f = 0 \) indefinitely, no log of \( f \) can have finite order. Hence, we may state

**Theorem 38.** If \( f \) has extended order, and is continuous, then \( f > 0 \) or \( f < 0 \).

In the following theorems, we assume for clarity that \( f > 0 \).

**Theorem 39.** If \( n > 1 \), or \( n = 1 \) and \( r > 0 \), then \( \theta(f) = \pm(n,r) \)

\[ \iff \theta(\log f) = (n-1,r) \]

This follows directly from the definition. 

Definition. Let \( e(n,r) \) stand for any function, (such as the scale function, \( \exp^n(x^r) \)) having order \( (n,r) \).

**Theorem 40.** If, for all \( \epsilon > 0 \), \( e(n,r-\epsilon) < f < e(n,r+\epsilon) \), then \( \theta(f) = (n,r) \).

Proof. Let \( n \) be \( > 0 \). \( x^{r-\epsilon} < n^{th} \log f < x^{r+\epsilon} \), \( \implies \theta(f) = (n,r) \). Similarly for \( n < 0 \).

**Theorem 41.** \( \theta(f) \geq \theta(g) \implies \theta(f+g) = \theta(f) \). \( \theta(f) > \theta(g) \)

\[ \implies \theta(f-g) = \theta(f) \]. \( \theta(f) = \theta(g) \) and \( f-g \) has extended order

\[ \implies \theta(f-g) \leq \theta(f) \]. If, in addition, \( f \) is not \( \sim g \), then \( \theta(f-g) = \theta(f) \).

The proof is just the same as for (9), with the Cauchy definition replaced by Theorem 40.
Theorem 42. If \( \Theta(f) \) and \( \Theta(g) \) are not both finite, and if
\(|\Theta(f)| > |\Theta(g)|\), or \( \Theta(f) = \Theta(g) \), then \( \Theta(fg) = \Theta(f) \). If \( \Theta(f) = -\Theta(g) \),
and \( fg \) has extended order, then \(|\Theta(fg)| < |\Theta(f)|\).

Proof. We apply (41) to \( \log f + \log g \). //

Theorem 43. \( \Theta(f) = (n,r) \) and \( n \neq 0 \Rightarrow \Theta(f^\alpha) = (n,r) \) for \( \alpha > 0 \),
and \( \Theta(f) = -(n,r) \) for \( \alpha < 0 \).

Proof. \( \Theta(\log f^\alpha) = \Theta(\alpha \log f) = \Theta(\log f) \). (39) completed the proof. //

Theorem 44. \( f \) has extended order and \( |\Theta(f)| = \infty \Rightarrow \Theta(f) = \Theta(f) \).

Proof. Let \( e(n,r) \) be a scale function with \( n \) and \( r > 0 \).
\( e'(n,r) = e(n,r) \cdot e(n-1,r) \cdot \ldots \cdot e(1,r) \cdot r^{x-1} \). By (42), \( \Theta[e'(n,r)] = (n,r) \Rightarrow e'(n,r+\epsilon) = o(f) \) and \( f = o[e'(n,r+\epsilon)] \Rightarrow e(n,r-\epsilon) = o(F) \) and \( f = o(e(n,r+\epsilon)) \), (16). (40) completes the proof. //

Theorem 45. If \( f \) has extended order and \( a > 0 \), then \( \Theta(f(ax+b)) = \Theta(f) \).

The proof follows from (11.2) and (40). //

Theorem 46. \( \Theta(f/F) = (n,r) \) and \( n > 0 \Rightarrow \Theta(F) = \Theta(f) = \pm (n+1,r) \).
\( \Theta(f/F) = (0,r) \) and \( r > -1 \Rightarrow \Theta(F) = \Theta(f) = \pm (1,r+1) \). This is also true if \( r = -1 \) and \( \Theta(f) = \pm \infty \).

Proof. Suppose \( n \neq 0 \). \( \Theta(f/F) = \Theta(\log F) \), (44), = (n,r),
\( \Rightarrow \Theta(F) = \pm (n+1,r) \), (39). \( \Theta(f) = \Theta(F \cdot f/F) = \Theta(F) \), (42). The proof of the second case is similar. //
CHAPTER III

ERRORS AND BOUNDS

The usefulness of any integral approximation depends not only on the
closeness of the approximation, but also on whether there exist close
bounds for the error involved. In this chapter, it will be shown that
there exist useful and simple bounds for any approximation, \( F_1(x) \), to a
tail integral, \( F(x) \). The relation between two kinds of error is discussed,
and the general properties of an asymptotic sequence of approximations
derived.

The purpose of this dissertation is to discuss approximations that
get increasingly accurate as \( x \to \infty \). When we seek a measure of accuracy,
it will not do to choose the absolute error, \( F_1 - F \), since if \( F \to 0 \),
any function, \( F_1 \), approaching 0 will serve as an increasingly accurate
approximation to \( F \). Rather, we should consider the relative error.

**Definition.** By the relative error of \( F_1 \), written \( E(F_1) \) or simply,
\( E_1 \), is meant \( (F_1 - F)/F \).

\( E_1 \) is not known in cases of interest. Otherwise, \( F = F_1/(1+E_1) \),
would then be known exactly and there would be no need to approximate.
However, a similar error is obtainable.

**Definition.** By the relative frequency error of \( F_1 \), written, \( e(F_1) \), or
simply, \( e_1 \), is meant \( (f_1 - f)/f \); where \( f_1 = -F_1' \), to conform to the
relation, \( f = -F' \). \( f_1 = +F_1' \) if \( F \to \infty \).

It follows from these definitions that \( F_1 \sim F \iff E_1 \to 0 \), that
\( f_1 \sim f \iff e_1 \to 0 \), and that if \( f > 0 \) and \( F_1 = F/\) a constant, then
\( e_1 \to 0 \implies E_1 \to 0, \) \((17)\). If \( E_1 \to 0 \), we will call \( F_1 \) asymptotic.

We can immediately obtain the following bounds, which are especially useful if \( F_1 \) is asymptotic.

**Bounds \(47\).** If \( f(x) > 0 \), and \( F_1 \) and \( F \to 0 \), then
\[
\min e_1(x) \leq E_1(a) \leq \max e_1(x), \quad \text{for} \quad x \geq a.
\]
Hence, if \( e_1(x) \to 0 \) monotonically for \( x \geq a \), then \( E_1(a) \) is bounded by \( e_1(a) \).

This theorem, written in the following form, can be derived from a proof by Lettenmeyer, [27], of a theorem similar to \((17)\).

**Bounds \(48\).** If \( f_1(x) > 0 \), and \( F_1 \) and \( F \to 0 \), and \( b_1 \leq f(x)/f_1(x) \leq b_2 \) for \( x \geq a \), then
\[
b_1 F_1(a) \leq F(a) \leq b_2 F_1(a).
\]

Bounds \(47\) and \(48\) apply equally well to proper integrals, replacing \( F(a), F_1(a) \), \( E_1(a) \), and \( x \geq a \), by \( F(a,b), F_1(a,b) \), \( E_1(a,b) \), and \( a \leq x \leq b \), respectively. \( F_1(a,b) \) is an approximation to \( F(a,b) \) in the form of a known integral, from \( a \) to \( b \), of \( f_1(x) \). The bounds should be used in this proper integral form if \( F(x) \to \infty \).

The reasoning of \((47)\) and \((48)\) was used by Williams, [57], in discussing an approximation to the normal distribution.

**Proof of \(48\).** \( b_1 f_1(x) \leq f(x) \leq b_2 f_1(x) \) for \( x \geq a \). Hence, the same inequality holds for the integral from \( a \) to \( \infty \). // To prove \((47)\), we interchange the roles of \( f \) and \( f_1 \). \( \min e_1(x) \leq f_1(x)/f(x) - 1 \leq \max e_1(x) \) for \( a \leq x \), \( \implies \) \( \min e_1(x) + 1 \leq f_1(x)/f(x) \leq \max e_1(x) + 1 \),
\[
\implies \min e_1(x) + 1 \leq F_1(a)/F(a) \leq \max e_1(x) + 1, \quad (48), \quad \implies \min e_1(x) \leq E_1(a) \leq \max e_1(x). \quad //
\]

If a good asymptotic formula, \( F_1 \), is known, it can be combined with mechanical quadrature to get an error as small as we like. We use a quadrature formula, \( F_2 \), from \( a \) to \( b \), and \( F_1 \) from \( b \) to \( \infty \), calling the combined approximation, \( F_3 \).
\[ E_3 = \frac{F_2(a,b) + F_1(b) - F(a,b) - F(b)}{F(a,b) + F(b)} = \frac{F(a,b) E_2(a,b) + F(b) E_1(b)}{F(a,b) + F(b)} \]

\[ \leq \frac{|E_2(a,b)| + \frac{F(b)}{F(a)} |E_1(b)|}{F(a,b) + F(b)}. \]

We may presume that \( E_2 \) is very small, and that \( F(b) \) is considerably less than \( F(a,b) \) in the case when \( F(x) \to 0 \), so that little accuracy was lost in this step. The last expression

\[ = \frac{|E_2(a,b)| + \frac{F_1(b)}{F_3(a)} \cdot \frac{1 + E_2(a)}{1 + E_1(b)} \cdot |E_1(b)|}{F_3(a)} \cdot E_1(a) \]

will tend to have the same sign as \( E_1(b) \) and be smaller. Hence, \( [1+E_3(a)]/[1+E_1(b)] \) will be between 1 and \( 1/[1+E_1(b)] \), and we get

Bounds 49. If \( F(x) \) and \( F_1(x) \) are positive and \( \to 0 \), then \( E_1(x) \)

\[ \geq 0 \text{ for } x \geq b \implies |E_3(a)| \leq |E_2(a,b)| + \frac{F_1(b)}{F_3(a)} \cdot E_1(b); \text{ and } E_1(x) \leq 0 \]

for \( x \geq b \implies |E_3(a)| \leq |E_2(a,b)| + \frac{F_1(b)}{F_3(a)} \cdot |E_1(b)|. \]

Upper and lower bounds for \( E_1(b)/[1+E_1(b)] \) can be obtained by substituting upper and lower bounds for \( E_1(b) \) in that expression, which is a monotonic function of \( E_1(b) \) since \( E_1(b) > -1 \). If \( E_1(b) \) is very small, the factor \( 1+E_1(b) \) can be neglected. The formula increases in accuracy two ways when \( b \) increases: \( E_1(b) \to 0 \) and \( F_1(b)/F_3(a) \to 0 \).

Bounds 49 apply to an \( F(x) \to \infty \), if \( F_3(a,c), E_2(a,c), F_1(b,c) \) and \( E_1(b,c) \) are substituted for \( F_3(a), E_3(a), F_1(b) \) and \( E_1(b) \).

Theorem 50. Let \( f \) be \( > 0 \).

1) \( g/f \) is monotonic \( \implies \int_c^x g/f \) is monotonic.

2) \( g/f \) is monotonic \( \implies \int_x^\infty g/f \) is monotonic, provided that both integrals converge.
3) Let \( e_1 \) be \( g/f \) and \( E_1 \) be \( \int_x^\infty g/\int_x^\infty f \) if both integrals converge, and be \( \int_c^x g/\int_c^x f \) otherwise. Then \( e_1 \to 0 \) monotonically \( \implies E_1 \to 0 \) monotonically.

Proof of 2). Suppose \( G \) and \( F \) converge.

\[
\frac{G}{F} = \frac{\int_a^b g + \int_\infty^b g}{\int_a^b f + \int_\infty^b f} = \frac{\int_a^b f \cdot \frac{g}{f} + \int_\infty^b f \cdot \frac{g}{f}}{\int_a^b f + \int_\infty^b f} = \frac{e_1(x_1) F(a,b) + e_1(x_2) F(b)}{F(a,b) + F(b)}
\]

where \( a \leq x_1 \leq b \leq x_2 \). This is a weighted average of \( e_1(x_1) \) and \( e_1(x_2) \). Hence, sign \( [e_1(x_1) - e_1(x_2)] = \text{sign} [E_1(a) - E_1(b)] \),

= sign \( [E_1(a) - E_1(b)] \), since \( e_1(x_2) = E_1(b) \). Thus, the monotonicity of \( e_1(x) \) implies that of \( E_1(x) \). The proof of 1) is analogous. To prove 3), we infer from 1) and 2) that \( E_1 \) is monotonic, and from (17) that \( g/f \to 0 \implies G/F \to 0 \). //

This theorem is useful for extending tables beyond the range for \( a \). If one knows \( E_1(a_o) \), say, and can verify that \( e_1(x) \) monotonically \( \to 0 \) for \( x \geq a_o \), \( E_1(x) \) is bounded by \( E_1(a_o) \). We will use this theorem extensively in Chapter IX for bounding the errors of approximations that extend common statistical tables.

Even more satisfactory than a good approximation to \( F \) is a procedure for obtaining a sequence of approximations, \( \{ F_n \} \), that get more and more accurate as \( n \to \infty \). The next three chapters will consider procedures of this kind. It would be nice if, for every \( x \geq a_o \), say, \( E_n(x) \to 0 \), and, moreover, did so in a rapid way, with \( \theta(E_n(x)) \to -\infty \) as \( n \to \infty \). The first property, convergence, is complicated to apply, and seems to depend on a lot of special knowledge about \( f \). The second, asymptoticity, has a more general application and will be given more careful consideration here.
Cramer points out the value of the property of asymptoticity in this way, [6, p. 224], "In practical applications, it is in most cases only of little value to know the convergence properties of our expansions. What we really want to know is whether a small number of terms—usually not more than two or three—suffice to give a good approximation to \( f(x) \) and \( F(x) \)."

We have defined \( E_1 \) to be asymptotic if \( E_1 \to 0 \). It would be more asymptotic, so to speak, if \( \theta(E_1) \leq -1 \), and still more so if \( \theta(E_1) \leq -2 \), and so on. This idea underlies the following:

**Definition.** A sequence of approximations, \( \{F_n\} \), is asymptotic I, if there exists \( \varepsilon \), such that for all \( n \), \( \theta(E_{n+1}/E_n) < -\varepsilon < 0 \).

This concept of asymptoticity generalizes Poincare's asymptotic series, [39]. He defined an asymptotic expansion for the function, \( f(x) \), to be a series, \( a_0 + a_1x^{-1} + a_2x^{-2} + \ldots \) with the property that \( x^n[f(x) - (a_0 + a_1x^{-1} + \ldots + a_nx^{-n})] \to 0 \) as \( x \to \infty \), another way of saying that the absolute error for the \( n^{th} \) partial sum = \( o(x^{-n}) \).

Another concept of asymptoticity was given by Erdelyi, [9], who defined an asymptotic sequence to be a sequence of functions, \( \{\phi_n\} \), for which \( \phi_2 = o(\phi_1) \), \( \phi_3 = o(\phi_2) \), and so on. This concept underlies the following:

**Definition.** A sequence of approximations is asymptotic II if \( E_2 = o(E_1) \), \( E_3 = o(E_2) \), and so on. A sequence of approximations asymptotic I is also asymptotic II, but the reverse is not true, as for example, when \( \{E_n\} = \{1/[x(\log x)^n]\} \). This is an asymptotic II sequence where the order of \( E_n \) is bounded by \( -1 \). A sequence, \( \{F_n(x)\} \) could be asymptotic I and yet converge to \( F(x) + e^{-x} \). The familiar series for the normal integral, \( \phi(x) (x^{-1} - x^{-3} + 3x^{-5} - 3.5 x^{-7} + \ldots) \) is a
well known example of a procedure that is asymptotic but not convergent, as will be shown in Example 89 of Chapter V.

The following is another example emphasizing the lack of relationship between convergence and asymptoticity.

**Example 51.** Let $E_n(x) = 2 - x$ for $x \leq 1$, and $-x^n$ for $x \geq 1$. Let $E(x) = E_n(x-n)$. Then $\theta(E_n) = -n$, and the procedure is asymptotic I and II. But for every $x$, $E_n(x) \to \infty$ as $n \to \infty$.

![Graph](image)

Theorem 52. Asymptotic II is invariant under a monotonic transformation of $x$ taking $\infty$ into $\infty$, but asymptotic I is not.

Proof. Let $x = \phi(y)$ be a monotonic transformation carrying $\infty$ into $\infty$. Let $G(y) = F(x)$, where $x = \phi(y)$ and $G_n(y) = F_n(x)$. Let $E_n = G_n(y)/G(y) - 1 = E_n$. Now $E_{n+1} = o(E_n) \iff E_{n+1} = o(E_n)$. Hence, $\{G_n\}$ is asymptotic II $\iff \{F_n\}$ is. Now let $E_n = 1/(\log x)^n$. $\{F_n\}$ is not asymptotic I. Let $\phi(y) = e^y$. $E_n = E_n = 1/(\log e^y)^n = y^{-n}$. Hence, $\{G_n(y)\}$ is asymptotic I. //

The relative frequency error gives us a simple test of whether a procedure is asymptotic II.

**Theorem 53.** If $F_1$ and $F_2$ are approximations to $F$ and neither $F_1-F$ nor $F_2-F$ a constant, (which will be the case if $F_1$, $F_2$, and $F \to 0$,) and $e_2/e_1 \to$ a limit, then $E_2/E_1 \to$ the same limit.

When $F(x) \to \infty$, $F(x)$ and $E_i(x)$ should be replaced by $F(a,x)$ and $E_i(a,x)$, $(i = 1$ or $2)$. 
Proof. \( \frac{e_2}{e_1} = \frac{f_2 - f}{f} \cdot \frac{f_1 - f}{f_1} = \frac{f_2 - f}{f_1 - f} \rightarrow \) a limit, \( L \). By (17), \( L \) is the limit of \( \frac{F_2}{F_1} = \frac{E_2}{E_1} \). //

**Theorem 54.** \( E[cF_1 + (1-c)F_2] = cE_1 + (1-c)E_2 \). \( E[cF_1 + (1-c)F_2] = ce_1 + (1-c)e_2 \).

The proof is direct. //

Approximations that are asymptotic II have the virtue that each is an infinite improvement over its predecessor. If we have two approximations such that \( e_2/e_1 \rightarrow \) a non-0 finite constant, \( L \), then (53) says that \( E_2/E_1 \rightarrow L \). By using (54), we can construct a linear combination, \( F_{21} \), of the two that is an infinite improvement over either; namely,

\[
\frac{L}{L-1} \cdot F_1 - \frac{1}{L-1} \cdot F_2.
\]

Then \( e_{21}/e_1 = c + (1-c)e_2/e_1 = \frac{L}{L-1} - \frac{1}{L-1} \cdot \frac{e_2}{e_1} \rightarrow 0. \)

We can summarize:

**Theorem 55.** If \( e_2/e_1 \rightarrow L \), a constant, and if neither \( F_1 - F \) nor \( F_2 - F \) \( \rightarrow \) constants, (which will be the case if \( F, F_1 \), and \( F_2 \rightarrow 0 \),) and

\[
F_{21} = \frac{L}{L-1} \cdot F_1 - \frac{1}{L-1} \cdot F_2,
\]

then \( \lim E_{21}/E_1 = \lim E_{21}/E_2 = 0. \)

In Chapters V and VI, we will show how some procedures that are not asymptotic II are made so by taking linear combinations.

The following theorems will show further the close relationship between the relative error and the relative frequency error. They will show that they have the same order, and are, under some circumstances, asymptotic to each other. Because the theorems have application beyond the context of errors, we will write \( g/f \) and \( G/F \) instead of \( e \) and \( E \). This implies the restriction that \( G \), which \( = F_1 - F \), cannot \( \rightarrow \) a constant.
Lemma 56. If $f(x) \geq 0$ and $g/f$ has order, then

1) $g/f \to 0$ and $\theta(g/f) < \rho \implies |G|/F < x^\rho$.

2) $g/f \to \infty$ and $\theta(g/f) > \rho \implies G/F > x^\rho$.

Proof of 1). Case 1: $\theta(g/f) = 0$. $g/f \to 0 \implies G/F \to 0$, (17),

$\implies |G|/F < x^\rho$. Case 2: $\theta(g/f) < \rho < 0$. Then $|g| < x^\rho f$, $\implies |G|

\leq \int_x^\infty |g| < \int_x^\infty t^\rho f(t) < t^\rho F$, for $x > 1$. Proof of 2). $f \geq 0$ and

$g/f \to \infty \implies g \geq 0$. $\theta(g/f) > \rho$ and $g/f \to \infty \implies F/g \to 0$ and $\theta(f/g)

<- \rho$, $\implies F/G < x^{-\rho}$, by 1), $\implies G/F < x^\rho$. //

Theorem 57. If $f \geq 0$ and $g/f$ has order, then

1) $\theta(f)$ is finite $\implies \theta(G/F) = \theta(g/f)$.

2) $\theta(f) = \pm \infty$ and $F/f$ has order, (or $F/(xf) \to 0$) $\implies \theta(G/F)

= \theta(g/f)$. We must remember that these theorems depend on the assumption that $G$ and $F/f$ constants.

Proof of 1). Case 1: $\theta(g/f) = \pm \infty$. Then by (56), $\theta(G/F) = \pm \infty$.

Case 2: $\theta(g/f)$ is finite, $\implies \theta(g) = \theta((g/f) \cdot f)$, is finite, (12),

$\implies \theta(F)$ and $\theta(G)$ are finite, (18), $\implies \theta(G/g) = \theta(F/f) = 1$, (18),

$\implies 0 = \theta(G/g) - \theta(F/f) = \theta(G/g)/(F/f)$, (15), $\implies \theta(G/F) = \theta(g/f)$, (15).

Proof of 2). Case 1: $\theta(g/f) = \pm \infty$. By (56), $\theta(G/F) = \pm \infty$. Case 2:

$\theta(g/f)$ is finite and $\theta(f) = \infty$. $F/f$ has order, $\implies \theta(F/f) \leq 1$,

(otherwise $F/(xf) \to \infty$, $\implies \theta(f) = 0$ by (24),) $\implies 1 - \rho F/(xf) < x^\rho$,

for all $\epsilon > 0$, $\implies g > x^\rho f[1 - \rho F/(xf)]$, for every $\rho < \theta(g/f)$,

$= x^\rho f - \rho x^\rho - 1 F$, $\implies G > x^\rho F$. ($\theta(f) = \theta(F) = \theta(g) = \theta(G) = \infty$, so that

all terms $\to 0$ and we have no trouble with constants.) By similar reasoning, $g < x^\rho f$ $\implies G < x^\rho F$. Hence, $\theta(G/F) = \theta(g/f)$. The proof for

the case $\theta(f) = \infty$ is analogous. //

Theorem 58. If $f$ has order and $f'/f$ has extended order, then

$\theta(f/F) = \theta(f'/f)$. 
Proof. Case 1: $\theta(f) = r$, finite. Then $\theta(f'/f) = -1$

$= \theta(f) - \theta(F) = \theta(f/F)$. Case 2: $\theta(f'/f)$ is finite, and $\theta(f) = \pm \infty$.

Then $\theta(F/f) = \theta(f'/f')$ by (57.2). Case 3: $\theta(f'/f)$ is $\infty$. Then

$\theta(f'/f) = -\infty$. Let $u(x)$ be a scale function with $\theta(f'/f') < \theta(u) < 0$.

$f'/f < u$, $\Rightarrow f < -uf'$, $\Rightarrow F < \int uf' < u f' = uf$, $\Rightarrow F/f < u$.

Now let $w(x)$ be a scale function with $\theta(w) < \theta(f'/f')$, but with

$\theta(w'/w) < \theta(f'/f)$, (46). Then $1 + \frac{w'/w}{f} \to 1$, $\Rightarrow -f'/f' > w(1 + \frac{w'/w}{f})$,

$\Rightarrow f > -wf'(1 + \frac{w'/w}{f})$, $\Rightarrow f > -(wf' + w'f)$, $\Rightarrow F > wf$,

$\Rightarrow F/f > w$, $\Rightarrow F/f$ has extended order $= \theta(f'/f')$ by (40). //

Theorem 59. If $F/f$ (or $f'/f'$) and $g/f$ have extended order and if $f$ has order, then $\theta(G/F) = \theta(g/f)$.

Proof. By (58), $\theta(F/f) = \theta(f'/f')$. (57) has disposed of the case

where $\phi(f)$ is finite. Case 1: $\theta(g/f) = -\infty$. Let $\theta(g/f) < \theta(u)$.

< 0, as in (58), $\Rightarrow g < uf$, $\Rightarrow G < \int uf < uF$, $\Rightarrow G/F < u$. Now

let $\theta(w)$ be $< \theta(g/f)$. Then $\theta(1 - \frac{w'}{w} \cdot \frac{F}{f}) < \theta(w)$,

$\Rightarrow \theta[w(1 - \frac{w'}{w} \cdot \frac{F}{f})] = \theta(w)$, (42, 46), $\Rightarrow g/f > w(1 - \frac{w'}{w} \cdot \frac{F}{f})$,

$\Rightarrow g > wf - w'F$, $\Rightarrow G > wf$, $\Rightarrow G/F > w$. Hence, by (40), $\theta(G/F) = \theta(g/f)$.

The case $\theta(g/f) = \infty$ is exactly analogous. We have supposed $F > 0$. If $F > \infty$, an analogous proof holds. //

Theorem 60. 1) If $xf/F$ and $xg/G$ $\to$ limits, and $\theta(G)$ or $\theta(F)$ is

finite, but they are not both 0, then $\frac{g}{f} \sim \frac{\theta(G)}{\theta(F)} \cdot \frac{G}{F}$.

2) If $\theta(f) = \pm \infty$, and $\theta(G/F)$ is finite, and $\frac{xf}{F}$ and

$x(\frac{G'}{F'})/(\frac{G}{F})$ (which is $\frac{xf}{G} - \frac{xG'}{F'}$) $\to$ limits, then $g/f \sim G/F$.

3) If $g$ and $f$ have order, and $g'$ and $f'$ are of constant

sign, and $\theta(g)$ or $\theta(f)$ is finite, but both are not 0, and if

$\frac{g'/f'}{f} \to$ a limit, then that limit is $\theta(g)/\theta(f)$.

Proof of 1). Let $G/F = H$. $g = fh \cdot \frac{xG}{G} \cdot \frac{F}{xf}$. Regardless of

whether $G \to \infty$ or 0, and $g = G'$ or $-G'$, $xg/G$ always $\to |\theta(G)|$. 
and similarly \( xf/F \to |\theta(F)| \), (24). Hence, \( g \sim fh \cdot |\theta(g)|/|\theta(F)| \).

Proof of 2). \( G = FH \implies g = Hf \pm H'F \), (since \( \theta(F) > \pm \infty \) and \( \theta(G/F) \) is finite, both \( G \) and \( F \to 0 \) or both \( \to \infty, \) = \( Hf(1 \pm \frac{xH'}{H} \cdot \frac{F}{xf}) \)).

Now \( xH'/H \to \) a finite number and \( F/xf \to 0 \), (24). Hence, \( g \sim Hf \).

Proof of 3). We suppose that \( g \) and \( f \) are \( > 0 \). (If \( g < 0 \), say, we let \( g^* = -g \). Then \( \frac{\lim g^*/g_f}{g} = \frac{\lim g^*/g_f}{g_f} \), and we are reduced to the above case.) \( \lim g^*/g_f = \lim g^*/g_f = \lim \frac{\log g}{\log f} \), (17) = \( \theta(g)/\theta(f) \), (7). (17) holds unless \( \log g \) or \( \log f \to \) a non-0 constant. \( \log g \) and \( \log f \) \( \to \) limits, because \( f' \) and \( g' \) are of constant sign. They both can't \( \to \) constants, or by (7), \( \theta(f) = \theta(g) = 0 \). So suppose \( \log g \to \) a constant, c. \( \lim \frac{\log g}{\log f} = \lim \frac{\log g - c}{\log f} \), (17), = 0, because \( \log f/\log x \) \( \to \) a non-0 limit. Now suppose \( \log f \) \( \to \) a constant, d.

\( \lim \frac{g^*/g_f}{g} = \lim \frac{\log g}{\log f} \to d = \pm \infty \), since \( \log g/\log x \to \) a non-0 limit, and the limit exists by hypothesis. //

This theorem can be written in the following slightly more restrictive, but more useful, form that caters to the probability of our knowing about \( g \) and \( f \) than about \( G \) and \( F \).

**Theorem 60.** (practical version.) 4). If \( xf'/f \) and \( xg'/g \to \) limits, and \( \theta(f) \) or \( \theta(g) \) is finite, then \( \frac{G}{F} \sim \frac{\theta(f)+1}{\theta(g)+1} \cdot \frac{g}{f} \).

5) If \( \theta(f) = \pm \infty \), if \( \theta(g/f) \) is finite, and if \( xf'/f \) and \( x(g/f)'/(g/f) \), (which is \( xf'/f - xg'/g \),) \( \to \) limits, then \( G/F \sim g/f \).

Proof. We use (35) to put 4) back into the form of 1). To prove 5), we let \( h = g/f \). \( g = hf \). \( G = xh'f \). \( h/H = h/f \). \( H/f \). \( \lim \frac{h}{H} = \lim \frac{hH}{hf} \), (17) = \( 1 \pm \frac{xf'}{h} \cdot \frac{F}{xf} \). \( \frac{xf'}{h} \to \) a finite limit and \( F/xf \to 0 \), (24, 35), \( \implies \lim h/H = 1 \). //

**Theorem 61.** Let \( H = G/F \). If \( f/F \) and \( H'H/F \), (which is \( g/G - f/F \)) have extended order, and if \( \theta(H) = \pm \infty \), and if \( |\theta(H)| < |\theta(F)| \) or
$|\Theta(G)|$, then $G/F \sim g/f$ and $\Theta(F) = \Theta(G)$. The theorem also holds if $\Theta(H)$ is finite and $\Theta(F) \neq (1,0)$.

Proof. $|\Theta(G/F)| < |\Theta(F)| \implies \Theta(G) = \Theta((G/F) \cdot F) = \Theta(F)$, (42).
$|\Theta(G/F)| < |\Theta(G)| \implies \Theta(F) = \Theta(G/(G/F)) = \Theta(G)$, (42). Thus, $\Theta(F) = \Theta(G)$ and $F$ and $G$ both $\to \infty$ or $0$. $G = FH \implies g = FH + FH' = FH(1 + \frac{H'}{H} \cdot \frac{F}{f})$. $|\Theta(H')/H| < |\Theta(f/F)|$, $\implies \frac{H'}{H} \cdot \frac{F}{f} \to 0$, $\implies g \sim FH$.

Theorem 61.2 (practical version.) Let $h = g/f$. If $f'/f$ and $h'/h$ have extended order, if $\Theta(h) = \pm \infty$, and if $|\Theta(h)| < |\Theta(f)|$ or $|\Theta(g)|$, then $G/F \sim g/f$ and $\Theta(f) = \Theta(g)$. The theorem also holds if $\Theta(h)$ is finite and $\Theta(f) \neq (1,0)$.

Proof. We have proved $\Theta(f) = \Theta(g)$, (61.1). As in the proof of (60.5), $\lim h/H = \lim \left(1 + \frac{h'}{H} \frac{F}{f}\right)$. Now $|\Theta(H)| < |\Theta(F)| = |\Theta(F)|$, (59,44). $F/f$ has extended order by (58). Hence, $\frac{h'}{h} \cdot \frac{F}{f} \to 0$ and $\lim h/H = 1$. //

We have shown how, in many cases, $e_1 \sim E_1$. The closeness of this asymptotic relationship is discussed in the next theorem.

Theorem 62. 1) If $\Theta(e_1)$ is finite and $\neq 0$, and $\Theta(f) = \pm \infty$, and $f/f'$ has order, and $x e_1'/e_1$ and $x f'/f$ limits, then

$\Theta(\frac{e_1}{E_1} - l) = \Theta(\frac{f'}{f} - l)$.

2) If $f/f'$, $e_1/e_1'$, and $x \frac{e_1'}{e_1} + \frac{f'}{f}$ have extended order, and $f$ and $e_1/f$ constants, then $\Theta(\frac{e_1}{E_1} - l) = \Theta(\frac{e_1'}{e_1} \cdot \frac{F}{f})$, $= \Theta(F - l)$ if $\Theta(e_1)$ is finite, and $= \Theta(F/f)$ if $|\Theta(e_1)| < |\Theta(F)|$ and $\Theta(F/f)$ is infinite.

Proof of 1). $\Theta(f/f') - l = \Theta(F/f) - l$, (57.2, $f$ is of constant sign by 29) = $\Theta([e_1'/e_1]) (F/f)$, (18) = $\Theta([e_1 f + e_1' F - e_1 f]/e_1 f]$, = $\Theta([e_1 f - f e_1 f]/e_1 f]$, (57.2), = $\Theta(e_1 [f/f e_1 f - l])$, = $\Theta(e_1/E_1 - l)$. 

To apply (57.2) the second time, we had to know that \((e_1 f)'/(e_1 f)\) has order. \((e_1 f)'/(e_1 f) = (e_1 f + e_1 f')/e_1 f = e_1'/e_1 + f'/f.\) Now \(x e_1'/e_1 \to \theta(e_1)\) and \(x f'/f \to +\infty, \implies (e_1'/e_1 + f'/f) \sim f'/f,\) which has order and constant sign, since \(x f'/f \to +\infty.\)

Proof of 2). \(\theta(e_1^F_1 - 1) = \theta(e_1 f - \int e_1 f), = \theta(e_1 f + e_1 F - e_1 f),\) \(= \theta(e_1 f e_1^F_1).\) (18) and (42) complete the proof. To apply (59), we had to know that \((e_1 f)'/(e_1 f), = e_1'/e_1 + f'/f,\) has extended order. //

When approximations are applied to families of integrals, much time may be saved if an overall error bound is obtained for the whole family. The following theorems may be helpful in this connection.

**Theorem 63.** Let \(F_1\) and \(G_2\) be approximations to \(F\) and \(G,\) all positive and \(\to 0,\) and let \(e_1(x,f) \geq e_2(x,g)\) for every significance level, \(\epsilon < \epsilon_0,\) i.e., where \(x\) is determined by the equation, \(F(x) = \epsilon,\) in the first case and by \(G(x) = \epsilon\) in the second. Then \(E_1 \geq E_2\) at level \(\epsilon_0.\)

For example, if we have a family, \(\{x_n\},\) like the \(t\) distribution, with \(f_n(x) \to\) the normal frequency for every \(x\) as \(n \to \infty,\) and an approximation formula, and can show that \(e_n \leq e(\text{normal})\) for levels \(< \epsilon_0,\) then the maximum relative error for the whole \(t\) family can be read off from calculations for the normal distribution.

This theorem is not easy to use, because it requires a knowledge of the significance level, which is what we wanted to know in the first place. The theorem can be rewritten in terms of the significance level of \(F_1\) and \(G_2,\) rather than \(F\) and \(G,\) to overcome this difficulty.

**Theorem 64.** Let \(F_1\) and \(G_2\) approximate \(F\) and \(G,\) all positive and \(\to 0,\) and let \(e_1(x,f) \geq e_2(x,g),\) where \(x\) is determined by the
equation, \( F_1(x) = \epsilon \), in the first case, and by \( G_2(x) = \epsilon \) in the second, hold for \( \epsilon < \epsilon_0 \). Then \( E_1(x,f) \geq E_2(x,g) \) at \( \epsilon_0 \).

The following theorem is even easier to use, when the conditions apply, because no calculation of significance levels is necessary.

**Theorem 65.** Let \( F_1 \) and \( G_2 \) be approximations to \( F \) and \( G \), all positive and \( \rightarrow 0 \), let \( f \) and \( g \) be \( > 0 \), and let \( f/g \) decrease monotonically for \( x \geq a \).

1) If \( e_1(x) \geq e_2(x) \) and decreases monotonically for \( x \geq a \), then \( E_1(a) \geq E_2(a) \).

2) If \( e_1(x) \leq e_2(x) \) and increases monotonically for \( x \geq a \), then \( E_1(a) \leq E_2(a) \).

The proof of (63) is little more than knowing what you are talking about. Let \( y = 1 - F(x) \). Let \( F(x) = H(y) = \int_y^1 h(t) dt \). Then \(-F'(x) = -H'(y)y'(x)\), or \( f(x) = h(y) \cdot f(x) \). Hence, \( h(y) = 1 \) between 0 and 1. Let \( F_1(x) = H_1(y) = \int_y^1 h_1(t) dt \). Then \( E_1(x) = H_1(y)/H(y) - 1 \).

\( h_1(y) = -H_1'(y) = -F_1'(x) x'(y) = f_1(x) \cdot x'(y) \). Similarly, \( h(x) = f(x) \cdot x'(y) \). Then \( e_1(x) = f_1(x)/f(x) - 1 = h_1(y)/h(y) - 1 \).

We make the analogous transformation for \( g \) and get \( E_2(x) = H_2(y)/H(y) - 1 \) and \( e_2(x) = h_2(y)/h(y) - 1 \). The theorem is so stated that for \( y \geq 1 - \epsilon_0 \),

\( e_1(x) \geq e_2(x) \), \( \implies h_1(y) \geq h_2(y) \), \( \implies H_1(y) \geq H_2(y) \),

\( \implies E_1(x) \geq E_2(x) \), for \( \epsilon < \epsilon_0 \). (We remember that the x's above are each determined by \( \epsilon \).) //

(64) is proved in just the same way, except that the transformations, \( y = 1 - F_1(x) \) and \( y = 1 - G_2(x) \) are used. //

Proof of (65.1). Let \( f^*(x) = f(x)/F(a) \) and \( g^*(x) = g(x)/G(a) \).

\( f^*/g^* = \text{constant} \cdot f/g \), which decreases monotonically. Since \( \int_a^\infty f^* = 1 = \int_a^\infty g^* \), \( f^* \) must start \( > g^* \) and end up \( < g^* \). (Otherwise \( \int_a^\infty f^* \neq \int_a^\infty g^* \).)
Hence, there exists $b$ such that $f^* \geq g^*$ for $x < b$ and $f^* \leq g^*$ for $x > b$. 

\[ \int_a^b (f^* - g^*) - \int_b^\infty (g^* - f^*) = 1 - 1 = 0, \implies \int_a^b (f^* - g^*) = \int_b^\infty (g^* - f^*), \implies \int_a^b e_1(x) [f^*(x) - g^*(x)] \geq \int_b^\infty e_1(b) [f^*(x) - g^*(x)] = \int_b^\infty e_1(b) [g^*(x) - f^*(x)] \geq \int_a^b e_1(x) f^*(x) \geq \int_a^\infty e_1(x) f^*(x) \geq \int_a^\infty e_2(x) g^*(x) \implies E_1(a) = \int_a^\infty e_1(x) f^*(x) \geq \int_a^\infty e_2(x) g^*(x) = E_2(a). \]

The proof for 2) is exactly analogous. //

The problem can also be approached by considering a family of integrals, $(F(x, \alpha))$, depending on the continuous parameter, $\alpha$. The maximum error for the family can be obtained by observing the sign of the derivative of $E_1(x, \alpha)$ with respect to $\alpha$.

**Theorem 66.1.** Let the subscript, $\alpha$, be short for $\frac{\partial}{\partial \alpha}$. 

\[ E_{1\alpha} = \frac{1}{F} (F_{1\alpha} - F_1 - \frac{\alpha}{F}). \]

Let $F$ and $F_1$ be positive and $\to 0$. If $F_{1\alpha}(a)$ - $F_1(a) \cdot \sup(f_\alpha(x)/f(x); x \geq a)$ and $F_{1\alpha}(a) - F_1(a) \cdot \inf(f_\alpha(x)/f(x); x \geq a)$ have the same sign, then it is the sign of $E_{1\alpha}(a)$.

**Proof.** The calculation of $E_{1\alpha}$ is direct. The sign is unaffected by removing $1/F$. $F_{1\alpha} - F_1 (F/\alpha)F$ is a monotonic function of $F/\alpha$, hence upper and lower Bounds for $F/\alpha$ can be substituted. //

**Theorem 66.2.** If $F$ and $F_1$ are positive and $\to 0$, and if $x$ is held at level $\epsilon$, i.e., determined by the equation, 

\[ F(x, \alpha) = \epsilon, \text{ then } \frac{dE_1}{d\alpha} = \frac{1}{\epsilon} \cdot \left( F_{1\alpha} - F_1 \frac{f_1(x, \alpha)}{f(x, \alpha)} \right). \]

To make this theorem practical, we observe whether $f_{1\alpha}(x)$ - $f_\alpha(x)f_1(a, \alpha)/f(a, \alpha)$ is of constant sign for $x > a$. If so, then as before, $E_{1\alpha}$ has the same sign. In both (66.1) and (66.2), we must assume that the functions are well-behaved enough so that differentiation under the integral sign is possible. This merely requires that $F$ and $F_1$ converge uniformly in some neighborhood of every $\alpha$. 
Proof of (66.2). $E_1(x, \alpha) = \frac{F_1(x, \alpha)}{F(x, \alpha)} - 1$. We have determined $x$ so that $F(x, \alpha) = \epsilon$. Hence, $\epsilon \frac{dE_1}{d\alpha} = \frac{dF_1}{d\alpha} = \frac{\partial F_1}{\partial x} \frac{dx}{d\alpha} + \frac{\partial F_1}{\partial \alpha} = F_1 \alpha - f_1 \frac{dx}{d\alpha}$.

To get $\frac{dx}{d\alpha}$, we observe that $F(x, \alpha) \equiv \epsilon$. Hence, $\frac{dF}{d\alpha} = 0 = \frac{\partial F}{\partial x} \frac{dx}{d\alpha} + \frac{\partial F}{\partial \alpha}$,

$\Rightarrow \frac{dx}{d\alpha} = \frac{\partial F}{\partial x} \frac{dx}{d\alpha} = F_1 \alpha \mid_a - F_\alpha \mid_a \frac{f_1(a)}{f(a)}$

$= \int_a^\infty f_1 \alpha(x) dx - \int_a^\infty f_\alpha(x) dx \frac{f_1(a)}{f(a)} = \int_a^\infty [f_1 \alpha(x) - F_\alpha(x) \cdot \frac{f_1(a)}{f(a)}] dx.$

If the latter integrand is of constant sign, then the integral has the same sign. //
CHAPTER IV

THE TAYLOR EXPANSION

When one thinks of approximation procedures, one thinks of a Taylor series. Much is known about this simple and basic procedure; and if it were sufficient to meet the needs of the tail integral problem, it would be a waste of time to explore more complicated, uncommon procedures. In this chapter, we will discuss applications of the Taylor series and its limitations. We will suppose that $F(x) \to 0$.

If one tries to apply the series directly, one gets the relation,

$$F(x) \approx F(a) - f(a) (x-a) - f'(a) (x-a)^2/2 + \ldots.$$  Hence,

$$F(a) - F(x) \approx f(a) (x-a) + f'(a) (x-a)^2/2 + \ldots.$$  The sequence of approximations would then be obtained by stopping at a finite number of terms and letting $x \to \infty$, which reduces to the worthless sequence,

$$[+\infty, +\infty, \ldots].$$  This difficulty might be overcome, if we were to make a monotonic transformation, $y = F_1(x)$, carrying $\infty$ into 0. If we define $G(y)$ to be $F(x)$, when $y = F_1(x)$, we may then hope to expand $G(y)$ about 0 or $F_1(a)$, (= b, say).

$G(y)$ is an increasing function of $y$, $= \int_0^y g(t)dt$. $g(y) = G'(y)$.

Expanding $G(y)$ about b, we get $G(y) \approx G(b) + g(b) (y-b) + g'(b) (y-b)^2/2 + \ldots$.

This expansion is much more satisfactory from the standpoint of asymptoticity, since $y-b < 1$, for large a, and hence $(y-b)^n \to 0$ faster and faster as $n$ increases. Letting $y \to 0$, $G(y) \to 0$, and we get an expansion for $G(b)$: $bg(b) - b^2g'(b)/2 + b^3g''(b)/3! - \ldots$. To evaluate
\( g^{(n)}(b) \), we observe that \( \frac{d^{(n)} g}{dy^{(n)}} / \frac{dy}{dx} = g(y) = \frac{dx}{dy} / \frac{d^{(n)} f}{dx^{(n)}} = f(x) \).

Procedure A. Thus, \( F(a) \approx F_1(a)u_1(a) + [F_1(a)]^2u_2(a)/2! + \ldots \) where \( u_1(a) = f(a)/f_1(a) \) and \( u_{n+1}(a) = u_n'(a)/f_1(a) \). (We lost the minus signs because \( f_1(x) = -dy/dx \).

If we expand about 0, we obtain the formal expansion,
\( G(b) \approx G(0) + g(0) \cdot b + g'(0) \cdot b^2/2! + \ldots \).
\( G(0) = 0. \) Let \( U_n = \lim u_n(x) \) as \( x \to \infty \). When these limits exist and are finite, we obtain

Procedure B. \( F(a) \approx F_1(a) \cdot U_1 - [F_1(a)]^2U_2/2! + \ldots \).

If \( F_1(a) \) is some simple function like 1/a, say, this series satisfies one’s intuitive desire to express \( F(a) \) in a series of powers of 1/a. Procedure B can be a satisfactory expansion for \( F(x) \) only if all the \( u_n(x) \)'s \( \to \) finite limits, not all of which are 0. This property implies conditions on \( F \) and \( F_1 \) that will be discussed later in the chapter.

The application of A and B is facilitated by the usual bound for the remainder term. Regardless of the convergence or asymptotic properties of the series, we can always take terms until the remainder bound is as small as is desired, provided only that the functions involved remain differentiable.

For A, the exact remainder after \( n \) terms is
\( (-)^n \int_o^b g^{(n)}(y) y^n dy/n! \), which becomes \( [F_1(a)]^{n+1}u_{n+1}(a) \) where \( a < a_1 < \infty \),
\( (n+1)! \)
using the Lagrange form. The remainder after no terms is
\( F_1(a)f(a_1)/f_1(a_1) \), which is a way of writing Bounds 48.

The remainder for B is the similar formula,
\( \int_o^b g^{(n)}(y) (b-y)^n dy/n! = (-)^n[F_1(a)]^{n+1}u_{n+1}(a_2)/(n+1)! \), where \( a < a_2 < \infty \). \( a_2 \) is not the same as \( a_1 \). If it were, we could average
A and B for one term, the remainder would cancel out, and an exact expression for F would be achieved. The exact remainder for A is a weighted average of \( g^{(n)}(y) \) on the interval, \([0,b]\), with the weight concentrated on the b end. The exact remainder for B is the same with an opposite sign and the weight concentrated on the 0 end. If \( F_1(x) \) could be determined so that \( g'(y) \) were relatively stable between 0 and b, the average of A and B for one term; namely, \( F_2(a) = F_1(a) [u_1(a) + U_1] / 2 \), would be a good approximation to F.

Carrying this idea out, we try to find an \( F_1 \) such that \( g'(y) \cong 1 \), \( g(y) \cong y \), \( G(y) \cong y^2 / 2 \), \( F(x) \cong [F_1(x)]^2 / 2 \). Let \( F_1 \) be a pretty good approximation to F. Then we should let \( F_1(x) = \sqrt{2F_1} \). After algebra, we obtain the formula, \( F_2 = F_1 f / f_1 \), which is Procedure D considered in Chapter VI.

Another compromise is obtained by the simplifying assumption that \( g'(t) \) lies on a straight line between \( g'(0) \) and \( g'(b) \). Then

\[
\text{error A/error B comes out } - \frac{g'(0) + 2g'(b)}{2g'(0) + g'(b)} = -\alpha, \text{ say. } \frac{F_A - F}{F_B - F} = -\alpha, \text{ and hence } F = \frac{F_A + \alpha F_B}{1 + \alpha}.
\]

When the assumption is removed, we get the approximate

\[
\text{Formula 67. } F(a) \cong \frac{u_1(a) + \alpha U_1}{1 + \alpha} F_1(a), \text{ where } \alpha = \frac{2u_2(a) + U_2}{u_2(a) + 2U_2}.
\]

B converges for a large enough \( \iff G(y) \) continued into the complex plane, is analytic at 0. (Proof. If \( G(y) \) is analytic at 0, the series holds for all \( b \) with \(|b| < \epsilon\). Conversely, if the series converges for some \( b_0 \), it converges for all complex \( b \) with \(|b| < b_0\), and the continuation of \( G(y) \) is provided by the series.)
If \( G(y) \) has a singularity at 0, it is possible that \( A \) can still converge absolutely. For example, let \( G(y) = \int_1^y -\log z \, dz \).

\[
-\log z = (1-z) + \frac{(1-z)^2}{2} + \frac{(1-z)^3}{3} + \cdots, \quad G(y) = -\frac{(y-1)^2}{1 \cdot 2} + \frac{(y-1)^3}{2 \cdot 3} - \frac{(y-1)^4}{3 \cdot 4} + \cdots,
\]

integrating term by term. \( G(y) \) cannot be analytic at 0, since its derivative is not. Yet the series converges absolutely at \( y = 0 \) by comparison with the series, \( \sum 1/n^2 \).

A Taylor series that converges does not always converge to the right value. Courant, [5, p. 336], gives the example, \( f(x) = e^{-1/x^2} \). \( f \) and all its derivatives are continuous everywhere on the real line, and all \( = 0 \) at 0. Hence, the Taylor series about 0 converges identically to 0, which \( \neq f(x) \) for \( x \neq 0 \). If you think there is something wrong with all those zeros, let \( h \) be \( f + \) an entire, analytic function, \( g \). Then \( h \) has a Taylor series about 0 that converges everywhere to \( g \).

If \( A \) converges absolutely, that shows there is a function, \( G^a(y) \) analytic in a region containing the interval \((0,b)\). If \( G(y) \) is that function, then we can show that \( A \) converges to \( G(b) \), regardless of the possible non-analyticity of \( G(y) \) at 0.

**Theorem 68.1.** If \( G(y) \) is analytic in a region containing \((0,b)\), and if \( A \) converges absolutely, then \( A \) converges to \( G(b) \).

**Proof.** \( G(y) = G(b) + g(b) (y-b) + g'(b) (y-b)^2/2! + \cdots \) for \( y \in (0,b) \). The series converges absolutely for \( y = 0 \), by hypothesis. For every \( \epsilon \), there exists \( n \) such that \( \sum_{n+1}^\infty |G^{(i)}(b)| |y-b|^i < \epsilon \).

Let \( e(y) = G(y) - \sum_{1}^n G^{(i)}(b) (y-b)^i = \sum_{n+1}^\infty G^{(i)}(b) (y-b)^i \) for \( y \in (0,b) \). Since \( G(0) = \lim G(y) \) as \( y \to 0 \) by definition, \( e(y) \) is continuous, being the difference of two continuous functions. \( |e(y)| < \epsilon \), \( \implies |e(0)| \leq \epsilon \), \( \implies \) the series converges to \( G(0) \), i.e., \( A \) converges to \( G(b) \).
However, when $G(y)$ has a singularity at $0$, $A$ can converge only "infinitely slowly," that is to say, $\frac{\text{improvement of approximation}}{\text{error of approximation}} \to 0$. This idea is made specific by the following theorem.

**Theorem 68.2.** If a real power series, $f(x)$, converges at a singularity, and if the ratio of succeeding terms $\to$ a limit, $L$, then $L = 1$. If, also, $|f(x) - S_n| \to 0$, monotonically at the singularity, for $n$ large enough, as $n \to \infty$, then $\frac{|f(x) - S_n| - |f(x) - S_{n+1}|}{|f(x) - S_n|} \to 0$.

Proof. Let $f(x) = a_0 + a_1x + a_2x^2 + \ldots$, converge at the singularity, $z$. Let $b_n = a_nz^n$. $f(z) = \sum_0^\infty b_n$. $|\lim b_{n+1}/b_n| = |L|$, is not $> 1$ or else the series would diverge. If

$|b_{n+1}/b_n| = |a_{n+1}| \cdot |z|/|a_n|, \to |L| < 1$, then

$|a_{n+1}| (|z| + \delta)/|a_n| \to$ a limit $< 1$, for some $\delta > 0$, $\implies f(|a| + \delta)$

converges, $\implies f$ is analytic in $\{z; |z| < |z| + \delta\}$, $\implies z$ is not a singularity, a contradiction. Hence $|L| = 1$ and there exists $N$ such that for $n > N$, $1-\varepsilon < |b_{n+1}/b_n| < 1+\varepsilon$. Case 1: every $b_n$ is positive (or negative,) for $n > N$. $\sum_0^\infty b_{n+1} > \sum_0^\infty b_n (1-\varepsilon) = b_n/\varepsilon$.

$\frac{|f(z) - S_n| - |f(z) - S_{n+1}|}{|f(z) - S_n|} \leq \frac{b_{n+1}}{b_{n+1}/\varepsilon} = \varepsilon$.

Case 2: There exists $n > N$ such that $b_n$ and $b_{n+1}$ are of opposite sign. Say $b_n > 0$, $b_{n+1} < 0$. (No $b_n$ can $= 0$ for $n > N$ or $|b_n/b_{n-1}|$ would $= 0$ and would not be $> 1-\varepsilon$.) $f(z) < S_n$, (otherwise $|f(z) - S_n| < |f(z) - S_{n+1}|$, contradicting the assumption that $|f(z) - S_n| \to 0$ monotonically as $n \to \infty$) $f(z) > S_{n-1}$, (for if not, $|f(z) - S_{n-1}| < |f(z) - S_n|$, a contradiction.) $f(z) > S_{n+1}$. (Suppose
not. \( S_{n-1} < f(z) \leq S_{n+1} \), \( \implies |f(z) - S_{n-1}| \leq |S_{n+1} - S_{n-1}| \)
\( = |b_n + b_{n+1}| < |b_n - b_n(1 + \varepsilon)| = \varepsilon |b_n| \). \( |f(z) - S_n| > |S_n - S_{n+1}| = b_n \),
\( \implies |f(z) - S_{n-1}| < |f(z) - S_n| \), a contradiction.) So we have
\( S_{n+1} < f(z) < S_n \). \( b_{n+2} \) is positive, (otherwise \( |f(z) - S_{n+1}| < |f(z) - S_{n+2}| \)) \( \implies f(z) \) alternates after the \( n^{th} \) term by the same reasoning.
\( f(z) - S_{n+1} < \frac{1}{2} |b_{n+1}| \), (or else \( |f(z) - S_{n+1}| \) would be
\( \implies |f(z) - S_n| > \frac{1}{2} |b_{n+1}| \) \( \implies |f(z) - S_{n+1}| > \frac{1}{2} |b_{n+2}| \),
by the previous reasoning, \( \frac{1}{2} |b_{n+1}| < (1 - \varepsilon) \), \( \implies |f(z) - S_n| \sim \frac{1}{2} |b_{n+1}| \)
\( \sim \frac{1}{2} |b_{n+2}| \sim |f(z) - S_{n+1}| \), as \( n \to \infty \), \( \implies \frac{|f(z) - S_n| - |f(z) - S_{n+1}|}{|f(z) - S_n|} \)
\( \to 0. // \)

It is thus of interest to know, when using \( A \) or \( B \), whether \( G(y) \) is analytic at \( 0 \). The following theorems throw light on this question.

The basic fact is that \( F(x) = G[F_1(x)] \). For \( G(y) \) to be analytic at \( 0 \), \( F(x) \) must be an analytic function of \( F_1(x) \) continuable to a neighborhood of \( F_1(\infty), (= 0) \). Let us suppose that \( G(y) \) is analytic at \( 0 \). \( G(y) = a_0 + a_1(y) + a_2y^2 + \ldots \) in a neighborhood of \( 0 \).
\( G(y) = F(x), \to 0 \) as \( y \to 0 \). Hence \( a_0 = 0 \). Let \( a_n \) be the first non-\( 0 \)
\( a_1 \). Then \( G(y) \sim a_n y^n \) as \( y \to 0 \). Hence, \( F(x) \sim a_n [F_1(x)]^n \) as \( y \to 0 \), and we obtain

Theorem 69. 1) If \( G(y) \) is analytic at \( 0 \), then
\( F(x) \sim \text{constant} \cdot [F_1(x)]^n \) as the real variable, \( x, \to \infty \), where \( n \) is
\( \geq 1. \)

2) If \( G(y) \) is analytic at \( 0 \), then \( f(x)/f_1(x) \) is an analytic function of \( F_1(x) \) in a neighborhood of \( F_1(\infty), (= 0) \) and
\( f(x)/f_1(x) \sim \text{constant} \cdot [F_1(x)]^{n-1}. \)
Proof of 2). \( G(y) = a_1 y + a_2 y^2 + \ldots \), where \( y = F_1(x) \).

\( g(y) = a_1 + 2a_2 y + 3a_3 y^2 + \ldots \) in a neighborhood of 0. Now

\[ F(x) = G[F_1(x)]. \quad f(x) = g[F_1(x)] f_1(x). \]

Hence \( f(x)/f_1(x) = g[F_1(x)] \), where \( g \) is analytic in a neighborhood of 0 and is \( \sim n_{a_1} y^{n-1} \).

Since we must find an \( F_1 \) such that \( F \sim a_1 F_1^n \), we might as well try to find one with \( F \sim F_1 \), which would require, by (69.2) and (16), that \( f \sim f_1 \). Of course, the perfect \( F_1 \) would be \( F \) itself. Then the \( A \) and \( B \) expansions would consist of the one term, \( F_1(a) \). These facts suggest that the proper choice of a transformation, \( y = F_1(x) \), for generating \( A \) and \( B \), is a close approximation to \( F \).

**Theorem 70.** 1) If \( G(y) \) is analytic at 0, then \( F_1(x) \) is analytic at \( \infty \) \( \iff \) \( F(x) \) is analytic at \( \infty \).

2) \( F \) and \( F_1 \) are analytic at \( \infty \) \( \implies \) \( G(y) \) is analytic at 0.

In other words, any two of the statements, "\( F_1(x) \) is analytic at \( \infty \)", "\( F(x) \) is analytic at \( \infty \)"; and "\( G(y) \) is analytic at 0" imply the third.

3) \( F(x) \) is analytic at \( \infty \) \( \iff \) \( f(x) \) is analytic at \( \infty \) and has order \( \leq -2 \). Hence, if \( F_1 \) is analytic at \( \infty \) and \( f(x) \) has a singularity at \( \infty \), then \( G(y) \) is not analytic at 0.

This theorem can be proved by the following lemma from function theory, which states, roughly, that an analytic function of an analytic function is an analytic function.

**Lemma 71.** 1) If the non-constant function \( f(z) \) is analytic in the region, \( R \), then \( f(R) \) is a region.

2) If, also \( g(w) \) is analytic in \( f(R) \), then \( h(z), = g[f(z)] \), is analytic in \( R \).

3) Conversely, if \( h \) and \( f \) are analytic in \( R \), then \( g \) is analytic in \( f(R) \).
4) If \( h \) is analytic at \( z_0 \) and the non-constant function, \( g \), is analytic at \( f(z_0) \), then \( f \) is analytic at \( z_0 \).

In other words, if \( f \), \( g \), and \( h \) are non-constant functions, and \( h = g(f) \), then any two of the statements, "\( f \) is analytic at \( z_0 \)," "\( g \) is analytic at \( f(z_0) \)," and "\( h \) is analytic at \( z_0 \)," imply the third.

Proof of 1). This lemma is a consequence of the inverse theorem as stated by Knopp, [24, p. 135 and the exercise on p. 137]. These theorems state that if \( f(z) - f(z_0) \) has an \( n \)-fold zero at \( z_0 \), then the image of any neighborhood of \( z_0 \) covers a small neighborhood of \( f(z_0) \) exactly \( n \) times. It follows that if \( z_0 \) is any interior point of \( R \), \( f(z_0) \) is an interior point of \( f(R) \). Thus, \( f(R) \) contains only interior points and is a region.

Proof of 2). Now we assume that \( f(z) \) is analytic in \( R \) and \( g(z) \) is analytic in \( f(R) \). Let \( w = f(z) \).

\[
\frac{h(z) - h(z_0)}{z - z_0} = \frac{g(w) - g(w_0)}{w - w_0} \cdot \frac{f(z) - f(z_0)}{z - z_0}. \quad \text{As } z \to z_0, \ f(z) \to f(z_0),
\]

\( \implies \) the left factor \( \to g'(w_0) \) and the right factor \( \to f'(z_0) \). Thus, \( h \) is differentiable at any point, \( z_0 \), of \( R \).

To prove 3), we write the same identity and suppose first that \( f'(z_0) \neq 0 \). Then \( \frac{\Delta g/\Delta w}{f'(z_0)} = h'(z_0)/f'(z_0) \) by the quotient of limits theorem, so \( g \) is differentiable at \( w_0 \). Thus \( g(w) \) is analytic at all points, \( f(z) \), except where \( f'(z) = 0 \). Now suppose \( f'(z_0) = 0 \). \( f'(z) \neq 0 \), or \( f(z) \) would be a constant. By the identity theorem, there exists a neighborhood, \( U \), of \( z_0 \) where \( f'(z) \neq 0 \) except at \( z_0 \). \( \implies g(w) \) is analytic in \( f(U) - w_0 \). Given \( \epsilon \), there exists \( \delta \) such that \( |z - z_0| < \delta \implies |h(z) - h(z_0)| < \epsilon \), since \( h \) is continuous.
By the Knopp theorems, again, there exists $\delta_1$ such that $|w-w_0| < \delta_1 \implies |z-z_0| < \delta$. Hence, $|w-w_0| < \delta_1 \implies |h(z) - h(z_0)| = |g(w) - g(w_0)| < \epsilon$. A theorem of function theory states that a function analytic in a region except for one point and continuous at that point, is analytic at that point also. Hence, $g(w)$ is analytic in $f(U)$. We have thus shown $g(w)$ analytic in $f(R)$.

Proof of 4). We suppose $h$ analytic in $U_1$, a neighborhood of $z_0$, and $g$ analytic in $U_2$, a neighborhood of $w_0$. We can make $U_2$ small enough so that $g'(w) \neq 0$ except possibly at $w_0$, as in the proof of 3), and we can then make $U_1$ small enough so that, by the continuity of $h$ at $z_0$, $h(U_1) \subset$ the region, $g(U_2)$. Let $h(z_1)$ be any point of $h(U_1)$ except $g(w_0)$. Let $g(w_1) = h(z_1)$. There exists a neighborhood of $w_1$, $U_3 \subset U_2 - w_0$, where $g(w)$ has an analytic inverse, and where $g(U_3) \subset h(U_1)$ by continuity. There exists a neighborhood, $U_4$, of $z_1$ with $h(U_4) \subset$ the region, $g(U_3)$, by continuity. Then $f(z) = g^{-1}[h(z)]$ in $U_4$ and is analytic there by 2). Hence, $f(z)$ is analytic in $U_1 - z_0$. Now by the Knopp theorems, there exists $\delta$ such that $|g(w) - g(w_0)| < \delta \implies |w-w_0| < \epsilon$. By continuity, $\exists \delta_1$ such that $|z-z_0| < \delta_1 \implies |h(z) - h(z_0)| = |g(w) - g(w_0)| < \delta$.

Thus, $f(z)$ is continuous at $z_0$, and, as before, is analytic there. //

Proof of (70). Let $F(z) = F(1/z)$ and $F_1(z) = F_1(1/z)$. $F$ is analytic at $\infty \iff F$ is analytic at 0, by definition, and $F_1$ is analytic at $\infty \iff F_1$ is analytic at 0. $F(1/z) = G[F_1(1/z)]$, hence, $F(z) = G[F_1(z)]$. $F$ and $F_1$ are not constant, hence $G$ is not constant, by the identity theorem. We can now apply Lemma 71, and know that if any two of the functions, $F$, $F_1$, or $G$ are analytic at 0, then the other is.
Proof of (70.3). \( F(x) \) is analytic at \( \infty \) if and only if \( \overline{F}(x) \) is analytic at 0, \( \iff \overline{F}'(x) \) is analytic at 0. \( \overline{F}'(x) = \frac{d}{dx} \overline{F}(1/x) = \frac{f(1/x)}{x^2} = \frac{\overline{f}(x)}{x^2} \). \( \overline{f}(x)/x^2 \) is analytic at 0 \( \iff \overline{f}(x) \) is analytic at 0 and \( \theta(f) \leq -2 \). //

If you should be looking for a transformation, \( F_1(x) \), that is analytic and biunique at \( \infty \), your choice is rather severely limited. For those requirements are equivalent to demanding that \( \overline{F}_1(z) \) be analytic and biunique at 0. Hence, by the implicit function theorem, \( \overline{F}_1'(z) \neq 0 \). So the expansion of \( \overline{F}_1(z) \) is \( a_1 z + a_2 z^2 + \ldots \), with \( a_1 \neq 0 \). Thus, \( \overline{F}_1(z) \sim a_1 z \) as \( z \to 0 \), and we get

**Theorem 72.** \( F_1 \) is analytic and biunique at \( \infty \) and in a neighborhood of \( \infty \), \( F_1(x) \sim x^{-1} \cdot \text{constant} \), as \( x \to \infty \).

To apply (70), we want to know whether \( f(z) \) is analytic at \( \infty \). The following theorem helps to decide this question.

**Theorem 73.** 1) If \( F(z) \) is analytic in the region, \( \{z; |z| > R\} \), and if \( F(z) \to 0 \) when the complex variable \( z \to \infty \), then \( F(z) \) is analytic at \( \infty \).

2) In order for \( F(z) \) to be analytic at \( \infty \), \( f(z) \) must be \( \sim a_n z^{-n} \) as \( z \to \infty \), for some \( n \geq 1 \) and \( a_n \neq 0 \).

3) In order for \( F(z) \) to be analytic at \( \infty \), \( f(z) \) must be \( \sim c_m z^{-m} \) as \( z \to \infty \), for some \( m \geq 2 \) and \( c_m \neq 0 \).

Proof. Let \( U \) be a neighborhood of 0. If \( \overline{F}(z) \) is analytic in \( U \) and continuous at 0, then \( \overline{F}(z) \) is analytic in \( U \), \( \iff 1 \).

Now if \( \overline{F}(z) \) is analytic at 0, \( \overline{F}(z) = a_0 + a_1 z + a_2 z^2 + \ldots \). Let \( a_n \) be the first non-0 \( a_1 \). \( \overline{F}(z) \sim a_n z^{-n} \) as \( z \to 0 \), \( \iff F(z) \sim a_n z^{-n} \) as \( z \to \infty \). //

If \( G(y) \) is analytic at 0, then the B series,

\[
G(y) = y \lim u_1(x) - y^2 \lim u_2(x)/2! + \ldots,
\]

converges for \( x \) large
enough. Since all these limits must exist and be finite, and not all 0, we can deduce the following order conditions:

**Theorem 74.** Let \( \theta(f_1) = r \) and \( \theta(u_1) = \theta(f/f_1) = s \). If \( G(y) \) is analytic at 0, then

1) \( s \) and \( r \) are \( \leq 0 \).

2) \( \theta(u_{n+1}) = \theta(u_n) - r \), unless \( \theta(u_n) = 0 \).

3) \( s \) is finite and \( \neq 0 \) \( \Rightarrow \) \( r \) is also and \( s/r \) is a positive integer.

4) If \( s \) is infinite, so is \( r \).

This theorem follows directly from (69.2). However, since it also applies to asymptotic expansions, a non-analytic proof will be given.

Proof. \( s \) cannot be \( > 0 \) or \( u_1(x) \to \infty \), \( r \) cannot be \( > 0 \) or \( F_1(\infty) = \infty \), \( u_{n+1} = u_n'/f_1 \), \( \Rightarrow \) \( \theta(u_{n+1}) = \theta(u_n) - 1 - (r-1) \), unless \( \theta(u_n) = 0 \), (18). If \( s \) were finite and \( r \) were \( -\infty \), \( \theta(u_2) \) would be \( \infty \) and \( u_2(x) \to \infty \), a contradiction. If \( r \) and \( s \) are finite, and \( s \) is not divisible by \( r \), then after a finite number of terms, \( \theta(u_n) \) would be \( > 0 \) by 2), a contradiction. Finally, if \( \theta(s) = -\infty \) and \( \theta(r) \) is finite, then \( \theta(\text{every } u_1) = -\infty \), all the coefficients are 0, and you get no series at all. //

When the series converges, \( \theta(u_n) \) gets to be zero every now and then, and then \( u_{n+1} \) gets new life from \( u_n' \) having order \( < -1 \).

Some of the previous reasoning is incorporated in the following theorem.

**Theorem 75.** If \( G(y) \) is analytic at 0, and \( f_1(x) \) exists, then

1) For all \( n \), \( u_n(x) \) exists and \( \lim u_n(x), (= U_n, \text{ say}) \) exists and is finite.

2) There exists \( n \) such that \( U_n \neq 0 \).
3) \( F(x) \sim (-)^{n+1} \frac{[F_1(x)]^n}{n!} \cdot (\text{first non-0 } U_n). \)

4) Let \( n + r \) be the index of the first non-0 \( U_i \) with \( r \geq 0. \)

Then \( u_n(x) \sim (-)^r \frac{U_{n+r} [F_1(x)]^r}{r!}. \)

We will show that this theorem applies also to asymptotic expansions.

Proof. \( G(y) = U_1 y - U_2 y^2/2! + U_3 y^3/3! - \ldots, \ u_1(x) = G'(y), \) which exists. Let \( u_2(x) = -G''(y), \) which exists, etc. \( u_n'(x) = \frac{d[u_n(x)]}{dy} \cdot \frac{dy}{dx} = -G^{(n+1)}(y) \cdot f_1(x). \) Hence, all the \( u_n'(x) \)'s exist and the recursion formula holds. Now, all the \( G^{(n)}(y) \)'s are analytic where \( G(y) \) is, \( \Rightarrow \) they are continuous and finite in a neighborhood of \( 0, \Rightarrow \) the limit of every \( u_n(x) \) exists and is finite, proving 1).

2) follows from the fact that the expansion of \( G(y), \) which \( = F(x) \neq 0. \)

As \( y \to 0, \ G(y) \sim (-)^{n+1} U_n y^n/n!, \) where \( U_n \) is the first non-0 limit, proving 3).

To prove 4), we differentiate the series term by term \( n \) times, getting \( G^{(n)}(y) = (-)^{n+1} [U_n - U_{n+1} y + U_{n+2} y^2/2! - \ldots], \)

\( \Rightarrow u_n(x) = U_n - U_{n+1} y + U_{n+2} y^2/2! - \ldots, \)

\( \Rightarrow u_n(x) \sim (-)^r \frac{U_{n+r} [F_1(x)]^r}{r!}. \)

The asymptotic properties of \( B \) are summed up in the following theorem.

Theorem 76. If all the \( u_n(x) \)'s exist for \( a < x < \infty, \) and \( \to \) finite limits not all \( 0, \) then \( B \) is asymptotic II. If, further, \( \theta(F_1) < 0, \) then \( B \) is also asymptotic I.
The condition for (76) is simply that a meaningful $B$ expansion exist. In order to prove the theorem, it will be necessary to prove a few lemmas first.

We recall from Chapter III the Poincare definition of asymptotic expansion, and rewrite it in a form discussed extensively by Erdelyi, [9].

**Definition.** $G(y)$ has the asymptotic expansion, $a_0 + a_1 y + a_2 y^2 + \cdots$, as $y \to 0$ from the right, if $G(y) \to a_0'$ and for all $n$,

$$G(y) - \left[ a_0 + a_1 y + \cdots + a_n y^n \right] \frac{y^{n+1}}{y^n} \to a_{n+1}'.$$

(The previous definition, as applied to $G(y)$, was that

$$G(y) - \left[ a_0 + a_1 y + \cdots + a_n y^n \right] \frac{y^n}{y^n} \to 0 \text{ as } y \to 0,$$

$$\Rightarrow \quad G(y) - \left[ a_0 + a_1 y + \cdots + a_{n-1} y^{n-1} \right] \frac{y^n}{y^n} \to a_n,$$

the two definitions are equivalent.)

The following theorem is proved by Erdelyi, [9, p. 13].

**Theorem 77.** $G(y)$ can have a most one asymptotic expansion as $y \to 0$ from the right.

The proof follows inductively from the definition. //

**Theorem 78.** If $G(y)$ has right-handed derivatives of every order, $n$, continuous on an interval, $[0, y_n]$, then $G(y)$ has an asymptotic expansion which is its Taylor expansion.

**Proof.**

$$\frac{G(y) - \sum_{k=1}^{n} \frac{G^{(k)}(0)}{k!} y^k}{y^{n+1}} = \frac{G^{(k)}(y)}{(n+1)!}, 0 \leq y \leq y_n, (65),$$

$- g^{(k)}(0)/(n+1)!$. //
Corollary 79. If \( G(y) \) is analytic at 0, its analytic expansion is its asymptotic expansion.

There are many useful functions that have an asymptotic expansion at 0 but are not analytic at 0. Courant's example, \( e^{-1/x^2} \), referred to earlier, is one. The asymptotic series, (Example 89), for the ratio of the normal tail integral to the normal density,

\[
\Phi(x)/\phi(x) \cong x^{-1} - x^{-3} + 3 x^{-5} - 3.5 x^{-7} + \ldots,
\]

is another. We make the transformation, \( x = 1/y \), to obtain an asymptotic expansion of \( G(y) = \Phi(1/y)/\phi(1/y) \), at 0.

Theorems 74 and 75 apply without change if we assume that a meaningful \( B \) expansion exists, rather than that \( G(y) \) is analytic at 0, (i.e., we require that every \( u_n(x) \rightarrow \) a finite limit, and that not all these limits are 0.)

Proof. (75.3) is a direct consequence of the definition of asymptotic expansion. We have

\[
F(x) \sim \frac{(-)^{n+1} [F_1(x)]^n}{n!} \quad \text{(first non-0 \( U_n \))},
\]

To prove (75.4), we remember that \( u_n(x) = (-1)^{n+1} G^{(n)}(y) \).

\[
g^{(n+r)} \sim (-)^{n+r+1} U_{n+r} \quad \text{as} \quad y \to 0. \quad g^{(n+r-1)} \sim (-)^{n+r+1} U_{n+r} \quad \text{by the finite analogue of (16) as explained in Chapter VII.}
\]

\[
g^{(n+r-2)} \sim (-)^{n+r+1} U_{n+r} \quad y^2/2! \quad \text{Keep this up.}
\]

\[
g^{(n)} \sim (-)^{n+r+1} U_{n+r} \quad y^r/r! \quad u_n(x) \sim (-)^{n+1} (-)^{n+r+1} U_{n+r} \quad y^r/r! \quad / /
\]
Corollary 80. If a B series, finite or infinite, exists, or, indeed, if some \( u_n(x) \to a \) constant, then \( f(x) \sim c F_1^{n-l} f_1 \), where \( n \) is the index of the first non-0 \( U_1 \), and \( c = (-1)^{n-l} U_n / (n-1)! \).

Proof. \( u_1(x) = f(x)/f_1(x) \to c F_1^{n-l} \) by (75.4). //

Proof of (76). Let \( a_n = (-1)^{n+1} U_n / n! \). \( a_n \neq 0 \iff U_n \neq 0 \). The existence of the \( u_n(x) \)'s and their finite limits guarantees a Taylor expansion of \( G(y) \) about 0 which is its asymptotic expansion about 0.

Let \( a_{n1}, a_{n2}, \ldots \) be the non-0 coefficients of that expansion.

\[
G(y) = F(x) = a_{n1} y^{n1} + a_{n2} y^{n2} + \ldots \quad F(x) - \sum_{1}^{k-1} a_{ni} y^{ni} \sim y^{nk} a_{nk},
\]

by the definition of asymptotic expansion, \( \implies F(x) \sim y^{n1} a_{n1} \),

\[
\implies E_{nk} \sim y^{nk} a_{nk}/F, \sim y^{nk-l} a_{nk}/a_{n1}. \quad \text{Since } y \to 0, \quad E_{n2} = o(E_{n1}),
\]

\( E_{n3} = o(E_{n2}) \), and so on. Thus, \( B \) is asymptotic II, and if \( F_1(x) = y \), has order \( < 0 \), \( B \) is also asymptotic I. //

Of course, \( B \) may be a finite series ending in \( a_{nk} y^{nk} \). Then, if \( m > nk \), we have \( \frac{F - \sum_{i=1}^{k} a_{ni} y^{ni}}{y} \to 0, \implies E_{nk}/y^{m-nl} \to 0, \)

\( \implies \theta(E_{nk}) = -\infty \).

Procedure A seems to have the advantage that it exists in many cases where \( B \) does not. The following theorem shows that under suitable regularity conditions, \( A \) is only asymptotic when an infinite \( B \) series exists, (and is therefore asymptotic.)

Theorem 81. If \( F_1 \) has order and \( u_{n+1}(x)F_1(x)/u_n(x) \to \) a limit for every \( n \) as \( x \to \infty \), then \( A \) is asymptotic II [and I] \( \iff \) an infinite sequence of \( u_n(x) \)'s \( \to \) constants [and \( \theta(F_1) < 0 \).]

Proof. We have defined a constant to be a finite, real number \( \neq 0 \).

Throughout this proof, we will use order theorems in the finite sense, as
described in Chapter VII. \( E_n(a) = \pm \int_0^b G^{(n+1)}(y) y^n \, dy/[n! \, F(a)]. \)

\[
\frac{E_{n+1}(a)}{E_n(a)} = - \int_0^b G^{(n+2)}(y) y^{n+1} \, dy \quad \frac{\lim_{x \to \infty} u_{n+2}(x) F_1(x)}{u_{n+1}(x)} =
\]

\[
\lim_{y \to \infty} - \frac{G^{(n+2)}(y) y}{G^{(n+1)}(y)} = - \theta(G^{(n+1)}), \quad (24), \quad (n+1) \lim_{b \to \infty} \frac{E_{n+1}(a)}{E_n(a)}, \quad (17).
\]

\( \theta(G^{(n+1)}) = \theta(G^{(n)}) - 1, \) unless \( G^{(n)} \to \) a constant.

Case 1: \( \theta(G) = \infty. \) (This means that \( G \to 0, \) because \( y^n \) is the typical function of order \( n \) in the finite sense. When \( n > 0, y^n \to 0. \) When \( n < 0, y^n \to \infty. \)) Then \( \theta(G) = \theta(G') = \theta(G'') = \ldots = \infty, \)

\[ \Rightarrow \frac{E_{n+1}}{E_n} \to -\infty \text{ as } b \to \infty, \text{ and } A \text{ is asymptotic II in reverse.} \]

Case 2: for \( n \geq N, \) no \( u_n(x) \to \) a constant. Then no \( G^{(n)}(y) \) \( \to \) a constant for \( n \geq N. \) If \( \theta(G^{(N)}) = \pm \infty, \) then \( A \) is asymptotic II in reverse, as in Case 1. If \( \theta(G^{(N)}) = r, \) finite, then \( \theta(G^{(N+m)}) = r-m. \)

\[
\frac{E_{N+m+1}}{E_{N+m}} \to - \frac{(r-m)}{N+m+1}, \text{ which } \to 1 \text{ as } m \to \infty, \text{ and } A \text{ is neither asymptotic II nor II. Infinitely slow convergence is not ruled out by this proof, however.}
\]

Case 3: An infinite sequence of \( u_n(x) \)'s \( \to \) constants. \( \lim_{b \to \infty} \frac{E_n/E_{n-1}}{E_{n+1}/E_{n-1}} \)

\[
= - \lim_{y \to \infty} \frac{G^{(n+1)}(y) y}{G^{(n)}(y)} = \lim_{x \to \infty} \frac{u_{n+1}(x) F_1(x)}{u_n(x)} \sim \text{constant} \cdot \frac{[F_1(x)]^s F_1(x)}{[F_1(x)]^r},
\]

by (75.4), = constant \([F_1(x)]^{s+1-r}\). Now \( s = r - 1 \) unless \( u_n(x) \to \) a constant, when \( s \geq r = 0, \) (75.4). Thus, \( E_n/E_{n-1} \to \) a constant unless \( u_n(x) \to \) a constant, when \( E_n/E_{n-1} \to 0 \) and \( \theta(E_n/E_{n-1}) \) is an integral multiple of \( \theta(F_1) \). Thus, if you always stop \( A \) after a term where \( u_n(x) \to \) a constant, you get a procedure that is asymptotic II, and, if \( \theta(F_1) < 0, \) is asymptotic I. //
The dependence of the asymptoticity of $A$ and $B$ on an infinite sequence of $u_n(x)$'s to constants severely restricts the usefulness of these procedures. We have seen that a necessary condition for this property is that $F$ be $\sim c F_1^n$, (75.3), and that $f$ be $\sim c' F_1^{n-1} f_1$, (80). Even if $F$ and $F_1$ are L-functions, there are many cases where the condition fails, as for example, when $F = e^{-x}$ and $F_1 = 1/x$. There is no $F_1$ that is universally applicable, but in each case, $F_1$ must be carefully tailored to fit $F$.

If a suitable $F_1$ is found, with $F \sim c F_1^n$, we'd have to differentiate $u_1(x)$ $n$ times to get the first coefficient of the $B$ series. As has been observed, labor would be saved if an $F_1$ could be found with $F \sim F_1$, which would require that $f \sim f_1$, (80).

**Example 82.** Such an improved $F_1$ cannot always be obtained by raising the old $F_1$ to the $n$th power.

Let $F = F_1^2 + F_1^3$. Then $F \sim F_1^2$. Let $F_2 = F_1^2 + F_2^3$. Now suppose an infinite $B$ series exists. Then $F$ has the asymptotic expansion, $F_2 + a_2 F_2^2 + a_3 F_2^3 + \ldots$, $\Rightarrow \frac{F - F_2}{F_2} \rightarrow a_2$. But

$$\frac{F - F_2}{F_2} = F_2^{-\frac{1}{2}} \rightarrow \infty, \text{ contradiction.}$$

Let us now consider the suitability of various likely choices for $F_1$.

$F_1(x) = 1/x$. The $A$ and $B$ series will be useful only if $f(x) \sim c x^{-n}$, (80). If $n$ is large, much differentiation will be needed to obtain even the first $B$ coefficient. The differentiation can be speeded by a pattern similar to Pascal's triangle.

$$u_n(x) = a_{n1} x^{n+1} f + a_{n2} x^{n+2} f' + \ldots + a_{nn} x^{2n} f^{(n-1)}$$

where the
coefficients are obtained from the triangle,

\[
\begin{array}{cccc}
  a_1 & a_2 & a_3 & a_4 \\
  u_1 & 1 & & \\
  u_2 & 2 & 1 & \\
  u_3 & 6 & 6 & 1 \\
  u_4 & 24 & 36 & 12 & 1 \\
\end{array}
\]

Each table entry = above (row + column of above) + above left. In other words, \( a_{ij} = a_{i-1,j} (i-1+j) + a_{i-1,j-1} \). \( G(y) \) is analytic at 0 \( \iff \) \( F(x) \) is analytic at \( \infty \) \( \iff \) \( f(x) \) is analytic at \( \infty \) and has order \( \leq -2 \), (70.3).

The \( 1/x \) transformation makes a good combination with mechanical quadrature.

\[ F(x) = G(y) = \int_0^y g(t) \, dt, \quad g(y) = F'(x)/y'(x) = x^2 f(x). \]

\[ g'(y) = (x^2 f(x))'/y'(x), = -x^2[x^2 f' + 2xf], \quad = -x^4 f' - 2x^3 f, \to 0 \quad \text{when} \quad y \to 0 \quad \text{if} \quad \theta(f) < -3. \] So we don't have trouble with infinite derivatives at end points. The \([0, 1/a]\) interval can be cut up into equal intervals by points \( k/na, \) \( k = 0, 1, \ldots, n. \) \( g(y) \) at these points, \( = x^2 f(x) \) for \( x = na/k. \) A good quadrature formula, like Weddell's rule, [20], can now be applied to get \( G(1/a), \) which = \( F(a) \).

\[ F_1(x) = e^{-x} \]. We can use this transformation only if \( f(x) \sim c \, e^{-nx}, \) (80).\[
\frac{u_n(x)}{u_n(x)} = e^{nx} \left[ a_{n1} f(x) + a_{n2} f'(x) + \ldots + a_{nn} f^{(n)}(x) \right]. \] The \( a_{ij} \)'s can be obtained from the recursion formula, \( a_{ij} = (i-l) a_{i-1,j} + a_{i-1,j-l} \)
which can be represented by the triangle

\[
\begin{array}{cccc}
  a_1 & a_2 & a_3 & a_4 \\
  u_1 & 1 & & \\
  u_2 & 1 & 1 & \\
  u_3 & 2 & 3 & 1 \\
  u_4 & 6 & 11 & 6 & 1 \\
\end{array}
\]

Each table entry = above • (above row number) + above left. \( G(y) \) is analytic at 0, (and hence \( B \) converges) \( \iff f(x) \) is an analytic function of \( e^{-x} \) at \( e^{-x} = 0 \), (69.2).

\[
F_1(x) = f(x). \text{ This transformation is only useful if } F \sim cf^n. \text{ If } \theta(f) \text{ is finite, then } \theta(f^n) < \theta(F), \text{ and } F \text{ cannot be } \sim cf^n. \text{ If } \theta(f) = -\infty \text{ and } F \sim cf^n, \text{ then } \log F \sim n \log f, \quad \frac{\log f}{\log F} \to \frac{1}{n}, \quad \implies n = 1, (30.2),
\]

\[
\implies F \sim cf, \quad \implies f/F \sim 1/c, \quad \implies -\log F \sim x/c, (16),
\]

\[
\implies \log f \sim -x/c, (30.2). \text{ Let } f(x) = e^{-x/c} h(x). \quad u_1(x) = f/f_1 \to \text{ a limit, } L, \quad \implies F/F_1 \to L, \quad \implies L = c, \quad \implies -f/f' \to c.
\]

\[
f' = e^{-x/c} \left[ h'(x) - \frac{1}{c} h(x) \right]. \quad \frac{f'}{f} = \frac{1}{c} \frac{h'(x) - h(x)}{h(x)} = \frac{1}{c} - \frac{h'(x)}{h(x)} \to \frac{1}{c},
\]

\[
\implies h'/h \to 0. \quad u_2 = u_1'/f'. \quad u_1' \sim \text{ constant } (1/u_1)', (8.3), = \text{ constant } \cdot (h'/h)', \quad \implies \lim u_2(x) = \lim u_1'/f' = \lim (h'/h)' / f', \text{ except for a constant factor, } = \lim (h'/h)/f, (17). \text{ This limit cannot be finite unless } \theta(h'/h) = -\infty, \implies h \to \text{ a constant, since, for any function, } g,
\]

\[
\theta(g'/g) \geq -1 \text{ unless } g \to \text{ a constant. We conclude that when } F_1 = f,
\]

A and B are useful only if \( f(x) \sim c_1 e^{-c_2 x} \).

\[
F_1(x) = xf(x). \text{ For this transformation to be useful, } F \text{ must be } \sim cf^nf^n,
\]

\[
\implies \log F \sim n \log x + n \log f. \text{ Suppose } \theta(f) = \infty. \text{ Log } x = o(\log f), (7),
\]

\[
\implies \log F \sim n \log f, \implies n = 1, (30.2), \implies F \sim cf, \implies f/f \sim 1/cx,
\]

\[
\implies -\log F \sim (\log x)/c, \implies \theta(F) = -1/c, \text{ a contradiction. Thus, } \theta(f) \text{ must be } r, \text{ finite, (presuming that } f \text{ has order at all.)}
\]
\[ u_1(x) = \frac{f}{f_1} = \frac{f}{-(xf' + r)} = \frac{-1}{xf' + 1}, \] 

a finite limit which must be 

\(-1/(r+1), (24). \) So we must have \( f \sim xf'/r. \) As before, \( u_1' \sim (1/u_1)', \) (except for a constant, by 8.3, ) \( = (xf'/f)' = (xf'/f - r)' \)

\[ u_2 = u_1'/f_1 \sim (xf'/f - r)'/f_1. \] \( \text{Lim } u_2 = (xf'/f - r)/F_1, (17), \) \( \Theta(u_3) = \Theta(u_2) - (r+1), \) and so on. We can conclude that \( A \) and \( B \) are only useful under the transformation, \( F_1(x) = xf(x) \) if \( \Theta(f) = \text{a finite number}, \ r, \ \text{and } \Theta(xf'/f - r) \) is a multiple of \( r+1. \)

\( F_1 = vf. \) The motivation for this transformation is Formula 34:

\[ F \sim \frac{v}{r+1} \text{vf. (v = -f'/f'). Thus } F \sim \text{constant} \cdot F_1, \text{ if } f \text{ is at all well behaved, and so the first requirement of a meaningful B expansion is satisfied. } \]

\[ f_1 = -(v'f + f'v) = f'(1-v'), \] \( 1/u_1(x) = 1-v', -1 + 1/r, \) \( (30.1), \) if \( f \) is well behaved. \( u_2(x) = u_1'/f_1. \) \( u_1'(x) \sim (1/u_1)', \) except for a constant factor, \( (8.3). \) Thus \( u_2(x) \sim v''/f_1, \) and 

\[ \lim u_2(x) = \lim v'/F_1, \text{ if } \Theta(f) = \infty, (17), = \lim v'/F_1, \text{ unless } v \to \text{a constant. Supposing that the functions appearing here have extended order, we find that } \Theta(f) = \Theta(F) = \Theta(F_1) = \Theta(F_1), (44), \text{ and that } \]

\[ |\Theta(v)| < |\Theta(f)|, (46). \text{ Hence, } |\Theta(v)| < |\Theta(F_1)|, \Rightarrow u_2(x) \to \infty. \] We must conclude that \( \Theta(f) \) is finite or that \( v \to \text{a constant. In the latter case, } f \sim \text{constant} \cdot f', \) and we have \( f \) bounded by the family \( (e^{-C}) \). If \( \Theta(f) \) is finite, \( v' \to -1/r, \) and 

\[ \lim (v''/f_1) = \lim [(v' + 1/r)/F_1]. \] We will see that \( \Theta(v' + 1/r) = \Theta(xf'/f - r). \)

We conclude that this transformation is only useful if \( \Theta(f) = \infty \) and \( f \) is bounded by the family, \( (e^{-C}) \), or if \( \Theta(f) = r, \) finite, and \( \Theta(v' + 1/r) \) is a multiple of \( r+1. \)

**Expansion of \( f^{-1}. \)** Another way one might want to apply the Taylor series is to expand the integral of \( g(y) = f^{-1}(y), \) when \( f(x) \) is monotonic
for \( x \geq a \), \( \int_a^\infty f(x)\,dx = \int_0^{f(a)} g(y)\,dy - a f(a) \). The asymptotic properties of this expansion are the same as those of \( A \) when no \( u_n(x) \rightarrow \) a constant. We can show, as with \( A \), that
\[
\lim_{y \to \infty} \frac{E_{n+1}(y)}{E_n(y)} = -\frac{\theta(g^{(n+1)}(y))}{(n+1)}
\]
= \(-\theta(g^{(n)}(y))\), presuming that limit exists. \( \theta(g) \) in the finite sense = \( 1/\theta(f) < 0 \). \( \theta(g') = \theta(g) - 1 \), and by induction, \( \theta(g^{(n)}) = 1/\theta(f) - n \). Hence, \( \lim_{n \to \infty} \lim_{y \to \infty} \frac{E_{n+1}/E_n}{1} = 1 \). We conclude that the procedure is neither asymptotic I nor II, and that if it converges, it does so infinitely slowly.

**Expansion of \( F/v\phi \).** An application of Procedure B is the expansion of \( F/v\phi \) in powers of \( 1/x \). We can subtract \( \gamma, = r/(r+1) \), where \( r = \theta(f) \), from \( F/v\phi \) to make \( F/v\phi \) correspond to the \( F \) of B, and then \( F_1 \) is \( 1/x \). We must require that \( (F/v\phi - \gamma)^r \) have integral order \( \leq -1 \). \( \theta(F/v\phi - \gamma)^r \) can be evaluated by considering the function, \( G = F/f + x/(r+1) \). \( G' = F/v\phi - \gamma \) and \( G'' = (F/v\phi - \gamma)' \). If \( r = \infty \)
\[
G = F/f, \quad 1/G = (-\log F), \quad -\theta(G) = \theta(\log F)' \quad \text{and} \quad -\theta(G) + 1 = \theta(\log f), \quad (30.2) \quad \theta(F/v\phi - \gamma)^r = \theta(G) - 2 = -\theta(\log f) - 1 \quad \text{for} \quad \theta(f) = -\infty \]
This, we have as a necessary condition for this expansion that \( \theta(\log f) \) be finite, a far less restrictive condition than others we have encountered. Indeed, this is the first procedure considered that yields an asymptotic series for the normal distribution. In that example, \( F/v\phi = x\Phi/\phi = 1 - x^2 + 3x^{-4} - 3.5x^{-6} + \ldots \) (See Example 89.)
The expansion is not easily obtained, since the $u_n(x)$'s will be expressions containing $F$; however, a method of undetermined coefficients may be applied. We write the expansion as \( \gamma + a_1x^{-1} + a_2x^{-2} + \ldots \). The relative frequency error for \( \nu f(\gamma + a_1x^{-1}) \) is calculated, and $a_1$ chosen to minimize its order. If the absolute frequency error is a simpler expression, that may be minimized asymptotically, instead. Then $a_2$ is chosen to minimize the error of \( \nu f(\gamma + a_1x^{-1} + a_2x^{-2}) \), and so on. If the order of the relative frequency error decreases, then the same must be true of the relative error, and the expansion is asymptotic I and II.

Since $A$ and $B$ are only useful when $B$ exists, the question arises as to which is better. We have seen that

\[
E_{nA} = (-1)^n \int_0^b G^{(n+1)}(y)y^n dy/n!/[n! F(a)] \quad \text{and}
\]

\[
E_{nB} = \int_0^b G^{(n+1)}(y)(b-y)^n dy/n!/[n! F(a)].
\]

\[
\frac{E_{nA}}{E_{nB}} = (-1)^n \frac{\int_0^b G^{(n+1)}(y)y^n dy}{\int_0^b (b-y)^n dy} \frac{\int_0^b y^n dy}{\int_0^b (b-y)^n dy}.
\]

This is the ratio of two weighted averages of $G^{(n+1)}(y)$, the top weighting the $b$ end of the interval, and the bottom the $0$ end. If $B$ exists, and $u_N(x) \to$ a constant, then $u_n(x) \to 0$ or a constant for $n > N$ by (75.4). Thus $G^{(n+1)}(y) = (-1)^n u_n(x) \to 0$ or a constant if a term of $B$ exists with degree $\geq n+1$. When $G^{(n+1)}(y) \to a$ constant, there would seem no general reason to believe one series more accurate than the other. If $G^{(n+1)}(y) \to 0$, then the integrand would tend to be smaller on the $0$ end of the interval, and the $B$ series would tend to be more accurate. If $B$ is a finite series, stopping
at the $n^{th}$ degree term, and $u_{n+1}(x) \to \pm \infty$, then the $A$ series would tend to be more accurate. The $B$ series may be handier to use, since it may contain fewer terms.

**Summary of Chapter IV.**

In order to apply a Taylor series to approximate $F(x)$, we must first apply a transformation, $y = F_1(x)$ carrying $\infty$ into 0. The series will not converge, except, possibly, "infinitely slowly," unless $G(y)$, which $= F(x)$, is analytic at 0. Conditions for this analyticity have been derived. The $B$ series, and expansion of $G(y)$ about 0, is asymptotic when it exists. But this existence requires a very fortunate choice of $F_1$. The $A$ series, an expansion about $F_1(a)$, exists under more general conditions, but is only asymptotic, and hence only useful, when the $B$ series exists. The most applicable Taylor procedure, an expansion of $F/vf$ in powers of $1/x$, is also the most difficult to apply, since a method of undetermined coefficients must be used. Mechanical quadrature can be simply applied to $G(y)$ after the transformation, $y = 1/x$, has been applied, if $\theta(f) < -3$. Unless the $B$ approximation is the whole of a finite $B$ series, the $B$ approximation is asymptotically no worse than the corresponding $A$ approximation, and is often better.
CHAPTER V

THE LAPLACE - WINCKLER EXPANSION

An expansion of more general applicability to tail integral approximation than the Taylor series is the Laplace-Winckler expansion. It was discovered by Laplace, [26, p. 88], in 1814 and given a simpler derivation and a remainder term by Winckler, [58], in 1871. As far as I know, it has not been referred to since that time.

Winckler derived the series using an integration by parts. He wrote, \( f = -f'v_1 \), where \( v_1 \) is what we have called \( v \), namely, \(-f/f'\). Then, integrating by parts, he obtained,
\[
\int_a^\infty f = \int_a^\infty -f'v_1 = v_1(a)f(a) + \int_a^\infty fv_1'.
\]
He then defined \( v_2 \) to be \( v_1v_1' \), so that \( f v_1' = -f'v_2 \). Then
\[
\int_a^\infty fv_1' = \int_a^\infty -f'v_2 = v_2(a)f(a) + \int_a^\infty fv_2'.
\]
The definition of \( v_3 \) was \( v_1v_2' \), and the derivation continued in this way. Thus the Expansion 83,
\[
\int_a^\infty f = f(a)[v_1(a) + \ldots + v_n(a)] + \int_a^\infty fv_n',
\]
was obtained. An implied assumption is that \( v_1f, \ldots, v_nf \to 0 \) as \( x \to \infty \) This will be the case when \( \int_a^\infty f \) exists, and regularity conditions hold, as the following theorem shows. (This point did not come up in Winckler's discussion, since he used the limits \( a \) and \( b \) rather than \( a \) and \( \infty \).)

Theorem 84. If \( \theta(f) = r, \neq 0 \) but possibly infinite, and \( v_1', v_2', \ldots, v_n' \) all \to \text{limits}, finite or infinite, then these limits are
\[
-1/r, 1/r^2, \ldots, 1/r^n.
\]

Proof. Case 1: \( r \) is finite. \( v_2/v_1 = v_1' \to -1/r \), (30.1).
\[ \lim v_2' / v_1' = \lim v_2' / (-1/r), \quad \text{and} \quad \lim v_2' / v_1', \quad (17), \quad = -1/r. \]

(Neither \(v_2\) nor \(v_1\) \(\to\) constants, since \(v_1' \to -1/r\) and \(v_2/v_1 \to -1/r\).) Hence \(\lim v_2' = 1/r^2\). \(\lim v_3'/v_1' = \lim v_3' / (-1/r), = \lim v_3'/v_1 = 1/r^2\). Hence \(\lim v_3' = -1/r^3\), and so on. Case 2: \(r\) is infinite.

\(v_2/v_1 = v_1' \to 0\). Suppose \(v_2' \neq 0\). Then \(v_1'/v_2' \to 0\), and \(v_1/v_2 \to 0\), (17), a contradiction. (17) fails only if \(v_1 \to \) a constant. Then \(v_2 = v_1 v_1' \to 0\), \(\Rightarrow v_2' \to 0\) in this case, too. Suppose \(v_3' \neq 0\).

Then \(v_1'/v_3' \to 0\), \(\Rightarrow v_1/v_3 \to 0\) or \(v_1 \to \) constant. The first alternative is a contradiction, and the second \(\Rightarrow v_3 = v_1 v_2' \to 0\), another contradiction. Thus \(v_3' \to 0\), and so on. //

The derivation of (83) holds under the same conditions with the added assumption that \(\int_0^\infty f \) exists. This, and the fact that \(v' \to -1/r \Rightarrow F \sim \frac{x}{r+1} \) \(vf\), \(\Rightarrow vf \to 0\), \(\Rightarrow (v_1 + \ldots + v_n)f\),

\[ = (1 + v_1' + \ldots + v_{n-1}') \) \(vf, \to 0\), (84), and the derivation is completed. //

If \(F \to \infty\), the original formulation of Laplace and Winckler is appropriate:

\[ \int_a^x f(t)dt = \int_a^x[v_1(t) + \ldots + v_n(t)] + \int f v_n'. \]

This formula reduced to the previous one when \(F \to 0\) and \(x = \infty\).

Laplace's derivation of this series is an interesting example of how an uninhibited handling of mathematical formulae can lead to correct results, [26, p. 88].

He let \(t(x) = \log [f(a)/f(x)]\), so that \(f(x) = f(a) e^{-t}\), and he expanded \(x\) in a Taylor series about \(a\): \(x = a + t \frac{dx}{dt} a + \frac{t^2}{2!} \frac{d^2x}{dt^2} a + \ldots\). Let \(v_1(x) = -f(x)/f'(x)\) and \(v_{n+1}(x) = v_n'(x) v_1(x)\), as before. \(\frac{dt}{dx} = -\frac{f'(x)}{f(x)} = \frac{1}{v(x)} \Rightarrow \frac{dx}{dt} = v\).
\[
\frac{d^2 x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx} \frac{dx}{dt} = vv' = v_2, \quad \text{and so on, so that } x = \\
a + tv_1(a) + \frac{t^2}{2!} v_2(a) + \ldots. \quad dx = dt [v_1(a) + tv_2(a) + \frac{t^2}{2!} v_3(a) + \ldots]. \\
\int_a^\infty f(x)dx = f(a) \int_a^\infty e^{-t} [v_1(a) + tv_2(a) + \frac{t^2}{2!} v_3(a) + \ldots] dt.
\]

Substituting \(\int_0^\infty e^{-tn} = n!\), he obtained

\[
\int_a^\infty f(x)dx = f(a) [v_1(a) + v_2(a) + \ldots].
\]

Laplace used a finite integral in his series. I have changed the notation somewhat to be consistent. //

**Bounds** 85. As with the Taylor series, the remainder term can be written in a convenient form: \(\int_a^\infty f(x) = v_n(x_1) \int_a^\infty f\), where \(a \leq x_1 < \infty\) by a mean value theorem. (We are assuming that \(f > 0\).) Then

\[
F(a) = f(a) [v_1(a) + \ldots + v_n(a)]/[1 - v_n'(x_1)].
\]

By substituting \(\sup(v_n'(x); x \geq a)\) and \(\inf(v_n'(x); x \geq a)\) for \(v_n'(x_1)\) in the above formula, provided that they are not on opposite sides of 1, one obtains bounds for \(F(a)\). If \(v_n'(x)\) is monotonic for \(a \leq x\), the values are \(v_n'(a)\) and \((-)^n/r^n\), (84).

If \(F(x) \to \infty\), the remainder = \(\int_a^X f v_n' = v_n'(x_1) \int_a^X f\), where

\[
a < x_1 < x. \quad F(a,x) = \frac{|f(t)|v_1(t) + \ldots + v_n(t)|a}{1 - v_n'(x_1)}.
\]

Sup \(v_n'(t)\) and \(\inf v_n'(t), a < t < x\), if not on opposite sides of 1, can be substituted to obtain bounds for \(F(a,x)\).

**Bounds** 86. If \(v_n\) and \(f \to 0\) monotonically, which occurs only if \(\theta(f) = -\infty\), then even simpler bounds are obtainable. \(\int_a^\infty f v_n'\) lies between \(f(a) v_n'\) and 0, that is, between \(-f(a) v_n(a)\) and 0.

Thus \(F(a)\) lies between \(f(a) [v_1(a) + \ldots + v_{n-1}(a)]\) and \(f(a) [v_1(a) + \ldots + v_n(a)]\).

The conditions for these bounds are not as remote as they might
seem. We will see that the expansion is asymptotic only when \( \theta(f) = -\infty \).

If \( \theta(f) \) is finite, a modification of the expansion is appropriate, and will be discussed later. If the \( v(x)'s \) have order, \( v_n \to 0 \) if \( \theta(v_n) < 0 \). We will see that when \( \theta(f) = \pm \infty, \theta(v_n) = n\theta(v') + 1 \).

In practice, \( \theta(v') \) is usually a negative integer. \( \theta(v') \) is only 0 if \( \theta(f) = \pm (1, 0) \), (46), which is a rare occurrence. (See Example 88.)

**Theorem 87.**

1) \( e_n = -v_n' \).

2) If \( f, v_1', v_2', \ldots, \) have order, \( \theta(v_1) = 1-\epsilon \), (which means that \( \theta(f) = \pm \infty \)) then (83) is asymptotic I and II; and \( \theta(E_n) = \theta(v_n') = -n\epsilon \) if \( \epsilon \neq 1 \), and \( = -n + 1 + \theta(v_1') \) if \( \epsilon = 1 \).

We must also require that \( v_i \neq 0 \) a constant for \( i > 1 \), and that when \( v_1 \to \) a constant, \( \theta(v_1') \) is finite. It could otherwise happen that \( |\theta(v_1')| > |\theta(f)| \), which would make (83) asymptotic in reverse, (94.1).

3) If \( \theta(v_1) = 1 \), and \( v_1' \) and \( v_{i+1}'/v_i' \) limits for \( i = 1, \ldots, n \), then \( E_{i+1}/E_i \to -1/r \), for \( i = 1, \ldots, n \), 2), and 3) apply if \( F(x) \to \infty \).

**Proof of 1) and 2).** Suppose \( \epsilon \neq 1 \), \( \theta(v_1) = 1-\epsilon, \theta(v_1') = -\epsilon. \)
\( \theta(v_2) = 1-2\epsilon \), \( \theta(v_2') = -2\epsilon. \) And so on. \( \theta(v_n') = -n\epsilon \) if \( \epsilon = 1 \), or \( \theta(v_1) = 0, \theta(v_2) = \theta(v_1'), \theta(v_3) = \theta(v_2') = \theta(v_1') - 1 \), (16), and so on. \( \theta(v_n') = \theta(v_{i+1}') - n+1 \).

Now \( E_n = \int_a^\infty f v_n'/\int_a^\infty f \). \( e_n = -v_n' \).

Proving 3). \( \lim E_{i+1}/E_i = \lim \int_a^\infty f v_{i+1}'/\int_a^\infty f v_1' \).

\( = \lim v_{i+1}'/v_1' \), (17) = \( \lim v_{i+1}'/v_1' \), (17), (since, by the reasoning of 2), \( \theta(v_1') = 0 \) and \( v_{i+1} \) cannot therefore \( \to \) a constant.)

\( = \lim v_1'/v_i', \lim v_1'/v_i', = \lim v_1/v_i' \), = \( \lim v_1/v_i', = \ldots = \lim v_2/v_1 \), = \( \lim v_1' = -1/r. // \)

Thus, the Laplace-Winckler series is, under regularity conditions, asymptotic II if \( \theta(f) = -\infty \), asymptotic I if, in addition, \( \theta(v) < 1 \),
and shows a regular improvement, though not an asymptotic one, if 
\[ |r| > 1. \]

We have seen that \( v' \to -1/r \) if any limit. Hence, \( \theta(v) < 1 \),
\( \iff \theta(v') < 0 \), only if \( r = \pm \infty \). It is still possible to have
\( \theta(v) = 1 \) and \( \theta(f) = -\infty \).

**Example 88.** (Hardy, [15, p. 23]). This would be the case if
\( \theta(f) = -(1,0) \), since \( \theta(\log f) = 0 \iff \theta(f'/f) = -1, \iff \theta(v) = 1. \)

Hardy's example of a function \(< \text{any } x^{-n} \text{ and }> \text{any } e^{-x/n} \text{, fits this description. It is } e^{-(\log x)^2} \text{, which is the same as } x^{-\log x}. \ ////

**Example 89.** (Laplace, [26, p. 103]). The well-known asymptotic series
for the normal distribution,
\[
\int_{-\infty}^{\infty} \varphi(x) \, dx = \varphi(a) \left( a^{-1} - a^{-3} + 3a^{-5} - 3\cdot5a^{-7} + \ldots \right),
\]
is a neat example of the Laplace-Winckler expansion. \( v_1(x) = x^{-1} \), \( v_2(x) = x^{-3} \), and so on. \( \varepsilon = 2 \), so that \( \theta(E_2) = -2n \), and the series is asymptotic I and II. It diverges for all \( a \), since the ratio of succeeding terms \( \to \infty \).

**Example 90.** The Laplace-Winckler series for \( f(x) = x^r \) is convergent
but not asymptotic.

It is \( a^{r+1}(-a/r + a/r^2 - a/r^3 + \ldots) \) which converges to
\( a^{r+1} \cdot \frac{-1}{r}/(1 + \frac{1}{r}) = a^{r+1}/(r+1) = F(a). \) \( v_n(x) = x^r/n \), so that
\( v_n'(x) \) not only \( \to \nabla/r^n \), but \( \equiv \frac{1}{r^n} \). \ ////

The Laplace-Winckler series is a special case of a general iterative
procedure. Let us derive it by writing, \( F(x) = u(x) \cdot f(x) \), (or simply,
\( F = uf \).) \( f' = -f = u'f + f'u, \iff f = -f'u/(1+u'), \iff u = (1+u') \cdot v. \)

If we solve this equation iteratively, we get

**Procedure C:** \( u_2 = (1 + u_1') \cdot v. \) If we start with \( u_1 = v \), we get
the Laplace-Winckler series. \( F_1 = vf. \) \( F_2 = f' \cdot (v + v_2). \)
\( F_3 = f(v + vv' + vv_2') = f(v + v_2 + v_3). \) And so on. If we start with some
other $u_1$, we get $F_1 = u_1f$, $F_2 = f'(v + u_1v)$, $= f'(v + w_2)$, say. 
$F_3 = f'(v + v_2 + w_2v)$, say, and so on. Thus, 
$F_n = f'(v_1 + v_2 + ... + v_{n-1} + v_n)$, where $w_2 = u_1'v$ and $w_{n+1} = v_n'v$. 
So the series is the same as the Laplace-Winkler series except for the last term.

If $F(x) \to \infty$, the approximations are all taken between the limits $a$ and $x$, so that $u_1f = u_1(a) f(a) - u_1(x) f(x)$. If $u_1 = v$, we get the Laplace-Winkler series in the $a$ and $x$ form.

The remainder term $= F - u_n f = \int_a^x f + (u_n f)'$ 
$= f + u_n f + f' u_n$, $=-f'(v + u_n'v - u_n)$, $=-f'(u_{n+1} - u_n)$, 
$=-f'(w_{n+1} + v_{n-1'}, v_n - v_{n-1})$. Thus, the remainder 
$= \int_a^x f(w_1' + v_{n-1'} - v_{n-1})$, where $v_1 = u_1$ and $w_{n+1} = v_n'v$.

Generalizations of Bounds 85 and 86 apply to Procedure C.

Bounds 85. The remainder $= \int f'(w_n' - w_{n-1'} + v_{n-1})$, $= \int f(u_n' - u_{n-1})$.
If $f$ is of constant sign, this can be written as $g'(x_1) F(a)$, 
a $\leq x_1 < \infty$, where $g'(x) = w_n'(x) - w_{n-1}'(x) + v_{n-1}'(x)$, 
$= u_1'(x) - u_{n-1}'(x)$. Then $F(a) = F_0(a)/[1 - g'(x_1)]$, and if the sup and inf of $[g'(t); a \leq t \leq x]$ are not on opposite sides of 1, they can be substituted for $g'(x_1)$ to obtain generalized Bounds 85.

If $F(x) \to \infty$, the remainder $= \int_a^x f g'$. $F(a, x) = F_0(a, x)/[1 + g'(x_1)]$, 
where $a < x_1 < x$; and sup and inf of $[g'(t); a \leq t \leq x]$ can be substituted, if not on opposite sides of 1, to get Bounds 85.

Bounds 86. 1) If $f$ and $u_1 - u_2$ are of constant sign, and $\to 0$ monotonically or both $\to +\infty$ monotonically, then $F$ lies between $F_1$ and $F_2$ of Procedure C.

2) If $F$ and $F_1$ converge, and $f \to 0$ monotonically, and $u_1 - u_2$ is of constant sign and $\to +\infty$ monotonically, then $F_2$ lies between $F_1$ and $F$. 
3) A special case of 2) is when \( \theta(f) \) is finite and \(<0\), \( v' \rightarrow \) a limit, \( f \) and \( u_1-u_2 \) are monotonic and of constant sign, and \( u_1f \sim cF \), \( c \neq 1 \).

Parts 2 and 3 lead us to believe that Procedure C may often be neither asymptotic I nor II, but converge monotonically and fairly rapidly with an asymptotic error ratio of \(-1/\theta(f)\), (92). A good example of this would be (90).

Proof of (96). \( F_1(a,x) - F(a,x) = \int_a^x (u_1f' - f) \, dx = \int_a^x (-u_1f' - u_1f' - f) \, dx \)
\[ = \int_a^x f' \cdot (u_2 - u_1) = [u_2(x) - u_1(x)] \cdot [f(x) - f(a)], \text{ where } a \leq x \leq x, \]
\[ = \psi(x) f(x) - \psi(x) f(a), \text{ writing } \psi(t) \text{ for } u_2(t) - u_1(t). \text{ Proof of } 1). \]
Case 1: \( \psi(x) \) and \( f(x) \) are \( >0 \) and increase to \( \infty \). Then \( 0 < \psi(x) f(x) - \psi(x) f(a) < \psi(x) f(x) - \psi(a) f(a) \), \( \Rightarrow 0 < F_1F \leq F_1 F_2 \), \( \Rightarrow F_2 < F < F_1 \). The other cases of 1) are analogous. If \( F \) and \( F_1 \) converge, the same proof becomes simpler. Proof of 2). \( F_1F \)
\[ = \int_a^\infty (-u_1f') \, dx = -\psi(x) f(a). \text{ Case 1: } f \text{ decreases to } 0 \text{ and } \psi \text{ increases to } \infty. \]
\[ -\psi(x) f(a) < -\psi(a) f(a), \Rightarrow F_1F < F_1 F_2 \text{ and } \]
\( F_1F < 0 \). Since \( u_2 - u_1 > 0 \), \( F_2 > F_1 \). Hence, \( F_1 < F_2 < F \). The case where \( u_2 - u_1 \) decreases to \( -\infty \) is analogous. Proof of 3). Suppose \( u_1 - u_2 \rightarrow \) a finite limit. Since \( v \to \infty, (u_1-u_2)/v, = e_1 \to 0, \Rightarrow E_1 \to 0, \)
\( (17), \Rightarrow u_1f/F \to 1, \Rightarrow c = 1, \) a contradiction. Thus, \( u_1 - u_2 \to +\infty, \)
\( \Rightarrow \text{ Part 2) applies. } // \)

Series 91. In order to improve the performance of C for finite \( r \),
we could start with \( \gamma v \) rather than \( v \), motivated by Formula 34:
\( F \sim \gamma v F' \), where \( \gamma = r/(r+1) \). This reduces to the Laplace-Winkler series
when \( r = -\infty \). When \( r \) is finite, \( F_n = f \cdot (v_1 + v_2 + \ldots + v_{n-1} + v^r_n) \).
The error term is \( \int_a^\infty f \cdot (\gamma v_n' + v_n' - \gamma v_n') = \int_a^\infty f \cdot (\frac{r}{r+1} v_n' + \frac{1}{r+1} v_n'). \)
The (91) error is an infinite improvement over the (83) error, since
\[
\lim E_{91}/E_{83} = \lim \left( \frac{r}{r+1} v_n' + \frac{1}{r+1} v_{n-1}' \right)/\lim v_n', \quad (17),
\]
\[
= \left[ \frac{(-1)^n}{(r+1)r^{n-1}} \right] + \frac{(-1)^n}{(r+1)r^{n-1}} / (-)^n r^n = 0.
\]

However, (83) had the desirable feature that each iteration was an improvement over the previous one. This is only true of (91) if \(|1 + \theta(v' + 1/r)| < |r|\). This fact is a special case of the following general theorem about the progression of errors in Procedure C.

**Theorem 92.** Let \( F_1' = u_1f \), be an approximation to \( F \), an integral that exists; let \( u_2', u_3', \ldots \), be obtained by Procedure C; let \( v' \to \) a limit; and let neither \( F_1', F_2', F_3' \), nor \( F_3' - F \to \) a constant.

1) \( e_1 = (u_1 - u_2)/v. \)

2) If \( f \) and \( u_1 - u_2 \) have order and not both orders are infinite, then \( E_2/E_1 \to -\theta(u_1 - u_2)/\theta(f) = [1 + \theta(e_1)]/\theta(f) \), if it \( \to \) a limit at all. \( [1 + \theta(e_1)]/\theta(f) \) is taken to mean 0 if \( \theta(f) = -\infty \).

3) \( E_3/E_2 \to \) the same limit, if it \( \to \) a limit at all, and if neither \( u_2 - u_3 \) nor \( u_1 - u_2 \) \( \to \) a constant.

4) If \( F_1 = yvf \), so that \( C \) becomes Series 91, then \( e_1 = -\gamma(v' + 1/r) \) and \( E_2/E_1 \to -[1 + \theta(v' + 1/r)]/r. \)

This whole theorem applies to the case, \( F(x) \to \infty. \) \( F(x) \) becomes \( F(a,x). \) \( F_1(x) = u_1(a) f(a) - u_1(x) f(x), \) so that \( f_1(x) = -[u_1(x) f(x)]'. \) \( E_1(x) \) becomes \( E_1(a,x). \)

Proof. The proof of 1) is direct. Proof of 2). \( e_2/e_1 \)

\[
= (u_2 - u_3)/(u_1 - u_2), = \frac{v(u_1' - u_2')}{u_1' - u_2'}, = \frac{x(u_1 - u_2)' / xf'}{u_1' - u_2'}, \to \text{a limit},
\]

L. \( v' \to \) a limit \( \implies xf'/f \to \theta(f), \) (30.1, 27, 24). Let \( \theta(f) = r \) and \( \theta(u_1 - u_2) = s. \) Case 1: \( r = +\infty. \) Then by hypothesis, \( s \) is finite. If \( L \neq 0, \) \( x(u_1 - u_2)'/(u_1 - u_2) \to \infty \implies s = \infty, \) a contradiction. Hence, \( L = 0, \) proving this case. Case 2: \( r \) is finite. Then
\[ x(u_1 - u_2)/ (u_1 - u_2) \to -L. \] Suppose \( L \) is finite. Then, by (24), \(-L = s, \iff \lim E_2/E_1 = \lim e_2/e_1, (53), = L = s/r. \) If \( L \) is infinite, then either \( s \) is infinite or \( r = 0 \). In either case, \( L = s/r. \)

\[ 1 + \theta(e_1) = \theta(u_1 - u_2) \quad \text{by 1) if } \theta(f) \text{ is finite. Otherwise,} \]
\[ \theta(f) = +\infty \iff L = 0, \text{ so the theorem still holds.} \]

Proof of 3). \( \lim E_3/E_2 = \lim e_3/e_2, (53), = \lim \frac{u_3 - u_4}{u_2 - u_3}, \)
\[ = \lim \frac{v(u_2' - u_3')}{v(u_1' - u_2')}, = \lim \frac{u_2 - u_3}{u_1 - u_2}, = \lim (u_2 - u_3)/(u_1 - u_2), (17). \]

4) is a special case of 2). //

This theorem shows that if \( \theta(f) \) is finite, Procedure C can only be asymptotic II if \( \theta(e_1) = \theta(e_2) = \ldots = -1. \) Since \( \theta(E_n) = \theta(e_n), \) the procedure can never be asymptotic I in this case. The same comments, of course, apply to the special case, Series 91, so that we will attempt to find a better modified procedure for finite \( r. \)

The condition in 3) that \( u_1 - u_2 \neq \) a constant is necessary, since the following theorem shows that Procedure C places virtually no restriction on \( u_1 - u_2. \)

Theorem 93. 1) Given any positive, integrable function, \( f, \) including functions that \( \to \infty, \) and any function, \( g, \) then there exists \( u_1 \) such that \( u_1 - u_2 = g. \)

2) If we further require that \( g = o(v), \) then there exists \( u_1 \) such that \( u_1 - u_2 = g \) and \( u_1 f \sim F. \) Conversely, if \( u_1 f \sim F, \) then \( u_1 - u_2 = o(v), \) provided that \( (u_1 - u_2)/v \to \) a limit at all.

Thus, if \( f \) is finite, so that \( \theta(v) = 1, \) one can find a \( u_1 \) such that \( u_1 f \sim F \) and \( u_1 - u_2 \to \) a constant, (or = a constant, for that matter.)

Proof. Let \( u_1 = \int_c^x f'((x + 1)dt - L)/f, \) where \( L = \lim_{x \to \infty} \int_c^x f'((x + 1)dt \)
if it converges, and = 0 otherwise. \(- (u_1 f)' = f' (\frac{\dot{g}}{v} + 1)\). \(- (u_1 f)' / f - 1 = g / v\) and also \(e_1 = (u_1 - u_2) / v\). Hence, \(g = u_1 - u_2\). Now suppose \(g = o(v)\). Then \(e_1 \to 0\), \(\Rightarrow E_1 \to 0\), (17), \(\Rightarrow F_1 \sim F\). Conversely, if \(u_1 f \sim F\), then \(E_1 \to 0\), so that \(e_1 = (u_1 - u_2) / v, \to 0\) if any limit, by (17). (17) doesn't apply if \(F_1 - F\) a constant. If \(F(x) \to 0\), we have defined \(u_1 f\) so that it / a constant, and therefore (17) applies. If \(F \to \infty\), \(E_1\) still \(\to 0\), even if \(F_1 - F\) does \(\to\) a constant. //

When \(\Theta(f) = - \infty\), we have seen that the Laplace-Winclker series is asymptotic. The asymptoticity of the general Procedure C in this case depends, paradoxically, on making a poor enough initial approximation, \(F_1\). For a run-of-the-mill \(F_1\) with \(\Theta(E_1) > \Theta(F)\), C is asymptotic II, but if \(F_1\) is so good that \(\Theta(E_1) < \Theta(F)\), then C turns out to be asymptotic II in reverse, as the following theorem shows.

**Theorem 94.1.** If \(|\Theta(e_1)| > |\Theta(f)|\), and \(\Theta(f) = \pm \infty\) and \(v\) has extended order, and neither \(F_1 - F\) nor \(F_2 - F\) \(\to\) constants, and if \(e_1 / e_2 \to \) a limit, then \(E_1 / E_2 \to 0\). If, in addition, \(e_2 / e_3 \to \) a limit and \(e_2\) has extended order, and \(F_3 - F\) / a constant, then \(E_2 / E_3 \to 0\). Similarly for \(e_3, e_4, \ldots\), under analogous regularity conditions, so that C is asymptotic II in reverse.

**Proof.**

\[
\frac{e_1}{e_2} = - \frac{f'}{f} \frac{(u_1 - u_2)'}{(u_1 - u_2)} \to L, \text{ say, as in the proof of (92.2),} \Rightarrow \frac{\log f}{\log (u_1 - u_2)} \to L, (17). \text{ Now } |\Theta(e_1)| > |\Theta(f)| > |\Theta(v)|,
\]

\(46\), \(\Rightarrow |\Theta(e_1)|, = |\Theta(\frac{u_1 - u_2}{v})|, = |\Theta(u_1 - u_2)|, (42), > |\Theta(f)|,
\]

\(\Rightarrow \log f / \log (u_1 - u_2) \to 0, (39, 37), \Rightarrow L = 0.
\]

\[\Theta(e_2) = \Theta(\frac{u_2 - u_3}{v}), = \Theta(u_1' - u_2'), = \Theta(u_1 - u_2), (44), = \Theta(e_1),\]

so that \(|\Theta(e_2)| > |\Theta(f)|\). Then the same reasoning applies as before,
so that \( e_2/e_3 \to 0 \). \( \lim E_1/E_2 = \lim e_1/e_2 \), and \( \lim E_2/E_3 = \lim e_2/e_3 \), (53). We could apply (17), because both \( f \) and \( u_1 - u_2 \) have order \( \pm \infty \), and so neither of them, nor their logs, \( \to \) constants. //

**Theorem 94.2.** If \( |\Theta(e_1)| < |\Theta(f)| \), and \( \Theta(f) = \pm \infty \), and \( v \) has extended order, and neither \( F_1 = F \) nor \( F_2 = F \to \) constants, and if \( e_2/e_1 \to \) a limit, then \( E_2/E_1 \to 0 \). If, in addition, \( u_1 - u_2 \not\to \) a constant, and \( e_3/e_2 \to \) a limit, and \( e_2 \) has extended order, and \( F_3 = F \not\to \) a constant, then \( E_3/E_2 \to 0 \). A similar result holds for \( E_4/E_3, E_5/E_4, \ldots \), under analogous regularity conditions, so that \( C \) is asymptotic II.

Proof. \[
\frac{e_2}{e_1} = -\frac{(u_1 - u_2)'}{u_1 - u_2} \frac{f'}{f} \to L, \quad \implies \frac{-\log(u_1 - u_2)}{\log f} \to \frac{-\log(u_1 - u_2)}{\log f} = \lim [\frac{-\log(u_1 - u_2)}{\log f} - c] = 0, \quad \text{since} \quad \log f \to \pm \infty \]

\( \log e_1 + \log v. \) Since both \( \log e_1 \) and \( \log v = o(\log f) \), (37, 45, 46), \( \log(u_1 - u_2)/\log f \to 0, \implies L = 0 \). By (53), \( \lim E_2/E_1 = \lim e_2/e_1 = 0 \). Now we assume \( e_2 \) has extended order. \( \Theta(e_2) = \Theta(u_1 - u_2)' = \Theta(u_1 - u_2) - 1 \), (18), if \( u_1 - u_2 \) is finite, \( \Theta(u_1 - u_2) \) otherwise, \( \implies |\Theta(e_2)| < |\Theta(f)| \), \( \implies E_3/E_2 \to 0 \) by the first part of the proof. // If we hadn't assumed that \( u_1 - u_2 \not\to \) a constant, then \( |\Theta(u_1 - u_2)'| \) could be \( > |\Theta(f)| \), and the asymptoticity of the procedure would go into reverse.

We have seen, (87), that if \( F_1 = v f \), then \( C \) is asymptotic II. This agrees with (94.2), since \( |\Theta(e_1)| = |\Theta(v')|, < |\Theta(f)| \). If \( \Theta(v') < 0 \), then \( C \) in this case is asymptotic I, since, under suitable regularity conditions, \( \Theta(E_{n+1}/E_n) = \Theta(e_{n+1}/e_n) \), (57.2), \( \Theta(v'_{n+1}/v'_n) = \Theta(v'_{n+1}/v'_n) \), (57.2), \( \Theta(v'_{n+1}/v'_{n-1}) = \cdots \), \( \Theta(v'_2/v'_1) = \Theta(v') \).
Theorem 92 has shown that when \( \theta(f) \) is finite, \( \frac{E_{n+1}}{E_n} \neq 0 \), regardless of the initial approximation chosen, unless \( \theta(u_n - u_{n+1}) = 0 \).

A modification of Procedure C is therefore desirable for finite \( \theta(f) \).

**Procedure Cl.** Such a procedure is obtained by the use of Theorem 55, which says that if \( \frac{E_2}{E_1} \to L \), then the linear combination,

\[
F_{21} = \frac{L}{L-1} F_1 - \frac{1}{L-1} F_2
\]

has an error, \( \frac{E_2}{E_1} \), that is \( o(E_1) \). We have seen that in C, \( \frac{E_2}{E_1} \to -[1 + \theta(e_1)]/r \), (where \( r = \theta(f) \)). Let that limit be \( L \). Then construct \( F_{21} \) as in (55), so that

\[
u_{21} = \frac{L}{L-1} u_1 - \frac{1}{L-1} u_2,
\]

and \( \frac{E_{21}}{E_1} \to 0 \). \( u_{21} \) is obtained by C, and

\[
u_{22}' = (1 + u_{21}') \nu_2 \to E_{22}/E_{21} \to -[1 + \theta(e_{21})]/r = L_2, \text{ say. Then}
\]

\[
u_{31} = \frac{L_2}{L_2-1} u_{21} - \frac{1}{L_2-1} u_{22},
\]

and \( \frac{E_{31}}{E_{21}} \to 0 \). Thus Cl is a modification of C that is asymptotic II as far as it goes.

It could happen that \( \theta(e_{n1}) = -\infty \) for some \( n \). Then

\[
L_n = \infty, u_{n+1,1} = u_{n1},
\]

and the procedure stops at the \( n \)th term, as is shown in Example 96, below. Since \( \theta(e_{n+1,1}) \leq \theta(e_{n1}) \), it could happen that \( \theta(e_{n1}) \to -\infty \) with \( n \), and Cl would be asymptotic I as well as II. A neat example of this is the Cl expansion of the t distribution discussed in Chapter IX. Finally, it could happen that

\[
\theta(e_1) = \theta(e_{21}) = \theta(e_{31}) = \ldots,
\]

as in the following example, so that the procedure is asymptotic II but not I.

**Example 95.** This is the one case when C can be asymptotic II for finite \( r \); namely, when \( \theta(e_n) \equiv -1 \). The C and Cl procedures coincide, since

\[
L_1 = 0, \text{ and hence, } u_{21} = u_2, u_{31} = u_3, \text{ etc.}
\]

Let \( u_1 - u_2 \) be a well-behaved function \( \to 0 \) and of order \( O \).

\[
\theta(u_2 - u_3) = \theta(v(u_1 - u_2)) = \theta(u_1 - u_2) - 1 + 1 = 0.
\]

And \( u_2 - u_3 \to 0 \), since \( e_2/e_1 \to 0 \) by (92). Thus, \( \theta(e_1) = \theta(e_2) = \ldots = -1 \), and Cl = Cl is asymptotic II but not I.
Now the only question is whether we can find a $u_1$ such that $u_1 - u_2 \to 0$ and has order 0. We have seen, (93), that $u_1 - u_2$ can be made to order and still have $u_1 f \sim F$, provided only that $u_1 - u_2 = o(v)$, which would be true in this case.

In this example, we will go a little further, and suppose that $u_1 = \gamma v$, so that we have a Series 91. $u_1 - u_2$ comes out $-\gamma v (v' + l/r)$. Let $f(x) = \int_c^x \frac{dt}{e^t \log \log t - t/r}$. $(\log f)' = \frac{1}{\log \log \frac{x}{x/r}}$.

$v = -l/(\log f)' = \log \log x - x/r$. $v' = l/(x \log x) - l/r$. Since $v' \to -l/r, \theta(f) = r, (30.1)$. $\theta(v' + l/r) = \theta(e_1) = -l$. $(u_1 - u_2)' \text{constant} = v(v' + l/r), = \frac{\log \log x - x/r}{x \log x} \sim -\frac{1}{r \log x} x \to 0$. Thus, $u_1 - u_2 \to 0$

and has order 0. //

Remembering that when $\theta(f) = \infty$, C was asymptotic I if $\theta(v') < 0$, one might speculate that Cl would be asymptotic I if $\theta(v' + l/r)$ were < 0. However, the previous example refutes that conjecture. I have investigated some other seemingly favorable initial conditions, but have not found any that made Cl necessarily asymptotic I.

Example 96. A Cl procedure that terminates.

1) The simplest example is a $u_1 - u_2$ of order $-\infty$. By (92), $E_2/E_1$ would $\to \infty$, $F_2 \to F_1$, and Cl consists of the one approximation, $F_1$. As in the previous example, we may find a Cl with this property starting with $\gamma v$. We require that $\theta(v' + l/r) = -\infty$, so we'll let $v' + l/r = e^{-x}$, which happens if $v = -e^{-x} - x/r$, i.e., if $f'/f = 1/(e^{-x} + x/r)$, i.e., if $\log f = \int_c^x \frac{dt}{e^t + t/r}$, i.e., if $f = e^{\int_c^x \frac{dt}{e^t + t/r}}$. //

2) A series that stops at two terms may be obtained as follows.

Let $f = x^r$, $\implies$ $xf'/f = r$. $\frac{f_2}{f_1} = \frac{g'}{g}$ / $\frac{f'}{f} = \frac{xe^r}{xg^r}$, where $g = u_1(x) - u_2(x)$. Now we choose $g$ so that $\theta(\frac{xg^r}{g} - s) = -\infty$. //
where \( s = \theta(g) \). (We want \((\log g)'x = s + e^{-x}\), i.e., \((\log g)'

= (s + e^{-x})/x, i.e., g = e^\int_c^x \frac{s + e^{-t}}{t} \, dt.\) Hence, \(\theta(e_2/e_1 - L) = -\infty\).

Now \(e_{21} = \frac{L}{L-1} e_1 - \frac{1}{L-1} e_2\), \((54)\), \(\implies \frac{e_{21}}{e_1} = \frac{1}{L-1} (L - \frac{e_2}{e_1})\)

\(\implies \theta(e_{21}/e_1) = -\infty = \theta(e_{21}) - \theta(e_1), \quad (15)\). Since \(\theta(e_1) = s-1\), finite, \(\theta(e_{21}) = -\infty\). Thus, \(E_{22}/E_{21} \to \infty\) and Cl is stalled on the second term.

A convenient form of Cl is the iterative formula,

\[ u_{21} = u_{11} + \frac{e_{11}v}{L-1}, \]

which is \(u_{21} = u_{11} - \frac{e_{11}v}{1 + \theta(e_1)} \). It is obtained from the relation, \(u_{12} = u_{11} - e_{11}v\).

The next theorem simplifies the calculation of \(\theta(v' + 1/r)\). We have seen that this is the order of the relative error of Formula 34, \(F \sim \gamma v f\), and is the constant order of the relative error of (91), provided that it is not \(-1\), \((92)\). It also determines whether the successive terms of (91) provide, asymptotically, an improvement in the approximation.

**Theorem 97.** Let \(f\) have finite order \(r\). \(f(x) = x^r g\), where \(\theta(g) = 0\).

Let \(f'\) and \(g'\) have order, and let \(g \to a\) limit, \(g_o\).

1) \(\theta(x f'/f - r) = \theta(g-g_o) = \theta(g') + 1\). (If \(g_o = \pm \infty\), substitute \(g\) for \(g-g_o\).)

2) If \(\theta(g-g_o) \neq -1\), then \(\theta(v' + 1/r) = \theta(g-g_o) = \theta(g') + 1\).

This is also true if \(\theta(g-g_o) = -1\), \(x^2 g' \to 0\) or \(\pm \infty\), and \(r \neq 1\).

The assumption that \(r \neq 0\) is implied in this part of the theorem and the next.

3) If \(\theta(g-g_o) = -1\) and \(x^2 g' \to a\) constant, \(L\), and \(\theta(x^2 g' - L) = \alpha\), and \(\theta(f) \neq 1\), then \(\theta(v' + 1/r) = \alpha - 1\) for \(\alpha > -1\), \(= -2\) for \(\alpha < -1\), and \(\leq -2\) for \(\alpha = -1\).
Proof. \( f = x^r g \iff f' = rx^{r-1} g + x^{r-1} g' = x^{r-1} (rg + xg') \).
\( v = -f/f' = -xg/(rg + xg') \). \( v + x/r \) comes out \( \frac{1}{r} x^2 g'/ (rg + xg') \).
Now \( \theta(rg + xg') = 0 \), since \( \theta(f') = r-1 \), \( \iff \theta(v + x/r) = \theta(x^2 g') \).
\( = \theta(g') + 2 \). \( \theta(x f'/r - r) = \theta(x + v)/v, = \theta(g') + 1, \) proving 1).
\( \theta(v' + 1/r) = \theta(g') + 1, (18) \), unless \( v + x/r \to a \) constant. This can only occur if \( \theta(g') = -2 \), which implies \( \theta(g - g_o) = -1 \), proving most of 2). The rest follows from 3).

Proof of 3). Now we suppose \( \theta(v + x/r) = 0 \).
\[ v + x/r = \frac{1}{r} x^2 g'/ (rg + xg') \] 
\( \theta(v + x/r) = \theta(\frac{x}{v + x/r})' \), (8.2),
\[ = \theta(\frac{1}{x} + \frac{r(g - g_o)}{x^2 g'} + \frac{rg_o}{x g'} \) \( g - g_o \to -1 \), if any limit, (27, 24). Hence,
\[ \theta(\frac{1}{x} + \frac{r(g - g_o)}{x^2 g'})' = -2, (9, 18) \] 
\( \theta(\frac{rg_o}{x^2 g'})' = \theta(x^2 g')', (8.2), \)
\[ = \theta(x^2 g' - L) - 1 = \alpha - 1 \). So by (9), \( \theta(v + x/r)' = \theta(\frac{1}{x} + \frac{r(g - g_o)}{x^2 g'} + \frac{rg_o}{x g'})' \]
\[ = \alpha - 1 \text{ for } \alpha > -1, = -2 \text{ for } \alpha < -1, \leq -2 \text{ for } \alpha = -1. // \]

Summary of Chapter V.

The Laplace-Winkler series, \( \int_a^\infty f(x)dx = f(a) [v_1(a) + \ldots + v_n(a)] + \int_a^\infty f(x)v_n'(x)dx \), where \( v_1 = -f/f' \) and \( v_{n+1} = v_n' v_1 \), was derived, along with an analogous expansion for an \( F(x) \) approaching \( \infty \). The series is asymptotic II, under regularity conditions, if \( \theta(f) = \pm \infty \), and also asymptotic I if \( \theta(v_1) < 1 \). If \( \theta(f) = r \), finite, the ratio of succeeding errors \( \to -1/r \). Bounds for the remainder term analogous to those of the Taylor series were obtained. The series is a special case of the iterative Procedure C: \( u_2 = (1 + u_1')v_1 \), where \( F_1 = u_1 f \). The Laplace-Winkler series is obtained by starting C with \( u_1 = v_1 \). C may be improved when \( \theta(f) \) is finite by starting with \( u_1 = \frac{r}{r+1} v_1 \).
rather than $v_l$. However, the procedure fails to be asymptotic for finite $r$ regardless of the choice of $u_l$, since the ratio of succeeding terms $\to -[1+ \theta(e_l)]/r$. (The one exception is when $\theta(e_n) \equiv -1$, an example of which was given. Then the procedure is asymptotic II but not I.) A modified procedure, Cl, was developed by taking linear combinations of $u_l$ and $u_2$. This had the effect of making Cl asymptotic II, but not necessarily asymptotic I.
CHAPTER VI
ANOTHER ASYMPTOTIC PROCEDURE

Procedure D. We recall that if we use an approximation, \( F_1 \), to find a transformation that will make the errors of \( A \) and \( B \) approximately cancel each other out, that we are led to the iterative formula,

\[ F_2 = F_1 f / f_1. \]

This is Procedure D.

This procedure can also be derived directly from Procedure A. When the transformation, \( y = F(x) \), is used, \( A \) contains just one term. One might hope that when a good approximation, \( F_1(x) \), serves as the transformation, the first term of \( A \), which is \( F_1(a)f(a)/f_1(a) \), will overshadow the rest of the series. If it be used as a better transformation to get a new first term, Procedure D is obtained.

Another justification may be derived from the relation,

\[ F = F_1 / (1 + E_1). \]

We don't know \( E_1 \) exactly, but Theorems 60 and 61 say that in many circumstances, \( E_1 \) may be approximated by \( e_1 \). This leads to the iterative formula, \( F_2 = F_1 / (1 + e_1) \), which \( = F_1 f / f_1 \).

And there is perhaps some heuristic justification in the idea that \( F : F_1 \approx f : f_1 \). This would be exactly true if \( f_1 = kf \), and might be approximately true if \( f_1 / f \) changed very slowly.

**Bounds.** Convenient bounds for Procedure D hold when \( f_1 \approx f \). If \( F \to 0 \), and \( f \) and \( f_1 \) are positive, and \( f_1 / f \to 1 \) monotonically for \( x \geq a \), (i.e., if \( e_1 \to 0 \) monotonically,) then \( F \) lies between \( F_1 \) and \( F_2 \) of Procedure D.
Proof. \( e_1(x) \to 0 \) monotonically \( \Rightarrow \) \( F_1 \) lies between \( e_1 \) and 0, (47), \( \Rightarrow \) \( 1/(1 + e_1) \) lies between \( 1/(1 + e_1) \) and 1, \( (e_1 \) can never be \( \leq -1, \) since \( f \) and \( f_1 \) are positive,\( ) \Rightarrow F_1/(1 + e_1) \) lies between \( F_1/(1 + e_1) \) and \( F_1/1, \) i.e., \( F \) lies between \( F_2 \) and \( F_1. // \)

A measure of the accuracy of these bounds is the difference between their ratio and 1. The smaller the difference, the more accurate the bounds. This accuracy index is \( \frac{F_1}{F_2} - 1, \) which reduced to \( e_1. \)

If \( F \) diverges, and \( f_1/f \) is monotonic for \( a \leq x \leq b, \) (i.e., \( e_1 \) is monotonic,) and \( f \) and \( f_1 \) are positive, then \( F(a,b) \) lies between \( F_2(a) \) and \( F_2(b). \) The proof is analogous.

Formula 99. If \( D \) starts with the initial approximation, \( F_1 = f, \) then \( F_2 = |vf| \) and \( F_3 = |vf/(1-v')|. \) \( F_2 \) is Formula 34, and \( F_3 \) will be called Formula 99. When \( \theta(f) = r, \) finite, \( \gamma vf \) is used instead of \( vf. \) \( \gamma = r/(r+1), \) cancels out when \( F_3 \) is obtained from \( \gamma vf. \)

These formulas are handy, both as approximations and bounds for \( F. \)

Bounds 100. If \( F \to 0 \) and \( v' \to 0 \) its limit, \(-1/r, \) monotonically, then \( F \) is bounded by \( \gamma vf \) and \( vf/(1-v'). \)

Proof. These bounds are a direct application of (85) with \( n = 1, \) and of (98), letting \( F_1 = \gamma vf, \) since \( e_1 = -\gamma(v' + 1/r). // \)

The smallness of \(-\gamma(v' + 1/r) \) measures the accuracy of these bounds. This expression becomes \(-v' \) when \( \theta(f) = -\infty. \)

Theorem 101. Procedure \( D \) is invariant under a monotonic transformation carrying \( \infty \) into \( \infty, \) but Procedure \( C \) is not.

Proof. Let \( x = \phi(y), \) with \( \phi(\infty) = \infty. \) Let \( G(y) = F(x) \) and \( G_1(y) = F_1(x), \) \( g(y) = f(\phi(y)) \phi'(y) \) and \( g_1(y) = f_1(\phi(y)) \phi'(y). \)

\[ \frac{G_2(y)}{g_2(y)} = \frac{F_1(x)}{f_1(x)} \frac{\phi'(y)}{\phi'(y)} = F_2(x). \] Thus, \( D \) leads from
$F_1$ to $F_2$ regardless of a monotonic transformation of the variable. Almost any example will show that $C$ is not invariant. Let $f = x^{-2}$ and $u_1 = x + \log x$. $u_2$ comes out $x + 1/2$. Applying the transformation, $x = e^y$, $u_1 f$ becomes $e^{-y} + ye^{-2y}$, $G(y) = e^{-y}$, $g(y) = e^{-y}$, and $v_y = 1$. Letting $u_1 f = v_1 g$, we find that $w_1 = 1 + ye^{-y}$, $w_2 = (1 + e^{-y} - ye^{-y}) \cdot 1$, and $w_2 g = e^{-y} + e^{-2y} - ye^{-2y}$. But $u_2 f = e^{-y} + e^{-2y}/2$, so $C$ applied with respect to $y$ does not take $u_1 f$ into $u_2 f$. Nor does the transform of $u_1$ lead to the transform of $u_2$ in this example. Thus, $C$ is in no sense invariant under monotonic transformation. //

A family of transformed procedures may be obtained from $C$ by applying a transformation, $x = \phi(y)$, carrying $\infty$ into $\infty$, $F(x)$ into $G(y)$, and $F_1(x)$ into $G_1(y)$, and letting $F_n(x) = G_n(y)$.

**Theorem 102.** $C$ and $D$ can coincide only if $F_1 = aF + b$. (We suppose that $f$ and $f_1$ are $> 0$.) Hence, $D$ is not a transformed procedure $C$.

**Proof.** $F_1 = u_1 f$, $F_2 = (1 + u_1')vf$, by $C$, and $= F_1 f/f_1$ by $D$, which comes out $(1 + \frac{u_1'}{1 + \frac{u_1'}{u_2}})$vf after algebra. Thus, $u_1'' = \frac{u_1''}{(1 + \frac{u_1'}{u_2})}$, $\implies u_1 = u_2$ at all points, $x$, where $u_1'(x) \neq 0$. $F_1 = F_2 = F_1 f/f_1$ at these points, $\implies f = f_1$ at these points. $F$ and $F_1$ are continuous by implication, since $C$ and $D$ exist. $u_1'(x)$ is continuous, because $u_2' = (1 + u_1')v$, is differentiable. Hence, $u_1'(x) \neq 0$ on a countable collection of open intervals, $\{I_i\}$, on which $F_1 = F_2 = F + b_i$. $\{x; u_1'(x) = 0\}$ is a closed set containing open intervals, $\{J_i\}$, (perhaps,) and boundary points.

$F_1 = F + b_i$, for some $i$, on the boundary points by continuity. On $J_i$, say, $u_1'(x) = 0$, $\implies u_1(x) = a$ constant on $J_i$, $\implies F_1 = k_i f$, $\implies F_2 = vf$ by $C$, $\implies F_3 = (1 + v')vf = vf/(1 - v')$ by $C$ and $D$,
$\implies l + v' = 1/(1-v')$, since $v_f \neq 0$, $\implies v'^2 = 0$, $\implies v' = 0$,
$\implies F_3 = v_f$ (by C) $= F_2$ on $J_1$. Hence, $F_3 = F_2$ everywhere, since
$F_3 = F_2$ on $(I_1)$ and the other $J$'s. Hence, $F_2f/f_2 = F_2$ for all $x$, 
$\implies f = f_2$ for all $x$, $\implies F_2 = F + b$ for all $x$, $\implies F_1f/f_1 = F + b$,
$\implies f_1/F_1 = f/(F + b) \implies \log F_1 = \log (F + b) + \text{constant}$,
$\implies F_1 = aF + b$. $a = 1$, unless $F$ is a special kind of function, like $e^{-x}$.

It now follows that D is not a transformed Procedure C. For suppose it were. There exists a transformation, $x = \varphi(y)$, carrying $\infty$ into $\infty$, $F(x)$ into $G(y)$, and $F_1(x) = G_1(y)$. We suppose that
$F_1 \neq aF + b$. Then $G_1 \neq aG + b$. The transformed Procedure C requires
that C be applied to $G$ and $G_1$, and since D is invariant, this procedure coincides with D applied to $G$ and $G_1$, contradicting the previous theorem. //

The next theorem will show that the asymptotic properties of D are strikingly similar to those of C.

**Theorem 103** 1) Let $F_1$ be $\sim F$ and $f_1 \sim f$. Let $F_2 = F_1f/f_1$,
therefore also $\sim F$). Then $E_2/E_1 \sim E_1$, and $E_2/E_1 \to \text{a limit}$
$\implies 1 - \frac{E_1}{E_1} \to \text{the same limit}$.

2) The same result holds if $F_1/F_1 \sim f_1/f_1$, which, by (60.1) happens when $\theta(f_1) = \theta(f)$, finite, and $xf/F$ and $xf_1/F_1 \sim \text{limits}$;
or when, by (60.2), $\theta(f) = \pm \infty$ and $\theta(f_1/f)$ is finite and $xf/F$ and
$xf_1/F_1 \sim \text{limits}$; or when, by (61), $|\theta(f_1/f)| < |\theta(f)|$,
$\theta(f) = \pm \infty$ and $f/F$ and $f_1/F_1 \not\sim f/F \text{ have extended order}$. We must
further assume that $F_1/F \not\sim \text{a constant}$.

3) If $F_1$ and $F \to 0$, and any of the three conditions listed
in 2) hold, then $E_2/E_1 \to -\theta(e_1)/(r + 1)$. 
To apply part 3) to a divergent $F_1$ we need the additional assumption that $F_1/F \neq$ a constant. Otherwise $\theta(E_1) \neq \theta(e_1)$, necessarily.

For divergent $F$, $E_2/E_1 \rightarrow 1 - \frac{\theta(e_1)}{r+1} + 1$.

This theorem is the analogue of (92), except that here $E_2/E_1 \rightarrow -\theta(e_1)/(r+1)$ rather than $-\frac{1 + \theta(e_1)}{r}$.

Proof of 1) and 2).

\[
\frac{E_2}{E_1} - 1 = \frac{\frac{F_1^f}{F_1} - 1 - \frac{F_1}{F} - 1}{\frac{F_1}{F} - 1} = \frac{\frac{F_1}{F} \cdot \frac{f_1}{F} - 1}{\frac{E_1}{F}} \sim -e_1/E_1. \text{ If either side } \rightarrow \text{ a limit, then the other side must } \rightarrow \text{ the same limit.}
\]

Proof of 3). If $r$ is finite, then $e_1/E_1 \rightarrow |\theta(F_1-F)/\theta(F)|$,

(60.1) $= |[\theta(E_1) + r+1]/(r+1)| = |\theta(E_1)/(r+1) + 1| \rightarrow E_2/E_1$

$\rightarrow 1 - |\theta(e_1)/(r+1) + 1|$, by 1), $= \theta(e_1)/(r+1)$ when $r \leq -1$,

since $\theta(e_1) \leq 0$. [Proof that $\theta(e_1) \leq 0$: $\frac{F_1}{F} \sim \frac{f_1}{F}$, $
\Rightarrow \log F_1 \sim \log F$, (16), $\Rightarrow \theta(F_1) = \theta(F)$, (24),

$\Rightarrow \theta(-\frac{1}{F}) \leq 0$, (9), $\Rightarrow \theta(e_1) \leq 0$, (57). We have enough assumptions to equate $\theta(E_1)$ with $\theta(e_1)$.] If $r = \pm \infty$ and $\theta(e_1)$ is finite, then $e_1 \sim E_1$, (60.2), and $E_2/E_1 \sim 1 - 1 = 0 = -\theta(e_1)/(r+1)$. A similar proof holds when $|\theta(e_1)| < |\theta(f)|$ and $r = \pm \infty$.

The next theorem is analogous to (94), and we again have the paradoxical situation that the procedure is only asymptotic if the initial approximation is bad enough.

Theorem 104. 1) Let $g = f_1 - f$. If $|\theta(E_1)| > |\theta(f)|$, and $\theta(f) = \pm \infty$ and $g/G$ and $f/F$ have extended order, and $F_1/F \sim f_1/f$,

(some conditions for which are given in the preceding theorem,) then $E_2/E_1 \rightarrow \infty$. 

Under the same regularity conditions for $F_2$, $F_3$, ..., 
$E_{n+1}/E_n \to \infty$, and hence $D$ is asymptotic II in reverse. $E_1$ may be replaced by $e_1$ throughout the statement of the theorem.

2) If $|\theta(E_1)| < |\theta(f)|$, but otherwise the conditions of 1) hold, and $E_1'/E_1$ has extended order, then $E_2/E_1 \to 0$. (We must also require that $\theta(E_1) = \pm \infty$, or that $xf/F$ and $xG/G - xf/F \to$ limits, in order to apply (60) or (61) at a crucial stage.) Similarly for $E_3$, $E_4$, ..., under the same regularity conditions, so that $D$ is asymptotic II. We must assume, however, that $E_1 \neq$ a constant other than $-1$. $E_1$ may be replaced by $e_1$ throughout.

3) If $(f_1/F)/(f_1/f) \to$ some limit, $L$, other than 1, then $F_1/F \to 0$ or $\infty$. If 0, then $E_2/E_1 \to L$; if $\infty$, then $E_2/E_1 \to 0$. When $F_1/F \to \pm \infty$ and $L = \pm \infty$, we need the additional assumptions that $f_1/f \to$ a limit and that $F_1 \neq$ a constant.

Proof of 1). $\theta(E_1) = \theta(G/F) = \theta(G), (42)$. $|\theta(G)| > |\theta(f)| = |\theta(f)|$

(44), $\Rightarrow \theta(G/F) > \theta(f/F)$, (46), $\Rightarrow e_1/E_1 = \frac{G/F}{G/F} \to \pm \infty$ (37, 38).

Now $E_2/E_1 \sim E_1$ (103.1), $\to \pm \infty$ $\Rightarrow E_2/E_1 \to \pm \infty$. $E_2/E_1 \sim E_1$, $E_2 \sim e_1$. Now $\theta(e_1) = \theta(E_1)$, (59),

$\Rightarrow |\theta(E_2)| = |\theta(E_1)| > |\theta(f)|$, and the same proof applies as before.

H remains to show that $e_1$ has extended order.

$(G/F)^{f_1/f_2} = \theta(G/F)^{f_1/f_2}$, (42). $\theta(e_1) = \theta(G \cdot \frac{G/F}{f_1/f_2}) = \theta(G/F)$, (42).

Proof of 2). $|\theta(E_1)| < |\theta(f)| = |\theta(f)|$, $\Rightarrow e_1 \sim E_1$, (60 or 61), $\Rightarrow E_2/E_1 \to 1$. (103.1), $\Rightarrow E_2/E_1 \to 0$. To complete the proof, we must show that $|\theta(E_2)| < |\theta(f)|$. Let $F_1 = KF$. Suppose $F$ and $F_1 \to 0$. Let $k = -K' = -E_1'$. $F_1 = KF + KF$. $F/F_2 = 1 + \frac{kF}{K/F}$. The same result obtains if $F$ and $F_1 \to \infty$. Then $k$ is
defined as $K'$. Now $F_2 = F_1 \cdot \frac{f}{f_1} \sim F$ by hypothesis. Hence,

$$E_2 = \frac{F_2}{F} - 1 \sim -\frac{(F/F) - 1}{F}, \quad (6.3), \quad \frac{k}{K F}.$$  

Case 1: $K \to l$. 

$$|\theta(E_2)| = |\theta(k/F)| \leq \max |\theta(k), \theta(f/F)| < |\theta(f)|, \quad (42),$$  

since $k = -E_1'$, and $\theta(E_1') = \theta(E_1) - 1$, if $\theta(E_1)$ is finite, and $= \theta(E_1)$ otherwise. Case 2: $K$ does not $\to$ a constant. $\frac{f}{F_1} \to 1$, $\implies \log F/\log F_1 \to 1$. (Log $F_1 \not= 0$ a constant, since $\log F \to \pm \infty$.) Hence, $\theta(F) = \theta(F_1)$, $\implies |\theta(k)| \leq |\theta(F)|, \quad (42), \quad \implies \theta(k/k) = \theta(f/F) < |\theta(F)|, \quad \implies |\theta(E_2)| < |\theta(F)|.$

If $E_2 \to l$, say, we would have no assurance that $|\theta(E_1')|$ would be $< |\theta(f)|$. An example would be $F_1 = 2F + \epsilon$, where $\theta(\epsilon) < \theta(f)$, and $\theta(f) = -\infty$. Log $F_1 \sim \log F + 2 \sim \log F$, so we could easily have $f/F \sim \frac{f_1}{F_1}$. $K = 2 + \epsilon$, and $k = \epsilon'$.

$\theta(f/F) = \theta(k)$ by (42), and we would get $|\theta(E_2)| > |\theta(F)|$. But

$E_1 = 1 + \epsilon$ and $|\theta(E_1)| = 0 < |\theta(f)|$.

Proof of 3). $E_2/E_1 = (\frac{F_1}{F} \cdot \frac{f}{f_1} - 1)/(\frac{F_1}{F} - 1)$, which $\to (1-1)/-1$ if $F_1/F \to 0$, and $\to 0$ if $F_1/F \to \infty$ and $l$ is finite, and is

$\sim \frac{F_1}{F} \cdot \frac{f}{f_1} \cdot \frac{f}{F_1}$, when $\frac{F_1}{F} \to \infty$ and $l = \pm \infty$, $f_1/f \to 0$ by (17). //

As with C, when we start D with $vf$, and $\theta(f)$ is infinite, the procedure is asymptotic, as the following corollary of (104.2) shows.

Theorem 105. 1) If $\theta(f)$ is infinite, and $F_1 = vf$, and the regularity conditions of (104.2) hold, and $v'$ has extended order, and $v \not= 0$ a constant, then D is asymptotic II.

2) If, in addition, $\theta(v') < 0$, (which $\implies \theta(v) < l$), then $\theta(E_2/E_1) = \theta(E_3/E_2) = \cdots = \theta(v')$, so that D is asymptotic I.

Proof. 1) is a special case of (104.2), since $\theta(v') = \theta(v) - 1 < \theta(f)$ in absolute value. To prove 2), we observe
that $E_2 \sim \frac{k/f}{K/F}$, as in the proof of (104.2), where $k = -E'_1$ and $K = F_1/F$. $K \to 1$ in this case, since $\theta(E_1) = \theta(e_1) < 0$. Hence,$E_2/E_1 \sim \frac{E_1'}{E_1}$. Now $vf/F \to 1$, since $e_1 = -v' \to 0$, (30.1). Hence,$E_2/E_1 \sim E_1'v/E_1$. Case 1: $\theta(v)$ is finite. $\theta(E_1) = \theta(e_1) = \theta(v')$.

$\theta(E_2/E_1) = \theta(v') - 1 - \theta(v') + \theta(v) = \theta(v) - 1 = \theta(v')$. Case 2:

$\theta(v)$ is infinite. $|\theta(E_1'/E_1)| < |\theta(E_1)|$, (46), $= |\theta(v')|$, $= |\theta(v)|$, 

$\Rightarrow \theta(v + E_1'/E_1) = \theta(v) = \theta(v')$. The same reasoning applies to $E'_2, E'_3, \ldots$.

When $\theta(f)$ is finite, we can make Procedure D asymptotic II by taking linear combinations as with C.

**Procedure Dl.** By (55), if $E_2/E_1 \to$ a constant, $L$, then $F_{21}$, which $= \frac{L}{L-1} F_1 - \frac{1}{L-1} F_2$, has an error, $E_{21}$, that goes to 0 with $E_1$. Thus, if $\theta(f) = r$, finite, $L = -\theta(e_1)/(r+1)$, and $E_{21}/E_1 \to 0$.

If $\theta(E_{21})$ is still finite, we can obtain $E_{22}$ by D, and take the linear combination, $F_{31} = \frac{L_2}{L_2-1} F_1 - \frac{1}{L_2-1} F_2$. Then $E_{31}/E_{21} \to 0$, and we have a procedure that is asymptotic II as long as it lasts. If any $\theta(E_{n1})$ is infinite, the procedure stops. As with Cl, we have no guarantee that the procedure will be asymptotic I as well as II.

**Example 106.** A Dl procedure that is asymptotic II but not I.

All we have to do is find a well-behaved $a_1 \to 0$ and of order 0. Then, by the proof of (105), $E_2/E_1 \sim \frac{E_1'}{E_1}$. By requiring that $\theta(f)$ be finite, we obtain $\theta(E_2/E_1) = \theta(E_1'/E_1) + 1$, (18), $= 0$, (18), since $\theta(E_1) = 0$ and $E_1 \to 0$ by (47), and hence $\theta(E_1'/E_1) = -1$, (18).

And finding such an $e_1$ is no problem, since $f$ and $f_1$ are at our disposal. If it be insisted that $F_1 = \gamma vf$, we must find an $f$ such that $e_1$ which $= -\gamma(v' + 1/r)$, $\to 0$ and has order 0. By (97), this can be accomplished by letting $f = x^r g$, where $g \to 0$, has order 0, and is well-behaved enough so that $v'$ a limit, which must be $-1/r$ by (30.1). $f(x) = x^{-2}/\log x$ would do nicely. //
Procedure 107. Procedure D was obtained by writing $F = F_1/(1 + E_1)$ and replacing $E_1$ by $e_1$. A more general procedure for improving approximations is to replace $E_1$ by an approximation, $\tilde{E}_1$, obtaining the iterative formula, $F_2 = F_1/(1 + \tilde{E}_1)$.

Formula 108. If $F$ is approximately a normal integral, $\Phi$, and $F_1$ has an analogue, $\Phi_1$, for the normal distribution, then it may happen that $E_1(F) \sim E_1(\Phi)$, and we get the improved approximation, $F_2 = F_1/[1 + E_1(\Phi)]$. This reduces to the formula, $F_2 = F_1 \cdot \Phi/\Phi_1$. The latter factor depends only on the normal distribution and could be tabulated, for some useful $F_1$ like $vf$. Sheppard, [45], has already tabulated $F/vf$ for the normal distribution. $F/vf$ is obtained by multiplying by $x$.

Formula 109. Another normal fudge formula can be obtained from the approximate proportion, $E_1(F) : e_1(F) \sim E_1(\Phi) : e_1(\Phi)$. Then $\tilde{E}_1(F) \sim e_1(F) E_1(\Phi)/e_1(\Phi)$. This tends to be more accurate and harder to calculate than (108).

Theorem 110. If $\tilde{E}_1 \sim E_1$ in Procedure 107, and $E_1 \rightarrow 0$, then $E_2/E_1 \rightarrow 0$, and $\theta(E_2/E_1) = \theta(\tilde{E}_1/E_1 - 1)$.

Proof. $E_2 = \frac{\frac{F_1}{1 + E_1} - 1}{1 + \tilde{E}_1 - 1} = \frac{1 + E_1 - \tilde{E}_1}{1 + \tilde{E}_1} \sim \frac{E_1 - E_1}{1 + \tilde{E}_1}$.

$E_2/E_1 \sim 1 - \frac{E_1}{E_1}$. Hence, $E_2/E_1 \rightarrow 0$ and has the same order as $1 - \tilde{E}_1/E_1$.

Procedure D2. Since the way to make (107) asymptotic II was to replace $E_1$ by an asymptotic expression, $\tilde{E}_1$, and since $E_1 \sim \frac{\theta(F)}{\theta(F_1-F)} e_1$, we should have let $\tilde{E}_1$ be that expression rather than $e_1$ when deriving D for finite $\theta(f)$. After substituting in (107), this comes out $F_{21} = \frac{\frac{F_1}{e_1}}{1 + \frac{\theta(e_1)}{1+r}} = \lim_{r \rightarrow 0} \frac{E_2}{E_1}$. This is a procedure
similar to D1 but not just the same. It is like a harmonic, rather than an arithmetic mean of succeeding approximations of D, since \( F_1/l \) is \( F_1 \), and \( F_1/(1 + e_1) \) is \( F_2 \). D2 prescribes multiplying \( e_1 \) by just such a constant, in the latter formula, that \( E_{21}/E_1 \to 0 \).

**Procedure D3.** D3 is a procedure that is in much the same spirit as D, but depends on two previous approximations rather than one. D, essentially, was of the form \( F_2 = aF_1 \), where \( a \), considered as a constant, was chosen to make \(- (aF_1)' = f \). In Procedure D3, we let \( F_{21} \) be a weighted average of two previous approximations, \( aF_1 + (1-a)F_2 \), with \( a \) chosen so that \(-F_{21}' = f \), considering \( a \) as a constant.

(a actually isn't a constant any more than with D, since the choice of \( a \) varies with \( x \).)

Solving the equation, we get \( a = \frac{f - f_2}{f_1 - f_2} \), so that D3 is the procedure, \( F_{21} = \frac{(f - f_2)F_1 + (f_1 - f)F_2}{f_1 - f_2} \). D3 is an interpolation between \( F_1 \) and \( F_2 \), using the relative closeness of \( f_1 \) and \( f_2 \) to \( f \) as a guide.

We can specialize D3 and get D by interpolating between \( F_1 \) and \( kF_1 \). D3 can also be obtained from the identity \( F = \frac{E_2}{E_1} F_1 - F_2 \)/\( \frac{E_2}{E_1} -1 \). If we replace \( E_2/E_1 \) by \( e_2/e_1 \) in that expression, we get an approximation to \( F \) that is an equivalent form of D3.

The following bounds hold for D3.

**Bounds III.** If \( e_2/e_1 \to a \) limit, \( L \), monotonically, and \( e_2/e_1 \) and \( L \) are not on opposite sides of \( 1 \), and \( F_1 - F \) and \( F_2 - F \to 0 \), and \( F_1 - F \) is of constant sign, then \( F \) lies between \( \frac{e_2}{e_1} F_1 - F_2 \) and \( \frac{E_2}{E_1} F_1 - F_2 \)/\( \frac{E_2}{E_1} -1 \), i.e., between \( F_{21}(D3) \) and \( F_{21}(55) \).
To prove this theorem, we need the following lemma.

**Lemma 112.** \( \frac{x a - b}{x - 1} \) is a monotonic function of \( x \) on either of the intervals, \( \{x; x > 1\} \) or \( \{x; x < 1\} \).

Proof of (112). \( \frac{x}{x-1} \) is monotonic on either of these intervals, because the derivative, \(-1/(x-1)^2\), is of constant sign there.

\[
(xa-b)/(x-1) = \lambda a + (1-\lambda)b, \quad \text{where} \quad \lambda = \frac{x}{x-1}.
\]

\((xa-b)/(x-1) = b + \lambda(a-b)\).

This is a monotonic function of \( \lambda \). //

Proof of (111). \[
\frac{e_2}{e_1} = \frac{\frac{f_2-f_1}{f_1-F_2}}{\frac{E_2}{E_1}} = \frac{F_2-F_1}{F_1-E_2}. \quad \text{By (48)},
\]

\[
\min \frac{f_2-f_1}{f_1-F_2} \leq \frac{F_2-F_1}{F_1-E_2} \leq \max \frac{f_2-f_1}{f_1-F_2}, \quad \text{and hence} \quad \frac{E_2}{E_1} \text{ lies between}
\]

\(e_2/e_1\) and \(L\). Now \( F = (\frac{E_2}{E_1} F_1 - F_2)/(\frac{E_2}{E_1} - 1) \). Hence, by the lemma, \(F\) lies between the linear combinations of \(F_1\) and \(F_2\) generated by \(e_2/e_1\) and \(L\).

D3 can be continued either by getting \(F_{22}\) from D and interpolating again, or by combining with C the same way, or by interpolating between \(F_{21}(D3)\) and \(F_{21}(55)\), or directly by using \(F_{21}\) and \(F_2\).

In order to discuss the asymptotic properties of D3, we shall first prove a theorem similar to (107).

**Theorem 113.** If an approximation, \(F_{21}\), be obtained by replacing \(E_2/E_1\) by an expression, \(\alpha(x)\), asymptotic to it in the identity,

\[
F = (\frac{E_2}{E_1} F_1 - F_2)/(\frac{E_2}{E_1} - 1), \quad \text{and if} \quad \frac{E_2}{E_1} \to \text{a limit,} \quad L, \quad \text{other than 1, then} \quad E_{21}/E_1 \quad \text{and} \quad E_{21}/E_2 \to 0. \quad \text{If} \quad \frac{E_2}{E_1} \quad \text{and} \quad \alpha(x) \to 0, \quad \text{but are asymptotic, then} \quad E_{21}/E_1 \to 0, \quad \text{but} \quad E_{21}/E_2 \quad \text{need not} \to 0.
\]

Proof. Let \(\beta = \alpha/(\alpha-1)\). Suppose \(L\) is finite.

\[
F_{21} = \beta F_1 + (1-\beta) F_2, \quad E_{21} = F_{21}/F - 1 = \beta E_1 + (1-\beta) E_2, \quad \Longrightarrow (54)
\]
holds for $E_1$ and $E_2$ even if $\beta$ is a variable.

$E_{21}/E_1 = \beta + (1-\beta) E_2/E_1$. $\beta \to L/(L-1)$, $\Rightarrow (1-\beta) E_2/E_1 \to -L/(L-1)$,

$\Rightarrow E_{21}/E_1 \to 0$. $E_{21}/E_2 = \beta E_1/E_2 + (1-\beta)$. $\beta E_1/E_2 \to L/(L-1)$ and 

$1-\beta \to \frac{1}{L-1}$, $\Rightarrow E_{21}/E_2 \to 0$. If $L = 0$ and $\alpha \sim E_2/E_1$, then

$\beta = \frac{\alpha}{\alpha-1}$, $\sim \alpha$, $\sim E_2/E_1$, $\Rightarrow \beta E_1/E_2 \to -1$ and $1-\beta \to 1$,

$\Rightarrow E_{21}/E_2 \to 0$. If $E_2/E_1 \to \pm \infty$, then $E_1/E_2 \to 0$, and $1/\alpha \sim E_1/E_2$.

$F_{21} = \frac{\frac{E_1}{E_2} - \frac{F_1}{1}}{\frac{1}{\alpha-1}}$ and $F = \frac{\frac{E_1}{E_2} - \frac{F_1}{F}}{\frac{E_2}{E_1} - 1}$. Thus, the part of the theorem

we have proved applies, so that $E_{21}/E_2 \to 0$ and $E_{21}/E_1 \to 0$. We only used the fact that $\alpha \sim E_2/E_1$ in proving that $E_{21}/E_2 \to 0$ when $L = 0$.

The result, $E_{21}/E_1 \to 0$, holds if only $\lim \alpha(x) = \lim E_2/E_1 = 0$. //

This theorem shows that the asymptotic properties of D3 are very similar to those of D1 and C1. If $E_2/E_1 \to$ a constant other than 1, then, under suitable regularity conditions, $e_2/e_1 \sim E_2/E_1$, (60), and hence $E_{21}/E_1$ and $E_{21}/E_2 \to 0$, and the procedure is asymptotic II, (for the first step anyway). If $E_2/E_1 \to 0$ or $\infty$, it still may very well happen that $e_2/e_1 \sim E_2/E_1$, (60, 61), with the same result.

In this case, D3 is an improvement over D1, since $E_{21}(D1) \sim E_1$ or $E_2$. If $(e_2/e_1)/(E_2/E_1) \to c$, finite or infinite, then $E_{21}/E_2 \to 1-c$, by the proof of (113). Since $\lim e_2/e_1 = \lim E_2/E_1$, (unless $F_1-F$ or $F_2-F \to$ a constant, (53)), it always follows that $E_{21}/E_1 \to 0$ when $E_2/E_1 \to 0$, and $E_{21}/E_2 \to 0$ when $E_1/E_2 \to 0$.

Although D3 seldom has an asymptotic advantage over D1, it seems to do much better in practice for small $x$. It may be that $F_2$ is asymptotically infinitely better than $F_1$, say, and that $E_{21} \sim E_2$, and $F_{21}$ would seem to be no better than $F_2$; yet $E_1$ and $E_2$ may
be of comparable magnitude and opposite sign for small $x$, and the interpolation procedure finds a happy medium between them.

Summary of Chapter VI.

Heuristic motivations for the iterative Procedure D,

$$F_2 = F_1 f/f_1,$$

were given. $F_1$ and $F_2$ are bounds for a convergent $F$ if $f \sim f_1$ and $f/f_1$ is monotonic. In particular, $F_1 = \gamma f$ and $F_2 = \gamma f/(1-\gamma)$ are bounds for $F$ if $\gamma$ is monotonic. A distinctive feature of D is its invariance under transformation, which sets it apart from previously considered procedures. D is asymptotic, under regularity conditions, if $F_1 = \gamma f$ and $\theta(f) = \pm \infty$ or more generally, if $\theta(f)$ is infinite and $|\theta(e_1)| < |\theta(f)|$. If $\theta(f)$ is finite, then D is not asymptotic, except in the special case, $\theta(e_n) \equiv 0$.

Several modifications were considered to make D asymptotic II for finite $\theta(f)$. D1, like C1, was obtained by linear combinations. D2 was obtained by replacing $e_1$ by $ke_1$ in the identity,

$$F_2 = F_1/(1 + e_1),$$

where $ke_1 \sim E_1$. This is a special case of the general Procedure 107, that consists of replacing $E_1$ by an approximation, $\tilde{E}_1$, in the identity, $F = F_1/(1 + E_1)$. If $\tilde{E}_1 \sim E_1$, this results in an infinitely improved approximation, asymptotically. If $f$ is approximately normal, approximations to $E_1$ may be derived from the normal distribution.

D3 is an interpolation procedure that can be used for finding a middle ground between any two initial approximations, $F_1$ and $F_2$, whether obtained by D or not. It consists of replacing $E_2/E_1$ by $e_2/e_1$ in the identity, $F = (E_2/E_1 F_1 - F_2)/(E_2/E_1 -1)$. If $e_2/e_1 \rightarrow$ a limit asymptotically, then $F$ is bounded by $F_{21}(D3)$ and $F_{21}(55)$. 
Like D1, D3 shows an infinite asymptotic improvement when \( \frac{E_2}{E_1} \to a \) constant other than 1, but unlike D1, it also has this property when \( \frac{E_2}{E_1} \to 0 \) or \( \pm \infty \) and \( \frac{e_2}{e_1} \sim \frac{E_2}{E_1} \).
CHAPTER VII
EXTENSION OF THE THEORY TO FINITE TAILS

In statistical research, we often wish to evaluate the integral of a density function with a finite tail, such as the gamma distribution. The preceding theory that was developed for infinite tails can be applied by analogy to this problem.

We suppose for simplicity that the tail ends at 0. \( \int_{0}^{a} f(x)dx \) plays the role of \( \int_{0}^{\infty} f(x)dx \) in the previous discussion, and we now seek approximations asymptotic as \( a \to 0 \) rather than \( \infty \). We shall write \( F(a) \) for \( \int_{0}^{a} f(x)dx \) if that integral converges, and for \( \int_{a}^{\infty} f(x)dx \) otherwise. This means that \( f(a) = F'(a) \) if \( F \to 0 \) and \( = -F'(a) \) if \( F \to \infty \), reversing the previous rule.

We again define order by making \( x^r \) the typical function of order \( r \), but note that this entails a reversal of magnitude. \( x^r \) now \( \to 0 \) if \( r > 0 \), and \( \to \infty \) if \( r < 0 \), and the greater \( \theta(f) \) is, the smaller \( f \) is. Specifically, the Cauchy definition of order now reads, "\( \theta(f) = r \) means that \( x^{r+\epsilon} < f < x^{r-\epsilon} \) or the same is true of \( -f \), \( \theta(f) = \infty \) means that \( |f| < x^N \), and \( \theta(f) = -\infty \) means that \( f > x^{-N} \) or the same is true of \( -f \)." Thus, the order class, \( \infty \), now plays the role \( -\infty \) did in infinite tail theory. The notation, \( f < g \), is now used with the meaning "there exists \( \delta > 0 \) such that for \( x < \delta \), \( f(x) < g(x) \)."

With these interpretations and exceptions borne constantly in mind, the theorems, proofs, and examples of the preceding chapters all hold.
In Chapter IV, the transformation, \( y = F_1(x) \), should now be considered a monotonic increasing function carrying 0 into 0. The direct application of the Taylor series, which is now possible, is included in this theory by use of the identity transformation, \( y = x \).

The fact that \( f \) and \( f_1 \) are now \( +F' \) and \( +F_1' \) changes a few signs in the formulas. The comments about the analyticity of \( F \) and \( F_1 \) at \( \infty \) now apply to their analyticity at \( 0 \).

The following is a table of substitutions required to make each previous theorem and example hold exactly in the finite tail case. Theorems and examples not included in this list hold as stated.

<table>
<thead>
<tr>
<th>Theorem</th>
<th>Old Expression</th>
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<td>All theorems</td>
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<td>( f_0 \int f(x)dx )</td>
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<td></td>
<td>( f_c \int^a f(x)dx )</td>
<td>( f_0 \int f(x)dx )</td>
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<td>( x \geq a )</td>
<td>( 0 \leq x \leq a )</td>
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<td>a or ( x \to \infty )</td>
<td>a or ( x \to 0 )</td>
</tr>
<tr>
<td>1</td>
<td>first coordinate of each point</td>
<td>its reciprocal</td>
</tr>
<tr>
<td>2</td>
<td>(</td>
<td>f</td>
</tr>
<tr>
<td></td>
<td>( f = o(g) )</td>
<td>( g = o(f) )</td>
</tr>
<tr>
<td></td>
<td>(</td>
<td>f</td>
</tr>
<tr>
<td>2</td>
<td>(-\infty )</td>
<td>( \infty )</td>
</tr>
<tr>
<td>after 4</td>
<td>( e^{-x} \sin x )</td>
<td>( e^{-1/x} \sin \left(\frac{1}{x}\right) )</td>
</tr>
<tr>
<td>5</td>
<td>( \sin x + 2 )</td>
<td>( \sin \left(\frac{1}{x}\right) + 2 )</td>
</tr>
<tr>
<td>Theorem</td>
<td>Old Expression</td>
<td>Replaced by</td>
</tr>
<tr>
<td>---------</td>
<td>----------------</td>
<td>-------------</td>
</tr>
<tr>
<td>6</td>
<td>$\infty$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>7</td>
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<td>$\infty$</td>
</tr>
<tr>
<td>9</td>
<td>All inequalities are reversed.</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>max</td>
<td>min</td>
</tr>
<tr>
<td></td>
<td>$\leq$</td>
<td>$\geq$</td>
</tr>
<tr>
<td>11</td>
<td>$r &gt; 0$</td>
<td>$r &lt; 0$</td>
</tr>
<tr>
<td>13</td>
<td>$g \to \infty$</td>
<td>$g \to 0$</td>
</tr>
<tr>
<td>19</td>
<td>lub</td>
<td>glb</td>
</tr>
<tr>
<td>20</td>
<td>$1/\log a$</td>
<td>$-1/\log a$</td>
</tr>
<tr>
<td>21</td>
<td>$x = n$ and $n^p$</td>
<td>$x = l/n$ and $n^{-p}$</td>
</tr>
<tr>
<td></td>
<td>$&lt; \text{ and } &gt;$</td>
<td>$&gt; \text{ and } &lt;$</td>
</tr>
<tr>
<td>23</td>
<td>lub and glb</td>
<td>glb and lub</td>
</tr>
<tr>
<td>24</td>
<td>$\infty$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>25</td>
<td>$e^{-x}$</td>
<td>$e^{-1/x}$</td>
</tr>
<tr>
<td></td>
<td>$x = n$</td>
<td>$x = l/n$</td>
</tr>
<tr>
<td></td>
<td>$\infty$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>26</td>
<td>$x^{-5}$ and $x^{-6}$</td>
<td>$x^{5}$ and $x^{6}$</td>
</tr>
<tr>
<td>28</td>
<td>$f(x) = 1 + e^{-x} \sin x$</td>
<td>$f(x) = 1 + e^{-1/x} \sin (1/x)$</td>
</tr>
<tr>
<td>32</td>
<td>$x^{-5}$ and $x^{-6}$, etc.</td>
<td>$x^{5}$ and $x^{6}$, etc.</td>
</tr>
</tbody>
</table>

The new definition of extended order is as follows:

$\Theta(f) = (n,r)$ means that the $(n-1)$st log of $-\log f$ has order $-r$. $\Theta(f) = -(n,r)$ means that the $n$th log of $f$ has order $-r$. The rest of the definition follows analogously.

| 37      | $f < g$         | $f > g$     |
|         | $f = o(g)$      | $g = o(f)$  |
### TABLE 1 (Continued)

<table>
<thead>
<tr>
<th>Theorem</th>
<th>Old expression</th>
<th>Replaced by</th>
</tr>
</thead>
<tbody>
<tr>
<td>39</td>
<td>((n-1,r))</td>
<td>(-(n-1,r))</td>
</tr>
<tr>
<td>40</td>
<td>(e(n,r-\epsilon) &lt; f &lt; e(n,r+\epsilon))</td>
<td>(e(n,r+\epsilon) &lt; f &lt; e(n,r-\epsilon))</td>
</tr>
<tr>
<td>41</td>
<td>All inequalities are reversed.</td>
<td></td>
</tr>
<tr>
<td>46</td>
<td>((n,r))</td>
<td>-(n,r)</td>
</tr>
<tr>
<td>49</td>
<td>Let 0 be (&lt; b \leq a ) and make the appropriate changes.</td>
<td></td>
</tr>
<tr>
<td>51</td>
<td>This example is not directly analogous. A similar example could easily be constructed, however.</td>
<td></td>
</tr>
<tr>
<td>52</td>
<td>(\infty ) into (\infty)</td>
<td>0 into 0</td>
</tr>
<tr>
<td>56</td>
<td>(\theta(g/r) &lt; \rho ) ([&gt; \rho])</td>
<td>(\theta(g/r) &gt; \rho ) ([&lt; \rho])</td>
</tr>
<tr>
<td>65</td>
<td>decrease and increase</td>
<td>increase and decrease</td>
</tr>
<tr>
<td>Definition of asymptotic I</td>
<td>(\theta(E_{n+1}/E_n &lt; -\epsilon &lt; 0)</td>
<td>(\theta(E_{n+1}/E_n) &gt; \epsilon &gt; 0)</td>
</tr>
<tr>
<td>Ch. IV</td>
<td>(G_n(y) = (-)^n u_n(x))</td>
<td>(G_n(y) = u_n(x))</td>
</tr>
<tr>
<td>A and B</td>
<td>Change signs of even-powered terms.</td>
<td></td>
</tr>
<tr>
<td>Remainder for A</td>
<td>Precede by ((-)^n).</td>
<td></td>
</tr>
<tr>
<td>Remainder for B</td>
<td>Omit ((-)^n).</td>
<td></td>
</tr>
<tr>
<td>Discussion preceding (69)</td>
<td>neighborhood of (F_1(\infty))</td>
<td>neighborhood of 0</td>
</tr>
<tr>
<td>69.1</td>
<td>(x \to \infty)</td>
<td>(x \to 0)</td>
</tr>
<tr>
<td>69.2</td>
<td>neighborhood of (F_1(\infty))</td>
<td>neighborhood of 0</td>
</tr>
<tr>
<td>70</td>
<td>analytic at (\infty)</td>
<td>analytic at 0</td>
</tr>
<tr>
<td>73</td>
<td>This theorem reduces to the trivial statement,  &quot;(F(x)) is analytic at 0 (\iff) (f(x)) is analytic at 0.&quot;</td>
<td></td>
</tr>
<tr>
<td>72</td>
<td>(\infty)</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>(x^{-1})</td>
<td>(x)</td>
</tr>
<tr>
<td>Theorem</td>
<td>Old Expression</td>
<td>Replaced by</td>
</tr>
<tr>
<td>---------</td>
<td>----------------</td>
<td>-------------</td>
</tr>
<tr>
<td>73</td>
<td>$\infty$</td>
<td>0</td>
</tr>
<tr>
<td>1)</td>
<td>$</td>
<td>z</td>
</tr>
<tr>
<td>2)</td>
<td>$z^{-n}$</td>
<td>$z^n$</td>
</tr>
<tr>
<td>3)</td>
<td>$z^{-m}$</td>
<td>$z^m$</td>
</tr>
<tr>
<td></td>
<td>$m \geq 2$</td>
<td>$m \geq 0$</td>
</tr>
<tr>
<td>74.1</td>
<td>$s$ and $r$ are $\leq 0$</td>
<td>$s$ and $r$ are $\geq 0$</td>
</tr>
<tr>
<td>75.3</td>
<td></td>
<td>omit $(-)^{n+1}$</td>
</tr>
<tr>
<td>75.4</td>
<td></td>
<td>omit $(-)^r$</td>
</tr>
<tr>
<td>80</td>
<td></td>
<td>omit $(-)^{n-1}$</td>
</tr>
</tbody>
</table>

$F_1(x) = 1/x$

This transformation is not permissible, since it does not carry 0 into 0.

$F_1(x) = e^{-x}$

This transformation is not permissible, but an analogous discussion applies to the permissible transformation, $y = e^x$.

$F_1 = f$

$\theta(f) = -\infty$

$f(x) \sim c_1 e^{-c_2 x}$

expansion of $f^{-1}$

does not apply.

expansion of $F/vf$

powers of $1/x$

powers of $x$

83, 85, and 86.1

$f(a)[v_1(a) + \ldots + v_n(a)]$

$-f(a)[v_1(a) + \ldots + v_n(a)]$

87

$\theta(v_1) = 1-\epsilon$

$\theta(v_n) = 1+\epsilon$

$\theta(E_n) = -n\epsilon$

$\theta(E_n) = n\epsilon$

case where $\epsilon = 1$

omits this case

discussion following 87

$\theta(f) = -\infty$

$\theta(f) = \infty$

88

$e^{-x/n} < e^{-(\log x)^2} < x^{-n}$

$e^{-n/x} < e^{-(\log x)^2} < x^n$
<table>
<thead>
<tr>
<th>Theorem</th>
<th>Old Expression</th>
<th>Replaced by</th>
</tr>
</thead>
<tbody>
<tr>
<td>89</td>
<td>does not apply</td>
<td></td>
</tr>
<tr>
<td>90</td>
<td>$r &lt; 0$</td>
<td>$r &gt; 0$</td>
</tr>
<tr>
<td></td>
<td>$a^r(-\frac{a}{r} + \frac{a}{r^2} - \frac{a}{r^3} + \ldots)$</td>
<td>$-a^r(-\frac{a}{r} + \frac{a}{r^2} - \frac{a}{r^3} + \ldots)$</td>
</tr>
<tr>
<td></td>
<td>$F(a) = -a^{r+1}/(r+1)$</td>
<td>$F(a) = a^{r+1}/(r+1)$</td>
</tr>
<tr>
<td>C</td>
<td>$u_2 = (1+u_1')v$ for $F \to 0$</td>
<td>$u_2 = (u_1'-1)v$ for $F \to 0$</td>
</tr>
<tr>
<td></td>
<td>$u_2</td>
<td>^a</td>
</tr>
<tr>
<td></td>
<td>Start with $u_1=v$ to get (83)</td>
<td>Start with $u_1=-v$ to get (83)</td>
</tr>
<tr>
<td></td>
<td>$F_n = f^c(v_1 + v_2 + \ldots + v_n)$</td>
<td>$F_n = f^c(-v_1 - v_2 - \ldots + v_n)$</td>
</tr>
<tr>
<td></td>
<td>remainder =</td>
<td>remainder =</td>
</tr>
<tr>
<td></td>
<td>$\int_a^\infty f^c(v_1' + v_{n-1}' - v_n')$</td>
<td>$\int_a^c f^c(v_1' + v_{n-1}' - v_n')$</td>
</tr>
<tr>
<td>85</td>
<td>$g(x) = w_n(x) - w_{n-1}(x) + v_{n-1}(x)$</td>
<td>$g(x) = w_{n-1}(x) - v_n(x) + v_{n-1}(x)$</td>
</tr>
<tr>
<td>91</td>
<td>Start with $\gamma v$</td>
<td>Start with $-\gamma v$</td>
</tr>
<tr>
<td>92.1</td>
<td>$e_1 = (u_1-u_2)/v$</td>
<td>$e_1 = (u_2-u_1)/v$</td>
</tr>
<tr>
<td>92.2</td>
<td>$-\infty$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>95</td>
<td>does not apply. This example will be discussed later.</td>
<td></td>
</tr>
<tr>
<td>96.1</td>
<td>$\theta(u_1-u_2) = -\infty$</td>
<td>$\theta(u_1-u_2) = \infty$</td>
</tr>
<tr>
<td></td>
<td>$f(x) = e^c e^{-t} + t/r$</td>
<td>$f(x) = e^c e^{-1/r} + t/r$</td>
</tr>
<tr>
<td></td>
<td>$v' + 1/r = e^{-x}$</td>
<td>$v' + 1/r = e^{-1/x/x^2}$</td>
</tr>
<tr>
<td>96.2</td>
<td>$-\infty$</td>
<td>$\infty$</td>
</tr>
<tr>
<td></td>
<td>$g = e^c \frac{s + e^{-t}}{t} dt$</td>
<td>$g = e^c \frac{s + e^{-1/t}}{t} dt$</td>
</tr>
<tr>
<td>97.3</td>
<td>All inequalities are reversed.</td>
<td></td>
</tr>
</tbody>
</table>
TABLE 1 (Continued)

<table>
<thead>
<tr>
<th>Theorem</th>
<th>Old Expression</th>
<th>Replaced by</th>
</tr>
</thead>
<tbody>
<tr>
<td>99</td>
<td>$\frac{v}{(1-v')} \quad \left</td>
<td>\frac{v}{(1-v')} \right</td>
</tr>
<tr>
<td>100</td>
<td>$\gamma v f, \frac{v}{(1-v')} \quad \left</td>
<td>\frac{v}{(1-v')} \right</td>
</tr>
<tr>
<td>105.2</td>
<td>$\theta(v') &lt; 0 \quad \theta(v') &gt; 0$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\theta(v) &lt; 1 \quad \theta(v) &gt; 1$</td>
<td></td>
</tr>
<tr>
<td>106</td>
<td>$f(x) = x^{-2}/\log x \quad f(x) = x^2 \log x$</td>
<td></td>
</tr>
<tr>
<td>108 and 109</td>
<td>do not apply.</td>
<td></td>
</tr>
</tbody>
</table>

The definition or order is closely related to the notion of order of contact, which is useful in differential geometry. It is usually considered, (e.g., in [55, p. 76],) that $f(x)$ has order of contact $r$ if $f(0), f'(0), \ldots, f^{(r-1)}(0)$ are 0 and $f^{(r)}(0) \neq 0$. By (18), if $f(x)$ has integral order, $r$, and order of contact, $s$, then $r = s$.

Example 95 does not apply, because if $F_1 \sim F$, then $u_1 - u_2 = o(v)$, and $\theta(v) \geq 1$. Hence, $\theta(u_1 - u_2) \geq 1$ and cannot be 0. However, if we drop the restriction that $F_1$ be $\sim F$, then the latitude afforded by (93) enables us to choose a well-behaved $u_1 - u_2$ approaching 0 and of order 0 which would make $C_1$ asymptotic II.

The identity transformation, $y = x$, becomes one of the most important to be considered in Chapter IV. The convergence of $A$ and $B$, (other than infinitely slow convergence,) depends upon $f(x)$ being analytic at 0, (70.3). $u_n(x) = f^{(n)}(x)$. Hence, for the procedure to be useful, the $f^{(n)}(x)$'s must $\rightarrow$ finite limits not all 0. This means that $f(x)$ has to have an asymptotic expansion at 0, (78). In particular, $f(x)$ must be $\sim cx^n$ for $n > 0$. 
Now that we have defined order in both senses, Theorem 13 can be stated and proved in full generality.

**Theorem 13, (general form.)** \( \theta(f(g)) = \theta(f) \cdot \theta(g) \), provided that \( g \to \pm \infty \) or 0, and \( f \) and \( g \) have order in the sense that applies, \( f \neq 0 \) and \( \pm \infty \) nor \( \pm \infty \) and 0.

The proof is the same as for (13). //

**Theorem 11.** If \( f \) has order and \( \to 0 \) or \( \pm \infty \) monotonically, then \( \theta(f^{-1}) = 1/\theta(f) \).

Proof. If \( f \) decreases to 0, then \( f < g \implies f^{-1} < g^{-1} \).

\[ \begin{align*}
\theta(f) = r & \implies x^{r - \varepsilon} < f < x^{r + \varepsilon}, \implies x^{\frac{1}{r - \varepsilon}} < f^{-1} < x^{\frac{1}{r + \varepsilon}}, \implies \theta(f^{-1}) = 1/r. \\
\theta(f) = -\infty & \implies f < x^{-N}, \implies f^{-1} < x^{-N} \text{ and } \to \infty, \implies \theta(f^{-1}) = 0.
\end{align*} \]

If \( f \) increases to \( \infty \), then \( f < g \implies f^{-1} > g^{-1} \). As before,

\[ \begin{align*}
\theta(f) = r & \implies \theta(f^{-1}) = 1/r. \quad \theta(f) = \infty & \implies f > x^N, \implies f^{-1} < x^{-N} \text{ and } \to \infty, \implies \theta(f^{-1}) = 0. //
\end{align*} \]
CHAPTER VIII

EXTENSION OF THE THEORY TO THE SUM OF AN INFINITE SERIES

We may use the theory of the preceding chapters to obtain approximations to the sum of an infinite series. Let us say we wish to approximate \( F(a) \), which is \( \sum_{n=a}^{\infty} f(n) \) if that series converges and is \( \sum_{n=c}^{a-1} f(n) \) otherwise. We can define \( f(x) = f(n) \) for \( n \leq x \leq n + 1 \), and then \( F(a) = \int_{a}^{\infty} f(x) \) or \( \int_{c}^{a} f(x) \), preserving the previous notation. Thus, any theorem about \( f(x) \) and \( F(x) \) applies to \( f(n) \) and \( F(n) \). The difficulty comes with the theorems that require \( f(x) \) to have a derivative. The nearest thing \( f(n) \) has to a derivative is a difference, \( \Delta f = f(n+1) - f(n) \). If we read \( \Delta \) everywhere a derivative appears, (using the basic analogy of the finite calculus,) then we can prove, with a few changes, most of the theorems of the preceding chapters. This analogy was used by Hardy, [15], to carry over some of his theorems from the continuous to the discrete case.

The following collection of lemmas will be needed to effect this transition. So that the formulae won't be cluttered with parentheses, \( f_n \) will be written instead of \( f(n) \). When it won't be confusing, the subscript, \( n \), will be left out.

The well-known relations, \( \Delta (fg) = f \Delta g + g \Delta f + \Delta (f/g) \) = \( (g \Delta f - f \Delta g) / (g \Delta g) \) will often be used, as will the summation by parts formula, \( \sum_{m}^{n} a_{i} \Delta b_{i} = a_{n+1} b_{n+1} - a_{m} b_{m} - \sum_{m}^{n-1} b_{i+1} \Delta a_{i} \). This reduces to \( -a_{n+1} b_{n+1} + \sum_{m}^{n} b_{i+1} \Delta a_{i} \) when \( n = \infty \) and \( a_{n+1} b_{n+1} \rightarrow 0 \).

**Lemma 115.** If \( f \sim g \) and \( h \sim k \), and \( |f/h + 1| > \epsilon \), then \( f + h \sim g + k \).
Lemma 116. 1) If \( f'(x) \) is monotonic, then \( \Delta [f(g_n)] \) lies between \( f'(g_n) \Delta g_n \) and \( f'(g_{n+1}) \Delta g_n \). This lemma has the following corollaries:

2) Let \( \alpha \) and \( h \) be constants, positive or negative. Let
\[ \Delta h f = f(n+h) - f(n). \Delta h n^\alpha/h \text{ lies between } \alpha n^{\alpha-1} \text{ and } \alpha(n+h)^{\alpha-1}. \]

3) \( \Delta n^\alpha \sim \alpha n^{\alpha-1}, \sum_{n=1}^{\infty} 1/n^\alpha \text{ or } \sum_{n=1}^{\infty} 1/n^{\alpha+l}/(\alpha+1), \text{ for } \alpha \neq -1, \text{ and } \sum_{n=1}^{\infty} n^{-1-\alpha} \sim \log n. \) (These results are well known. See Hardy [15].)

4) \( \Delta (f_n^n) \) lies between \( \alpha f_n^n \Delta f_n \) and \( \alpha f_{n+1}^{n-1-1} \Delta f_n \).

5) \( \Delta \log f \) lies between \( \Delta f_n/f_n \) and \( \Delta f_{n+1}/f_{n+1} \), i.e., between \( \Delta f/f \) and \( \Delta f/f (1 + \Delta f/f) \).

Proof of (116.1). \( \Delta [f(g_n)] = \Delta [f(g_n)] / \Delta g_n \Delta g_n, = f'(x) \Delta g_n \)
where \( x \) is between \( f'(g_n) \) and \( f'(g_{n+1}) \) by the law of the mean. //

Two other useful facts in working with logs are that \( \log (1 + \Delta f/f) = \Delta \log f \) and \( \log (1 + \Delta f/f) \sim \Delta f/f \) when \( \Delta f/f \to 0 \). Theorems 11.2 and 45, which say that \( \theta(f_{n+m}) = \theta(f_n) \) and \( \theta(f_{n+m}) = \theta(f_n) \), will often be used in this chapter.

Lemma 117. If \( f \) and \( g \) are positive, then

1) \( \Delta f/f + 1 < \Delta g/g + 1, \implies f < kg; \)

2) \( \Delta f/f + 1 \to 0 \) and \( \Delta f/f + 1 = o(\Delta g/g + 1) \implies f = o(g). \)

Proof of 1). \( \Delta f/f + 1 < \Delta g/g + 1 \implies \Delta \log f < \Delta \log g, \implies \log f < \log g + \log k, \implies f < kg. \)

Proof of 2). \( \Delta f/f + 1 < N (\Delta g/g + 1) \) for all \( N \), \implies \Delta \log f < \Delta \log g + \log N, \implies \log f < \log g + \log k + \log N, \) for some constant \( K, \implies f < Kn, \implies f = o(g). //

The following table describes the series analogues to the theorems of Chapter II.
<table>
<thead>
<tr>
<th>Theorem</th>
<th>Old Expression</th>
<th>Replaced by</th>
</tr>
</thead>
<tbody>
<tr>
<td>All theorems</td>
<td>$x$</td>
<td>$n$</td>
</tr>
<tr>
<td></td>
<td>$f(x)$ and $F(x)$</td>
<td>$f_n$ and $F_n$</td>
</tr>
<tr>
<td></td>
<td>$\int_a^\infty$</td>
<td>$\sum_a^\infty$</td>
</tr>
<tr>
<td></td>
<td>$\int_c^a$</td>
<td>$\sum_c^{a-1}$</td>
</tr>
<tr>
<td></td>
<td>the operation, derivative</td>
<td>the operation, difference</td>
</tr>
<tr>
<td></td>
<td>$f''(x)$</td>
<td>$\Delta^2 f_n$</td>
</tr>
<tr>
<td>13</td>
<td>$f(x)$ and $g(x)$</td>
<td>$f(x)$ and $g_n$</td>
</tr>
<tr>
<td>21</td>
<td>$f(x)$</td>
<td>$f(2n) = 1, f(2n+1) = 1/n$</td>
</tr>
<tr>
<td>24</td>
<td>If $f'(x)$ is continuous</td>
<td>If $f_n$ is positive</td>
</tr>
<tr>
<td></td>
<td>If $r \neq 0$ or $-\infty$</td>
<td>If $r \neq 0$</td>
</tr>
<tr>
<td>25</td>
<td>Does not apply.</td>
<td></td>
</tr>
<tr>
<td>27</td>
<td>$f'$ is continuous</td>
<td>$f$ is monotonic</td>
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<tr>
<td>29</td>
<td>$f'$ is continuous</td>
<td>$f$ is positive</td>
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<tr>
<td>30.1</td>
<td>$f'^2/f''$</td>
<td>$f_{n+1} \Delta^2 f_n / (\Delta f_{n+1} \Delta f_n)$</td>
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<tr>
<td></td>
<td>$v(x) = -f(x) / f'(x)$</td>
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<tr>
<td></td>
<td>$f''$ is continuous</td>
<td>$\Delta^2 f$ is of constant sign</td>
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<td>30.2</td>
<td>$\log</td>
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<td>32</td>
<td>$f'^2/f''$</td>
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<td>( f''/f^{1.2} )</td>
<td>( f_{n+1} \Delta^2 f_n / (\Delta^2 f_{n+1} \Delta f_n) )</td>
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<td>( F \sim \left</td>
<td>\frac{f^2}{f_{n+1} (2-t)} \right</td>
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46 This theorem does not hold.

The proofs of most of the theorems of Chapter II are analogous to the original proofs, with occasional help from Lemmas 115-117 required. Theorem 17 becomes the Cauchy-Stolz theorem, [47], referred to by Hardy, [15].

Proof of (24). Case 1: \( n \Delta f / f \to r \), finite. Then \( \Delta f / f \to 0 \),
\[ \Rightarrow \Delta f / f \sim \Delta \log f, \Rightarrow \lim \frac{\Delta f / f}{n} = \lim \log f / \log n = \theta(f). \]
Case 2:
\[ n \Delta f / f \to -\infty. -1 < \Delta f / f < 0. \]
It can be shown that \( \log x \leq x-1 \) for \( x > 0 \),
\[ \Rightarrow \log (1 + \frac{\Delta f}{f}) \leq \frac{\Delta f}{f} < 0. \]
Hence \( \frac{1}{n} = o\left(\frac{\Delta f}{f}\right) \Rightarrow \frac{1}{n} = o(\Delta \log f), \]
\[ \Rightarrow \log n = o(\log f), \Rightarrow \theta(f) = -\infty. \]
Case 3: \( n \Delta f / f \to \infty. \) Then \( 0 < \Delta f / f \) and \( 1/n < \epsilon \Delta f / f \) for all \( \epsilon \). Let \( \alpha = \log 2. \)
\[ \alpha(x-1) \leq \log x \leq x-1 \text{ when } 1 \leq x \leq 2. \]
Hence, \( \alpha \Delta f / f \leq \Delta \log f \leq \Delta f / f \) when \( 0 \leq \Delta f / f \leq 1. \) When \( \Delta f / f > 1, \Delta \log f > \alpha. \) Thus \( 1/n < \frac{\epsilon}{\alpha} \Delta \log f \) for all \( \epsilon \), \( \Rightarrow 1/n = o(\Delta \log f), \Rightarrow \log n = o(\log f), \Rightarrow \theta(f) = \infty. \)

The proof that \( \theta(\Delta f) = r-1 \) is the same as before, except that the theorem also holds when \( r = -\infty. \) \( n \Delta f / f \to -\infty \) and \( f_n > 0 \Rightarrow \Delta f < 0, \)
\[ \Rightarrow |\Delta f| < |f|, \Rightarrow \theta(\Delta f) = -\infty. \]

If we don't require \( f \) to be monotonic in (27), but only positive, then the theorem doesn't hold. Let \( f(2n) = e^{-2n} \) and \( f(2n+1) = 2e^{-2n}. \)
\( \theta(f) = -\infty. \) \( 1/2 < |\Delta f| < 1. \) Hence \( f/(n \Delta f) \to 0. \) But \( n \Delta f / f \to \) both \( \infty \) and \( -\infty. \)
Proof of (30.1). Let \( v_n = -f_n/\Delta_f^2 \), \( \Delta v_n = \frac{f_{n+1}}{\Delta f_{n+1}} \frac{\Delta^2 f_n}{\Delta f_n} - 1 \). If we assume \( \Delta v_n \to \) a non-0 limit, \(-1/r\), then \( v_n \to -1/r \), (17), \( \implies \theta(f) = r \). Now suppose \( \Delta v_n \to 0 \). \( \Delta^2 f \) is of constant sign \( \implies \Delta f \) is of constant sign \( \implies f \) is of constant sign \( \implies \Delta v_n \) is of constant sign \( \implies v_n/\sqrt{n} \) is of constant sign and \( \to 0 \), (17), \( \implies n/v_n \to \infty \) or \(-\infty\), \( \implies \theta(f) = \pm \infty \), (24). //

Proof of (30.2). The only case that is different is \( r = +\infty \).

Case 1: \( r = -\infty \). -1 < \( \Delta f \)/\( f \) < 0, \( \implies \log |\Delta f|/\log f > 1 \). Suppose \( \lim \log |\Delta f|/\log f > 1 + \varepsilon \). Then \( |\Delta f| < f^{1+\varepsilon} \), \( \implies \Delta f/f < f^\varepsilon \)

\( \implies n\Delta f/f < nf^\varepsilon \). \( \theta(f^\varepsilon) = -\infty \), \( \implies \theta(nf^\varepsilon) = -\infty \), \( \implies n\Delta f/f \to 0 \),

\( \implies \theta(f) = 0 \), (24), a contradiction. The proof for the case \( r = \infty \) is analogous. //

Formula 34, \( F_n \sim |\gamma v_{n-1} f_n| \), has a paradoxical relationship with the formula \( |\gamma v_{n} f_{n}| \). The two formulas appear different; yet if \( \theta(f) = \pm \infty \), they come to the same thing in practice. Suppose

\( \theta(f) = -\infty \). \( \sum_{n=1}^{\infty} f_i \sim v_n f_{n+1} \) (34). Hence \( \sum_{n=1}^{\infty} f_i \sim f_n + v_n f_{n+1} 

= v_n f_n \). So if we use the last two terms as the basis for an approximation to finish off a series, the two formulas give identical results.

A similar relationship holds when \( \theta(f) = +\infty \). If \( -v_{n-1} f_{n-1} \) is considered an approximation to \( F_{n-1} \), which we have defined to be \( \sum_{c=2}^{n-2} f_i \), then it works out the same in practice to approximate \( \sum_{c=1}^{n-1} f_i \) by \( -v_{n-1} f_{n-1} + f_{n-1}' = -v_{n-1} f_n \). Another practically equivalent formula

is \( -v_{n-1} f_n + f_n = f_n^{2/\Delta f_{n-1}} \) as an approximation to \( F_{n+1} \). This one is preferable, since it uses the last two terms of the sum.

Although \( |\gamma v_{n-1} f_n| \) is always \( \sim F_n \) under regularity conditions, this is not true of \( |\gamma v_{n} f_{n}| \) when \( \theta(f) \) is large enough.
Proof. $e(v f_n) \sim (\Delta v/v - \Delta v) \cdot v \to 0$ when $\theta(f)$ is large enough. We can define $f$ inductively, making $f/\Delta f$ as small as we please. Thus, we have $\theta(v) < -(1,1)$, $\Rightarrow \Delta v/v \to -1$ if any limit (118), $\Rightarrow e(v f_n) \to -1$, $\Rightarrow v f_n \nRightarrow F_n$. $e(\gamma v f_{n-1} f_n) = -\gamma(\Delta v_{n-1} - 1/r), \Rightarrow e(\gamma v f_{n-1} f_n) \to 0$ under regularity conditions (30.1). //

If $f$ has finite order, the two approximations are asymptotic.

Proof. $\gamma(v f_n) = \gamma(v f_{n+1} + f_n) \sim \gamma(v f_{n+1})$. This is because $f_n = o(F_n) \Rightarrow F_n \sim F_{n+1} \Rightarrow f_n = o(F_{n+1})$, and $\gamma v f_{n+1} \sim F_{n+1}, \gamma(v f_{n+1}) \sim F_n$. //

(46) fails to hold, and the following lemmas take its place.

**Lemma 118.** If $f$ has extended order, then

$$\Delta f/f \to 0 \Rightarrow |\theta(f)| \leq (1,1).$$

$$\Delta f/f \to -1 \text{ or } \infty \Rightarrow |\theta(f)| \geq (1,1).$$

$$\Delta f/f \to \text{ a constant } \neq -1 \Rightarrow |\theta(f)| = (1,1).$$

Proof. If $\Delta f/f \to \text{ a constant } \neq -1$, then $\log \left( \frac{\Delta f}{f} + 1 \right) = \Delta \log f \to \text{ a constant}, \Rightarrow \theta(\log f) = 1, \Rightarrow |\theta(f)| = (1,1)$. If $(\frac{\Delta f}{f} + 1) \to 0$ or $\infty$, then $\Delta \log f \to \pm \infty \Rightarrow \theta(\log f) \geq 1, \Rightarrow |\theta(f)| \geq (1,1)$. If $\Delta f/f \to 0$, then $\Delta f/f \sim \Delta \log f \to 0, \Rightarrow \theta(\log f) \leq 1, \Rightarrow |\theta(f)| \leq (1,1)$. //

**Lemma 119.** Let $f$, $\Delta f/f$, and $\Delta f/f + 1$ have extended order, and let $|\theta(f)| = (n,r)$. $|\theta(f)| \geq (2,0) \Rightarrow \theta(\Delta f/f)$ or $\theta(\Delta f/f + 1) = \theta(f)$. $|\theta(f)| > (1,1) \Rightarrow \theta(\Delta f/f)$ or $|\theta(\Delta f/f + 1)| = (1,r-1)$. $|\theta(f)| \geq (1,0) \Rightarrow \theta(\Delta f/f) = r-1$. $\theta(f)$ is finite

$\Rightarrow \theta(\Delta f/f) = -1$, unless $f \to \text{ a constant}$, when $\theta(\Delta f/f) \leq -1$. $\theta(f) = (1,1)$ and $\Delta f/f \to \infty \Rightarrow 0 \leq \theta(\Delta f/f) \leq (1,0)$. $\theta(f) = -(1,1)$ and $\Delta f/f \to -1 \Rightarrow -(1,0) \leq \theta(\Delta f/f + 1) \leq 0$. $|\theta(f)| = (1,1)$ and $\Delta f/f \to \text{ a finite limit } \neq -1 \Rightarrow \theta(\Delta f/f) = 0$. //
Proof. \( \log (\Delta T/f + 1) = \Delta \log f \), has extended order.

\( |\theta(f)| \leq (2,0) \implies \theta(\log f) \geq (1,0), = \theta(\Delta \log f), (44), \)

\( \implies \theta(f) = \theta(\Delta T/f + 1). (2,0) > |\theta(f)| > (1,1) \implies \theta(\Delta \log f) \)

\( = \theta(\log f) - 1, (18), \implies |\theta(\Delta T/f + 1)| = (1,r-1). \) In each of these

cases, when \( f \to \infty, \theta(\Delta T/f) > 0, \implies \Delta T/f \sim \Delta T/f + 1. \)

\( (1,1) > |\theta(f)| \geq (1,0) \implies \theta(\Delta \log f) = r-1 < 0, \implies \Delta T/f \to 1, \)

\( \implies \Delta T/f \sim \Delta \log f. \) The fourth statement recapitulates Theorem 18.

\( \theta(f) = (1,1) \) and \( \Delta T/f \to \infty \implies \Delta T/f \sim \Delta T/f + 1. \) Log (\( \Delta T/f + 1 \)),

which = \( \Delta \log f \), has order 0, \( \implies (1,0) \geq |\theta(\Delta T/f + 1)| \geq 0. \)

If \( \Delta T/f \to \) a constant \( \neq -1 \), then \( \theta(\Delta T/f) = 0. \) If \( \Delta T/f \to 0 \), then

\( \Delta T/f \sim \Delta \log f, \implies \theta(\Delta T/f) = \theta(\Delta \log f) = 0. \) //

A corollary of this theorem is that \( |\theta(v)| \) and \( |\theta(v-1)| \) are

\( \leq |\theta(f)|. \) We note that when \( \theta(f) \) is finite, \( f = o(T) \), and

\( F_{n+1} \sim F_n. \)

In Chapter III, the theorems hold as stated, with the following

exceptions. The procedure referred to in Bounds 49 is greatly

simplified in the discrete case. In one has an asymptotic approxi-
mation, \( F_1, \) it may be improved by summing a finite number of terms
first and applying \( F_1 \) to the remainder. This is Procedure 124,

which will be described at the end of this chapter. Bounds 49 apply

with the first term omitted.

Theorems 60.2 and 61 require a few changes.

**Theorem 61.1.** Let \( H = G/F. \) If \( F, f/F, \Delta H/H, \) and \( f/F - 1 \) have ex-
tended order and \( \to \) limits, and if \( |\theta(H)| < |\theta(F)| \) and if \( \theta(F) = \pm \infty \)

then

\( g/f \sim G/F \) if \( F/f \) a non-0 limit,

\( g/f \sim G/F [1 - \lim (\Delta H/H)] \) if \( F/f \to 0, \)
\[
g/f \sim G_{n+1}/F_{n+1} \quad \text{if} \quad F/f \to 0 \quad \text{and} \quad \Delta H/H \to \infty.
\]

If \( F/f \to 1 \) and \( \theta(f) = -\infty \), then \( g/f \sim G/F \) even if \( \theta(H) \leq \theta(F) \).

Proof. The first part of the original proof holds.

\[ G = FH, \quad \Rightarrow \quad g = fH + F_{n+1} \Delta H, = fH \left[ 1 + \frac{\Delta H}{H} \left( \frac{F}{f} - 1 \right) \right]. \]

Case 1: \( F/f \to 1 \), and \( \theta(f) = -\infty \). If \( \lim \frac{\Delta H}{H} \) is finite, then

\[ \frac{\Delta H}{H} \left( \frac{F}{f} - 1 \right) \to 0, \quad \Rightarrow \quad g \sim fH. \]

If \( \frac{\Delta H}{H} \to \infty \), then \( \frac{\Delta H}{H} \left( \frac{F}{f} - 1 \right) \sim \left( \frac{\Delta H}{H} + 1 \right) \left( 1 - \frac{F}{f} \right) \).

\[ \log \left[ \frac{\Delta H}{H} \left( \frac{F}{f} - 1 \right) \right] \sim \Delta \log H + \Delta \log F \sim \Delta \log F, \quad \text{since} \quad \left( \frac{\Delta H}{H} + 1 \right), \]

\[ (1 - \frac{F}{f}), \quad \text{and therefore} \quad \Delta \log H \quad \text{and} \quad \Delta \log F \quad \text{have extended order}. \]

Thus, \( \log \left[ \frac{\Delta H}{H} \left( \frac{F}{f} - 1 \right) \right] \to -\infty, \quad \Rightarrow \quad \frac{\Delta H}{H} \left( \frac{F}{f} - 1 \right) \to 0, \quad \Rightarrow \quad g \sim fH. \]

Case 2: \( F/f \to a \text{ constant}, \neq 1 \), if \( \theta(f) = -\infty \). Then \( \frac{\Delta H}{H} \to 0 \), since \( |\theta(H)| < |\theta(F)| = (1, 1), (118), \quad \Rightarrow \quad \frac{\Delta H}{H} \left( \frac{F}{f} - 1 \right) \to 0, \quad \Rightarrow \quad g \sim fH. \)

Case 3: \( F/f \to \infty \). Then \( \frac{\Delta H}{H} \to 0 \), (118), and \( (F/f - 1) \sim F/f \),

\[ \Rightarrow \quad \frac{\Delta H}{H} \left( \frac{F}{f} - 1 \right) \sim \frac{\Delta H}{H} \frac{F}{F}, \sim - \Delta \log H/\Delta \log F \to 0, (8.1), \quad \Rightarrow \quad g \sim fH. \]

Case 4: \( F/f \to 0 \). \( \frac{\Delta H}{H} \left( \frac{F}{f} - 1 \right) \sim - \frac{\Delta H}{H} \). \( F/f \to 0 \implies F \) and \( G \to \infty \), (see the original proof of (61.1)), \( \Rightarrow \quad g = fH \left[ 1 + \frac{\Delta H}{H} \left( \frac{F}{f} - 1 \right) \right], \sim fH[1 - \frac{\Delta H}{H}], \sim fH[1 - \lim \frac{\Delta H}{H}], \) supposing that \( \lim \frac{\Delta H}{H} \neq 1 \) or \( \pm \infty \).

The expression, \( g/f \sim G/F[1 - \lim (\Delta H/H)] \), is interpreted as

\[ g/f = o[G/F] \quad \text{if} \quad \lim \Delta H/H = \infty, \quad \text{and} \quad G/F = o(g/f) \quad \text{if} \quad \lim \Delta H/H = 1. \]

The proof is similar in these cases.

Case 5: \( F/f \to 0 \) and \( \frac{\Delta H}{H} \to \infty \). \( G = FH \implies g = fH_{n+1} + F_{n+1} \Delta H, \)

\[ = fH_{n+1} \left( 1 + \frac{\Delta H}{H} \right) \frac{F}{f}. \quad \Delta H/H \to \infty \implies \Delta H/H_{n+1} \to 1 \implies g \sim fH_{n+1}. \quad // \]

Theorem (60.2) is essentially a special case of this theorem.

(1) The only difference is that we don't have to assume that \( F \) has extended order.) The original proofs of (60.1) and (60.3) still hold.

For Theorem 62, we obtain the result \( \theta(F/f - 1) = \theta(F_{n+1}/F). \)
If \( \theta(f) = -\infty \) and \( \theta(e_1) \) is finite, this reduces to \( \theta(f) - 1 - 1 \), 
\[ \Theta(v - 1) - 1, \] 
(59). We must assume that \( \Delta(e_1 f)/(e_1 f) \), 
\[ \frac{\Delta(f - f)}{f - f} \] 
has order. If \( \theta(f) = \infty \), 
\[ \Theta(e_1 f) = \Theta(f + 1) - 1. \]

Since it will, in general, be impossible to solve exactly the 
equation \( F_n = \epsilon \), Theorems 63, 64, and 66.2 do not apply.

The results of Chapter IV do not carry over to infinite series.

The Gregory-Newton formula (the finite analogue of the Taylor series,) 
leads to the same difficulty as with the untransformed Taylor series.

But now, no transformation of the variable of integration is possible. 
\( dx \) could be transformed to \( \phi'(y) dy \) when \( x = \phi(y) \), but \( \Delta x \)
always has to be 1 for a series.

**Expansion 83.**

The series analogue of the Laplace-Winkler expansion may be de-

rived by a summation by parts. Let \( F_a = \sum_{n=0}^{\infty} f_n \), a convergent, posi-
tive series. 
\[ f_n = -\Delta v_n, \implies F_a = v_a f_a + \sum_{n=0}^{\infty} f_{n+1} \Delta v_n, \]
summing by parts. Now let \( f_{n+1} \Delta v_n = -\Delta v_{2n}, \) i.e., 
\[ v_{2n} = f_{n+1} \Delta v_n/\Delta v_{n} = (v_{n+1} - v_n) \Delta v_n. \] 
Then we may sum by parts again, getting

\[ F_a = v_a f_a + v_{2a} f_{2a} + \sum_{n=0}^{\infty} f_{n+1} \Delta v_{2n} = (v_{n+1} - v_n) \Delta v_{2n}, \] 
and so on, 

obtaining the expansion, \( F_a = v_a \left( \sum_{j=1}^{k} \frac{v_j f_j}{a_j} \right) + \sum_{n=0}^{\infty} f_{n+1} \Delta v_{kn} \). To justify 
this expansion, we must show that \( v_{kn} f_{n+1} \to 0 \) as \( n \to \infty \).

\[ v_{kn} f_{n+1} \to \gamma v_{n+1} F_{n+1} \] 
if \( \gamma v_{n+1} F_{n+1} \to \) any limit. Under that regularity 
condition, \( v_{n+1} \to 0 \) since \( \gamma = r/(r+1), r \leq -1 \), and \( F_{n+1} \to 0 \).

\[ v_{kn} f_{n+1} = (v_{n+1} - v_n) \Delta v_{kn} \]. Since \( v_n \geq 1 \), \( |(v_{n+1} - v_n) f_{n+1}| < v_{n+1} f_{n+1}, \to 0 \).

Theorem 84 will show that \( \Delta v_{kn} \to \pm 1/r^n \), if any limit. Thus, 
under the regularity condition that \( \Delta v_{kn} \to \) a limit for \( k = 1,2, ..., m \), 
the Laplace-Winkler expansion is valid for \( m \) terms.
An alternative expansion is obtained by letting \( f_{n+1} \Delta v_n = -\Delta f_{n+1} v_{2n} \), so that \( v_{2n} = v_{n+1} \Delta v_n \), \( v_{3n} = v_{n+2} \Delta v_{2n} \), and so on.

\[
\sum_{a=0}^{\infty} f_n = v_{a+1} f_{a+1} + \cdots + v_{ka+k-1} f_{ka+k-1} + \sum_{n=a}^{\infty} f_{n+k} \Delta v_{kn}.
\]

The properties of these two expansions are similar.

A third, apparently different expansion is obtained by letting

\[
f_n = -\Delta f_{n-1} w_{2n}, \quad w_{2n} = w_{n} \Delta w_{2n}, \quad w_{3n} = w_{n} \Delta w_{2n}, \quad \text{and so on.}
\]

\[
\sum_{a=0}^{\infty} f_n = f_{a+1} (w_{a+1} + \cdots + w_{ka+k-1}) + \sum_{n=a}^{\infty} f_{n+k} \Delta v_{kn}.
\]

If we expand \( \sum_{n=a+1}^{\infty} f_n \) this way and add \( f_n \), we get the first expansion. The first term = Formula 34, \( v_{a+1} f_{a} \).

Theorems analogous to those of Chapter V will be derived for the first expansion. We will assume \( f_n \) is \( \geq 0 \) for the rest of this chapter.

**Theorem 84.** The statement and proof of Theorem 84 for the discrete case are analogous to the original. We have \( v_{2n}/(v_{n-1}) = \Delta v_{2n} = -\frac{1}{r} \).

\[
\lim_{n \to \infty} \Delta v_n = \lim_{n \to \infty} v_{2n}/(v_{n-1}) = -\frac{1}{r}. \quad \text{Hence } \lim_{n \to \infty} \Delta v_n = 1/r^2, \quad \text{and so on.}
\]

If \( F_n \) diverges, we have the expansion,

\[
\sum_{n=c}^{a-1} f_n = f_{a} \sum_{j=1}^{k} v_{ja} - \sum_{j=1}^{k} v_{ja} - \sum_{a=j+1}^{a-1} f_{a-j+1} + \sum_{n=c}^{a-k-1} f_{n} \Delta v_{kn}.
\]

Bounds 85.1 \( F_a = f_a (\sum_{j=1}^{k} v_{ja} - \delta)/(1-\delta) \), where \( \inf \Delta v_{kn} \leq \delta \leq \sup \Delta v_{kn} \) for \( n > a \). If the inf and the sup are not on opposite sides of 1, they may be substituted for \( \delta \) in this expression to get bounds for \( F_a \).

Proof. \( F_a = f_a \sum_{j=1}^{k} v_{ja} + \sum_{n=a}^{\infty} f_{n+1} \Delta v_{kn} = f_a \sum_{j=1}^{k} v_{ja} + \delta F_a - \delta f_a \)

\[
\implies F_a = f_a \left[ (\sum_{j=1}^{k} v_{ja} - \delta)/(1-\delta) \right]. \quad \text{The proof concludes with the following lemma: } (x-\delta)/(1-\delta) \text{ is a monotonic function of } \delta, \text{ provided that}
\]
\( \delta \) stays on one side of 1. [Proof. Let \( f(\delta) = (x-\delta)/(1-\delta) \). \( f'(\delta) = (x-1)/(1-\delta^2) \), is of constant sign for \( \delta \) on one side of 1.]
When we attempt to find similar bounds for divergent $F$, we get a complicated expression in $\delta$ that is not necessarily monotonic.

Bounds 86.1. If $f \to 0$ monotonically and $v_{kn} \to 0$, then $F_a$ lies between $f_{a}^{l}[v_{la}^{n} + \ldots + v_{ka}^{n}]$ and $(the\ same\ v_{ka}^{n}f_{a+1}^{n})$.

If $\theta(f) < -(1,0)$, and $k$ is large enough, and $v_{kn} \to$ a limit, then that limit is 0, as is shown in Theorem 87.

Bounds 86.2. If $\Delta v_{kn}$ is of constant sign, then

$$\sum_{a=1}^{M} f_{n} = \sum_{c=1}^{M} v_{c}^{k} f_{n} - \sum_{j=1}^{K} v_{j,a-j}^{k} f_{a-j+1} + 5 (v_{k,a-k} - v_{k,c}),$$

where $\inf f_{n} \leq \delta \leq \sup f_{n}$ for $c+1 \leq n \leq a-k$. Substituting the inf and the sup for $\delta$, we get bounds for $F(c,a)$.

Theorem 87.1. If $f$, $\Delta v_{1}$, $\Delta v_{2}$, ... have order and $\theta(v_{1}-l) = 1 - \epsilon$ (which means that $\theta(f) = +\infty$) and $v_{1} \to$ a limit, then

$$\theta(e_{k}) = -k \epsilon \text{ when } \epsilon < 1,$$

$$= -k + 1 + \theta(\Delta v_{1}) \text{ when } \epsilon = 1,$$

$$= 1 - (k+1) \epsilon \text{ when } \epsilon > 1, \text{ and } F \text{ converges},$$

$$= -k - (k-1) \epsilon \text{ when } \epsilon > 1 \text{ and } F \text{ diverges}.$$

Proof. $v_{2} = (v_{1}-l) \Delta v_{1}$. $\theta(v_{2}) = l - \epsilon - \epsilon = l - 2\epsilon$, etc.,

$$\Rightarrow \theta(\Delta v_{k}) = -k \epsilon \text{ for } \epsilon \neq 1. \text{ Similarly, } \theta(\Delta v_{k}) = -k + 1 + \theta(\Delta v_{1}) \text{ when } \epsilon = 1.$$

Case 1: $f \to 0$. $E_{k} = \sum_{a=1}^{n} f_{n}^{k-1} \Delta v_{kn}$, $e_{k} = -\frac{f_{n+1}^{k}}{f_{n}^{k}} \Delta v_{kn}$,

$$= - (1 + \frac{\Delta f}{f_{n}^{k}}) \Delta v_{kn} = (\frac{1}{v} - 1) \Delta v_{kn}. \text{ } 1/v \to \text{ a limit between 0 and 1}.$$

If $\epsilon < 1$, then $1/v \to 0$, (118), $\Rightarrow \theta(e_{k}) = \theta(\Delta v_{k})$. If $\epsilon > 1$, then $1/v \to 1$, $\Rightarrow \theta(1/v - l) = \theta(v - l)$, (8.3), $= l - \epsilon$,

$$\Rightarrow \theta(e_{k}) = 1 - \epsilon + \theta(\Delta v_{k}). \text{ If } \epsilon = 1, \text{ and } v \to 1, \text{ then } \theta(1 - \frac{1}{1/v})$$

$$= \theta(v - l)$$. (8.3) $= 0$. If $v \to$ another limit, $0 = \theta(v - l) = \theta(v) = \theta(1/v)$

$$= \theta(1 - 1/v).$$
Case 2: $F$ diverges. $E(c,a) = \frac{\sum_{n=c}^{a-k-1} f_{n+1} \Delta v_n}{\sum_{n=c}^{a-1} f_n}$

$e_n = \frac{f_{n-k+1} \Delta v_k}{f_n}, n-k \cdot \theta(\Delta v_k) = -k\epsilon$ if $e \neq 1$ and $= -k + 1 + \theta(\Delta v_1)$

if $e = 1$. $\theta \left( \frac{f_{n-k+1}}{f_n} \right) = -\theta \left( \frac{f_n}{f_{n-k+1}} \right) = \theta \left( \frac{f_n}{f_{n-1}} \cdot \frac{f_{n-1}}{f_{n-2}} \cdot \ldots \cdot \frac{f_{n-k+2}}{f_{n-k+1}} \right)$

$= \theta \left( \frac{n-1}{n-k+1} \right) (1 - \frac{1}{v_{11}}) = (k-1) \theta \left( 1 - \frac{1}{v} \right)$. If $e < 1$, then $1/v \to 0$,

(118), $\Longrightarrow \theta(e_n) = -k\epsilon$. If $e > 1$, then $1/v \to -\infty$ (118),

$\Longrightarrow \theta(1 - 1/v) = \theta(1/v) = -\theta(v), \epsilon - 1, \Longrightarrow \theta(e_n) = -k\epsilon + \epsilon - 1$.

If $e = 1$, then $\theta(v) = \theta(1/v) = \theta(1 - 1/v) = 0$, since $v \to$ a limit between 0 and $-\infty$. //

Theorem 97.2. The statement and proof are analogous to the original.

Procedure C. Let the convergent series, $F_n = u_n f_n$. Then after algebra,

$u_n = 1/n + (v_n - 1) \Delta u_n$. If we solve this equation iteratively, we obtain

Procedure C: $u_{2n} = v_n + (v_n - 1) \Delta u_{1n}$. By setting $u_1 = v_1$ we obtain the Laplace-Winckler series.

Theorem 92. The statement and proof are analogous to the original,

except Part 4, where if $u_1 = \gamma v$, $e_1 = -\gamma(\Delta v + 1/r - \Delta v/v)$.

Theorem 94. Let $u_1, u_2, \ldots$, be determined by Procedure C:

$u_2 = v + (v-1) \Delta u_1$. Let $\theta(f) = +\infty$, let $v$ have extended order and $\Rightarrow$ a limit, $v_0$; and let $e_2/e_1$, which $= \frac{(\Delta u_1 - u_2)}{u_1 - u_2} (v-1)$, $\Rightarrow$ a limit, $L$.

I. If $|\theta(e_1)| > |\theta(f)|$ and $\theta(e_1) < 0$, then $e_2/e_1 \to 1 - v_0$.

II. If $|\theta(e_1)| > |\theta(f)|$ and $\theta(u_1 - u_2) > 0$, then $e_2/e_1 \to +\infty$. In each case, if $e_2$ has extended order and $e_3/e_2 \Rightarrow$ a limit, then the limit is $L$. //
2) If \(|\Theta(e_1)| < |\Theta(f)|\) and \(F\) converges, then \(e_2/e_1 \to 0\).
If \(|\Theta(e_1)| < |\Theta(f)|\) and \((1,1)\), and \(F\) diverges, then \(e_2/e_1 \to 0\).
If \((1,1) < \Theta(e_1) < \Theta(f)\), then \(e_2/e_1 \to -\infty\). In each case, if \(u_1-u_2\)
and \(u_2-u_3\) are constants, and \(e_2\) has extended order and \(e_3/e_2 \to a\)
limit, then the limit is \(L\).

Proof of 1). \(\Theta(u_1-u_2) = \Theta(e_1 v) = \Theta(e_1), (42, 119)\).

Case 1: \(v \to 1\). This means \(\Theta(f) \leq -(1,1), (118)\). \(\lim -e_2/e_1 = \lim -\frac{\Delta(u_1-u_2)}{u_1-u_2} (v-1) = \lim [(-\frac{\Delta(u_1-u_2)}{u_1-u_2} + 1) (\frac{\Delta f}{f} + 1)], (8.3)\).

\(\lim \log \left(\frac{e_2}{e_1}\right) = \lim [\Delta \log (u_1-u_2) + \Delta \log f], = \lim \Delta \log (u_1-u_2), (\text{since} \frac{\Theta(u_1-u_2)}{\Theta(f)} > \Theta(u_1-u_2), \Theta(f) = -\infty \text{ if} \Theta(u_1-u_2) < \Theta(f) \text{ and} \infty \text{ if} \Theta(u_1-u_2) > -\Theta(f). \text{Thus} \ L = 0 \ \text{and} \ -\infty\) respectively.

Case 2: \(v \to -\infty\). Then \(\Delta f/f \to 0, \Rightarrow \Delta f/f \sim \log (1 + \Delta f/f), (8.1)\),

\(= \Delta \log f. \ e_2/e_1 \sim -\frac{\Delta(u_1-u_2)}{u_1-u_2} / \frac{\Delta f}{f} \to L, \text{since} \ v-1 \sim v. \ \text{Suppose} \ L \neq \pm \infty. \ \text{Then} \ \frac{\Delta(u_1-u_2)}{u_1-u_2} \to 0, \Rightarrow \frac{\Delta(u_1-u_2)}{u_1-u_2} \sim \Delta \log (u_1-u_2), \Rightarrow \lim e_2/e_1 = \lim \Delta \log (u_1-u_2) + \Delta \log f = \lim \log (u_1-u_2)/\log f, (17), = \pm \infty, \text{contradicting the supposition that} \ L \neq \pm \infty. \ \text{Thus} \ L = \pm \infty \ \text{in this case}.

Case 3: \(v \to \text{a constant,} v_o, \neq 1\). Then \(\Delta f/f \to \frac{-1}{v_o} \neq -1, \Rightarrow |\Theta(f)| = (1,1), (118). \ \Delta(u_1-u_2)/(u_1-u_2) \to L/(v_o-1). \ \text{Since} \ |\Theta(u_1-u_2)| > (1,1), \ L \ \text{must be either} + \infty \ \text{or} -(v_o-1) \ \text{according as} \ \Theta(u_1-u_2) > (1,1) \ \text{or} < -(1,1), (118)\).

Case 4: \(v \to 0\). Then \(\Theta(f) > (1,1), (118). \ e_2/e_1 \sim -\frac{\Delta(u_1-u_2)}{u_1-u_2}. \ \text{If} \ \Theta(u_1-u_2) > \Theta(f) \geq (1,1), \ \text{then} \ e_2/e_1 \to -\infty, (118)\).
If $\theta(u_1 - u_2) < -\theta(f) \leq -(1,1)$, then $e_2/e_1 \to 1$, (118). To get the limit of $e_3/e_2$, we observe that $\theta(e_2) = \theta[\frac{v-1}{v} \Delta(u_1 - u_2)]$, (92.1).

Suppose $|\theta(e_2)| < |\theta(f)|$. $\Delta(u_1 - u_2) = e_2/(\frac{v-1}{v})$. Let $|\theta(f)|$ be $<(n,r) < |\theta(u_1 - u_2)|$. $|\Delta(u_1 - u_2)| < e(n,r)$, since it is the quotient of two functions having extended order $<(n,r)$ in absolute value.

This contradicts the statement, $(n,r) < |\theta(u_1 - u_2)|$, (44).

Thus $|\theta(e_2)| > |\theta(f)|$, then the results obtained for $e_2/e_1$ apply to $e_3/e_2$. Furthermore, $\theta(e_2)$ has the same sign as $\theta(e_1)$, since $\theta(e_2) = \theta[\Delta(u_1 - u_2)] = \theta(u_1 - u_2) = \theta(e_1)$.

Proof of 2. Case 1: $v \to 1$, $e_2/e_1 \sim u_1 - u_2$. $e_2/e_1 = \frac{\Delta(u_1 - u_2)}{u_1 - u_2} (v-1)$

$\to L$. Suppose $L \neq 0$. $v - 1 \to 0$, $\Rightarrow \frac{\Delta(u_1 - u_2)}{u_1 - u_2} \to \pm \infty$,

$\Rightarrow e_2/e_1 \sim \frac{\Delta(u_1 - u_2)}{u_1 - u_2} + 1 \cdot \frac{\Delta f}{f} + 1$, (8.3),

$\Rightarrow \log \left(\frac{e_2}{e_1}\right) \sim \Delta \log (u_1 - u_2) + \Delta \log f$, $= -\infty$, since $\theta[\log (u_1 - u_2)] < -(\log f)$ and $\Delta \log f \to \infty$. Hence $e_2/e_1 \to 0$, contradicting the supposition that $L \neq 0$. Hence $L = 0$.

Case 2: $v \to$ a constant, $v_0 \neq 1$. Then $\theta(f) = \pm (1,1)$, (118).

$e_2/e_1 \sim \frac{\Delta(u_1 - u_2)}{u_1 - u_2} (v_0 - 1) \to L$. Suppose $L \neq 0$. Then $|\theta(u_1 - u_2)| \geq (1,1)$, (118). $e_1 \sim (u_1 - u_2)/v_0$, $\Rightarrow \theta(e_1) = \theta(u_1 - u_2)$

$\Rightarrow |\theta(e_1)| \geq (1,1) = |\theta(f)|$, a contradiction. Thus $L = 0$.

Case 3: $v \to -\infty$. $|\theta(u_1 - u_2)| = |\theta(e_1) v| < |\theta(f)|$, (42), since by (118) and (119), $|\theta(v)| < |\theta(f)|$ when $v \to -\infty$.

$e_2/e_1 \sim \frac{\Delta(u_1 - u_2)}{u_1 - u_2} / \frac{\Delta f}{f} \to L$, since $v \sim v - 1$. Suppose $L = \pm \infty$. 


Then $\frac{\Delta f}{f} = o\left(\frac{\Delta(u_1-u_2)}{u_1-u_2}\right)$. At this point we need a lemma.

**Lemma 120.** If $x \to 0$ and $x = o(y)$, then $\log(1+x) = o(\log(1+y))$.

**Proof.** There exists $\epsilon$ such that $|y| < \epsilon \implies |\log(1+y)| > |y/2|$. Hence $|\log(1+y)| > \min\{|y/2|, \log(1+\epsilon)\}$. $x \sim \log(1+x)$ and $\to 0$, $\implies \log(1+x) = o(y)$ and $\to 0$, $\implies \log(1+x) = o(\log(1+y))$. //

We continue with the proof of (94.2), Case 3. The lemma shows that $\Delta \log f = o[\Delta \log (u_1-u_2)]$, a contradiction, since $|\theta(u_1-u_2)| < |\theta(f)|$, (17). Hence $L$ is finite, $\implies \frac{\Delta(u_1-u_2)}{u_1-u_2} \to 0$, $\implies \frac{\Delta(u_1-u_2)}{u_1-u_2} \sim \Delta \log (u_1-u_2)$ and $\Delta f/f \sim \Delta \log f$. So $\Delta \log (u_1-u_2)/\Delta \log f \to L$, $\implies \log (u_1-u_2)/\log f \to L$, (17), $\implies L = 0$, since $|\theta(u_1-u_2)| < |\theta(f)|$.

The only possible failure of (17) could occur if $\log (u_1-u_2) \to$ a constant. But in that case, too, $L = 0$, since $\log f \to -\infty$. Thus, $L = 0$ in Case 3.

Case 4: $v \to 0$. $\frac{e_2}{e_1} - \frac{\Delta(u_1-u_2)}{u_1-u_2}$.

What this approaches depends on $\theta(u_1-u_2)$, (118). $|\theta(u_1-u_2)| = |\theta(e_1 v)| \leq |\theta(f)|$, but it still could be $> (1,1)$, in which case $e_2/e_1 \to -\infty$ or 1, (118). If $|\theta(u_1-u_2)| < 1$, then $e_2/e_1 \to 0$, (118), and if $|\theta(u_1-u_2)| = (1,1)$, then $e_2/e_1$ can $\to$ anything. The proof is concluded by observing that $\lim e_3/e_2$ $= \lim \frac{u_3-u_4}{u_2-u_3}$, $= \lim \frac{\Delta(u_2-u_3)}{\Delta(u_2-u_3)}$, $= \lim \frac{u_2-u_3}{u_1-u_2}$, (17), $= \lim e_2/e_1$. //

**Procedure Cl.** When $e_2/e_1 \to$ a constant, Procedure Cl can be obtained as before.

**Procedure D in Chapter VI,** carries over without change.

**Bounds 98** are directly analogous.
Formula 99. 1) If we start with \( F_1 = f, \quad F_2 = vF \) and
\[
F_3 = vF/[1 + (\frac{1}{v} - 1)Δv].
\]

2) If we start with \( F = f_{n-1}, \quad F_2 = v_{n-1} f, \) (Formula 34), and
\[ F_3 = v_{n-1} f/(1 - Δv_{n-1}). \] \( F_3 \) of (99.1) is not equivalent to \( F_3 \) of (99.2). Use of the formula, \( v_{n-1} f, \) is equivalent to the well known practice of finishing off a series with a fitted geometric series.

Bounds 100. If \( Δv \to a \) limit monotonically, then a convergent series, \( F_n \), is bounded by \( γ(v_n + \frac{1}{r}) f_n \) and \( \frac{v_{n-1} f_n}{1 - Δv_{n-1}}. \)

Proof. Let \( F_1 = γ v_{n-1} f_n, \quad e_1 = -γ(Δv_{n-1} + \frac{1}{r}) \) and
\[ F_2 = v_{n-1} f_n/(1 - Δv_{n-1}), \] after algebra. \( Δv \to a \) limit monotonically \( \implies e_1 \to 0 \) monotonically, (84), \( \implies F \) lies between \( F_1 \) and \( F_2 \), (98).

For the same reason, \( F_{n+1} \) is bounded by \( γ v_n f_{n+1} \), \( \implies F_n \) is bounded by \( f_n + γ v_n f_{n+1} = γ(v_n + \frac{1}{r}) f_n. \) // The bounds in this form use information through the \((n+1)\)st term.

Theorem 103. The statement and proof of this theorem are analogous to the original, except for a part of the proof of 3). We have \( f_1/F_1 \sim f/F_1 \) and wish to show that \( F_1 \to a \) constant. Suppose \( F_1 \to a \) constant, \( c. \) Then \( f_1/F_1 \to 0, \implies f/F \to 0, \implies Δ \log F_1 \sim Δ \log F, \) (since \( \log (1+x) \sim x \) when \( x \to 0), \implies \log F_1 - \log c \sim \log F, (16), \) a contradiction, since \( \log F_1 - \log c \to 0 \) and \( \log F \to -∞. \)

Theorem 104. Let the regularity conditions of the original theorem hold, let \( f \) and \( f_1 \) be positive, and let \( f/F \to a \) limit.

1) Let \( |θ(E_1)| \) be > \( |θ(F)|. \) If \( E_1 \to ±∞, \) then \( E_2/E_1 \to ±∞. \)

If \( E_1 \to 0 \) and \( f/F \to c, \) then \( E_2/E_1 \to 1 - 1/c. \) If all the previous regularity conditions hold for \( E_2, \) then \( \lim E_3/E_2 = \lim E_2/E_1. \) If \( f_{n+1}/f_n \to a \) limit, \( r, \) then \( f/F \to 1-r \) or \( r-1, \) depending on whether \( F \) converges or diverges.
2) Let $|\Theta(E_1)| < |\Theta(F)|$. If $\Theta(F) < (1,1)$ or $\Delta E_1/E_1 \to 0$, then $E_2/E_1 \to 0$. If $f/F \to \infty$, and $\Delta E_1/E_1$ a limit, then $\lim E_2/E_1 = \lim \Delta E_1/E_1$. If the previous regularity conditions hold for $E_2$, then $\lim E_3/E_2 = \lim E_2/E_1$.

Proof of (104). We know from (103.1) that $\lim \frac{E_2}{E_1} = 1 - 1 \lim \epsilon_1 \epsilon_2$.

Let $\text{G} = F_1 - F$. $E_1 = G/F$. $E_2/E_1 \to 1 - \lim \frac{G/F}{f/F}$.

Proof of 1). $|\Theta(G)| = |\Theta(E_1 F)| > |\Theta(F)|$, (42).

Case 1: $f/F \to 1$. Then $|\Theta(F)| \geq (1,1)$, (118), $\Rightarrow |\Theta(G) > (1,1)$,

$\Rightarrow \lim \frac{G/F}{f/F} = \lim \frac{G}{F} = 1$ or $\infty$, (118), according as $E_1 \to 0$ or $\infty$,

$\Rightarrow E_2/E_1 - 1 \to -1$ or $-\infty$, $\Rightarrow E_2/E_1 \to 0$ or $-\infty$.

Case 2: $f/F \to c$, $c \neq 1$. Then $|\Theta(F)| = (1,1)$,

$\Rightarrow |\Theta(G)| > (1,1) \Rightarrow \lim \frac{G/F}{f/F} = \lim \frac{G}{F} = c/lc$ or $\infty$,

$\Rightarrow E_2/E_1 - 1 \to l/c$ or $-\infty$ according as $E_1 \to 0$ or $\infty$.

Case 3: $f/F \to \infty$. Then $\Theta(F) > (1,1)$, (118). If $\Theta(G) > \Theta(F)$, then $\lim \frac{G/F}{f/F} = \lim (1 + \frac{G}{F})/(1 + \frac{F}{f})$. Log of this limit $= \Delta \log G - \Delta \log F$.

This can only be $\infty$, since $|\Theta(G)| > |\Theta(F)|$. $\Rightarrow E_2/E_1 \to -\infty$. If $\Theta(G) < \Theta(F)$ then $\frac{G/F}{f/F}$ can only $\to 0$. Otherwise $\frac{G}{F} \to \pm \infty$, which is impossible by (118). Hence $E_2/E_1 \to 1$.

Case 4: $f/F \to 0$. Suppose $\lim \frac{G/F}{f/F}$ a finite limit. Then $g/G \to 0$,

$\Rightarrow f/F \sim \pm \Delta \log F$ and $g/G \sim \pm \Delta \log G$, $\Rightarrow \Delta \log G/\Delta \log F$ a finite limit, $\Rightarrow \log G/\log F$ a finite limit, (17), contradicting the assumption that $|\Theta(G)| > |\Theta(F)|$. Hence $\frac{G/F}{f/F} \to \infty$, $\Rightarrow E_2/E_1 \to -\infty$.

(Neither $\log G$ nor $\log F$ a constant, since $\Theta(G)$ and $\Theta(F)$ are infinite). We now obtain the limit of $E_3/E_2$. If $E_2/E_1 \to \pm \infty$, the original proof that $E_3/E_2 \to \pm \infty$ still holds. If $E_2/E_1 \to$ a constant, then $\Theta(E_2) = \Theta(E_1)$, $\Rightarrow$ the same reasoning applies to $\lim E_3/E_2$ as to $E_2/E_1$, $\Rightarrow E_3/E_2$ the same constant. $E_2/E_1$ only $\to 0$ when
\[ \theta(E_1) < -|\theta(F)|, \quad \implies \theta(E_2) \leq \theta(E_1) < -|\theta(F)|, \quad \implies E_2/E_1 \to 0 \] by the same reasoning that proved that \( E_2/E_1 \to 0 \).

Proof of 2). By (6.1.1), \( e_1 \sim E_1 \) when \( f/F \) has a finite limit, and \( \sim E_1[1 - \lim (\Delta E_1/E_1)] \) when \( \lim f/F = \infty \). Hence, \( E_2/E_1 \to 0 \) in the first case and \( \lim \Delta E_1/E_1 \in the second.

It remains to show that \( \lim E_3/E_2 = \lim E_2/E_1 \). Case 1: \( E_2/E_1 \to 0 \).

If \( E_1 \to \infty \), then \( \theta(E_2) \leq \theta(E_1) < |\theta(F)| \), \( \implies \) the first part of the proof applies, \( \implies E_2/E_2 \to 0 \). If \( E_1 \to 0 \), we follow the original proof, concluding that \( E_2 \sim \pm K \left( \frac{f}{f+1} \right) \), \( + \) for increasing \( F \), \( - \) for decreasing \( F \), \( K = F_1/F_0 \), \( \implies |\theta(E_2)| \leq \max \{|\theta(F/f+1)|, |\theta(E_1)|\}, (42) \), or is finite. If \( \theta(F) > - (2,0) \), then \( |\theta(E_2)| < |\theta(F)|, (119) \), \( \implies \) the first part of the proof applies, \( \implies E_3/E_2 \to 0 \). If \( \theta(F) \leq - (2,0) \), then \( F/f \to 1 \), (118), and \( E_2 \to 0 \), \( \implies e_2 \sim E_2 \\
(6.1.1), \implies E_3/E_2 \to 0 \).

Case 2: \( E_2/E_1 \to \) a constant. Then \( |\theta(E_2)| = |\theta(E_1)| < |\theta(F)|. \)

\[ \lim \Delta E_2/E_1 = \lim \Delta E_1/E_1, \quad \text{since} \quad \lim \left( \Delta \log E_1/\Delta \log E_2 \right) = \lim \left( \log E_1/\log E_2 \right) = 1. \quad \text{Hence,} \quad \lim E_3/E_2 = \lim E_2/E_1. \]

Case 3: \( E_2/E_1 \to \infty \). This happens when \( f/F \to \infty \) and \( \Delta E_1/E_1 \to \infty \).

Since \( E_1 = o(E_2) \), \( \lim \Delta E_2/E_2 = \infty \), (117.1). //

Theorem 105.1. If \( \theta(f) \) is finite, and \( F_1 = vf \) or \( v_{n-1} f \), and the regularity conditions of (104.2) hold, and \( v \to 0, 1, \) or \( \infty \), and \( \Delta v \) has extended order, and \( f/F \to \) a finite limit, [which happens when \( \theta(F) < (1,1) \)], then \( D \) is asymptotic II.

Proof. \( e(vf) = \Delta v (1/v - 1) \) and \( e(v_{n-1} f) = - \Delta v_{n-1} \). Case 1: \( -(2.0) < \theta(F) \leq (1,1) \). Then \( |\theta(1/v - 1)| \), and hence \( |\theta(\Delta v)| < |\theta(F)|, (119) \), \( \implies |\theta(e_1)| < |\theta(F)|, \implies (104.2) \) applies, \( \implies D \) is asymptotic II. Case 2: \( \theta(F) \leq -(2.0) \). Then \( D \) is asymptotic II even if \( |\theta(e_1)| \geq |\theta(F)|. // \)
Theorem 105.2. If, in addition to the assumptions of (105.1), \(\Theta(\Delta v) < 0\), then \(\Theta(E_2/E_1)\), \(\Theta(E_3/E_2)\), ... are all \(< - \epsilon\), so that \(D\) is asymptotic

I. \(E_1\) refers both to \(E_1(vf)\) and \(E_1(v_{n-1}f)\).

Proof. Since \(\Delta v\) has extended order, \(v\) and \(v-1\) have extended order. \(E_2 = \frac{\Delta K}{K} \left(\frac{F}{F} - 1\right)\) if \(F \to 0\), \(= \frac{\Delta K}{K} \left(\frac{F}{F} + 1\right)\) if \(F \to \infty\), as in the proof of (104.2). \(K = F_1/F \to 1\), since \(E_1 \to 0\), \(\implies \frac{E_2}{E_1} \sim \frac{\Delta E_1}{E_1} \left(\frac{F}{F} - 1\right)\) or \(\frac{\Delta E_1}{E_1} \left(\frac{F}{F} + 1\right)\). \(\Theta(F/F) = \Theta(\frac{F}{F} + 1)\) by (59). Let \(\Theta(f) = -(n,r)\) when \(F \to 0\) and \((n,r)\) when \(F \to \infty\).

Case 1: \(-(n,r) \leq -(2,0)\). Then \(\Theta(F/F - 1) = -(n,r)\), (119),

\(= \Theta(\frac{F}{F} + 1) = \Theta(v-1) = \Theta(\Delta v)\). Hence, \(\Theta(E_1) = -(n,r) = \Theta(E_2/E_1)\), (42),

for both \(E_1\)'s, \(= \Theta(E_2)\), (42).

Case 2: \(-(2,0) \leq -(n,r) < -(1,1)\). Then \(\Theta(F/F - 1) = \Theta(\frac{F}{F} - 1)\)

\(- (1,r-1) = \Theta(\Delta v)\), \(\implies \Theta(E_1) = -(1,r-1)\), \(\implies \Theta(\Delta E_1/E_1) \leq 0\), (119),

\(\implies \Theta(E_2/E_1) = -(1,r-1)\), (42), \(\implies \Theta(E_2) = -(1,r-1)\), (42).

Case 3: \(-(1,1) \leq -(n,r) < -(1,0)\). Then \(\Theta(\frac{F}{F} - 1) = r - 1\), \(\implies \Theta(\Delta f) = 1-r\), \(\implies \Theta(\Delta v) = -r = \Theta(e_1)\) for both \(e_1\)'s, since \(1 - 1/v \sim 1\) in this case. Since \(\Theta(\Delta v) < 0\), \(r > 0\). \(\Theta(E_2/E_1) = -1 + 1 - r = -r\), and \(\Theta(E_2) = -2r\).

Case 4: \(-(n,r) = -(1,1)\) and \(\Delta f/F \to -1\), \(\implies -(1,0) \leq \Theta(\frac{F}{F} - 1) \leq 0\),

\(\implies -(1,0) \leq \Theta(\Delta v) \leq -1\), \(\implies \Theta(\Delta E_1/E_1) = -1\), (119), \(\implies -(1,0) \leq \Theta(E_2/E_1) \leq -1\). \(\Theta(E_2) = \Theta(E_1) + \Theta(E_2/E_1)\).

Case 5: \(-(n,r) = -(1,1)\) and \(\Delta f/F \to 0\). \(\Theta(\frac{F}{F} - 1) = 0\) and \(\to \infty\).

\(\Theta(\Delta v) = -1 = \Theta(e_1)\), for both \(e_1\)'s, since \(1 - 1/v \sim 1\). \(\Theta(F/F - 1) = \Theta(\frac{F}{F} + 1) = 0\), \(\implies \Theta(E_2/E_1) = -1\) and \(\Theta(E_2) = -2\).

Case 6: \((1,0) \leq (n,r) < (1,1)\). \(\Theta(E_1) = \Theta(E_2/E_1) = -r\) as in

Case 3. \(\Theta(E_2) = -2r\).

The properties of \(E_1\) necessary to make this proof work carry over to \(E_2\), and then to \(E_3\) and so on. //
Procedure 107 applies without change to series. Formulas 108 and 109 do not apply, since the normal distribution is continuous, but analogous formulae based on a discrete distribution would be appropriate. Theorem 110 holds as stated.

Procedures D2 and D3 apply without change, and Theorems 111, 112, and 113 hold as stated.

**Alternating Series**

Let \( \{ f_n \} \) be a sequence of terms alternating in sign. Such series are fully as common in practical applications as positive series. For convenience, we will write \( |f_n| \) as \( \ell_n \). We will suppose throughout the discussion that \( \ell_n \) converges, and that \( \ell_n \), which therefore decreases to 0, is monotonic.

\[
F_n = \pm (\Delta \ell_n + \Delta \ell_{n+2} + \Delta \ell_{n+4} + \ldots ),
\]

according as \( f_n \) is \( + \). Since \( \Delta \ell_n \) is of constant sign, \( F_n \) may be approximated by the methods for positive series we have just studied.

In the following pages, we will take another approach which uses fewer terms. It is to derive theorems for alternating series analogous to those for positive series.

**Theorem 57:** Let \( \{ g_n \} \) be another alternating sequence. Let \( g_n \) stand for \( |g_n| \). If \( \Delta g_n / \Delta \ell_n \) has order, then

1) \( \theta(\ell) \) is finite \( \Rightarrow \theta |G/F| = \theta |g/f| \);

2) \( \theta(\ell) = -\infty \) and \( \ell / \Delta \ell \) has order, (or \( f/(n\Delta f) \to 0 \).)

\( \Rightarrow \theta |G/F| = \theta |g/f| \).

Proof. \( \theta(\Delta g / \Delta \ell) = \theta |G/F| \) and \( = \theta |g/f| \), (57). //

If \( \ell \) and \( g \) are well-behaved, then the relation between \( G/F \) and \( g/f \) is even closer.

**Theorem 121.** If \( \Delta \ell \) and \( \Delta g \to 0 \) monotonically, \( F \) lies between \( f \) and \( f/2 \), and neither \( G/F \) nor \( g/f \) is more than twice the other.
Proof. \( f_n = \pm (\Delta \xi_n + \Delta \xi_{n+1} + \Delta \xi_{n+2} + \ldots ) \)
\[ F_n = \pm (\Delta \xi_n + \Delta \xi_{n+2} + \ldots ) \]

Since \( \Delta \xi_n \) is monotonic, \( F_n < f_n < 2F_n \), for \( f_n > 0 \), and
\( 2F_n < f_n < F_n \) for \( f_n < 0 \). Similarly for \( G_n \). Thus \( F_n \) and \( f_n \)
have the same sign, and so do \( g_n \) and \( G_n \). Suppose that \( f_n \) and \( g_n \)
are positive. \( g_n < 2G_n \) and \( F_n < f_n \implies g_n / f_n < 2G_n / F_n \). The other
cases are similar. //

Let \( F_1 \) approximate \( F \), and let \( \xi_1 \to 0 \) monotonically.

Theorem 57 says that \( \theta |E_1| = \theta |e_1| \). If in addition, \( \Delta \xi \) and
\( \Delta (\xi_1 - \xi) \) are monotonic, then (121) says that \( E_1 \) lies between
\( e_1 / 2 \) and \( 2e_1 \).

Lemma 118 can be paraphrased as follows: If \( \xi_{n+1} / \xi_n \to \) a limit, \( \lambda \), then
\[ \lambda = 1 \implies \theta(f) \geq -(1,1). \]
\[ \lambda = 0 \implies \theta(f) \leq -(1,1). \]
\[ \lambda = \text{constant} \neq 1 \implies \theta(f) = -(1,1). \]

Theorem 122. If \( \Delta \xi_{n+1} / \Delta \xi_n \to \lambda_f \), then \( F_n \sim f_n / (1 + \lambda_f) \). The relative
frequency error of this formula is \( \left( \frac{1}{1 + \lambda} \right) \left( \frac{\xi_{n+1}}{\xi_n} - \lambda \right) \). If also,
\[ \frac{\Delta \theta_{n+1}}{\Delta \theta_n} \to \lambda_g \] then \( \frac{G}{F} \sim \left( \frac{1 + \lambda_f}{1 + \lambda_g} \right) \frac{g}{f} \).

Proof. Write \( f_n \) and \( F_n \) as series of \( \Delta \xi \)'s. \( \Delta \xi_{n+1} \sim \lambda_f \Delta \xi_n \)
\[ \implies f_n \sim F_n + \lambda_f F_n \]. //

This theorem can be used as an approximation formula. Needless to
say, if \( \Delta \xi_{n+1} / \Delta \xi_n \to \lambda_f \), then \( \xi_{n+1} / \xi_n \to \lambda_f \), (17). If \( \theta(\xi) \) and
\( \theta(g) \) are both \( > -(1,1) \) or both \( < -(1,1) \), then \( G/F \sim g/f \).

Expansion 83. The Laplace-Winckler expansion is better, if anything,
for an alternating series. \( v_n = \xi_n / (\xi_n + \xi_{n+1}) \), is positive, and
\( \to a \) limit, if any, between \( 1/2 \) and \( 1 \). If \( \Delta v \) has order, it is \( \leq -1 \).
The validity of the expansion and the remainder did not depend on positive terms, and so it is applicable here.

\[ e_k = \frac{f_{n+1}}{f_n} \Delta v_{kn} \text{ as before. } \quad \theta(e_1) = \theta(\Delta v_1) \leq -1. \quad v_2 = (v_1-1) \Delta v_1, \]

\[ \implies \theta(v_2) \leq -1, \quad \implies \theta(\Delta v_2) \leq -2, \text{ and so on. } \]

Following the reasoning of (87), we find that the procedure is asymptotic I and II, under regularity conditions, but with no requirement for infinite order.

Procedure D shows a similar improvement for alternating series.

Under regularity conditions, \( E_1 \sim c_1 e_1 \), (122), which we can write \( \tilde{E}_1 \).

Procedure 107, a generalized form of D, can now be applied. By (110), it is asymptotic II.

**Theorem 123.** If \( F_1 \sim F \) and if \( \theta|E_1| \) and \( \theta(\bar{r}) \) are both \( > -(1,1) \), or if \( \theta(F) < -(1,1) \), then, under regularity conditions, D is asymptotic II, and if \( \theta|E_1| < 0 \), it is also asymptotic I.

**Proof.** \( E_2 \sim -\Delta E_1(F/f -1) \), as in the proof of (104.2). If \( E_1 \to 0 \) and \( \theta|F| < -(1,1) \), then \( F_1/F = o(F) \), \( \implies \theta|F_1-F| < -(1,1) \), \( \implies e_1 \sim E_1 \), (122), \( \implies E_2/E_1 \to 0 \), \( \implies E_2 \to 0 \), \( \implies e_2 \sim E_2 \) by the same reasoning, and so on.

If \( \theta(F) \) and \( \theta(E) > -(1,1) \), then \( \theta(F_1-F) = \theta(E_1-F) > -(1,1) \), (42), \( \implies e_1 \sim E_1 \) again. \( \theta(E_2) = \min \{ \theta(\Delta E_1), \theta(F/f -1) \} > -(1,1) \), so the same reasoning applies to \( e_2, e_3, \ldots \). Thus, D is asymptotic II.

Now assume \( \theta(E_1) < 0 \). \( \theta(E_2/E_1) = \min \{ \theta(\Delta E_1/E_1), \theta(F/f -1) \} \).

If \( \theta(E_1) \) and \( \theta(F) \) are \( > -(1,1) \), then \( \theta(\Delta E_1/E_1) < 0 \). \( \theta(F/f -1) \leq 0 \).

Hence, \( \theta(E_2/E_1) < 0 \) and is finite. If \( \theta|F| < -(1,1) \), then \( \theta(F/f -1) \) is infinite, since \( |F/f -1| < |\Delta k| + 1 \), which has infinite order by (119). \( \theta(\Delta E_1/E_1) \leq 0 \). Hence \( \theta(E_2/E_1) < 0 \) in either case, and the same conditions are perpetuated when \( E_2 \) is considered. Thus D is asymptotic I and II.
Theorem 105. If we start \( D \) with \( v_f \) or \( v_{n-1}f \), and if \( \theta(\Delta v) > -(1,1) \) when \( 1 < \lim v < 2 \), then, under regularity conditions, \( D \) is asymptotic I and II. (No requirement for infinite order is needed.)

Proof. \( e(v_f) = \Delta v (1 - 1/v) \). \( e(v_{n-1}f) = \Delta v_{n-1} \). \( \theta(E_1) \leq -1 \) in either case. The previous theorem now applies directly, except, possibly for the cases \( \theta|F| = -(1,1) \) and \( \theta|F| > -(1,1) \).

Case 1: \( \theta|F| > -(1,1) \). Then \( 1/v - 1 = \frac{\zeta_n + \zeta_{n+1}}{\zeta_n} - 1 \),

\[(118) \Rightarrow \Delta \zeta_n / \zeta_n \to 0. \quad \theta(\Delta \zeta_n / \zeta_n) > \theta(\Delta \zeta_n) > -(1,1), \Rightarrow \theta(e_1) > -(1,1), \text{ and Theorem 123 applies.} \]

Case 2: \( \theta|F| = -(1,1) \) and \( \zeta_{n+1}/\zeta_n \to 0. \quad 1 - 1/v = -\zeta_{n+1}/\zeta_n \).

\[\log |1 - 1/v| = \Delta \log \zeta. \quad \theta(\Delta \log \zeta) = 0, \Rightarrow \theta|1 - 1/v| \geq -(1,0), \Rightarrow \theta(E_1) \geq -(1,0). \quad \theta(E_2/E_1) = \min \{ \theta(\Delta E_1/E_1), \theta(\zeta_{n+1}/\zeta_n) \}. \]

\( \theta(E_1) = -1, (119), \Rightarrow -1 \geq \theta(E_2/E_1) \geq -(1,0), (42), \text{ and } \theta(E_2) \geq -(1,0), (42). \text{ The same argument applies to } E_2, E_3, \ldots, \Rightarrow D \text{ is asymptotic I and II.} \]

Case 3: \( \theta|F| = -(1,1) \) and \( \zeta_{n+1}/\zeta_n \to 1. \) Then \( 1/v - 2 = \Delta f/f \to 0. \quad \Delta \log f \sim \Delta f/f, \Rightarrow \theta(\Delta f/f) = 0. \text{ Hence } 1/v - 2 \to 0 \)
and has order 0, \( \Rightarrow \theta(\Delta v) = -1 = \theta(e_1) \) for both \( e_i \)'s.

\( \theta(E_2/E_1) = \min \{ \theta(\Delta E_1/E_1), \theta(F/f - 1) \}. \) Now \( F/f \to -1/2, (122), \Rightarrow \theta(E_2/E_1) = -1 \) and \( \theta(E_2) = -2. \)

Case 4: \( \theta(f) = -(1,1) \) and \( \zeta_{n+1}/\zeta_n \to \lambda, \) where \( 0 < \lambda < 1. \) There need be no restriction on \( \theta(\zeta_{n+1}/\zeta_n - \lambda), \Rightarrow \theta(\Delta v) \) could be anything \( \leq -1. \text{ Thus, we have to assume that } \theta(\Delta v) > -(1,1) \) in order to make \( \theta(E_1) > -(1,1), \Rightarrow \theta(\Delta E_1/E_1) < 0 \) and finite, \( \Rightarrow D \) is asymptotic I and II. //

Partial Summation

No iterative procedure for summing a series is of any value unless
it is an improvement over the convergent procedure, partial summation.

In making this comparison, we should equalize the amount of calcula-
tion. If the terms themselves are not difficult to calculate, we may
be able to take many more terms for partial summation than for a more
complicated procedure.

Partial summation is a special case of the following procedure,

letting $F_1 = 0$.

**Procedure 124:** Let $F_n$ be an approximation to $F$, a convergent series
with $E_n$ bounded: $F_{kn} = f_n + f_{n+1} + \ldots + f_{n+k-l} + F_{1,n+k}$.

$$E_k = \frac{F_{1,n+k}}{F_n} = \frac{F_{a+1}}{F_a} \frac{F_{a+2}}{F_{a+k-1}} \ldots \frac{F_{a+k}}{F_{a+k-l}} E_{1,a+k}.$$

$$\log \frac{F_{n+1}}{F_n} = \Delta \log F_n.$$ Let $\theta(F) = -(n,r)$.

$n > 1 \implies \theta(\Delta \log F)$

$$= \theta(\log F), (44), \implies \theta\left(\frac{F_{n+1}}{F_n}\right) = \theta(F), \implies \theta(E_k) = \min \{\theta(F), \theta(E_1)\}.$$ 

$n = 1$ and $r > 1 \implies \theta\left(\frac{F_{n+1}}{F_n}\right) = -(1,r-1),

\implies \theta(E_1) = \min \{(1,r-1), \theta(E_1)\}$. $n = 1$, $r < 1 \implies F_{n+1}/F \rightarrow 1$,

if any limit, (10), $\implies \theta(E_k) = \theta(E_1)$. If $n = 1$, $r = 1$, and

$\theta(E_1) \leq (1,1)$, then $\theta(E_k) = \min \{k \theta\left(\frac{F_{n+1}}{F_n}\right), \theta(E_1)\}$ if one of these

is infinite, and = the sum of the two orders if both are finite.

$$\theta\left(\frac{F_{n+1}}{F_n}\right) = \theta\left(\frac{F_{n+1}}{F_n}\right) \text{ under regularity conditions, and } = 0 \text{ unless}$$

$$\frac{F_{n+1}}{F_n} \rightarrow 0. \text{ Then } -(1,0) \leq \theta\left(\frac{F_{n+1}}{F_n}\right) \leq 0, (119).$$

$$\frac{E_{k+1}}{E_k} = \frac{F_{a+1}}{F_{a+k}} \frac{E_{1,a+k+1}}{F_{a+k}} \frac{E_k}{E_{1,a+k}} \theta\left(\frac{E_{k+1}}{E_k}\right) = \min \{\theta(F_{n+1}/F_n), \theta(E_1,n+1/E_1)\},$$

if either is infinite, and the sum of the two if they are finite.

$\theta\left(\frac{E_{k+1}}{E_k}\right) = -\infty$ if either $\theta(F)$ or $\theta(E_1)$ are $< -(1,1)$, under
regularity conditions. If both $\Theta(F)$ and $\Theta(E_1) > -(1,1)$, then

$$E_{k+1}/E_k \to l.$$ 

We conclude that partial summation is a highly effective procedure when $\Theta(F) < -(1,1)$, but is ineffective if $\Theta(F) > -(1,1)$. Procedure 124 works well if $\Theta(F)$ or $\Theta(E_1) < -(1,1)$, but is ineffective if both are $> -(1,1)$.

The previous reasoning applies also to alternating series.

**The Euler-Maclaurin Sum Formula**

A powerful method for approximating certain sums should not be overlooked. It is the Euler-Maclaurin sum formula:

$$\sum_{a}^{n} f_i = \int_{a}^{n} f + \frac{1}{2} (f_n + f_a) + \frac{B_2}{2!} (f_n' - f_a') +$$

$$+ \frac{B_4}{4!} (f_n(3) - f_a(3)) + \frac{B_6}{6!} (f_n(5) - f_a(5)) + \ldots.$$ 

$B_1 = -\frac{1}{2}, B_2 = \frac{1}{2},$ all subsequent odd $B_1$'s are 0, $B_4 = -1/30, B_6 = 1/42, B_8 = -1/30,$ $B_{10} = 5/66, B_{12} = -691/2730, B_{14} = 7/6.$

The remainder after the term, $\frac{B_{2k}}{(2k)!} (f_n(2k-1) - f_a(2k-1))$, 

$$= \int_{a}^{n} P_{2k+1}(x) f(2k+1)(x) \, dx,$$ 

where $P_k(x) = (x + B)^k / k!$, interpreting $B^i$ as $B_1$, for $0 \leq x \leq 1$. $P_k(x)$ is periodic, and repeats the same graph for $n \leq x \leq n + 1$, for all $n$.

If either the integral or the sum has a simple form, the other may be approximated easily by the use of this formula. It is very handy for such sums as $\sum_a^n 1/i$ and $\sum_{1}^{n} \log i$, which $= \log (n!)$.

The expansion very seldom converges for functions occurring in practical applications.

A good description of the formula, containing the foregoing and many other facts, is given by Knopp, [23, Ch. 14].
When all the derivatives of \( f \) are ultimately monotonic, as for instance, when \( f \) is one of Hardy's L-functions, [15], and \( \sum_1^\infty f_i \) converges, then we can let \( n \to \infty \) for any finite number of terms, and obtain the expansion, \( \sum_1^\infty f_i = \int_1^\infty f + f(x)/2 - B_2 f(x)^3/2 - B_4 f(x)^5/4! \ldots \).

We may derive the formula by letting \( F(x) = \int_0^x f(t) \, dt \), and expanding \( F(0) \) about 0 and \( F(1) \) about 0 in Taylor series, obtaining
\[
F(0) - F(1) = -f(1) + f'(1)/2! - f''(1)/3! + \ldots \quad \text{and}
\]
\[
F(1) - F(0) = f(0) + f'(0)/2! + f''(0)/3! + \ldots \quad \text{.}
\]
Averaging the two, we obtain the expansion \( F(1) - F(0) = \frac{1}{2} [f(0) + f(1)] + \frac{1}{4} [f'(0) - f'(1)] + \frac{1}{12} [f''(0) + f''(1)] + \ldots \). We may expand \( \frac{1}{6} [f'(1) - f'(0)] \) the same way, adding the left side to the second term and subtracting the right from the rest of the series. This cancels the second derivative term.

The other even derivative terms can be cancelled the same way, and the result is the Euler-Maclaurin expansion between the limits of 0 and 1.

The expansion is equally valid between the limits of 1 and 2, 2 and 3, and so on. By adding \( n \) such expansions together, we obtain the Euler-Maclaurin formula for \( n \) terms.

This derivation shows that the Euler-Maclaurin formula between 0 and 1 cannot be stretched over an infinite interval to obtain an asymptotic procedure. We run into the same difficulty of infinite terms as with the Taylor series.
CHAPTER IX
APPLICATIONS TO INTEGRALS OF INTEREST IN STATISTICS

The Normal Distribution.

The normal distribution plays a central role in statistics, both as a distribution in its own right and as the limiting form of many other important distributions. Methods of approximation that are poor for the normal distribution are not likely to be useful in statistics. This distribution thus provides a test run for the approximation methods previously discussed.

For the normal distribution, \( f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \), and is usually written, \( \phi(x) \). \( v = 1/x \), so that Formula 34 is the well known approximation, \( \phi(x)/x \).

Formula 99 becomes \( \frac{\phi(x)}{x} (1 - \frac{1}{x^2+1}) \). Gordon, [14], has shown how (34) and (99) bound the normal integral. Formula 132 becomes \( \frac{\phi(x)}{x} (1 - \frac{1}{x^2+3}) \), and the first two terms of C become \( \frac{\phi(x)}{x} (1 - \frac{1}{x^2}) \).

To find out which of these formulas is best, we multiply \( \phi(x)/x \) by a rational fraction of degree two with undetermined coefficients, and choose the coefficients that minimize \( e(x) \) as \( x \to \infty \).

Formula 132 is the winner, with a relative frequency error of \( 6/[x^2(x^2 + 3)^2] \). Since this expression is monotonic, it is a bound for \( E(x) \), (47). The order of this error is \(-6\), a great improvement over that for \( \phi(x)/x \), which is \(-2\).
In Table 3, we see that Formula 132 has an accuracy of 3 significant figures for \( x > 5 \) and 5 significant figures for \( x > 10 \). The formula can be modified slightly for the express purpose of extending a good, readily available table of the normal integral, [33, Table 1], past the point \( x = 5 \) with an accuracy of four significant figures.

The modified formula is \( \frac{\varphi(x)}{x} \left( 1 - \frac{1}{x^2+2.8} \right) \). Its relative frequency error is \( \frac{-2x^2 + 5.04}{x^2(x^2 + 2.8)^2} \).

Formula 132 for the normal distribution is not new, since it is a convergent of a continued fraction due to Laplace, [26]. The fraction is \( \varphi(x) = \int_a^\infty \varphi(x) \, dx = \frac{\varphi(x)}{x} \left( \frac{1}{1} + \frac{1}{x^2/1} + \frac{2}{x^2/1} + \frac{3}{x^2/1} + \ldots \right) \).

(132) \( = \frac{\varphi(x)}{x} \left( 1 + \frac{1}{x^2} + \frac{2}{x^2} \right) \).

A discussion of this continued fraction is given by Kendall, [21, pp. 129-30].

More complicated formulas in the preceding chapters turn out to be \( \phi(x)/x \) multiplied by a rational fraction of degree four. The best such fraction, found, as before, by a method of undetermined coefficients, is

Formula 125. \( \Phi(x) \approx \frac{\varphi(x)}{x} \left( 1 - \frac{x^2 + 7}{x^4 + 10x^2 + 15} \right) \).

This, too, is a convergent of Laplace's continued fraction; namely, \( \frac{\phi}{x} \left( \frac{1}{1} + \frac{1}{x^2} + \frac{2}{x^2} + \frac{3}{x^2} + \frac{4}{x^2} \right) \). It has a relative frequency error of \( 120/[x^2(x^4 + 10x^2 + 15)^2] \), which, being monotonic, is a bound for the relative error, (47).

It, too, can be modified to be more accurate for small \( x \), and we obtain \( \frac{\varphi(x)}{x} \left( 1 - \frac{x^2 + 4.6}{x^4 + 7.6x^2 + 8.1} \right) \). Its relative frequency error is

\[
\frac{3x^4 - 14.4x^2 + 28.35}{x^2(x^4 + 7.6x^2 + 8.1)^2}.
\]
The relative errors of these formulas are found in Table 3. The monotonicity of \( e(x) \) for (132) and (125) means that \( E(x) \), too, \( \rightarrow 0 \) monotonically.

Procedure C for the normal distribution is the familiar asymptotic expansion,

\[
\Phi(x) = \frac{\Phi(x)}{x} \left( 1 - x^{-2} + 3x^{-4} - 3\cdot5x^{-6} + 3\cdot5\cdot7x^{-8} + \ldots \right) .
\]

It has been recommended as an easy way to approximate the normal integral for large \( x \). However, Formula 125, which gives us accuracy of 7 significant figures for \( x > 9 \), would seem to be easier to use.

If a high degree of accuracy is desired, Laplace's continued fraction converges rapidly on the tails, and one given by Shenton, \([44]\), is especially efficient in the center.

### Table 3

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<td>-.000035</td>
<td>.0000 046</td>
<td>.037</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>.0000 95</td>
<td>.000035</td>
<td>.0000 096</td>
<td>.026</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>.0000 40</td>
<td>.000034</td>
<td>.0000 024</td>
<td>.020</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>.0000 19</td>
<td>.000023</td>
<td>.0000 0072</td>
<td>.015</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>.0000 098</td>
<td>.000020</td>
<td>.0000 0024</td>
<td>.012</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>.0000 051</td>
<td>.000018</td>
<td>.0000 0090</td>
<td>.0000 0016</td>
<td>.0098</td>
</tr>
</tbody>
</table>
Table 4 shows how $e(x)$ approaches $E(x)$ asymptotically and is a bound for $E(x)$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>12</th>
<th>125</th>
<th>$\phi(x)/x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.0010</td>
<td>0.0000 40</td>
<td>0.063</td>
</tr>
<tr>
<td></td>
<td>0.0008</td>
<td>0.0000 28</td>
<td>0.056</td>
</tr>
<tr>
<td>8</td>
<td>0.0000 209</td>
<td>0.0000 008 3</td>
<td>0.0156</td>
</tr>
<tr>
<td></td>
<td>0.0000 192</td>
<td>0.0000 007 2</td>
<td>0.0152</td>
</tr>
<tr>
<td>10</td>
<td>0.0000 056</td>
<td>0.0000 000 99</td>
<td>0.0100</td>
</tr>
<tr>
<td></td>
<td>0.0000 051</td>
<td>0.0000 000 90</td>
<td>0.0098</td>
</tr>
</tbody>
</table>

A convenient table of the normal integral is found in Biometrika Tables, [33, Table 1]. More extensive tables are given by Sheppard, [45], and by [52].

**Student's t distribution.**

Formulas that are just as accurate and handy as those for the normal distribution can be developed by the same methods for Student's t distribution.

$$ f(t) = \text{constant} \cdot (1 + \frac{t^2}{n})^{-\frac{n+1}{2}}. \text{ It has order } -(n+1). $$

$$ \gamma = \frac{n+1}{n}. \quad v = \frac{n}{n+1} \left( \frac{1}{t} + \frac{t}{n} \right), \text{ so Formula } 34 = \gamma v f = \left( \frac{1}{t} + \frac{t}{n} \right) f. $$

Formula 99 = $\gamma v f \left(1 - \frac{1}{t^2 + 1}\right)$ and Formula 132 = $\gamma v f \left(1 - \frac{1}{\frac{n+2}{n} t^2 + 3}\right)$.  

(34) and (99) are bounds for $\int_{-\infty}^{\infty} f(x)dx$, (100).

As before, we can find the rational fraction of degree two which, when multiplied by $vf$, minimizes the relative frequency error asymptotically. The result is
Formula 126. \( F(t) \sim \gamma \varphi \left( l - \frac{1}{at^2 + b} \right) \), where \( a = \frac{n+2}{n} \) and 
\[ b = 3\left(\frac{n+2}{n+4}\right) \] 
\( e(126) = \frac{b^2 - b}{t^2(at^2 + b)^2} \). \( E(126) \) is, of course, bounded by \( e(126) \). However, Theorem 65.1 provides a closer and even more convenient bound.

It is easy to show that \( e(126) \) increases with \( n \) to a limit of \( 6/[t^2(t^2 + 3)^2] \), which is \( e(132) \) for the normal distribution. Moreover, \( \phi(t)/f(t) \) decreases monotonically as \( t \) increases from 1 to \( \infty \).

By (65.1), for any \( t > 1 \), \( E(126) \leq E(132, \text{normal}) \), equating \( t \) with \( x \). The first column of Table 3 thus provides a table of bounds for \( E(126) \).

Proof. \( e(126) = \frac{3\left(\frac{n+2}{n+4}\right)(3\left(\frac{n+2}{n+4}\right) - 1)}{t^2(\frac{n+2}{n}t^2 + 3\left(\frac{n+2}{n+4}\right))^2} = \frac{3\left(\frac{n+4}{n+2}\right)}{t^2(\frac{n+4}{n}t^2 + 3)^2} \).

The numerator increases with \( n \), while the denominator decreases with \( n \). Hence, the whole function increases with \( n \). It remains to show that \( \phi(x)/f_n(x) \) decreases as \( x \to \infty \), i.e., to show that

\[ \log \phi(x) - \log f_n(x) \text{ decreases.} \]

\[ \log \phi - \log f_n = -\frac{x^2}{2} + \frac{n+1}{2} \log(1 + \frac{x^2}{n}) \]

+ constant. The derivative \( = -x + \frac{n + 1}{1 + \frac{x^2}{n}} \cdot \frac{x}{n} = x(-1 + \frac{1 + \frac{x^2}{n}}{1 + \frac{x^2}{n}}) \),

which is negative for \( x > 1 \).

When the best rational fraction of degree 4 is found, we obtain

Formula 127. \( F(t) \sim \gamma \varphi [1 - \left(\frac{n}{n+2}\right)(\frac{t^2 + a}{t^4 + bt^2 + c})] \), where

\[ a = \left(\frac{7n+16}{n+4}\right) \left(\frac{n}{n+8}\right) \]
\[ b = 10 \left(\frac{n}{n+8}\right) \]
\[ c = \left(\frac{15n}{n+8}\right) \left(\frac{n}{n+8}\right) \].

The relative frequency error is

\[ \frac{c^2 - acn}{t^2(\frac{t^2 + a}{t^4 + bt^2 + c})^2} \],

which is a bound for \( E(127) \).
As before, a more convenient bound is $E(125)$, which is recorded in Table 3. The proof is analogous.

Since $E_n(127)$ increases with $n$ to a limit of $E(125)$ as $t$ remains constant, it is all the more true that $E_n(127)$ increases with $n$ to a limit of $E(125)$ as the significance level remains constant, since the percentage point decreases with $n$. Thus, $E(127)$ at any given percentage point is bounded by $E(125)$ at that percentage point.

The usefulness of these formulas can be seen by comparing them with the normal approximation to the $t$ distribution, which is often recommended for a sample size $> 30$. The following is a table of the relative error of the normal approximation compared with $E(126)$ and $E(127)$ for a sample size of 120 at the commonly used levels of the two tailed test.

<table>
<thead>
<tr>
<th>Level of the two tailed test</th>
<th>.05</th>
<th>.01</th>
<th>.001</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E(\text{Normal approximation})$</td>
<td>.046</td>
<td>.113</td>
<td>.257</td>
</tr>
<tr>
<td>$E(126)$</td>
<td>.018</td>
<td>.0058</td>
<td>.0018</td>
</tr>
<tr>
<td>$E(127)$</td>
<td>.003</td>
<td>.00056</td>
<td>.000093</td>
</tr>
</tbody>
</table>

Procedure C is not appropriate for the $t$ distribution, since the order is finite. Instead, the modified Procedure C1 is used, resulting in the expansion, $\gamma \nu f \left[ 1 - \frac{n}{n+2} t^{-2} + 3\left(\frac{n}{n+2}\right)\left(\frac{4}{n+4}\right) t^{-4} - 3.5 \left(\frac{n}{n+4}\right)\left(\frac{n}{n+6}\right) t^{-6} + \ldots \right]$. If we write the term of degree $k$ as $b_k$, then $b_{k+2} = -b_k (k+1) \left(\frac{n}{n+k+2}\right)^2/t^2$, and $e_k = \frac{(k+1) b_k}{t^2}$.
Since \( e_k \) is monotonic for \( t > 0 \), \( E_k \) is bounded by 0 and \( e_k' \), (49). And since \( e_k \) alternates in sign, it follows that \( E_k \) alternates is sign. This means that any two succeeding partial sums of the series are bounds for \( F(t) \).

The ratio of succeeding terms \( = -(k+1) \left( \frac{n}{n+k+2} \right)/t^2 \), which \( \to \frac{n}{t^2} \) as \( k \to \infty \). Thus, for \( t^2 > n \), the series converges, and since it also has the enveloping property just discussed, it must converge to \( F(t) \).

This expansion provides a useful complement to an expansion given by Hotelling and Frankel, [19], which is most accurate for \( t^2 \leq n \).

Our series was derived by writing \( Cl \) in the form,

\[
u_{21} = u_{11} + e_1/v(1-l), \quad \text{where} \quad e_1 = e(u_{11}) \quad \text{and} \quad L = \lim E_2/E_1.
\]

By (92.2), \( L = -(1 + e_1)/(n + 1) \). By writing \( u_{11} = \gamma v/(1-h) \) and simplifying, we obtain the recursive relation,

\[
u_{21} = \gamma v(1 - h - \frac{e_1}{1 - \theta(e_1)}).
\]

Thus, when \( b_k \) is obtained, \( b_{k+2} = -e_k/[1 - \theta(e_k)] \). After some algebra, and a mathematical induction, the expansion is established.

A table of the \( t \) distribution may be found in [12] or [33, Tables 9 and 12].

The Incomplete Beta Function.

Using Karl Pearson's notation, [38], we will define the incomplete beta function, \( I_x(a,b) \), to be \( \int_0^x t^{a-1} (1-t)^{b-1} \, dt/B(a,b) \).

\( B(a,b) \), the complete beta function = \( \Gamma(a+b)/\Gamma(a)\,\Gamma(b) \),

\[
= \frac{(a+b-1)!}{(a-1)!(b-1)!} = (a+b-1)/a-b. \quad \text{By making the substitution} \quad u = 1 - t,
\]

we find that \( I_x(a,b) = 1 - I_{1-x}(b,a) \).

The incomplete beta function has many uses in statistics. The following well known results are listed for convenient reference:

It is, first of all, a probability distribution in its own right.

As one of the family of Pearson curves, it is sometimes fitted to
empirical data. It has a mean of $\frac{a}{a+b}$, a variance of $\frac{ab}{(a+b)^2 (a+b+1)}$, and a mode of $\frac{a-1}{a+b-2}$, [29, p. 117].

The cumulative binomial distribution $\sum_{r=0}^{n} \binom{n}{i} p^i q^{n-i}$, measuring the probability of getting $r$ or more successes in $n$ independent trials with constant probability $p = \frac{r}{p}(n-r+1)$. [21, p. 120].

The cumulative negative binomial distribution, $\sum_{r=0}^{\infty} \binom{i-1}{r-1} p^r (1-p)^{n-r}$, measures the probability that it takes $n$ or more trials to achieve $r$ successes. This is the same as the probability of getting $< r$ successes in $n-1$ trials, which is $1 - I_p(r, n-r)$. (This relation is proved mathematically in [37] and [59].)

The $F$ ratio of mean squares may be simply reduced to an incomplete beta function. $F$ is defined as $\frac{\chi^2_m / \chi^2_n}{m/n}$, where $\chi^2_m$ and $\chi^2_n$ are independent chi-square variates with $m$ and $n$ degrees of freedom respectively. The probability that $F > F_0 = I_\frac{1}{2} \left( \frac{n}{2}, \frac{m}{2} \right)$, [49, also 6, p. 241].

Let $r$ be the coefficient of correlation between two normal samples of size $n$ under the null hypothesis of independence. The probability that $r^2 > r^2_0 = 1 - I_{r_0^2} \left( \frac{1}{2}, \frac{n}{2} - 1 \right)$. Hence, the probability that $r > r_0 > 0$ (or the probability that $r < r_0 < 0$) is one half this number. [56, p. 120, also 21, p. 342].

Let $R$ be the coefficient of multiple correlation between a sample, $x_1$, and $k-1$ samples, $x_2, \ldots, x_k$, of size $n$, under the hypothesis that $x_1, \ldots, x_k$ have a multivariate normal distribution and $x_1$ is independent of the set $x_2, \ldots, x_k$. Then the probability that $R^2 > R^2_0 = 1 - I_{R^2_0} \left( \frac{k-1}{2}, \frac{n-k}{2} \right)$ [11, also 56, p. 244 and 21, p. 381].

The distribution of Hotelling's $T^2$ statistic for testing
hypotheses about the means of a multivariate normal sample of dimension k and size n can be obtained similarly. The probability that

\[ T^2 > t_0^2 = I \frac{1}{T_0^2} \left( \frac{n-k}{2}, \frac{k}{2} \right), \]

[18, also 22, p. 337]

\[ 1 + \frac{t_0^2}{n-1} \]

Since \( T^2 \) reduces to \( t^2 \), the square of Student's statistic, when \( k = 1 \), the probability that \( t^2 > t_0^2 = I \frac{1}{1 + \frac{t_0^2}{(n-1)}} \left( \frac{n-1}{2}, \frac{1}{2} \right) \).

The distribution of the \( i^{th} \) order statistic, \( x_i \), in a sample of size n can be obtained from an incomplete beta function. Let the distribution function of the probability distribution from which the sample was drawn be \( F(x) \). The probability that \( x_i < \lambda \) is the same as the probability of getting i or more successes out of n trials with constant probability, \( F(\lambda) \), which we have seen is

\[ I_F(\lambda)_{(i, n-i+1)}. \]

[7, p. 103, also 21, p. 211.] The distribution of the sample median, \( \tilde{x} \), is a special case of the preceding result.

Let us suppose that the size of the sample is 2n-1. The median is the \( n^{th} \) order statistic, and hence its distribution function = \( I_{F(\lambda)}(n,n) \).

If \( F(x) \) depends on a one-dimensional parameter \( \theta \), an estimate of \( \theta \) and a confidence interval for \( \theta \) may be obtained by solving

\[ I_\gamma_{1} (n,n) = .025, I_\gamma_{2} (n,n) = .50, \text{ and } I_\gamma_{3} (n,n) = .975 \]

for \( \gamma_{1}, \gamma_{2}, \text{ and } \gamma_{3} \), and then solving \( F(\tilde{x} | \theta) = \gamma_{1}, \gamma_{2}, \text{ and } \gamma_{3} \) for \( \theta \). (\( \gamma_{2} \), by symmetry, is .5.)

When the approximation methods of this dissertation are applied to the incomplete beta function, the nearest result is the expansion of \( F/\gamma \phi \) in a power series; namely,

\[ f = -\gamma \phi [1 - \frac{1}{a} - \frac{2(n-a)}{a(a+1)} - \frac{\gamma_1(n-a)(n+1)x^2}{a(a+1)(a+2)} - \frac{\gamma_2(n-a)(n+1)(n+2)x^3}{a(a+1)(a+2)(a+3)} - \ldots \]. \]

n here = a + b - 1 and \( -\gamma \phi \) comes out \( \frac{x(1-x)}{(a-1) - (n-1)x} \).
The frequency error, $e_k$, for the expansion stopping after the term of $k^{th}$ degree is

$$\frac{(n-a)(n+1) \ldots (n+k)[(k+2)(a-1)x^{k+1} - (k+1)(n-1)x^{k+2}]}{[(a-1) - (n-1)x]^2 a(a+1) \ldots (a+k)}.$$ 

The expansion is asymptotic I and II, since $\theta(e_k) = k+1$.

$e_k$ is a monotonic function of $x$ for $x < \frac{a-1}{n-1}$.

Proof. $[(a-1) - (n-1)x]^2$ decreases monotonically as $x$ increases. The numerator is 0 for $x = 0$. The derivative of the numerator $= (k+1)(k+2)x^k [\frac{a-1}{n-1}] > 0$, $\implies$ the numerator increases, $\implies e_k$ increases. //

Thus $e_k$ is a bound for $E_k$, (47), and if $x_1 < x_2$, $E_k(x_1)$ is bounded by $E_k(x_2)$, (50).

The expansion was obtained by a method of undetermined coefficients described in Chapter IV. It converges to $F(x)$ for $x < (a-1)(n-1)$ since $e_k(x) \to 0$ as $k \to \infty$ and $0 < E_k(x) < e_k(x)$. It also converges slowly to $F(x)$ for $x > (a-1)/(n-1)$, although this fact is of little importance, since it is more convenient to calculate $1 - I_{1-x} (b,a)$ in this case.

Proof. The series alone defines a function analytic inside the unit circle, as is shown by the ratio test. $F$ is analytic in a region containing the open-ended interval $(0,1)$. Furthermore, $-1/\nu F = [(a-1) - (n-1)x]/x^a(1-x)^b + 1$, is also analytic in such a region, $\implies F/\nu F$ is analytic in a region containing $(0,1)$, because the product of two analytic functions is analytic. Hence, by the identity theorem, $F/\nu F$ and the series are the same analytic function in a region containing $(0,1)$. //

An assessment of the usefulness of this expansion depends on what alternative methods of calculation are already available. Of the many
known methods of approximating the incomplete beta function, the best
seem to be given in [59], [1], [46], [30], [41], [4], and [32]. Of
these, the only procedure that has a comparable simplicity and accuracy
on the tail is a series given by Soper, [46]; namely,
\[ F(x) = \frac{x(1-x)}{a} f \left[ 1 + \frac{n+1}{a+1} x + \frac{(n+1)(n+2)}{(a+1)(a+2)} x^2 + \frac{(n+1)}{(a+1) \cdots (a+3)} x^3 \cdots \right]. \]
This series shares many of the properties of the expansion previously
discussed, being asymptotic I and II, and convergent for \(0 < x < 1\).
Soper has observed that the ratio of terms, \(\frac{n+k}{a+k} x\), changes slowly
and monotonically from \(\frac{n+1}{a+1} x\) to \(x\). This implies that the error of
stopping after the \(k\)th degree term is bounded by the \((k+1)\)st degree
term divided first by \(1 - \frac{n+k}{a+k} x\) and then by \(1-x\). Similar bounds
apply to the first series, provided that \(a < n-1\), since the ratio of
terms is \(\frac{(k+1)(n+k-1)}{k}(\frac{n+k-1}{a+k}) x\).
Since the ratios of succeeding terms is so similar for the two
series, a rough check on their comparative accuracy is afforded by the
ratio of corresponding terms. The ratio of the \(k\)th degree term of the
first series to that of the second is \(\frac{n-a}{[(a-1) - (n-1) x]}(\frac{1+k}{n+k})\). A
calculation of this number would show which series is better to use.
If \(n\) is large, and only a few terms will be used, or if \(a\) is nearly
as large as \(n\) while \(x\) is small, the first series would be appro-
priate. Under other conditions, the second might be better.
The ratio, \(r_k\), of succeeding terms, \(t_k\), may approach \(x\) very
slowly; hence, the bound depending on the observed ratio will usually
be a lot closer than that using \(x\). A closer bound is provided by
Bounds 100 of Chapter VIII. The \(v_{k-1} t_k\) bound is the same as the
fitted geometric series, since \(v_{k-1} = 1/(1-r_k)\). The other bound is
\(v_{k-1} t_k/(1-\Delta v_{k-1})\). The ratio of the error of the second bound to that
of the first approaches \(\frac{x}{x-1}\) as \(k \rightarrow \infty\), (104).
The most complete table of the incomplete beta function was given by Karl Pearson, [38]. A table of the percentage points of this distribution was computed by Thompson, [49]. This table is reprinted in [33, Table 16.] Tables of the F ratio are found in [33], and [12], [16], [51], and [42] are good tables of the cumulative binomial distribution.

The Incomplete Gamma Function.

We will define the incomplete gamma function, \( \Gamma(n,x) \), to be \( \int_x^\infty e^{-t} t^{n-1} dt / \Gamma(n) \). The integrand has its mode at \( n-1 \). The mean and variance of \( f(x) \) are both \( n \).

The following is a summary of well known uses for the incomplete gamma function in statistics.

It can be used directly as a probability distribution, usually in the following form: \( f(x) = \frac{x^{n-1} e^{-x}}{\Gamma(n) \beta^n} \), [29, p. 112]. Then
\[
\int f(t)dt = \Gamma(n, \frac{x}{\beta}).
\]
This \( f \) has a mean of \( n\beta \), a variance of \( n\beta^2 \), and a mode at \( (n-1)\beta \).

The \( \chi^2 \) distribution with \( k \) degrees of freedom is a special case of the gamma distribution with \( n = k/2 \) and \( \beta = 2 \). Hence, the probability that \( \chi^2 > \chi^2_0 = \Gamma(\frac{k}{2}, \frac{\chi^2_0}{2}) \).

The Poisson distribution is one of the basic statistical distributions. It arises naturally in the following situation. An experiment is performed whose outcome is a finite number of points in an interval, I. The distribution of outcomes of the experiment is such that if \( I_1 \) and \( I_2 \) are any two disjoint subintervals of I with equal length, then the number of points in \( I_1 \) and \( I_2 \) are independently and identically distributed. It can be shown that the number, \( k \), of
points falling in $I$ has a Poisson distribution; namely, $e^{-\lambda} \frac{\lambda^k}{k!}$, [10, pp. 116-117]. The probability that $k \geq n$ is $1 - \Gamma(n, \lambda)$ as is shown by integrating $\int_0^\infty e^{-x} x^{n-1} dx$ by parts.

The choice of methods for approximating the incomplete gamma function depends on which tail is being approximated. For the upper tail, formulas analogous to those for the normal and $t$ distributions are appropriate.

We have $f(x) = \text{constant} \cdot e^{-x} x^m, (m = n-1)$. $v = \frac{x}{x-m}$. $1 - v' = 1 + \frac{m}{(x-m)^2} = 1 + \frac{m}{y^2}$, letting $y = x-m$. Formula 99 is $vf(1 - \frac{m}{y^2 + m})$. Bounds 100 apply here: $vf > \Gamma(n, x) > \frac{vf}{1 + \frac{m}{y^2}}$. Formula 132 is $vf(1 - \frac{m}{y^2 + 2y + 3m})$.

To find the best formula of this type, we determine the coefficients of a rational fraction of degree two by minimizing the asymptotic frequency error. The result is Formula 128. $\Gamma(n, x) \sim vf(1 - \frac{m}{y^2 + 2y + 3m-2})$, which is almost the same as (132). To make the formula comparable to the normal distribution, we will write it in terms of $z = \frac{x-m}{\sqrt{m}}$. Then $\Gamma(n, x) = vf(1-h)$, where $h = 1/(z^2 + \frac{2z}{\sqrt{m}} + 3m-2)$. As $n \to \infty$, $z$ becomes asymptotically normal and (128) $\to$ (132) for the normal distribution.

$$e(128) = \frac{2(m-1) (h \sqrt{mz} + 3m-2)}{z^2 (mz^2 + 2\sqrt{mz} + 3m-2)^2} = h^2 \frac{2(m-1)}{mz^2} \left( \frac{h}{\sqrt{m}} + 3 - \frac{2}{m} \right).$$

If $m \geq 2/3$ and $z > m$, then $e(128)$ is monotonic, and hence is a bound for $E(128)$. This is seen when the top and bottom of the first form of $e(128)$ are divided by $z$. The top decreases while the bottom increases.
Formulas 128 and 99 are bounds for $F$ if $m \geq \frac{2}{3}$ and $z > m$.

Proof. (34) and (99) are bounds for $F$, since $v'$ is monotonic, (100). The sign of $E(134) = \text{the sign of } e(34)$, (47), = the sign of $-v' = +$. Hence, the sign of $E(99) = -$. The sign of $E(128) = \text{the sign of } e(128) = +$. //

Unfortunately, $e(128)$ tends to decrease with $m$, while $\Phi(x)$ decreases more rapidly than the corresponding expression for $z$. Thus (65) cannot be applied to get a convenient bound for $E(128)$.

The best approximation of the form, $vf(1-h)$, where $h$ is a rational fraction of fourth degree is a more complicated expression.

**Formula 129:**

$$h = \frac{z^2 + az + b}{z + cz^3 + dz^2 + ez + f}.$$ To obtain $h$ we compute

- $A = 2/\sqrt{m}$
- $B = 3 - 2/m$
- $C = -2A(B-1)$
- $D = B - 5A^2$
- $E = (114 + 7m)/(6m)$.

Then

- $a = AE/3$
- $b = -DE/6 + 2B + 3D/2$
- $c = a + A$
- $d = Aa + b + B$
- $e = Ba + Ab + C$
- $f = Ca + Eb - (B-1)D$.

Thus, $E(129) = -9$, which is almost as good as (125). (129) $\rightarrow$ (125) as $m \rightarrow \infty$, and $e(129) \rightarrow e(125)$.

To get an expansion for $\Gamma(n,x)$, we apply Procedure 83; namely,

$$f^\infty_a = f(a) \left[v_1(a) + \cdots + v_k(a)\right] + f^\infty_a f v_k', \text{ which is the same as } v(a) f(a) \left[1 + v_1'(a) + \cdots + v_{k-1}'(a)\right] + f^\infty_a f v_k', v_k' \text{ turns out to be } b_{kl}^m \frac{m}{y^{k+1}} + \cdots + b_{kk}^m \frac{m}{y^{2k}}, \text{ where } b_{kj} = -(k+j-1)(b_{k-1,j} + b_{k-1,j-1}).$$

The $b$'s are most easily obtained by making a table:
The rule is $-(\text{upper row number} + \text{column number}) \times (\text{upper} + \text{upper left})$.

A power series may be obtained from this expansion by collecting terms according to powers of $y$. The coefficients are of the form \[ a_{kl}m + a_{k2}m^2 + \ldots + a_{k,k/2}m^{k/2}, \] where the $a$'s are obtained from the following table:

<table>
<thead>
<tr>
<th>Power</th>
<th>Row</th>
<th>$m$</th>
<th>$m^2$</th>
<th>$m^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y^{-2}$</td>
<td>1</td>
<td>-1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$y^{-3}$</td>
<td>2</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$y^{-4}$</td>
<td>3</td>
<td>-6</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>$y^{-5}$</td>
<td>4</td>
<td>24</td>
<td>-20</td>
<td></td>
</tr>
<tr>
<td>$y^{-6}$</td>
<td>5</td>
<td>-120</td>
<td>+130</td>
<td>-15</td>
</tr>
</tbody>
</table>

The rule is $-(\text{row number}) (\text{upper} + \text{upper left})$.

Tricomi, [50], derived the first three terms of this expansion by a different method, and gave a complicated rule for obtaining succeeding terms. It can be shown that Expansion 83 diverges for all $m$ and $y$, and that the power series converges if and only if $m$ is an integer.
A well known series for the incomplete gamma function, [35, p. xv], has properties similar to those of the previous two expansions.

\[ \Gamma(a, n) = \int_a^\infty e^{-x} x^{n-1} = e^{-a} a^m + m \int_a^\infty e^{-x} x^{m-1}, \]

integrating by parts. A repetition of this operation leads to the expansion,

\[ \Gamma(a, n) = f(a) \left[ 1 + \frac{m}{a} + \frac{m(m-1)}{a^2} \right. \]
\[ + \ldots + \frac{m(m-1) \ldots (m-k+1)}{a^k} \left. \right] + m \ldots (m-k) \int_a^\infty e^{-x} x^{m-k-1}. \]

The relative frequency error = constant \( \frac{e^{-x} x^{m-k-1}}{e^{-x} x^m} \), = \( \frac{m \ldots (m-k)}{x^{k+1}} \), which is a convenient bound for \( E_k \). Thus, the expansion is asymptotic I and II. It is a finite series when \( m \) is an integer and therefore converges. When \( m \) is not an integer, the series diverges by the ratio test.

These facts show that the power series expansion of \( F/vf \) converges if and only if \( m \) is an integer, since \( F/vf = (1 - \frac{m}{x}) \cdot (F/f) \). The first factor is analytic except at 0. The product is therefore analytic when and only when the second factor is analytic.

Karl Pearson, [35], has recommended the third expansion when \( n > 50 \), the limit of his table. Enough terms can be calculated until the error term is within range of the table. If \( n \) were considerably \( > 50 \), this would be a tedious process and Bounds 100 might be used to bound the error term.

A rough idea about the relative accuracy of the first and third expansions may be obtained by the ratio of the errors after one or two terms. After the first term, the ratio of the error of the first to that of the third is

\[ \frac{\int_a^\infty f' v}{\int_a^\infty e^{-x} x^{m-1}}, \frac{-\int e^{-x} x^m (x-m)^2}{m \int e^{-x} x^{m-1}} = a \int_a^\infty e^{-x} x^{m-1}. \]
By (48), this ratio is less in absolute value than the ratio of the integrands at \( a \), which is \( a/(a-m)^2 \). Thus, the ratio is \( < 1 \) for \( a > m + \sqrt{a} \), which will happen approximately when \( a \) is more than one standard deviation larger than the mean. The ratio of the error after two terms is

\[
\frac{\int_a^\infty f v_2}{m(m-1) \int_a^\infty e^{-x} x^{m-2}} = \frac{\int_a^\infty e^{-x} x^{m} \left( \frac{-2m}{(x-m)^3} + \frac{3m^2}{(x-m)^4} \right)}{m(m-1) \int_a^\infty e^{-x} x^{m-2}}. \tag{48}
\]

By (48), this ratio is less than the ratio of the integrands at \( a \), which is

\[
\frac{a^2}{m-1} \left( \frac{2}{(a-m)^3} + \frac{3m}{(a-m)^4} \right). \tag{83}
\]

This is \( < 1 \) only for \( a \) somewhat larger than \( m + \sqrt{a} \), and the trend will be seen to continue. Expansion 83 is superior for large \( a \) where few terms will be used. The third expansion catches up with the first if enough terms are used. It also has the important advantage of ease of calculation.

Similar results are obtained when the second expansion is compared with the third.

The useful integral, \( Ei(x) = \int e^{-t} t^{-1} dt \), can be approximated by any of the previous formulas, letting \( m = -1 \) and \( f(x) = e^{-x} x^{-1} \).

The lower tail of the incomplete gamma function behaves very much like that of the incomplete beta function. The neatest approximation procedure again turns out to be a power series expansion of \( F/vf \); namely,

\[
l - \frac{1}{m+1} - \frac{2x}{(m+1)(m+2)} - \frac{3x^2}{(m+1)(m+2)(m+3)} - \cdots \]

The relative frequency error after the \( x^{k-1} \) term is

\[
\frac{(k+1) mx^k - kx^{k+1}}{(m-x)^2(m+1) \ldots (m+k)}.
\]

We can show, as with the beta distribution, that this expansion is monotonic as \( x \) increases from 0 to \( m \), so that \( F_k \) is
bounded by \( e_k \) in that domain. Since \( e_k \to 0 \) as \( k \to \infty \), we conclude that the series converges to \( F/nF \) for \( 0 < x < m \). By an argument analogous to that used for the beta distribution, we can continue this result to all positive, real \( x \). The series converges slowly, however, for \( x > m \). Since \( \theta(e_k) = k+1 \), the series is asymptotic \( I \) and \( II \).

Once again the rival for this approximation procedure is a well known series obtained by an integration by parts; namely,

\[
F = f \cdot \left( \frac{x}{m+1} + \frac{x^2}{(m+1)(m+2)} + \frac{x^3}{(m+1)(m+2)(m+3)} + \cdots \right), \quad [33, \text{ p. xv}].
\]

The relative frequency error for stopping after the term of \( k \)th degree is \( x^k/[(m+1) \cdots (m+k)] \). This is a convenient bound for \( E_k \) since it is monotonic.

By Theorem 47, if the ratio of relative frequency errors is monotonic for \( x > a \), then the ratio at \( a \) is a bound for the ratio of relative errors at \( a \). The ratio of the relative frequency error of the first series to that of the second is

\[
\frac{(k+1)m-kx}{(m-x)^2} = \frac{k}{m-x} + \frac{m}{(m-x)^2},
\]

which monotonically increases as \( x \) increases from 0 to \( m \). If \( x \) is small, and only a few terms will be taken, the first series is preferable. Otherwise, the second should be used.

A convenient table of the complete gamma function is the table of factorials given by Fisher and Yates, [12]. For an integer, \( \Gamma(x) = (x-1)! \); otherwise, \( \Gamma(x) \) may be obtained by interpolating in the table of log \( n! \). A more complete table, giving values of \( \Gamma(n) \) and \( \Gamma(n+\frac{1}{2}) \) up to \( n = 1000 \), has been published by the National Bureau of Standards, [43].

Percentage points of the \( x^2 \) distribution are available in many
textbooks including [12] and [33, Table 8]. A good table of the \( \chi^2 \) distribution which doubles as a table of the cumulative Poisson distribution is [33, Table 7]. [28] is a more complete table of the Poisson distribution. A thorough table of the incomplete gamma function was given by K. Pearson, [35].

A generally good approximation for the gamma distribution was given by Wilson and Hilferty, [21, pp. 294-297]. If \( f(y) = \text{constant} \cdot y^{n-1} e^{-y} \), then \((y/n)^{3/2}\) is approximately normal with mean \( 1 - \frac{1}{2n} \) and variance \( \frac{1}{2n} \). In other words, if \( x \) has the \( \chi^2 \) distribution with \( k \) degrees of freedom, \( (x/k)^{3/2} \) is approximately normal with mean \( 1 - \frac{2}{9k} \) and variance \( \frac{2}{9k} \).

**Pearson Type IV Distribution.**

This is one of a family of Pearson curves that are sometimes used as models for empirical distributions and difficult theoretical distributions. The frequency function, as given by Kendall, [21, p. 141],

\[ f(y) = \text{constant} \cdot (1 + \frac{x^2}{a^2})^{-m} e^{-y} \tan^{-1} \frac{x}{a}. \]

We will simplify it by letting \( y = \frac{x}{a} \) and \( m = \frac{n}{2} \). Then the frequency function,

\[ f(y) = \text{constant} \cdot (1 + y^2)^{-n/2} e^{-y} \tan^{-1} y. \]

This function cannot be integrated in terms of other easily accessible functions, and it has not been tabulated. When \( y = 0 \) it reduces to the \( t \) distribution. \( v = \frac{1+y^2}{ny+y} \), \( v' = \frac{ny^2 + 2vy - n}{(ny + y)^2} \),

\[ \frac{v'}{r} = -\frac{n(v^2 + v^2)}{n(ny + y)}, \quad \frac{v''}{r} = \frac{2(nv^2 + v^2)}{(ny + y)^3}, \quad \theta(x) = -n, \quad \text{and} \quad \gamma = n/(n-1). \]

The formulas listed in the Appendix can now be applied. If \( ny > -v \), the mode, then \( v' \) is monotonic and Bounds 100 apply; namely, \( F \) is bounded by \( \gamma v f \) and \( v f/(1-v') \). The smallness of
\[ \frac{\gamma(n^2 + v^2)}{n(ny + v)} \] measures the accuracy of these bounds, which is the difference between their ratio and 1.

Both (99) and (125) are of the form \( \gamma v (1-h) \), where

\[ h = \frac{1}{ay^2 + by + c}. \]

Formula 130. The constants, \( a, b, \) and \( c \) minimizing the relative frequency error asymptotically are \( a = \frac{n^2(n+1)}{n^2 + v^2} \), \( b = \left( \frac{2v}{n+2} \right) a \), and \( c = \frac{a}{n^2} \left[ 3 + \frac{v^2(n+6)}{(n+2)^2} \right] \). The relative frequency error is

\[ \frac{16avn^2 \left[ (n+2)^2 + v^2 \right] + [(n^2 + v^2) c^2 - cv^2 n - cn^2 - bn]}{(n+2)^2 (n+3)} \]

\[ \frac{n(ny+v)^2 (ay^2 + by + c)^2} {n(ny+v)} \]

This error can be put in the form, constant \( \cdot \frac{y + A}{(y+B)^2 [(y+C)^2 + D]} \).

As \( y \to \infty \), this expression is \( \sim \) constant \( \cdot y^{-5} \), \( \Rightarrow \) it eventually decreases in absolute value. To find out whether it is monotonic, we observe that if \( u, v, \) and \( w \), are three functions of \( y \), then \( u/vw \) is monotonic if \( (u/vw)' \) has a constant sign. Let us suppose that \( y \) is large enough so that \( u, v, \) and \( w \), are \( > 0 \). Then \( \text{sign}(u/vw) \)

\[ = \text{sign} \left( \frac{yvu' - uv'w - uvw'}{v^2w^2} \right), \]

\[ = \text{sign} \left( \frac{u' - v'w - uw'}{u} \right), \]

\[ = \text{sign} \left( \frac{1}{y^2a} - \frac{2}{y+B} - \frac{2}{y+C + \frac{D}{y+C}} \right) \]

in the case considered here. Thus, if \( y \) is large enough so that \( y+A, y+B, \) and \( (y+C)^2 + D \) are positive, and \( \frac{1}{y+A} < \frac{2}{y+B} + \frac{2}{y+C + \frac{D}{y+C}} \), then \( e(130) \) is monotonic and is a bound for \( E(130) \).
The effect of approximating the negative, rather than the positive
tail of the Type IV distribution is to change the sign of \( \nu \) since
\[
\tan^{-1}(-y) = -\tan^{-1} y.
\]

A table of percentage points for standardized Pearson curves as a
function of the third and fourth moments may be found in [33, Table 42].
This table is useful when the first four moments of a distribution are
known or estimated and a Pearson curve is to be fitted to it.

**Bivariate Normal Distribution.**

In this discussion, we will denote \( \frac{-x^2}{2} \) by \( \Phi(x) \), \( \int_{-\infty}^{\infty} \varphi(t) dt \) by 
\( \Phi(x) \) and \( \int_{-\infty}^{x} \varphi(t) dt \) by \( \Phi(x) \). \( x_1 \) and \( x_2 \) have a bivariate normal
distribution when
\[
f(x_1, x_2) = \frac{\frac{1}{2}}{2\pi \sqrt{\sigma_{i1} \sigma_{j1}}} \cdot \frac{1}{2} \Sigma \sigma_{ij} (x_1 - \mu_1) (x_j - \mu_j).
\]

Let \( x = (x_1 - \mu_1)/\sigma_1 \) and \( y = (x_2 - \mu_2)/\sigma_2 \). Then the new
\[
||\sigma_{ij}|| = ||\frac{1}{\rho} \sigma_1 ||, \quad ||\sigma_{ij}|| = \frac{1}{\rho} \frac{1}{\rho^2}, \quad \text{and } x \text{ and } y \text{ are said}
\]
to have the standard bivariate normal distribution,
\[
\phi(x, y) = \frac{1}{2(1-\rho^2)} \left( x^2 - 2\rho xy + y^2 \right) \cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(x, y) dy dx = \int_{-\infty}^{\infty} \phi(x) dx,
\]
where \( f(x) = \int_{-\infty}^{\infty} \phi(x, y) dy \),
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(x, y) dy dx = \int_{-\infty}^{\infty} \phi(x) dx,
\]
\[
= \Phi(b) \Phi(b - \rho x) \cdot \Phi(b) - \int_{-\infty}^{\infty} \phi(x) dx, \quad \text{since the marginal distributions are}
\]

properly, we have a choice of many other formulations of the problem
\[
\int_{a}^{b} \phi(x, y) = \Phi(b) - \int_{-\infty}^{\infty} \phi(x, y), \quad \text{since the marginal distributions are}
\]

\[
\Phi(b) - \int_{-\infty}^{\infty} \phi(x) dx.
\]
standard normal, \( \int_{-\infty}^{a} \phi(x) \Phi\left(\frac{b-\rho x}{\sqrt{1-\rho^2}}\right) = \int_{-\infty}^{a} \phi(z) \Phi\left(\frac{b+\rho z}{\sqrt{1-\rho^2}}\right) \),

letting \( z = -x \). Similarly, \( \int_{a}^{b} \int \phi = \Phi(a) - \int_{a}^{\infty} \phi \)

\[ \Phi(a) - \int_{a}^{\infty} \Phi\left(\frac{b-\rho x}{\sqrt{1-\rho^2}}\right) = \Phi(a) - \int_{a}^{\infty} \Phi\left(\frac{-b+\rho x}{\sqrt{1-\rho^2}}\right), \]

since \( \Phi(-x) = \Phi(-x) \). Another formulation is

\[ \int_{a}^{b} \int \phi(x,y) = \Phi(b) - \Phi(a) + \int_{-\infty}^{a} \int_{-\infty}^{b} \phi. \] The last double integral

\[ \int_{-\infty}^{a} \phi(x) \Phi\left(\frac{b-\rho x}{\sqrt{1-\rho^2}}\right) = \int_{-\infty}^{a} \phi(z) \Phi\left(\frac{b+\rho z}{\sqrt{1-\rho^2}}\right) \]

Four other formulations are obtained by interchanging the roles of \( x \) and \( y \). Further formulations may be obtained by integrating

by parts. \( \int_{a}^{\infty} \phi(x) \Phi_+(c-dx)dx = - \Phi(x) \Phi_+(c-dx)|_{a}^{\infty} + d \int_{a}^{\infty} \Phi(x)\phi(c-dx)dx, \)

\[ = \Phi(a) \Phi_+(c-da) - \int_{c-da}^{\infty} \Phi\left(\frac{c-y}{d}\right) \phi(y)dy, \] changing the variable of integration to \( y = c-dx \). Similarly, \( \int \phi(x) \Phi_-(c-dx)dx = \Phi_+(a) \Phi_-(c-da) \)

\[ + \int_{c-da}^{\infty} \Phi\left(\frac{c-y}{d}\right) \phi(y)dy, = \Phi(a) \Phi_+(da-c) + \int_{c-da}^{\infty} \phi(y) \Phi\left(\frac{-c+y}{d}\right)dy. \]

It is now possible to obtain \( v, v', \) and \( v'' \), and get approximations and bounds for the double integral. In the formulation,

\[ \int_{a}^{b} \int \phi(x,y)dydx = \int_{a}^{\infty} \phi(x) \Phi\left(\frac{b-\rho x}{\sqrt{1-\rho^2}}\right)dx, \]

\( v(x) \) comes out \( \frac{1}{x-d} \frac{\phi(c-dx)}{\Phi_+(c-dx)} \), letting \( c = \frac{b}{\sqrt{1-\rho^2}} \) and \( d = \frac{\rho}{\sqrt{1-\rho^2}} \).
Neither $c$ nor $d$ need be positive. Let $\psi(x) = \frac{\varphi(x)}{\Phi_+(x)}$.

$$
\psi'(x) = \frac{-x \Phi_+ + \varphi + \varphi^2}{\Phi_+^2}, = -x \psi(x) + [\psi(x)]^2. \text{ So } \psi' = \psi^2 - x\psi, \text{ and }
$$

$$
v(x) = \frac{1}{x-d \psi(c-dx)}. \quad v'(x) = -\frac{1 + d^2 \psi'}{(x-d \psi)^2}.
$$

The mode of the distribution can be found by observing where $v$ becomes infinite. This is the solution of the equation $x = d \psi(c-dx)$.

In order to apply even the simple formula, Bounds 100, we must show when $v'$ is monotonic.

**Theorem 131.** If $d > 0$ and $a > \alpha$, where $\alpha = \text{the mode}$, is the solution of $x = d \psi(c-dx)$, or if $d < 0$ and $a > \alpha > \frac{c+1}{d}$, then $v'(x)$ is monotonic for $x > a$.

**Proof.** Case 1: $d > 0$. Let $y = c-dx$. $v = \text{constant}/[c-y-d^2\psi(y)]$.

Let $c-y-d^2\psi(y)$ be $u(y)$. $-u' = 1 + d^2\psi(y-dy)$. A tabulation of values shows that $\psi(y)$ monotonically increases from 0 to 1 as $y$ increases from $-\infty$ to $\infty$, and that $[\psi(y)]'$, which = constant $\cdot u''$, monotonically decreases for $y > -1$. (The latter fact will be used in Case 2.) As $x$ increases, $y$ decreases, $\implies d^2\psi(y)$ decreases, $\implies u(y)$ increases. $-u'$ decreases as $y$ decreases, and $v' = u'/u^2$. Thus, the numerator of $v'$ decreases in absolute value while the denominator increases, $\implies v'$ is monotonic.

A missing step in the proof is to show that $\psi(y)$ is monotonic increasing. By (50), $\psi(y)$, which = $\frac{\varphi(y)}{\Phi_+(y)}$, is monotonic increasing if $\frac{\varphi'(y)}{\Phi_+(y)}$ is; i.e., if $\frac{-\psi}{-\varphi}$ is. But this is the increasing function, $y$. 
Case 2: \( d < 0 \). Then \( y \) increases as \( x \) increases. Let \( \beta \) be the mode with respect to \( y, = \) the solution of \( c-y-d^2\psi(y) = 0 \).

Let \( u = y + d^2\psi(y) - c \). Since \( u'' \) decreases and \( u' \) increases for \( y > -1 \), it follows that \( u''/u' \) decreases for \( y > -1 \), \[ u'(y) - u'({\beta})]/u(y) \] decreases for \( y > -1 \). It is also positive.

If we add to it the positive, decreasing function, \( u'({\beta})/u(y) \), we obtain the positive decreasing function, \( \frac{u'(y)}{u(y)} \).

Dividing that by the positive, increasing function, \( u(y) \), makes it decrease all the more. Hence, \( v'(y) \) is monotonic. We have implicitly used the relation \( y > \beta > -1 \). Since there can be no question of having \( v' \) monotonic where \( v \) becomes infinite, we must have \( y < \beta \) in the first case and \( > \beta \) in the second. The point, \( y = -1 \), is the same as the point, \( x = \frac{c+1}{d} \).

When \( d < 0 \) and \( \beta < -1 \), it is possible to construct a counterexample in which \( v' \) is not monotonic. \( v' \) is monotonic if \( (u'/u^2)' \) is of constant sign for \( u > 0 \). Sign \( \frac{u'}{u^2}' = \text{sign} \left( \frac{u'^2u'' - u'2uu'}{u^4} \right) = \text{sign} (uu'' - 2u'^2) \) for \( u > 0 \). Now as \( u \) decreases to \( 0 \), \( u'/u^2 \) becomes infinite. Thus if \( v' \) is monotonic, \( u'/u^2 \) must decrease, \( \Rightarrow 2u'^2 > uu'' \) for \( u > 0 \). \( u = y + d^2\psi(y) - c \). \( u' \) and \( u'' \) are independent of \( c \). Thus, if we fix everything but \( c \) and make \( c \) negative enough, \( uu'' > 2u'^2 \).

Of the previously known methods of calculating the bivariate normal integral, two of the neatest results are given by K. Pearson, [34], and Polya, [40]. Pearson expands the integral,

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(x,y)dydx, \quad \text{into the series}, \quad \phi(a) \phi(b) \sum_{r=0}^{\infty} \frac{p^2}{r!} H_{r-1}(a) H_{r-1}(b)
\]

\( H_i \) is the \( i^{th} \) Hermite polynomial. The series converges for all \( a \) and \( b \). This expansion is best in the center, and it is far from...
asymptotic. The expansion generalizes without difficulty to the normal multivariate distribution.

Polya expands the integral, \( \int \int \varphi(x) \varphi(y) \, dy \, dx \), into the series, \( \frac{1}{2} \int \frac{h}{k} \, \varphi(x) \, dx - \frac{1}{2\pi} \tan^{-1} \left( \frac{h}{k} \right) + \frac{h}{k} e^{-\frac{h^2}{2}} \) times \( \frac{1}{2\pi} \tan^{-1} \left( \frac{h}{k} \right) \),

\[
\left[ 1 - \left( \frac{1}{k^2} + \frac{2}{1^2} \right) + \left( \frac{1 \cdot 3}{k^4} + \frac{1 \cdot 4}{k^2 \cdot 1^2} + \frac{2 \cdot 4}{1^4} \right) - \left( \frac{1 \cdot 3 \cdot 5}{k^6} + \frac{1 \cdot 3 \cdot 6}{k^4 \cdot 1^2} + \frac{1 \cdot 4 \cdot 6}{k^2 \cdot 1^4} + \frac{2 \cdot 4 \cdot 6}{1^6} \cdots \right) \right],
\]

which, although divergent, is asymptotic and enveloping. \( (1^2 = h^2 + k^2.) \)

The similarity to the asymptotic series for the univariate normal distribution is evident. This integral, which we call \( V(a, b) \), will yield the former one through the relation,

\[
\int \int \varphi(x, y) \, dy \, dx = V(a, \frac{b - pa}{\sqrt{1 - p^2}}) + V(b, \frac{a - pb}{\sqrt{1 - p^2}}) + \frac{1}{2} \left[ \Phi(a) + \Phi(b) \right] - \frac{1}{4} \left[ \Phi(a) + \Phi(b) \right] - \frac{1}{2} \sin^{-1} \left( \frac{p}{\sqrt{1 - p^2}} \right),
\]

found on page vii of [53].

A thorough table of the bivariate normal integral was recently published by the National Bureau of Standards, [53].
APPENDIX

A LIST OF APPROXIMATION FORMULAS AND BOUNDS

In this appendix are listed some of the approximation formulas and bounds most likely to be useful.

Definitions.
\[ F(a) = \int_a^\infty f(x)\,dx \quad \text{for infinite tails,} \]
\[ = \int_0^a f(x)\,dx \quad \text{for finite tails.} \]

\( F_1 \) is an approximation to \( F \).
\( f_1 = -F_1' \) for infinite tails, and \( = F_1' \) for finite tails.

We will assume that \( F \) and \( F_1 \to 0 \) and that \( f(x) \) and \( f_1(x) > 0 \) for \( x > a \).

\[ v(x) = -f(x)/f'(x). \]

\( r = \theta(f) \) is the order of \( f \), which means that for any \( \epsilon \), \( f \) is bounded by \( x^{r+\epsilon} \) and \( x^{r-\epsilon} \) for \( x \) large enough.

\[ r = +\infty \] means that \( f \) is bounded by \( x^\alpha \) for all \( \alpha \), bounded below for infinite tails, bounded above for finite tails. The definitions are exchanged for \( r = -\infty \).

\[ \gamma = r/(r+1). \]

A limit can be any real number or \( \pm \infty \).

\[ E(F_1) = E_1 = \frac{F_1}{F} - 1. \quad e(F_1) = e_1 = \frac{f_1}{F} - 1. \]

Formulas 34 and 99. If \( v' \to \) a limit monotonically, then \( |\gamma v f| \) and \( |vf/(1-v')| \) are both asymptotic to and bounds for \( F \), (36, 100). The
smallness of \( \gamma(y' + \frac{1}{r}) \) measures the accuracy of these bounds, since that expression is the difference between their ratio and 1.

**Bounds 47.** Min \( e_1(x) \), for \( x \geq a \), \( E_1(a) \leq e_1(x) \) for \( x \geq a \). If \( e_1(x) \to 0 \) monotonically for \( x \geq a \), then \( E_1(a) \) is bounded by \( e_1(a) \).

**Bounds 48.** If \( b_1 \leq \frac{f(x)}{f_1(x)} \leq b_2 \) for \( x \geq a \), then \( b_1F_1(a) \leq F(a) \leq b_2F_1(a) \).

An asymptotic formula, that is, one where \( F_1(x) \sim F(x) \) as \( x \to \infty \), can be combined with quadrature to get an error as small as we like.

Let \( F_2(a,b) \) be the quadrature from \( a \) to \( b \), \( F_1(b) \), the formula that approximates \( F(b) \), and \( F_3(a) \), the combined approximation to \( F(a) \).

**Bounds 49.** If \( E_1(x) \geq 0 \) for \( x \geq b \), then
\[
|E_3(a)| \leq |E_2(a,b)| + \frac{F_1(b)}{F_3(a)} E_1(b).
\]
If \( E_1(x) \leq 0 \) for \( x \geq b \), then
\[
|E_3(a)| \leq |E_2(a,b)| + \frac{F_1(b)}{F_3(a)} \cdot \frac{E_1(b)}{1 + E_1(b)}.
\]

**Bounds 50.** If \( e_1 \to 0 \) monotonically, then \( E_1 \to 0 \) monotonically.

**Theorem 57.** Let \( e_1 \) have order. If \( \theta(f) \) is finite, or if \( \theta(f) = + \infty \) and \( f'/f \) has order, then \( \theta(E_1) = \theta(e_1) \).

**Theorem 60.** If \( r \) or \( e_1 \) is finite, then \( e_1 \sim [\frac{\theta(e_1)}{r+1} + 1] E_1 \), provided that \( xf'/f \) and \( xe_1'/e_1 \to \) limits.

**Formula 67.** Start with an approximation, \( F_1 \). Let \( u_1 = f/f_1 \) and \( u_2 = u_1'/f_1 \). Let \( U_1 \) and \( U_2 \) be the limits of \( u_1(s) \) and \( u_2(x) \) as \( x \to \infty \). \( F(a) \sim \frac{u_1(a) + \alpha U_1}{1 + \alpha} F_1(a) \), where \( \alpha = \frac{2u_2(a) + U_2}{u_2(a) + 2U_2} \).

The \( 1/x \) Transformation and Quadrature. \( F(a) = \int_{0}^{1/a} f(1/y) \frac{dy}{y^2} \). The
interval can be split up into equal subintervals, and a good quadrature formula like Weddell's rule, [20], applied. This quadrature is most successful if \( \theta(f) < -3 \).

Expansion of \( F/vf \). This is the power series expansion most likely to be useful for tail integral approximation. The first term is \( \gamma \). Following terms are powers of \( x^{-1} \) with coefficients chosen successively to minimize \( e(x) \) as \( x \to \infty \). For finite tails, \( F/vf \) is expanded in powers of \( x \).

**Laplace-Winckler Expansion.**

\[
F(a) = f(a) \left[ v_1(a) + \ldots + v_n(a) \right] + \int_a^\infty f(x) v_n'(x) \, dx, \quad \text{where} \quad v_1 = -f/f',
\]

and \( v_{i+1} = v_i v_1' \). For finite tails, replace \( f(a) \) by \(-f(a)\).

**Bounds 85.** If \( v_n'(x) \to \text{its limit, } (-1/r)^n, \) monotonically, then \( F(a) \) is bounded by \( f(a) \left[ v_1(a) + \ldots + v_n(a) \right]/(1-\alpha), \) where \( v_n'(a) \) and \( (-1/r)^n \) are substituted for \( \alpha \), provided that \( |v_n'(a)| < 1 \).

**Bounds 86.** If \( f \) and \( v_n \to 0 \) monotonically, then \( F \) is bounded by the expansion, first with \( n-1 \) terms, then with \( n \) terms.

The expansion is asymptotic only if \( \theta(f) \) is infinite. Otherwise, \( E_{n+1}/E_n \to 1/r \), under regularity conditions.

**Procedure C.** Start with an approximation, \( u_1 f \). A second approximation is obtained by the iterative formula, \( u_2 = (1 + u_1')v \). For finite tails, \( u_2 = (u_1' -1)v \).

If \( u_1 = v \), we get the Laplace-Winckler expansion.

\( E_2/E_1 \to [1 + \theta(e_1)]/r \), under regularity conditions. Thus, the procedure is only asymptotic if \( r \) is infinite. Bounds 85 and 86 can be generalized. In general, C improves if \( e_1 \) decreases more slowly than \( f \), and gets worse and worse if \( e_1 \) decreases more rapidly than \( f \).
Procedure C1. This modification of C is for use with finite r.
Start with $F_1$ and obtain F by C. Let $L = \lim E_2/E_1$. The improved approximation, $u_{21} = \frac{L}{L-1} u_1 - \frac{1}{L-1} u_2$. Then $E_2/E_1 \to 0$.

Procedure D. $F_2 = F_1 f/f_1$.

Bounds 98. If $f_1/f \to 0$ monotonically for $x \geq a$, then F lies between $F_1$ and $F_2$.

$E_2/E_1 \to -\theta(e_1)/(r+1)$ under regularity conditions, so D is asymptotic only if $r$ is infinite. As with C, D improves only if $e_1$ decreases more slowly than $f$. D can be modified for finite $r$ to make it asymptotic.

Procedure D3. Start with two approximations, $F_1$ and $F_2$.

$F_{21} = \frac{(f-f_2)F_1 + (f_1-f)F_2}{F_1 - F_2} = \frac{e_2}{e_1} \frac{F_1 - F_2}{F_1 - F_2}.$

Bounds 111. If $e_2/e_1 \to L$ monotonically and $e_2/e_1$ and L are not on opposite sides if 1, and $F_1$-F is of constant sign, then

F lies between $\frac{e_2}{e_1} F_1 - F_2$ and $\frac{L F_1 - F_2}{L - 1}$.

Finite Tails. Replace $x \geq a$ by $0 \leq x \leq a$, and $x \to \infty$ by $x \to 0$. Then all the previous formulas hold except the $1/x$ transformation.

Many handy formulas are obtained by taking one or two iterations of C, C1, D, D1, D2, and D3 in various combinations.

Example 132. One of the most useful combinations is D3 and C; namely, an interpolation between $vf$ and $vf(1+v')$. This comes out

$vf(1 + \frac{v'}{1-v'-\nu''/v'})$. Although $vf(1+v')$ is for $\theta(f) = -\infty$ infinitely
more accurate than $v_f$ as $a \to \infty$, the magnitude of the errors may be comparable for small values of $a$. Formula 132 then makes a good compromise between the two approximations. If $\theta(f)$ is finite, the formula becomes 

$$
\gamma v_f \left[1 + \frac{v' + 1/r}{1-v' - v''/v'/(v' + \frac{1}{r})}\right], 
$$

and represents an interpolation between $\gamma v_f$ and $v_f(1+\gamma v')$, the first two terms of (91). Another way of writing this formula is to let 

$$
\alpha(x) = \frac{e_2(x)}{e_1(x)} = \frac{(\gamma-1)v' - \gamma v'^2 - \gamma v''}{\gamma - 1 - \gamma v'}. 
$$

Then $132 = v_f[\alpha(x) - \frac{1 - \gamma v'}{\alpha(x) - 1}].$

**Formula 133.** If $\alpha(x)$ be replaced by its limit, $\alpha_0$, in the previous expression, we get Formula 133, which is the Cl compromise between $\gamma v_f$ and $v_f(1+\gamma v')$. If $\alpha(x) \to \alpha_0$ monotonically for $x \geq a$, and $\alpha(x)$ and $\alpha_0$ are not on opposite sides of 1, then (132) and (133) are bounds for $F$, (111). If $\theta(f) = -\infty$, then (126) is simply $v_f(1+v')$, (87.2).

If $\theta(f) = -\infty$, then the first three iterations of $C$; namely, $v_f$, $v_f(1+v')$, and $v_f(1+v'+v'^2+v'^2)$, are asymptotically increasingly accurate, (87). The same is true of the first three iterations of $D$; namely, $v_f$, $v_f/(1-v')$, and $v_f/(1-v' - v''/1-v')$, (105).

Bounds 85 and 86 apply to the $C$ formulas, and Bounds 98 apply to the $D$ formulas. In addition, the following theorems relate the two procedures.

**Theorem 134.1.** Let $u_2^f$ and $u_2^{*f}$ be obtained from $u_1^f$ by $C$ and $D$, respectively. Let $e_1^f$, $e_2^f$, and $u_1 - u_2 \to 0$ monotonically, and let $F$ and $u_1^f$ converge, so that $[u_1^f, u_2^f]$ and $[u_1^f, u_2^{*f}]$ are both bounds for $F$ by (86.1) and (98).

Then $u_1'(a) > 0 \implies u_2$ is better than $u_2^*$.

and $u_1'(a) < 0 \implies u_2$ is worse than $u_2^*$. 


Bounds 134.2. If the above conditions hold except that $u_1 - u_2 \to \pm \infty$ monotonically, then $F$ is bounded by $[u_2^f, u_2^* f]$.

Proof of 1). $u_2^* f = u_1 f / (1 + e_1)$, $\implies u_2^* = u_1 / (1 + \frac{u_1 - u_2}{v})$, (92.1).

Hence, $u_1 < u_2 \implies u_2^* < u_1$. Now $u_2 = v(l+u_1')$, and

$u_2^* = v[1 + u_1'/(1 + \frac{u_1 - u_2}{v})]$, after algebra. Case 1: $u_1' > 0$ and $u_1 - u_2 > 0$. Then $1 + \frac{u_1 - u_2}{v} > 1 \implies \frac{u_1}{1 + \frac{u_1 - u_2}{v}} < u_1'$,

$u_2^* < u_2 < u_1$, $\implies u_2^* f < u_2 f < F < u_1 f$. The other cases are analogous.

Proof of 2). $e_1 \to 0 \implies F$ lies between $u_1 f$ and $u_2^* f$, (98).

$u_1 - u_2 \to \pm \infty \implies u_2^* f$ lies between $u_1 f$ and $F$, (96.2). Hence, $u_1 f, u_2^* f, F$, and $u_2^* f$ are all in order, one way or the other.

The following is a useful special case of this theorem.

Bounds 135. Let $F_1 = \gamma v f$, let $F_1$ and $F$ converge, and let $e_1, ve_1$, and $f \to$ limits monotonically.

1) If $\theta(f)$ is finite and $\theta(e_1) > -1$, then $F$ lies between $vf/(1-v')$ and $vf(1+\gamma v')$.

2) If $\theta(f)$ is finite and $\theta(e_1) < -1$, then $[\gamma v f, vf(1+\gamma v')]$ are better bounds for $F$ than $[\gamma v f, vf/(1-v')]$ if $v'(a) > 0$; worse, otherwise.

3) If $\theta(f) = -\infty$ and $\theta(v) > \frac{1}{2}$ then $F$ lies between $vf/(1-v')$ and $vf(1+v')$.

4) If $\theta(f) = -\infty$ and $0 < \theta(v) < \frac{1}{2}$ then $[vf, vf(1+v')]$ are better bounds for $F$ than $[vf, vf/(1-v')]$.

5) If $\theta(f) = -\infty$ and $\theta(v) = 0$, then $[vf, vf(1+v')]$ are better or worse bounds for $F$ than $[vf, vf/(1-v')]$ according as $v'(a) > 0$ or $< 0$. 
6) If \( \theta(c) = -\infty \) and \( \theta(v) < 0 \), then \([vf, vf/(1-v')]\) are better bounds for \( F \) than \([vf, vf(l+v')]\).

Proof. When \( \theta(f) \) is finite, \( \theta(e_1) < \) or \( > -1 \) \( \implies u_1 - u_2 \to 0 \)
or \( \pm \infty \), since \( \theta(e_1) = \theta\left(\frac{u_1 - u_2}{v}\right) = \theta(u_1 - u_2) - 1 \). Hence, (134) applies to 1) and 2). When \( \theta(f) = -\infty \), \( \theta(u_1 - u_2) = \theta(vv) = \),
\( \implies u_1 - u_2 \to \pm \infty \) or \( 0 \) according as \( \theta(v) > \) or \( < \frac{1}{2} \). Thus 3) and 5) follow from (134). Since \( e_1 = -v' \), is monotonic, \( v'(a) < 0 \) when the positive variable, \( v \to 0 \), and is \( > 0 \) when \( v \to \infty \). Thus, 4) and 6) follow from (134.1). //

The following is a collection of useful formulas for series. We let
\[
F(a) = \sum_{n=0}^{\infty} f(n) \text{ and assume that series converges, and } f(n) > 0.
\]

\[ \Delta f = f(n+1) - f(n), \quad v = f/-\Delta f. \]

Bounds 100 for Positive Series. If \( \Delta v \to a \) limit monotonically, then
\[ |\gamma_{n-1} f| \text{ and } |v_{n-1} f/(1 - \Delta v_{n-1})| \]
are asymptotic to and bounds for the positive, convergent series, \( F \).

Bounds 47, 48, 49, and 50 all apply to positive series. Bounds 49 assume a greater importance. Instead of using quadrature, we sum a finite number of terms, and thus \( |E_2(a,b)| \) is omitted from the error bound.

The Euler-Maclaurin Sum Formula. To get such sums as \( \sum_{n=0}^{\infty} \frac{1}{n^2} \), a useful approximation is the following formula:
\[
\sum_{n=0}^{\infty} f(n) \sim \int_a^\infty f(x) dx + \frac{f(a)}{2} - \frac{f'(a)}{12}.
\]

If \( f^{(3)}(x) \) is of constant sign for \( x \geq a \), then the remainder is less in absolute value than \( .008 f''(a) \). If, further, \( f^{(3)}(x) \) is monotonic, then the remainder is less in absolute value than \( .003 f^{(3)}(a) \).

Other useful forms of the Euler-Maclaurin formula are found in Chapter VIII.
Proof of the bounds. Knopp, [23], gives the remainder as

\[ \int_{a}^{\infty} P_3(x) f^{(3)}(x) \, dx, \quad \text{where} \quad P_3(x) = \frac{x^3}{6} - \frac{x^2}{4} + \frac{x}{12} \quad \text{for} \ 0 \leq x \leq 1 \] and repeats with period 1 for \( x > 1 \). Max \( \{ |P_3(x)|; 0 \leq x \leq 1 \} \) is found by calculus to be .0080188. Hence, if \( f^{(3)}(x) \) is of constant sign,

\[ |\int_{a}^{\infty} P_3(x) f^{(3)}(x) \, dx| \leq .008 \quad |\int_{a}^{\infty} f^{(3)}(x) \, dx| = .008 \ f^{(2)}(x). \] This bound is actually quite a bit too wide, since \( P_3(x) \) is negative as much as it is positive. In fact, \( P_3(x) = -P_3(1-x) \), as can be seen by writing \( P_3(x) \) as \( (2x-1)(x-1) \ x/12 \). Moreover, \( P_3(x) \) is \( > 0 \) for \( 0 < x < .5 \) and \( < 0 \) for \( .5 < x < 1 \). This shows that

\[ \int_{a+5}^{a+1} P_3(x) f^{(3)}(x) \, dx + \int_{a+5}^{a+1} P_3(x) f^{(3)}(x) \, dx + \int_{a+5}^{a+1} P_3(x) f^{(3)}(x) \, dx + \cdots \]

forms an alternating series of terms decreasing in absolute value when \( f^{(3)}(x) \) is monotonic. Thus, the remainder is less in absolute value than

\[ |\int_{a+5}^{a+1} P_3(x) f^{(3)}(x) \, dx|, < \ |f^{(3)}(a)| \int_{0}^{\infty} P_3(x) \, dx \ < \ .003 \ |f^{(3)}(a)|. \]

For alternating series, the following simple formulas are available.

Let \( (n) = |f(n)| \).

**Bounds 121.** If \( \Delta f \to 0 \) monotonically, then \( F \) lies between \( f \) and \( f/2 \).

Furthermore, if \( F_1 \) is an approximation to \( F \) such that \( \Delta f \) and \( \Delta|f_f-f| \) \( \to 0 \) monotonically, then \( E_1 \) lies between \( \frac{e_1}{2} \) and \( 2e_1 \).

**Formula 122.** If \( \frac{\Delta f^{(n+1)}}{\Delta f(n)} \to \lambda \) then \( F(n) \sim \frac{f(n)}{1 + \lambda} \). The relative frequency error of this formula is \( \frac{1}{1 + \lambda} \ (\frac{f(n+1)}{f(n)} - \lambda) \).

Although Procedures C and D apply to series, even to the point of being asymptotic for finite \( r \) in the case of alternating series, it is usually easier to sum partially and then apply simple bounds.
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