ON A CLASS OF NON-PARAMETRIC TESTS

by

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CHAPTER I

DESCRIPTION OF A CLASS OF PROBLEMS

1. Characterization of a Class of Problems.

Let $F_1, \ldots, F_k$ be cumulative distribution functions (cdf's), and let $n = (n_1, \ldots, n_k)$ be a vector of positive integers. Let $G_{n_1, \ldots, n_k} = G_n$ represent the joint distribution of $(n_1 + \ldots + n_k)$ independent random variables of which n_i are distributed according to F_i $(i = 1, \ldots, k)$. Let $\mathcal{D}$ be a class of k-tuples $(F_1, \ldots, F_k)$, and let $\mathcal{D}_n = \left\{ G_n : (F_1, \ldots, F_k) \in \mathcal{D} \right\}$. The distributions F_i may be multivariate.

Let $\underline{x}_{i,n_i} = (x_{i1}, \ldots, x_{in_i})$, $i = 1, \ldots, k$, and $z_n = z_{n_1, \ldots, n_k}$ $= (\underline{x}_{1,n_1}, \ldots, \underline{x}_{k,n_k})$.

Suppose that for all $(F_1, \ldots, F_k) \in \mathcal{D}$ there is defined a real-valued functional

$$\theta(F_1, \ldots, F_k) = \int \phi(z_{r,r,\ldots,r}) \, dG_{r,\ldots,r}(z_{r,\ldots,r}), \tag{1.1}$$

where $r \geq 1$ is an integer. The kernel ϕ is understood to be a measurable function and integration is extended over kr-dimensional Euclidean space (in the case of univariate F_i). The integral is understood in the sense of Lebesgue-Stieltjes. Suppose the functional $\theta(F_1, \ldots, F_k)$ takes values in an interval ω which may be finite or infinite.

Many of the problems of statistical decision theory may be stated in these terms. We consider non-sequential two-decision problems which may be characterized as follows:

We are given a sample z_n consisting of n_i observations from each of k populations, and we assume that Z_n is distributed according to $G_n \in \mathcal{D}_n$. We wish to make one or the other of two decisions d_0, d_1, and we are given subsets ω_0, ω_1 of the interval ω (corresponding to subsets $\mathcal{D}_0$, $\mathcal{D}_1$ of $\mathcal{D}$), such that we prefer d_j if $\Theta(F_1,\ldots,F_k)\in\omega_j$ (j = 0,1). We will call this the problem $(\mathcal{D}, \omega_0, \omega_1)$. The object of theoretical investigations in this context is to find or determine the properties of procedures for selecting one of d_0, d_1 on the basis of the sample z_n, that is, for solving the problem $(\mathcal{D}, \omega_0, \omega_1)$.

This representation of a class of problems is not new; references to some of the contributions to the theory of certain problems of this type are included in Section 2 of this chapter.

It is assumed that the kernel ϕ of the functional (1.1) is symmetrical in each of its k sets of r arguments. This is not an essential restriction, except as noted below. If ϕ is not symmetrical, it is always possible to construct a symmetrical kernel ϕ'. Let

$$(r!)^k \phi' = \Sigma' \phi \left(\underset{\sim}{x}_{1,r}^{(j_1)}, \ldots, \underset{\sim}{x}_{k,r}^{(j_k)}\right),$$

where for i = 1, ..., k, $\underset{\sim}{x}_{i,r}^{(j_i)}$ is a permutation of $(x_{i1},\ldots,x_{ir})$ and j_i indexes the set of permutations of (1, 2, ..., r); Σ' denotes k-fold summation over the ranges of the indexes $j_1,\ldots,j_k$.

We have assumed that the kernel ϕ of (1.1) has the same number r of arguments in each of the k sets. This, again, is not an essential restriction. Suppose

$$\phi = \phi\,(\underset{\sim}{x}_{1,r_1}, \ldots, \underset{\sim}{x}_{k,r_k}) ,$$

and let $r = \max(r_1, \ldots, r_k)$. For $i = 1, \ldots, k$, ϕ may always be regarded as a function of $(x_{i1}, \ldots, x_{ir})$ which depends on these arguments only through the values of $(x_{i1}, \ldots, x_{ir})$. When regarded this way, of course, ϕ is not symmetrical.

We will give particular attention to certain decision problems for which the functional (1.1) is equal to the probability of an event; that is, for which ϕ is the characteristic function of a set in kr-dimensional Euclidean space (univariate F_i) and takes values zero or one only. In this case, the assumption we have made about the symmetry of the function ϕ is not trivial, since in general the symmetrical function ϕ' would not be a characteristic function. For the examples with which we will be concerned, however, ϕ satisfies all our assumptions.

2. A Class of Decision Rules for the Problem $(\mathfrak{D}, \omega_0, \omega_1)$.

The purpose of the present investigation is to develop and illustrate the application of methods for deriving properties of certain classes of decision rules which have been proposed for solving the problem $(\mathfrak{D}, \omega_0, \omega_1)$. Chapters II and III contain some general theorems about these classes of decision rules, with specializations and illustrations relevant to a particular class of two-sample problems. In Chapter IV, the properties of these classes of decision rules are examined in detail in the context of the same two-sample problems.

A fixed-sample-size decision rule based on the sample z_n is determined by, and will be identified with, a real-valued measurable function $\phi_n(z_n)$, $0 \le \phi_n(z_n) \le 1$, which gives the probability of making decision d_0, say, when z_n has been observed. Except in Sections 3 and 4 of Chapter IV, we will consider decision rules of the form

$$\phi_n(z_n) = \begin{cases} 1 & \text{if } t_n(z_n) < \lambda_n \\ a_n & \text{if } t_n(z_n) = \lambda_n \\ 0 & \text{if } t_n(z_n) > \lambda_n \end{cases}, \tag{1.2}$$

where the statistic $t_n(z_n)$ is a real-valued measurable function of the observations, λ_n is a constant, and a_n is a constant ($0 \le a_n \le 1$).

Consider the statistic $U_n(z_n)$ defined by

$$\binom{n_1}{r} \cdots \binom{n_k}{r} U_n(z_n) = \Sigma'' \phi(\underline{x}_{1,r}^{(j_1)}, \ldots, \underline{x}_{k,r}^{(j_k)}), \tag{1.3}$$

where for $i = 1, \ldots, k$, $\underline{x}_{i,r}^{(j_i)}$ is a vector of r distinct elements $(x_{i\ell_1}, \ldots, x_{i\ell_r})$ from the vector $\underline{x}_{i,n_i}$ and j_i indexes the class of sets of integers $(\ell_1, \ldots, \ell_r)$ with $1 \le \ell_1 < \ldots < \ell_r \le n_i$; Σ'' denotes k-fold summation over the ranges of $j_1, \ldots, j_k$.

Statistics of the form (1.3) have been considered in general by various authors, including Halmos [8], Hoeffding [10], Lehmann [16]. They are unbiased estimators of minimum variance for the functional (1.1); they have asymptotic normal distribution as $(n_1, \ldots, n_k)$ tend to infinity, under quite general conditions. A variety of examples is given by Hoeffding [10] for $k = 1$, and by Lehmann [16]

for k = 2. The general results obtained in Chapter III are especially applicable to a sub-class of the class of statistics of the form (1.3), namely, those for which the function ϕ takes values zero or one only. Various statistics belonging to this sub-class have been proposed for non-parametric decision problems, and their properties studied in some detail.

Among them are (1) the sign test for the median, see [6], [10], [11]; (2) Kendall's difference-sign rank correlation coefficient τ, see [15]; (3) the Wilcoxon-Mann-Whitney two-sample statistic, see for example [18], [16], [25]; (4) other two-sample statistics discussed by Lehmann [16], including in particular (5) a statistic for testing $F = G$ against $F \neq G$ which has been further investigated by Sundrum [26]. When it is assumed that $\mathcal{Q}$ contains only continuous distributions, these statistics are all based on functionals which are equal to the probability of an event. Corresponding functionals are : (1) for testing whether the median is at $x = 0$, $P(X < 0)$; (2) for a bivariate random variable (X, Y), a linear function of $P[(X_1 - X_2)(Y_1 - Y_2) > 0]$; for independent random variables X, Y, (3) $P(X < Y)$, and (5) $P(X_1, X_2 < Y_1, Y_2$ or $Y_1, Y_2 < X_1, X_2)$.

The following are additional examples of these statistics, not in the sub-class illustrated above, but of a closely related type. (6) A certain set of k-sample statistics which appears in the construction of a rank-order one-way analysis of variance test, see Andrews [1], is associated with the set of functionals (for equal sample sizes) $\sum_{i \neq j} P(X_i < X_j)$, $j = 1, \ldots, k$, where $X_1, \ldots, X_k$ are independent

random variables. (7) Spearman's rank correlation coefficient ρ (the product-moment correlation of the ranks) is a consistent estimator for a functional of the form (1.1), which is a linear function of $P(X_1 > X_2, Y_1 > Y_3)$, where (X,Y) is a bivariate random variable. The symmetrical kernel of this functional does not take the values zero and one only. The rank correlation coefficient is a linear function (with coefficients depending on n) of the "sample functional", and has the same limiting distribution as the statistic of form (1.3); see [10].

In Chapter II, we use Hoeffding's methods ([10], Section 5) to obtain the variance and certain other properties of $U_n(Z_n)$ in the case $k = 2$, and point out that the same methods may be applied for any k (although the notation would become increasingly complicated). Conditions are given under which $U_n(Z_n)$ converges uniformly to its limiting normal distribution, for $k \leq 2$, with the same possibility of generalization.

The principal result of Chapter III provides a method for obtaining an asymptotic expression for an index of efficiency for families of decision rules of the form (1.2) based on statistics (1.3), relative to certain decision problems of the form $(\mathfrak{D}, \omega_0, \omega_1)$.

We may mention one further property of $U_n(z_n)$ which recommends its consideration in connection with decision problems of the form $(\mathfrak{D}, \omega_0, \omega_1)$. A statistic of the form (1.3) is equivalent or closely related to the "sample functional" $\theta(z_n; S_1, \ldots, S_k)$, where

$$S_i = S_i(x; \underline{x}_{i,n_i})$$

$$= (\text{no. of observations } x_{ij} \leq x,\ j = 1,\ldots,n)/n_i;$$

$$n_1^r \cdots n_k^r\, \theta(z_n; S_1, \ldots, S_k)$$

$$= \Sigma\, \phi(x_{1j_{11}}, \ldots, x_{1j_{1r}}; \ldots; x_{kj_{k1}}, \ldots, x_{kj_{kr}}),$$

and Σ denotes rk-fold summation over the ranges $(1 \leq j_{i\ell} \leq n_i)$ of the indexes $j_{i\ell}$ $(i=1, \ldots, k;\ \ell = 1, \ldots, r)$. $\theta(z_n; S_1,\ldots,S_k)$ reduces to $U_n(z_n)$ if $r = 1$, and is a consistent, though not in general unbiased, estimator for $\theta(F_1, \ldots, F_k)$, if $r \geq 2$. Hoeffding [10] points out that the asymptotic distribution of $\theta(Z_n; S_1, \ldots, S_k)$ is the same as that of $U_n(Z_n)$.

3. "Optimum" Decision Rules for $(\mathcal{D}, \omega_0, \omega_1)$.

An important object of a general investigation of decision rules based on $U_n(z_n)$ is to determine whether such decision rules have optimum or "good" properties, for use in connection with problems $(\mathcal{D}, \omega_0, \omega_1)$. Even if $\mathcal{D}$ is taken to be a relatively small class of distributions, very little is known about the relative "goodness" of alternative decision procedures for non-parametric problems such as those cited in the preceding section. We will summarize some of the known "good" properties of statistics of the form (1.3), and then consider the question of "optimum" properties.

First, however, we introduce another class of statistics which is important to this discussion. Let $s = [\min(n_1,\ldots,n_k)/r]$, where

$\lfloor u \rfloor$ denotes the greatest integer $\leq u$. Let

$$(1.4) \qquad s\, W_n(z_n) = \sum_{j=0}^{s-1} \phi\left(\underset{\sim}{x}_{1,r}^{(j)}, \ldots, \underset{\sim}{x}_{k,r}^{(j)}\right),$$

where $\underset{\sim}{x}_{i,r}^{(j)} = (x_{i(rj+1)}, \ldots, x_{i(rj+r)})$, $i = 1, \ldots, k$. Thus, $W_n(Z_n)$ is a sum of s independent and identically distributed random variables.

Statistics of the form (1.3) have been called U-statistics, and we will call statistics of the form (1.4) W-statistics.

Consistency and unbiasedness are considered to be desirable properties of tests (decision rules). Lehmann [16] has demonstrated that two-sample U-statistics, for functionals (1.1) having ϕ equal to zero or one, may be used to construct consistent sequences of tests. It is frequently possible to construct uniformly consistent sequences of tests using U-statistics. For the present purpose, it is sufficient to say that a uniformly consistent sequence of tests (cf. [2]) is a sequence $\left\{\phi_n(z_n)\right\}$ such that

$$\lim_{n \longrightarrow \infty} \sup_{(F_1,\ldots,F_k)\in\omega_0} E_{F_1,\ldots,F_k}\, \phi_n(Z_n) = 0$$

$$\lim_{n \longrightarrow \infty} \sup_{(F_1,\ldots,F_k)\in\omega_1} E_{F_1,\ldots,F_k}\, \phi_n(Z_n) = 1,$$

where "$n \longrightarrow \infty$" is written to represent "$n_1,\ldots,n_k \longrightarrow \infty$". The following statement is proved by an easy application of the Tchebycheff inequality (see for example Lemma 4.3 in Chapter IV): If $\omega_0 = \left\{\theta: \theta \leq \theta_0\right\}$, $\omega_1 = \left\{\theta: \theta \geq \theta_1\right\}$, $\theta_0 < \theta_1$, and the variance of U_n tends to zero as $\min(n_1,\ldots,n_k)$ tends to infinity uniformly for

$(F_1, \ldots, F_k) \in \mathcal{D}$, then a sequence of tests of the form (1.2) with $t_n = U_n$ is uniformly consistent, if $\theta_0 < \liminf_{n \longrightarrow \infty} \lambda_n$, $\limsup_{n \longrightarrow \infty} \lambda_n < \theta_1$

In Lehmann's paper [16] it is further shown that tests based on U-statistics may not be unbiased. If the kernel ϕ of the functional (1.1) takes only the values zero and one, however, it is always possible to construct unbiased tests by using W-statistics which under this condition have binomial distribution with parameters (s, θ).

We turn now to the question of optimum properties. In this investigation, we consider the performance of tests under assumptions which imply that $\mathcal{D}$ may be a very large class of distribution. In Chapters II and III, certain restrictions are imposed on $\mathcal{D}$ in order to insure uniform convergence of the distribution of $U_n(Z_n)$ to its limiting distribution. In Chapter IV, for detailed consideration of the properties of two-sample U- and W-statistics for the functional $P(X < Y)$, it is assumed that $\mathcal{D}$ consists of pairs of continuous distributions (but see Section 8 of that chapter).

Under such broad conditions, the class of admissible tests for a given problem $(\mathcal{D}, \omega_0, \omega_1)$ will be very large; one cannot expect, for example, to obtain uniformly most powerful tests. A criterion of optimality proposed in this context by Hoeffding [11], which he there applied successfully in one case, is the following. A test ϕ_n is said to maximize the minimum power with respect to ω_1 among tests of fixed level of significance α for testing H_0: $\theta(F_1, \ldots, F_k) \in \omega_0$ if

$$\inf_{(F_1,\ldots,F_k)\in\mathcal{D}_1} E[1 - \phi_n(Z_n)] \geq \inf_{(F_1,\ldots,F_k)\in\mathcal{D}_1} E[1 - \phi_n'(Z_n)]$$

for all $\phi_n'(z_n)$ satisfying

$$\sup_{(F_1,\ldots,F_k)\in\mathcal{D}_0} E[1 - \phi_n'(Z_n)] \leq \alpha,$$

where $\mathcal{D}_0$, $\mathcal{D}_1$ are the subsets of $\mathcal{D}$ corresponding to the subsets ω_0, ω_1 of ω. (Recall that we defined $\phi_n(z_n)$ to be the probability of making decision d_0.)

For testing H_0: $P(X < 0) \leq \theta_0$ against H_1: $P(X < 0) \geq \theta_1$, $\theta_0 < \theta_1$, the statistic for the sign test is the proportion of the observations $x_1, \ldots, x_n$ less than zero. This is Hoeffding's case, mentioned above; in [11] it is shown that a test of the form (1.3) based on this statistic maximizes the minimum power.

A further successful application of this criterion of optimality is included in Chapter IV, where it is found that a two-sample test based on the W-statistic associated with $P(X < Y)$ maximizes the minimum power with respect to ω_1 among tests of level α for testing $P(X < Y) \leq \theta_0$, where $\omega_1 = \{\theta: \theta \geq \theta_1\}$, $\theta_1 > \theta_0$. This result is suggested by the observation that the problem stated is equivalent to testing the hypothesis that $P(Z < 0) \leq \theta_0$, where $Z = X - Y$. The U-statistic for $P(X < Y)$ does not in general provide tests which maximize the minimum power for testing $\theta \leq \theta_0$. Nor has it been possible to find a test having this property for testing the usual two-sample null hypothesis that the distributions of X and Y are identical.

The small-sample exact distribution of the U-statistic for $P(X < Y)$, that is, the Wilcoxon statistic, has been examined in some detail under the assumption that X and Y have continuous distributions. Sections 2 to 4 of Chapter IV summarize the few results which it has been possible to obtain. Except for certain significance levels and certain very small sample sizes, it appears unlikely that it will be possible to find optimum tests according to the criterion stated, when the class $\mathcal{D}$ is a large class. This is true even if we try only to find tests having optimum properties among the class of rank order tests. This investigation, in summary, makes negative rather than positive contributions to the attempt to determine "good" properties of tests based on U-statistics.

Results may be obtained if the class $\mathcal{D}$ is severely restricted. The work of I. R. Savage [21] is an example; Savage finds most powerful rank order tests for testing the hypothesis that two samples come from the same distribution, with $\mathcal{D} = \{(F, G): \; G = F^a, \; a > 0\}$. Savage's methods may be applied for any class $\mathcal{D}$ for which the probabilities of possible rankings in a sample may be calculated in manageable form; the methods will be successfully applied if these probabilities imply an ordering of the possible rankings.

CHAPTER II

SOME CONVERGENCE THEOREMS.

1. Introduction.

In this chapter, we consider a class of statistics with asymptotic normal distribution, namely the class of U-statistics, whose properties have been investigated by Hoeffding [10]. Theorem 2.1 gives conditions under which the distribution of a U-statistic converges uniformly to the normal distribution as sample size tends to infinity. In Section 3, a two-sample U-statistic is defined and certain properties of its variance are derived, by methods analogous to those of Section 5 of [10]. Theorem 2.2 (Section 4) is the two-sample analog of Theorem 2.1.

The two-sample U-statistic is an obvious generalization of the single-sample statistic. Its variance has been found for certain cases by various authors [20, 26]. Lehmann [16] has shown that its distribution is asymptotically normal. It is evident that the methods employed in Sections 3 and 4 of this chapter could be extended to obtain the analogous results for k-sample U-statistics.

Theorems 2.1 and 2.2 are required for the proofs of Theorems 3.3 and 3.4 of Chapter III. Theorem 2.3 (Section 5 of this chapter) shows how Theorem 2.1 or 2.2 may be used to obtain bounds for the distributions, and is used in Chapter IV.

2. Uniform Convergence: Single-Sample Case.

Given a cdf $F(x)$, let the functional

$$\theta(F) = \int \cdots \int \phi(x_1, \ldots, x_r) dF(x_1) \cdots dF(x_r) \tag{2.1}$$

be defined for that class $\mathcal{D}$ of cdf's for which the integral exists and equals the repeated integral. We will assume that the kernel $\phi(x_1, \ldots, x_r)$ is symmetric in its r arguments. F may be a univariate or multivariate distribution.

Let X be a random variable having cdf F(x), $F \in \mathcal{D}$. Let $\underline{x} = (x_1, \ldots, x_n)$ be a sample of n independent observations on X, $n \geq r$. The U-statistic associated with the functional $\Theta(F)$ is defined by

$$U_n(\underline{x}) = \binom{n}{r}^{-1} \sum \phi(x_{i_1}, \ldots, x_{i_r}), \tag{2.2}$$

where summation is taken over all sets of r distinct integers, $1 \leq i_1 < \ldots < i_r \leq n$. It is obvious that

$$E_F U_n(\underline{X}) = \Theta(F) .$$

Hoeffding [10] makes the following definitions:

$$\begin{aligned} \Psi(x_1, \ldots, x_r) &= \phi(x_1, \ldots, x_r) - \Theta \\ \Psi_c(x_1, \ldots, x_c) &= E_F \Psi(x_1, \ldots, x_c, X_{c+1}, \ldots, X_r), \\ & \qquad c = 1, \ldots, r, \\ \Psi_0 &= 0, \end{aligned} \tag{2.3}$$

and

$$\zeta_c = \zeta_c(F) = E_F \Psi_c^2(X_1, \ldots, X_c); \tag{2.4}$$

and derives

$$\sigma^2(U_n) = \mathrm{Var}_F U_n(\underline{X}) = \binom{n}{r}^{-1} \sum_{c=1}^{r} \binom{r}{c} \binom{n-r}{r-c} \zeta_c . \tag{2.5}$$

Let

$$Y_n = (U_n - \Theta)/\sigma(U_n). \tag{2.6}$$

Hoeffding shows that Y_n has limiting (standard) normal distribution, provided $\zeta_1 > 0$ and $\zeta_r < \infty$. By a slight modification of his proof [10], we will show that this convergence is uniform in any class of distributions F having ζ_1 bounded away from zero and ζ_r, μ_3 bounded, where

$$\mu_3 = E|\Psi_1(X)|^3. \tag{2.7}$$

Consider the random variable $\Psi_1(X)$; $E\,\Psi_1(X) = 0, E\,\Psi_1^2(X) = \zeta_1$. Let

$$W_n = \sum_{i=1}^{n} \Psi_1(X_i)/n^{1/2}\zeta_1^{1/2}. \tag{2.8}$$

If $0 < \zeta_1 < \infty$, then clearly the distribution of W_n converges to $\Phi(u) = (2\pi)^{-1/2} \int_{-\infty}^{u} \exp\left\{-\frac{1}{2}t^2\right\} dt$ as n tends to infinity. Furthermore, letting $H_n(u) = P_F(W_n \le u)$, we may apply a theorem of A. C. Berry [5] to obtain:

Lemma 2.1. If $\zeta_1 > 0$, $\mu_3 < \infty$, then

$$\sup_{-\infty < u < \infty} |H_n(u) - \Phi(u)| \le C\mu_3/\zeta_1^{3/2}n^{1/2}$$

where C is an absolute constant.

Lemma 2.2. Choose $\epsilon > 0$. If $\zeta_1 > 0$, $\zeta_r < \infty$, $r \ge 2$, then

$$P(|Y_n - W_n| > \epsilon) \le \frac{1}{\epsilon^2}\frac{B}{r^2}\frac{\zeta_r - r\zeta_1}{\zeta_1}n^{-1}, \quad n \ge r, \tag{2.9}$$

where B is a constant depending only on r, given in (2.14).

Proof of Lemma 2.2. From (2.6) and (2.8) and the Tchebycheff-Bienaymé inequality, we obtain

$$P(|Y_n - W_n| > \epsilon) \le \frac{2}{\epsilon^2}(1 - EY_nW_n). \tag{2.10}$$

We have by an easy calculation

$$EY_n W_n = r\left(\frac{\zeta_1}{n}\right)^{1/2} / \sigma(U_n).$$

We will show that

$$(2.11) \qquad 0 \le \sigma(U_n) - r\left(\frac{\zeta_1}{n}\right)^{1/2} \le \frac{B}{2r} \frac{\zeta_r - r\zeta_1}{\zeta_1^{1/2}} n^{-3/2} .$$

Then

$$(2.12) \qquad 1 - EY_n W_n \le \frac{B}{r^2} \frac{\zeta_r - r\zeta_1}{2\zeta_1} n^{-1} ,$$

and we substitute (2.12) in (2.10) to obtain (2.9). It remains to establish (2.11). The first inequality of (2.11) is given in Theorem 5.2 of [10]. From Theorem 5.1 of the same paper, we obtain

$$r\zeta_2 \le c\zeta_r \qquad (c = 1, 2, \ldots, r)$$

and substitute in (2.5) to obtain

$$(2.13) \qquad \sigma^2(U_n) - r^2 \frac{\zeta_1}{n} \le B(\zeta_r - r\zeta_1)n^{-2},$$

where

$$B = \sup_{n \ge r} b_n(r) ,$$

$$b_n(r) = n^2 \binom{n}{r}^{-1} \left[\binom{n-1}{r-1} - \binom{n-r}{r-1} \right] .$$

The second inequality of (2.11) follows from (2.13). We assume $r \ge 2$ because $Y_n \equiv W_n$ for $= 1$.

It remains to establish

$$(2.14) \qquad B = \begin{cases} 4 & \text{if} \quad r = 2 \\ 12.6 & \text{if} \quad r = 3 \\ r(r-1)^2 & \text{if} \quad r \ge 4 \end{cases} .$$

We have

$$b_n(2) = 2n/(n-1) ,$$

$$b_n(3) = 6n(2n-5)/(n-1)(n-2),$$

and the first two parts of (2.14) are obtained by maximizing $b_n(r)$ with respect to integral values of $n \geq r$, $r = 2, 3$.

To establish the last part of (2.14), we first show that $r(r-1)^2$ is an upper bound for $b_n(r)$ when $n \geq r \geq 4$. For $r \leq n \leq 2(r-1)$, we have $b_n(r) = nr \leq r(r-1)^2$; for $n \geq 2r-1$, $r \geq 4$, we show $b_n(r) \leq r(r-1)^2$ by induction on r. Finally, $b_n(r) \longrightarrow r(r-1)^2$ as $n \longrightarrow \infty$.

Theorem 2.0. Let $X_1, X_2, \ldots$ be distributed according to $F(x)$. Let Y_n' and W_n' be functions of $X_1, \ldots, X_n$ and $\mathcal{C}$ a class of cdf's F such that as $n \longrightarrow \infty$

$$P_F(W_n' \leq u) \longrightarrow \overline{\emptyset}(u) \quad \text{uniformly for} \quad F \in \mathcal{C}$$

and

$$P_F(|W_n' - Y_n'| > \epsilon) \longrightarrow 0 \text{ uniformly for} \quad F \in \mathcal{C}$$

for any $\epsilon > 0$. Then as $n \longrightarrow \infty$

$$P_F(Y_n' \leq u) \longrightarrow \overline{\emptyset}(u) \quad \text{uniformly for} \quad F \in \mathcal{C}.$$

Proof: Choose $\epsilon > 0$. Now

$$P_F(Y_n' \leq u) = P_F(Y_n' \leq u,\ |Y_n' - W_n'| > \epsilon) + P_F(Y_n' \leq u,\ |Y_n' - W_n'| \leq \epsilon).$$

But

$$P_F(W_n' \leq u-\epsilon) - P_F(|Y_n'-W_n'| > \epsilon) \leq P_F(Y_n' \leq u, |Y_n'-W_n'| \leq \epsilon) \leq P_F(W_n' \leq u+\epsilon),$$

and

$$0 \leq P_F(Y_n' \leq u,\ |Y_n' - W_n'| > \epsilon) \leq P_F(|Y_n' - W_n'| > \epsilon).$$

Thus

$$P_F(W_n' \leq u - \epsilon) - P_F(|Y_n' - W_n'| > \epsilon) \leq P_F(Y_n' \leq u)$$

(2.15)

$$\leq P_F(W_n' \leq u + \epsilon) + P_F(|Y_n' - W_n'| > \epsilon).$$

We have, therefore, for any $\epsilon > 0$

$$\underline{\phi}(u - \epsilon) - \delta_n(F) \leq P_F(Y_n' \leq u) \leq \underline{\phi}(u + \epsilon) + \delta_n'(F)$$

where $\delta_n(F)$, $\delta_n'(F) \longrightarrow 0$ as $n \longrightarrow \infty$ uniformly for $F \in \mathcal{C}$.

The theorem follows by uniform continuity of $\underline{\phi}(u)$.

As a corollary to Theorem 2.0, we have

Theorem 2.1. Let Y_n be given by (2.6), $r \geq 2$. $P_F(Y_n \leq u) \longrightarrow \underline{\phi}(u)$ as $n \longrightarrow \infty$, uniformly in a class $\mathcal{D}$ of cdf's $F(x)$ such that $\zeta_1 \geq a > 0$, $\mu_3 \leq M$, $\zeta_r \leq M < \infty$.

Proof: Let W_n be given by (2.8). By Lemmas 2.1 and 2.2, the conditions of this theorem are sufficient for the uniform convergence hypotheses of Theorem 2.0.

3. Properties of Two-Sample U-Statistic.

Consider the functional

(2.16) $\theta(F,G) = \int \cdots \int \phi(x_1, \ldots, x_r; y_1, \ldots, y_r) dF(x_1) \ldots dG(y_r)$

defined for that class of pairs (F,G) of cdf's such that the integral exists. We suppose that ϕ is symmetric in each of its two sets of r arguments. Let $\underline{z} = (x_1, \ldots, x_n; y_1, \ldots, y_m)$, where n and m are not smaller than r, and

(2.17) $$U_{mn}(\underline{z}) = \binom{n}{r}^{-1} \binom{m}{r}^{-1} \sum \phi(x_{i_1}, \ldots, x_{i_r}; y_{j_1}, \ldots, y_{j_r}),$$

where summation is extended over all sets of integers $1 \le i_1 < \ldots < i_r \le n$ and $1 \le j_1 < \ldots < j_r \le m$.

Given (m + n) independent random variables $X_1, \ldots, X_n$, $Y_1, \ldots, Y_m$, distributed according to F(x) and G(y), respectively, we develop the properties of U_{mn} analogous to those which are known for the single-sample statistic U_n.

$$E_{FG} U_{mn}(\underline{Z}) = \theta(F,G) = \theta.$$

We define

$$\bar{\Psi}(x_1, \ldots, x_r; y_1, \ldots, y_r) = \phi(x_1, \ldots, x_r; y_1, \ldots, y_r) - \theta,$$

(2.18) $$\bar{\Psi}_{cd}(x_1, \ldots, x_c; y_1, \ldots, y_d) =$$

$$= E_{FG} \bar{\Psi}(x_1, \ldots, x_c, X_{c+1}, \ldots, X_r; y_1, \ldots, y_d, Y_{d+1}, \ldots, Y_r)$$

$$(c, d = 0, 1, \ldots, r)$$

$$\bar{\Psi}_{00} = 0,$$

and

(2.19) $$\zeta_{cd} = \zeta_{cd}(F,G) = E_{FG} \bar{\Psi}^2_{cd}(X_1, \ldots, X_i; Y_1, \ldots, Y_d),$$

and obtain

$$(2.20)\quad \sigma^2(U_{mn}) = \mathrm{Var}_{FG}U_{mn}(\underline{Z})$$

$$= \binom{m}{r}^{-1}\binom{n}{r}^{-1} \sum_{c=0}^{r}\sum_{d=0}^{r}\binom{r}{c}\binom{n-r}{r-c}\binom{r}{d}\binom{m-r}{r-d}\zeta_{cd} .$$

The following lemmas are two-sample analogs of certain of the results of Section 5 of [10].

First, it will be convenient to define

$$(2.21)\qquad \eta_{cd} = \zeta_{cd} - \zeta_{c0} - \zeta_{0d}$$

for $c,d = 1, 2, \ldots, r$. The η_{cd} are non-negative, since we have

$$\eta_{cd} = E[\bar{\Psi}_{cd}(X_1, \ldots, X_c; Y_1, \ldots, Y_d) - \bar{\Psi}_{c0}(X_1,\ldots,X_c) - \bar{\Psi}_{0d}(Y_1,\ldots,Y_d)]^2 .$$

We have then, from (2.20), an alternative expression for the variance:

$$\sigma^2(U_{mn}) = \binom{m}{r}^{-1}\binom{n}{r}^{-1} \sum_{c=1}^{r}\sum_{d=1}^{r}\binom{r}{c}\binom{n-r}{r-c}\binom{r}{d}\binom{m-r}{r-d}\eta_{cd}$$

$$+\binom{n}{r}^{-1} \sum_{c=1}^{r}\binom{r}{c}\binom{n-r}{r-c}\zeta_{c0} + \binom{m}{r}^{-1} \sum_{d=1}^{r}\binom{r}{d}\binom{m-r}{r-d}\zeta_{0d} .$$

Lemma 2.3. Let

$$\delta_{cd} = \sum_{i=0}^{c}\sum_{j=0}^{d}(-1)^{c+d-i-j}\binom{c}{i}\binom{d}{j}\zeta_{ij} .$$

Then

$$\zeta_{cd} = \sum_{i=0}^{c}\sum_{j=0}^{d}\binom{c}{i}\binom{d}{j}\delta_{ij},$$

and

$$\delta_{cd} \geq 0, \qquad 0 \leq c,d \leq r.$$

The method of proof is analogous to that given for Lemma 5.1 of [10].

Lemma 2.4. If $1 \le c \le g \le r$, $1 \le d \le h \le r$, then

$$\frac{\zeta_{0d}}{d} \le \frac{\zeta_{0h}}{h}, \quad \frac{\zeta_{c0}}{c} \le \frac{\zeta_{g0}}{g}, \quad \frac{\eta_{cd}}{cd} \le \frac{\eta_{gh}}{gh} .$$

The proof of Lemma 2.4 follows from application of Lemma 2.3.

Now, applying Lemma 2.4 to (2.22), we obtain:

Lemma 2.5.

$$(2.23) \qquad \sigma^2(U_{mn}) \le \frac{r^2}{mn}\eta_{rr} + \frac{r}{n}\zeta_{r0} + \frac{r}{m}\zeta_{0r},$$

$$(2.24) \qquad \sigma^2(U_{mn}) \ge \frac{r^4}{mn}\eta_{11} + \frac{r^2}{n}\zeta_{10} + \frac{r^2}{m}\zeta_{01} .$$

As a further consequence of Lemma 2.4, we have the following two-sample analog of the relation given in (2.13):

Lemma 2.6. If $n \le m$, then

$$(2.25) \quad \sigma^2(U_{mn}) - r^2\left(\frac{\zeta_{10}}{n} + \frac{\zeta_{01}}{m}\right) \le B(\zeta_{rr} - r\zeta_{10} - r\zeta_{01})n^{-2}$$

where B depends only on r; $B = 1$ if $r = 1$ and is given by (2.14) if $r \ge 2$.

If $n > m$, then n is replaced by m in the right-hand side of (2.25).

4. Uniform Convergence: Two-Sample Case.

Consider the functional (2.16) and the U-statistic (2.17) associated with it. Let $\underline{Z} = (X_1,\ldots,X_n; Y_1,\ldots,Y_m)$ be a vector of independent random variables X_i, Y_j distributed according to $F(x)$ and $G(y)$, respectively. Proceeding as in Section 2 of this chapter, consider the random variables $\bar{\Psi}_{10}(X)$ and $\bar{\Psi}_{01}(Y)$.

$$E\,\bar{\Psi}_{10}(X) = E\,\bar{\Psi}_{01}(Y) = 0;$$

$$E\,\Psi_{10}^2(X) = \zeta_{10};$$

$$E\,\Psi_{01}^2(Y) = \zeta_{01}\,.$$

Let

$$\mu_3 = E|\Psi_{10}(X)|^3\,,$$

(2.26)

$$\nu_3 = E|\Psi_{01}(Y)|^3\,.$$

Let

(2.27) $$Y_{mn} = (U_{mn} - \theta)/\sigma(U_{mn}).$$

We will show that the distribution of Y_{mn} converges to $\Phi(u)$ when $m,n \longrightarrow \infty$ $(m/(m+n) \longrightarrow \rho,\ 0 < \rho < 1)$, uniformly in the class of pairs of distributions (F,G) having μ_3, ν_3, ζ_{rr} bounded and at least one of ζ_{10}, ζ_{01} bounded away from zero.

Let

(2.28) $$W_{mn} = \frac{m\sum_{i=1}^{n}\Psi_{10}(X_i) + n\sum_{j=1}^{m}\Psi_{01}(Y_j)}{[mn(m\zeta_{10} + n\zeta_{01})]^{1/2}}\,.$$

$$EW_{mn} = 0, \qquad EW_{mn}^2 = 1.$$

Let $H_{mn}(u) = P_{FG}(W_{mn} \le u)$. Then, applying a result of Bergström [4], we have

Lemma 2.7. If $\zeta_{10} + \zeta_{01} > 0$, $\mu_3 < \infty$, $\nu_3 < \infty$, then

$$\sup_{-\infty < u < \infty} |H_{mn}(u) - \Phi(u)| \le C'\gamma'_{mn}$$

where C' is an absolute constant ($C' < 4.8$) and

$$\gamma'_{mn} = (n\mu_3 + m\nu_3)(n\zeta_{10} + m\zeta_{01})^{-3/2}\,.$$

By a method analogous to that followed in the proof of Lemma 2.2, and applying Lemma 2.6, we obtain

Lemma 2.8. Let $\epsilon > 0$. If $\zeta_{10} + \zeta_{01} > 0$, $\zeta_{rr} < \infty$, and $n \leq m$, then

$$P(|Y_{mn} - W_{mn}| > \epsilon) \leq \frac{1}{n}\frac{1}{\epsilon^2}\frac{B}{r^2}\frac{m(\zeta_{rr} - r\zeta_{10} - r\zeta_{01})}{m\zeta_{10} + n\zeta_{01}} ,$$

where $B = 1$ if $r = 1$ and B is given by (2.14) if $r \geq 2$.

The following theorem is a corollary of an obvious two-sample analog of Theorem 2.0.

Theorem 2.2. Let Y_{mn} be given by (2.27). $P_{FG}(Y_{mn} \leq u) \longrightarrow \overline{\Phi}(u)$ as $m,n \longrightarrow \infty$, uniformly in a class $\mathcal{D}$ of distributions (F,G) such that $\zeta_{10} + \zeta_{01} \geq a > 0$ and $\mu_3 < M$, $\nu_3 < M$, $\zeta_{rr} < M < \infty$, provided $m/(m + n) \longrightarrow p$, $0 < p < 1$.

Proof: Without loss of generality, we may assume $n \leq m$ and $\frac{1}{2} \leq p < 1$. Let W_{mn} be given by (2.28). By Lemmas 2.7 and 2.8, the uniform convergence conditions of Theorem 2.0 are satisfied for $(F,G) \in \mathcal{D}$.

This theorem could be proved using Berry's bound in Lemma 2.7. However, Lemma 2.7 will be used to obtain bounds on the deviation from normality of $P(Y_{mn} \leq u)$, and the Bergström bound is generally better.

5. Bounds for the Deviation from Normality.

Examination of the proofs of Theorems 2.1 and 2.2 and of the lemmas on which they are based enables us to obtain bounds on the difference between the distribution of a U-statistic and the normal approximation to it. In this section, we illustrate the method of obtaining bounds for a two-sample statistic in the case where the kernel ϕ of the functional $\theta(F,G)$ takes the values zero and one only.

Let

$$\epsilon_{mn}(u;\, F,G) = P_{FG}(Y_{mn} \le u) - \overline{\emptyset}(u).$$

Theorem 2.3. If $\emptyset = 0,1$ only and $\zeta_{10} + \zeta_{01} > 0$, then for $n \le m$, $|\epsilon_{mn}(u;\, F,G)|$ is bounded by

$$(2.29)\quad \frac{C'(n\mu_3 + m\,\nu_3)}{(n\zeta_{10} + m\zeta_{01})^{3/2}} + \min_{\epsilon > 0} \left\{ \epsilon(2\pi)^{-1/2}\rho(u,\epsilon) + \frac{1}{\epsilon^2}\frac{B}{r^2}\frac{m(\zeta_{rr} - r\zeta_{10} - r\zeta_{01})}{m\zeta_{10} + n\zeta_{01}}\, n^{-1} \right\},$$

$$-\infty < u < \infty,$$

where

$$(2.30)\quad \rho(u,\epsilon) = \begin{cases} 1 & \text{if} \quad |u| \le \epsilon \\ \exp\left\{-\frac{1}{2}(|u| - \epsilon)^2\right\} & \text{if} \quad |u| > \epsilon\,, \end{cases}$$

and $C' < 4.8$, and

$$(2.31)\quad B = \begin{cases} 1 & \text{if} \quad r = 1 \\ 4 & \text{if} \quad r = 2 \\ 12.6 & \text{if} \quad r = 3 \\ r(r-1)^2 & \text{if} \quad r \ge 4. \end{cases}$$

Proof: The conditions of Lemmas 2.7 and 2.8 are satisfied. From a two-sample analog of (2.15) and Lemma 2.7, we have for any $\epsilon > 0$

$$\overline{\emptyset}(u-\epsilon) - C'\gamma'_{mn} - b_{mn} \le P(Y_{mn} \le u) \le \overline{\emptyset}(u+\epsilon) + C'\gamma'_{mn} + b_{mn},$$

where C', γ'_{mn} are given in Lemma 2.7 and

$$b_{mn} = P(|Y_{mn} - W_{mn}| > \epsilon)$$

has the upper bound given by Lemma 2.8.

Now, for $\epsilon > 0$

$$\sqrt{2\pi}\,|\overline{\phi}(u \pm \epsilon) - \overline{\phi}(u)| \leq \epsilon \sup_{|t-u|\leq\epsilon} \exp\left\{-\tfrac{1}{2}t^2\right\} = \epsilon\rho(u,\epsilon).$$

Thus, from (2.32), for any $\epsilon > 0$

$$(2.33)\quad |\epsilon_{mn}(u; F,G)| \leq \epsilon(2\pi)^{-1/2}\rho(u,\epsilon) + C'\gamma'_{mn} + b_{mn} \quad .$$

The bound (2.29) is obtained by substituting γ'_{mn} from Lemma 2.7 and the bound for b_{mn} from Lemma 2.8, in (2.33).

Remarks.

(1) To use a similar bound in the one-sample case, we would need a numerical bound for the constant C of Lemma 2.1. The value ($C \leq 1.88$) given by Berry [5] has been disputed. See, for example, P. L. Hsu [14], or a footnote by K. L. Chung in [7], p. 201.

(2) The bound (2.29) can be improved; the form given here is relatively simplified. In particular, the bound given in Lemma 2.8 can be improved, since the inequality of Lemma 2.6 can be made more precise. For the case r = 1, see Chapter IV, Section 6.

(3) Lemma 2.7 could be replaced by an alternative lemma based on Berry's theorem. In a particular case, it would be advisable to investigate both bounds to see which is better.

(4) For the case of the Wilcoxon statistic, Stoker [22,23] has made a very extensive investigation of the bound for deviation from normality. His method is basically the same as that employed here. He mentions that the method can be applied to the more general case considered in this section.

Stoker [22] gives a bound similar to (2.29), for the Wilcoxon statistic only, using Berry's theorem. He also gives an improved bound valid for the tails of the distribution.

(5) An upper bound ($C < 3.2$) has been given by Bergström [3] for the constant of Berry's theorem.

CHAPTER III

ASYMPTOTIC EFFICIENCY.

1. Introduction.

The efficiency of a test procedure for a particular decision problem may be defined. In this chapter, an asymptotic expression for an index of efficiency is derived under conditions sufficiently general to include certain classes of tests based on U-statistics. In particular, a method is given for calculating the asymptotic efficiency of a decision rule based on a U-statistic relative to a decision rule based on a related statistic which has binomial distribution.

The concept of efficiency used here is that developed by Pitman and generalized by Hoeffding and the author in [13]; see also the references cited there. The definition given in Section 2 is extended slightly to cover k-sample decision procedures.

The asymptotic results are developed in Sections 3, 4, 5 and 7 and examples are given in Sections 6 and 8.

2. Definition of Efficiency.

Let $n = (n_1,\ldots,n_k)$ where the n_i are integers $(i = 1,\ldots,k)$; and let $\mathcal{N}$ be the set of k-vectors whose components are positive integers.

Let $X_i = (X_{i1},\ldots,X_{in_i})$ be distributed according to $G_{n_i}(x_i)$, $(i = 1,\ldots,k)$. Let $Z_n = (X_1,\ldots,X_k)$ be distributed according to

$$F_n(z_n) = \prod_{i=1}^{k} G_{n_i}(x_i), \qquad n \in \mathcal{N}.$$

Let the decision problem be given by the classes $\{\mathcal{D}_m\}$, $n \in \mathcal{N}$, of distribution functions F_n, and disjoint subclasses $\mathcal{D}_{1n}$,

$\mathcal{D}_{2n}$ of $\mathcal{D}_n$ such that we prefer alternative A_i when the distribution F_n is in $\mathcal{D}_{in}$ $(i = 1, 2)$. Generally, $\mathcal{D}_{1n}$ and $\mathcal{D}_{2n}$ will not exhaust $\mathcal{D}_n$, but are chosen so that we are indifferent between the two alternatives if F_n is in neither $\mathcal{D}_{1n}$ nor $\mathcal{D}_{2n}$.

Let ϕ_n be a decision rule based on a sample z_n from k populations. Let ϕ_n be defined for n in some subset of $\mathcal{N}$; e.g., all vectors n whose components are not less than some integer r; or all vectors n whose components are equal (equal sample sizes).

Let $\mathcal{J}$ be a family of decision rules ϕ_n, and let $\mathcal{N}(\mathcal{J})$ be the subset of $\mathcal{N}$ for which the family $\mathcal{J}$ is defined.

Let $P(A_i, \phi_n | F_n)$ represent the probability that a decision rule ϕ_n from $\mathcal{J}$, based on z_n, will select alternative A_i $(i = 1, 2)$ when z_n is distributed according to F_n. Given $\alpha_1 > 0$, $\alpha_2 > 0$, we say (cf. [13]) that the problem $(\{\mathcal{D}_{1n}\}, \{\mathcal{D}_{2n}\}, \alpha_1, \alpha_2)$ is solved by $\mathcal{J}$ if we can find a test in the family $\mathcal{J}$ such that

$$P(A_i, \phi_n | F_n) \geq 1 - \alpha_i \quad \text{when} \quad F_n \in \mathcal{D}_{in} \quad (i = 1, 2). \tag{3.1}$$

There are several possible ways of defining an index $N(\mathcal{J})$ of efficiency for the family $\mathcal{J}$ relative to this problem. We require an index representing in an appropriate sense the "least" among the set of vectors n for which the inequalities (3.1) are satisfied by some test ϕ_n in $\mathcal{J}$.

Let $\mathcal{M}(\mathcal{J}) \subset \mathcal{N}(\mathcal{J})$ be that set of vectors n for which (3.1) holds.

Let $c = (c_1, \ldots, c_k)$ where $c_i \geq 0$, $\Sigma c_i = 1$. If $\mathcal{M}(\mathcal{J})$ is not empty, let

$$N_c(\mathcal{J}) = \min_{n \in \mathcal{M}(\mathcal{J})} \sum_{i=1}^{k} c_i n_i. \tag{3.2}$$

If we suppose that cost of sampling is proportional to sample size and that c_i/c_j is the ratio of unit sampling costs for the i-th population relative to the j-th population, then $N_c(\mathcal{J})$ may be interpreted as an index of efficiency. Observe that $N_c(\mathcal{J})$ is not in general an integer.

To determine $N_c(\mathcal{J})$, we proceed as in Section 3 of [13] :

Let $\mathcal{J}_n^*$ be the sub-family of $\mathcal{J}$ containing tests ϕ_n based on z_n and having

$$P(A_1, \phi_n | F_n) \geq 1 - \alpha_1 \quad \text{for all } F_n \in \mathcal{D}_{1n}.$$

Let

$$M(\phi_n) = \sup_{F_n \in \mathcal{D}_{2n}} P(A_1, \phi_n | F_n)$$

$$M_n = \inf_{\phi_n \in \mathcal{J}_n^*} M(\phi_n).$$

Then $\mathcal{M}(\mathcal{J})$ is that set of vectors n for which $M_n \leq \alpha_2$ and either $M_n < \alpha_2$ or $M(\phi_n) = \alpha_2$ for some $\phi_n \in \mathcal{J}_n^*$. $N_c(\mathcal{J})$ is then determined from (3.2).

We might further define

$$N(\mathcal{J}) = \max_{c \in \mathcal{C}} N_c(\mathcal{J}) \tag{3.3}$$

where $\mathcal{C}$ stands for the set of all vectors c (with non-negative components adding to one).

The following are a few immediate properties of these indexes.

(I) If $\mathcal{M}(\mathcal{J})$ is determined by an integer n_0 and the inequalities $n_i \geq n_0$ (i = 1,...,k), then for every $c \in \mathcal{C}$

$$N_c(\mathcal{J}) = n_0 = N(\mathcal{J}).$$

(II) $$N(\mathcal{J}) \leq \min_{n \in \mathcal{M}(\mathcal{J})} \max(n_1,\ldots,n_k).$$

Let us suppose in the rest of this section that $\mathcal{M}(\mathcal{J})$ contains, with a point n, every n' such that $n' \geq n$, that is, $n_i' \geq n_i$ $(i = 1,\ldots,k)$. Then, if we let $m_n = \max(n_1,\ldots,n_k)$ we have $(m_n,\ldots,m_n) \in \mathcal{M}(\mathcal{J})$ whenever $n \in \mathcal{M}(\mathcal{J})$.

Let s be the least integer such that the point $(s,\ldots,s) \in \mathcal{M}(\mathcal{J})$. Then it is clear that

$$s = \min_{n \in \mathcal{M}(\mathcal{J})} m_n$$

and (II) may be stated:

$$N(\mathcal{J}) \leq s.$$

Let $\overline{\mathcal{M}}(\mathcal{J})$ denote the complement of $\mathcal{M}(\mathcal{J})$ in $\mathcal{N}(\mathcal{J})$. We will say (for brevity) that $\mathcal{M}(\mathcal{J})$ is convex when the convex hull of $\mathcal{M}(\mathcal{J})$ in E^k contains no point of $\overline{\mathcal{M}}(\mathcal{J})$.

(III) If $\mathcal{M}(\mathcal{J})$ is convex, $N(\mathcal{J}) > s - 1$.

We say $\mathcal{M}(\mathcal{J})$ is symmetrical if $n \in \mathcal{M}(\mathcal{J})$ implies $n' \in \mathcal{M}(\mathcal{J})$ for any n' obtained by a permutation of the coordinates of n.

(IV) If $\mathcal{M}(\mathcal{J})$ is symmetrical,

$$N(\mathcal{J}) = N_{c_0}(\mathcal{J})$$

where

$$c_0 = (1/k,\ldots,1/k).$$

These properties of the index $N(\mathcal{J})$ suggest that under fairly reasonable conditions, a "most economical" procedure is approximated

through taking samples of equal size from each of the k populations, when we have no information about the cost of sampling.

We may make some remarks as to the conditions under which Properties (I) - (IV) hold.

(1) The condition of Property (I) is satisfied for families of tests based on the W-statistics introduced in Chapter I, Section 2.

(2) It would not be reasonable to suppose that there might be for any of the k samples a greatest sample size such that (3.1) holds. We assume conversely that once a set of sample sizes has been found such that (3.1) holds, all vectors corresponding to sets of larger sample sizes will also be in $\mathcal{M}(\mathcal{J})$. Any family $\mathcal{J}$ containing only sequences $\{\phi_n\}$ of tests whose power functions are decreasing functions of the sample sizes will satisfy this condition.

(3) $\mathcal{M}(\mathcal{J})$ is symmetrical, and the condition for Property (IV) is satisfied, if the tests ϕ_n in $\mathcal{J}$ and the classes $\mathcal{D}_n$ of distributions have appropriate symmetry properties. For example, a family $\hat{\mathcal{U}}$ of tests based on the Wilcoxon two-sample statistic is defined in Chapter IV, Section 6, and has $\mathcal{M}(\hat{\mathcal{U}})$ symmetrical under conditions stated in Lemma 4.5 of that section.

(4) Property (II) always holds; and under the condition discussed in (2), we have Property (II) in the form $N(\mathcal{J}) \leq s$, where s is the smallest integer for which a vector representing equal sample sizes s is contained in $\mathcal{M}(\mathcal{J})$. If $\mathcal{M}(\mathcal{J})$ is also convex, so that Property (III) holds, then taking samples of equal size s is very close to a "most economical" procedure when we can make no assumptions about the cost of making observations.

(5) An example of a family $\mathcal{J}$ having $\mathcal{M}(\mathcal{J})$ convex is the following. Consider the usual test for testing the difference between the means of two normally distributed random variables, X_1 and X_2, having variance one and means μ and $\mu + \delta$ $(-\infty < \mu < \infty, \delta \geq 0)$, respectively. Let $\mathcal{J}$ be the family of tests $\{\phi_{n_1,n_2}\}$ defined as follows. Let

$$t_{n_1n_2} = \left(\frac{n_1n_2}{n_1+n_2}\right)^{1/2} (\bar{x}_2 - \bar{x}_1)$$

where n_1, n_2 are the two sample sizes and $\bar{x}_i$ is the mean of n_i observations on the random variable X_i $(i = 1, 2)$. For testing H: $\delta \leq 0$ against H': $\delta \geq \delta_1$, we use the test of level α

$$\phi_{n_1,n_2} = \begin{cases} 1 & \text{if} \quad t_{n_1n_2} \geq \lambda_\alpha \\ 0 & \text{if} \quad t_{n_1n_2} < \lambda_\alpha \end{cases},$$

where $\overline{\Phi}(\lambda_\alpha) = 1 - \alpha$, $\overline{\Phi}$ is the standard normal distribution function. The power function of this test is

$$\beta(\delta, n_1, n_2) = \overline{\Phi}\left[\delta\left(\frac{n_1n_2}{n_1+n_2}\right)^{1/2} - \lambda_\alpha\right].$$

Now $\mathcal{M}(\mathcal{J})$ is the set of pairs of integers (n_1,n_2) such that

$$\beta(\delta_1, n_1, n_2) \geq 1 - \beta.$$

Equivalently, $\mathcal{M}(\mathcal{J})$ is given by

$$\frac{n_1n_2}{n_1+n_2} \geq \left(\frac{\lambda_\alpha+\lambda_\beta}{\delta_1}\right)^2, \quad n_1 > 0, \; n_2 > 0 .$$

If n_1,n_2 are allowed to vary continuously, we obtain a set M whose

boundary is one branch of a hyperbola. The convex hull of $\mathcal{M}(\mathcal{J})$ must be contained in M, but every pair of integers (n_1,n_2) in M is in $\mathcal{M}(\mathcal{J})$.

If a second family $\mathcal{J}'$ of tests is given, then we define similarly the indexes of efficiency $N_c(\mathcal{J}')$ and $N(\mathcal{J}')$ for the family $\mathcal{J}'$. The relative efficiency of $\mathcal{J}'$ with respect to $\mathcal{J}$ is given by

$$\text{eff}^{(c)}(\mathcal{J}'/\mathcal{J}) = N_c(\mathcal{J})/N_c(\mathcal{J}'),$$

for a fixed vector c.

If we desire a measure of relative efficiency independent of the cost of sampling, we may use

$$\text{eff}^{(0)}(\mathcal{J}'/\mathcal{J}) = N(\mathcal{J})/N(\mathcal{J}').$$

Suppose a family of tests $\mathcal{J}$ is partitioned according to a parameter p (which may be a vector) having values in some set P. We may compute $N_c(\mathcal{J}_p)$ for each sub-family $\mathcal{J}_p$ of $\mathcal{J}$ and then we have

$$N_c(\mathcal{J}) = \inf_{p\in P} N_c(\mathcal{J}_p).$$

3. Asymptotic Expression for $N_c(\mathcal{J})$.

Let $\Theta(F_n)$ be a real-valued function defined for all $F_n \in \mathcal{D}_n$ and for every $n \in \mathcal{N}$. Assume that the function $\Theta(F_n)$ takes on values in an interval ω (finite or infinite). Let Θ_1 and Θ_2 be two numbers in ω, $\Theta_1 < \Theta_2$, and let

$$\mathcal{D}_{1n} = \{F_n : \Theta(F_n) \leq \Theta_1\}$$

$$\mathcal{D}_{2n} = \{F_n : \Theta(F_n) \geq \Theta_2\} .$$

Θ_1 and Θ_2 are assumed to be independent of n.

Let $\mathcal{J}$ be that family of tests based on the sequence of statistics $\{t_n\}$, with the probability of selecting A_1 when z_n is observed given by

$$(3.4) \qquad \phi_{1n}(z_n) = \begin{cases} 1 & \text{if} \quad t_n \leq \lambda \\ 0 & \text{otherwise,} \end{cases}$$

$-\infty < \lambda < \infty$; $n \in \mathcal{N}(\mathcal{J})$, and ϕ_{2n}, the probability of selecting A_2, given by $1 - \phi_{1n}$.

We will consider the asymptotic behavior of $N_c(\mathcal{J})$ as $\delta = \theta_2 - \theta_1 \longrightarrow 0$, with θ_1, α_1, α_2 fixed.

Assumption A.1: The functions $t_n(z_n)$ are independent of δ.

Assumption A.2:

$$\lim_{x \longrightarrow \infty} \inf_{F_n \in \mathcal{D}_{1n}} P(t_n \leq x | F_n) > 1 - \alpha_1.$$

This ensures that λ_n is finite, when λ_n is defined as the smallest number which satisfies

$$\inf_{F_n \in \mathcal{D}_{1n}} P(t_n \leq \lambda_n | F_n) \geq 1 - \alpha_1.$$

We have, then, $\mathcal{M}(\mathcal{J})$ given by the set of vectors n for which

$$\sup_{F_n \in \mathcal{D}_{2n}} P(t_n \leq \lambda_n | F_n) \leq \alpha_2.$$

$N_c(\mathcal{J})$ is then determined from (3.2).

Assumption A.3: $\alpha_1 + \alpha_2 < 1$.

Let $\mathcal{D}_n(\theta') = \{F_n : \theta(F_n) = \theta'\}$ for $\theta' \in \omega$.

Assumption A.4: For every $n \in \mathcal{N}(\mathcal{J})$ and every $\theta \in \omega$, there exists a distribution function $H_n^*(x,\theta)$ and a subset $\mathcal{D}_n^*(\theta)$ of $\mathcal{D}_n(\theta)$, such that

$$P(t_n \leq x \mid F_n) = H_n^*(x,\theta) \quad \text{when} \quad F_n \in \mathcal{D}_n^*(\theta),$$

where for every x $H_n^*(x,\theta)$ is continuous on the right in θ at $\theta = \theta_1$.

Assumptions A.1 - A.4 are essentially the same as assumptions A through D of [13] and are needed first to prove that $N_c(\mathcal{J}) \longrightarrow \infty$ as $\delta \longrightarrow 0$.

Lemma 3.1: If Assumptions A.1 - A.4 are satisfied and for every $n \in \mathcal{N}(\mathcal{J})$ there exists $n' \in \mathcal{N}(\mathcal{J})$ such that $n' > n$, then for any $c \in \mathcal{C}$, $N_c(\mathcal{J}) \longrightarrow \infty$ as $\delta \longrightarrow 0$.

Proof: The assumptions give us

$$\sup_{F_n \in \mathcal{D}_{2n}} P(t_n \leq \lambda_n \mid F_n) \geq H_n^*(\lambda_n, \theta_2),$$

$$1 - \alpha_1 \leq \inf_{F_n \in \mathcal{D}_{1n}} P(t_n \leq \lambda_n \mid F_n) \leq H_n^*(\lambda_n, \theta_1) .$$

Now, for any $n' \in \mathcal{N}(\mathcal{J})$ we can choose δ_1 so small that if $\delta < \delta_1$,

$$H_n^*(\lambda_n,\theta_1) - H_n^*(\lambda_n,\theta_1 + \delta) < 1 - \alpha_1 - \alpha_2$$

for every $n \in \mathcal{N}(\mathcal{J})$ with $n \leq n'$. Hence, if $\delta < \delta_1$ and $n \leq n'$,

$$\sup_{F_n \in \mathcal{D}_{2n}} P(t_n \leq \lambda_n \mid F_n > \alpha_2;$$

thus $n' \notin \mathcal{M}(\mathcal{J})$ and $N_c(\mathcal{J}) > \min(n_1', \ldots, n_k')$. This completes the proof.

Assumption A.5: There exist positive integers $s_1, \ldots, s_k$ such that $n \in \mathcal{N}(\mathcal{J})$ if and only if $n_i = s_i m$; $i = 1, \ldots, k$, $m = 1, 2, \ldots$.

(The following is unchanged if we require $m \geq r > 1$.)

Thus, we may replace the subscript n by the subscript m.

Now, for $a > 0$, $d \geq 0$, let $\ell_{m,c} = \sum_{i=1}^{k} c_i n_i = m \sum_{i=1}^{k} c_i s_i$, and

$$\mathcal{D}_{m,c}(a,d) = \left\{ F_m : \theta(F_m) \geq \theta_1 + \ell_{m,c}^{-a}\, d \right\} .$$

Assumption A.6: There exist numbers $a > 0$, $d_0 \geq 0$, and a function $H_c(d)$ defined for $d \geq 0$, $c \in \mathcal{C}$, such that if $d \geq d_0$,

$$\lim_{m \longrightarrow \infty} \sup_{F_m \in \mathcal{D}_{m,c}(a,d)} P(t_m \leq \lambda_m \mid F_m) = H_c(d).$$

Assumption A.7: $H_c(d)$ is a continuous, everywhere decreasing function and

(i) $\lim_{d \longrightarrow \infty} H_c(d) = 0$,

(ii) $H_c(d_0) > \alpha_2$.

By Assumption A.7, the equation in D, $H_c(D) = \alpha_2$, has a unique root $D_c = D_c(\alpha_1,\alpha_2) > d_0$.

Theorem 3.1: Let $\mathcal{J}$ be given by (3.4). If Assumptions A.1 - A.7 are satisfied, then for θ_1 and $c \in \mathcal{C}$ fixed and $\delta \longrightarrow 0$, we have asymptotically

$$N_c(\mathcal{J}) \sim \left[\frac{D_c(\alpha_1,\alpha_2)}{\delta} \right]^{1/a} .$$

Proof: Since $D_c > d_0 \geq 0$, we may choose $\epsilon > 0$ so that

(3.5) $$\epsilon \leq D_c - d_0.$$

Let

(3.6) $$\gamma = \min \left\{ H_c(D_c) - H_c(D_c + \epsilon),\ H_c(D_c - \epsilon) - H_c(D_c) \right\} .$$

By Assumption A.7, $\gamma > 0$. Let

$$\eta_m(d) = \sup_{F_n \in \mathcal{D}_{m,c}(a,d)} P(t_m \leq \lambda_m \mid F_m) - H_c(d).$$

By (3.5) and Assumption A.6, we may choose m_1 so large that

$$(3.7) \qquad |\eta_m(D_c \pm \epsilon)| < \gamma \qquad \text{if } m > m_1.$$

We may assume that

$$(3.8) \qquad \left(\frac{m}{m-1}\right)^a < 1 + \epsilon \qquad \text{if } m > m_1.$$

Since $N_c(\mathcal{J}) \longrightarrow \infty$ as $\delta \longrightarrow 0$, we may choose δ_1 so that

$$(3.9) \qquad N_c(\mathcal{J}) > m_1 + 1 \qquad \text{if } \delta < \delta_1.$$

By Assumption A.5, there exists an integer m_0 such that $n \in \mathcal{M}(\mathcal{J})$ only if $m \geq m_0$. Hence $n \in \mathcal{M}(\mathcal{J})$ implies $N_c(\mathcal{J}) = \ell_{m_0,c} = \sum_{i=1}^{k} c_i s_i^{m_0}$.

The theorem will be proved if we can show that for $\delta < \delta_1$

$$D_c - \epsilon \leq \delta N_c^a \leq (D_c + \epsilon)(1 + \epsilon).$$

By (3.8), (3.9) this will hold if $\delta < \delta_1$ and

$$(3.10) \qquad (D_c - \epsilon)N_c^{-a} \leq \delta \leq (D_c + \epsilon)(N_c - 1)^{-a}.$$

We establish (3.10) by contradiction. First, we have, by Assumption A.5

$$(3.11) \qquad \sup_{F_m \in \mathcal{D}_{2m}} P(t_m \leq \lambda_m \mid F_m) \begin{cases} \leq \alpha_2 & \text{if } m = m_0 \\ > \alpha_2 & \text{if } m < m_0 \end{cases}.$$

Now, suppose $\delta < \delta_1$ and

$$(3.12) \qquad \delta < (D_c - \epsilon)N_c^{-a} \quad .$$

This implies

$$\mathcal{D}_{m_0,c}(a,D_c-\epsilon) \subset \mathcal{D}_{2m_0} .$$

Thus (3.12) implies, if $m = m_0$,

$$\sup_{F_m \in \mathcal{D}_{2m}} P(t_m \le \lambda_m \mid F_m)$$

$$\ge \sup_{F_m \in \mathcal{D}_{m,c}(a,D_c-\epsilon)} P(t_m \le \lambda_m \mid F_m)$$

$$> H_c(D_c - \epsilon) - \gamma \qquad \text{by (3.9), (3.7)}$$

$$\ge H_c(D) \qquad \text{by (3.6).}$$

Hence (3.12) implies

$$\sup_{F_m \in \mathcal{D}_{2m}} P(t_m \le \lambda_m \mid F_m) > \alpha_2, \qquad \text{if } m = m_0 , \tag{3.13}$$

which contradicts (3.11). The second inequality of (3.10) is established in similar fashion.

Remarks.

(1) Theorem 3.1 holds also for the family of tests based on $\{t_n\}$ with

$$\phi_{1n} = \begin{cases} 1 & \text{if } t_n < \lambda \\ \text{arbitrary} & \text{if } t_n = \lambda \\ 0 & \text{if } t_n > \lambda . \end{cases}$$

(2) The class $\mathcal{D}_{m,c}(a,d)$, defined on the basis of the function $\ell_{m,c}^{-a}$, might be replaced by

$$\mathcal{D}_{m,c}(K,d) = \left\{ F_m : \theta(F_m) \ge \theta_1 + dK(\ell_{m,c}) \right\} ,$$

where $K(\ell_{m,c})$ is a strictly decreasing function tending to zero as $m \longrightarrow \infty$. We would have in this case

$$N_c(\mathcal{J}) \sim K^{-1}(\delta/D_c),$$

where K^{-1} is the inverse function.

(3) Assumption A.4 is stated in a form convenient for verification in the applications with which we are concerned at present. A.4 may be replaced (see [13], p.56) by any assumption sufficient to prove Lemma 3.1.

(4) Theorem 3.1 is proved essentially by a method used by Hoeffding in [12] to prove a less general theorem. Theorem 3.1 is slightly more general than Theorem 4.1 of [13], but Assumption A.6 does not immediately lend itself to verification.

(5) In the foregoing, we have suppressed a subscript $s = (s_1,\ldots,s_k)$ which would indicate the partition $\{\mathcal{J}_s\}$ of a family $\mathcal{J}$ given by (3.4) with $\mathcal{N}(\mathcal{J})$ equal to the set of vectors with positive integer components. Thus Theorem 3.1 gives us $N_c(\mathcal{J}_s)$ for such a partition, and $N_c(\mathcal{J})$ is found from

$$N_c(\mathcal{J}) = \min_{s \in S} N_c(\mathcal{J}_s),$$

where S is the set of all vectors s containing at least one pair of relatively prime integers. In general, of course, $D_c(\alpha_1,\alpha_2)$ will depend on s.

4. A Class of Statistics for which Theorem 3.1 Applies.

Let $\underline{\xi} = \underline{\xi}(F_n) = (\xi_1,\ldots,\xi_h)$ be a vector of real-valued functions

$$\xi_j = \xi_j(F_n), \qquad j = 1,\ldots,h,$$

with $\underline{\xi}(F_n) \in \Omega$ when $F_n \in \mathcal{D}_n$, $n \in \mathcal{N}$.

We will continue to assume A.5, so that $\mathcal{N}(\mathcal{J})$ may be indexed equivalently by the vector n and the integer m.

Assumption A.8: There exist $\underline{\xi}(F_n)$ and a family of continuous cdf's $H_n(x,\Theta,\underline{\xi})$ defined for every $\Theta \in \omega$, $\underline{\xi} \in \Omega$, $n \in \mathcal{N}(\mathcal{J})$, such that for every real x

$$P(t_m \leq x \mid F_m) = H_m(x,\Theta,\underline{\xi}) + \epsilon_m(x,F_m)$$

when $\Theta(F_m) = \Theta$ and $\underline{\xi}(F_m) = \underline{\xi}$; and $\epsilon_m(x,F_m) \longrightarrow 0$ as $m \longrightarrow \infty$ uniformly for $F_m \in \mathcal{D}_m$.

Assumption A.9: There exist $a > 0$, $d_0 \geq 0$, and a function $H_c(d)$ defined for $d \geq 0$, $c \in \mathcal{C}$, such that when $d \geq d_0$,

$$\lim_{m \longrightarrow \infty} \sup_{F_m \in \mathcal{D}_{m,c}(a,d)} H_m(\lambda_m,\Theta,\underline{\xi}) = H_c(d).$$

Evidently, Assumptions A.8 and A.9 are sufficient for Assumption A.6.

We will be particularly concerned with cases where t_n has normal limiting distribution. In the following, the function $H_c(d)$ and the number $D_c = D_c(\alpha_1,\alpha_2)$ are evaluated for a class of statistics including some U-statistics.

Assumption A'.8: There exist $\underline{\xi}(F_n)$, functions $m(\Theta)$ and $s_n(\Theta,\underline{\xi}) > 0$ defined for $\Theta \in \omega$ and $\underline{\xi} \in \Omega$, $n \in \mathcal{N}(\mathcal{J})$, and everywhere increasing functions $g_n(t_n)$ such that for every real x

$$P\left[\frac{g_m(t_m) - m(\Theta)}{s_m(\Theta,\underline{\xi})} \leq x \mid F_m\right] = \Phi(x) + \epsilon_m(x,F_m)$$

when $\Theta(F_m) = \Theta$ and $\underline{\xi}(F_m) = \underline{\xi}$; and $\epsilon_m(x,F_m) \longrightarrow 0$ as $m \longrightarrow \infty$, uniformly for $F_m \in \mathcal{D}_m$.

If Assumption A'.8 is satisfied, then clearly Assumption A.8 holds with

$$H_m(x,\theta,\underline{\xi}) = \underline{\Phi}\left[\frac{\bar{g}_m(x) - m(\theta)}{s_m(\theta,\underline{\xi})}\right] .$$

Lemma 3.2: If Assumption A'.8 is satisfied and $\alpha_1 < \frac{1}{2}$, then there exists m_1 such that

$$g_m(\lambda_m) > m(\theta_1) \qquad \text{if } m > m_1 .$$

Proof: From A'.8 and the definition of λ_n, we have

$$1 - \alpha_1 \leq \inf_{F_m \in \mathcal{D}_{1m}} P(t_m \leq \lambda_m \mid F_m) \leq \underline{\Phi}\left[\frac{g_m(\lambda_m) - m(\theta_1)}{s_m(\theta_1,\hat{\underline{\xi}})}\right]$$

$$+ \epsilon_m(\lambda_m, \hat{F}_m)$$

where $\hat{F}_m \in \mathcal{D}_m(\theta_1)$ has $\underline{\xi}(\hat{F}_m) = \hat{\underline{\xi}}$. Clearly, if $\alpha_1 < 1/2$, we may find m_1 so large that

$$1 - \alpha_1 - |\epsilon_m(\lambda_m, \hat{F}_m)| > 1/2 \qquad \text{if } m > m_1,$$

and hence the argument of $\underline{\Phi}$ must be positive for $m > m_1$.

Now, let $\Omega(\theta)$ be the subset of Ω such that $\underline{\xi}(F_n) \in \Omega(\theta)$ when $\theta(F_n) = \theta$.

Assumption A'.9: There exist $a > 0$, $s_c(\theta) > 0$ such that for every $\theta \in \omega$, $c \in \mathcal{C}$,

$$\sup_{\underline{\xi} \in \Omega(\theta)} \ell^a_{m,c}\, s_m(\theta,\underline{\xi}) = s_c(\theta).$$

Assumption A".9: The function $m(\theta)$ is non-decreasing for $\theta \in \omega$, and has a continuous and positive derivative at $\theta = \theta_1$. The

function $s_c(\theta)$ is bounded for $\theta \in \omega$ and is continuous at $\theta = \theta_1$. Furthermore, there exists $\eta > 0$ such that $[s_c(\theta)]^{-1}[u - m(\theta)]$ is a non-increasing function of θ for $\theta \in \omega$, if $m(\theta_1) < u < m(\theta_1) + \eta$.

We remark that examination of the proofs below shows that the continuity requirements of Assumption A".9 could be stated in weaker forms.

We remark also that Assumption A.2 follows from Assumptions A'.8, A'.9, A".9.

Lemma 3.3: If Assumptions A'.8, A'.9, A".9 are satisfied and $\alpha_1 < 1/2$, then given $\epsilon > 0$ there exists m_0 such that

$$g_m(\lambda_m) < m(\theta_1) + \epsilon \qquad \text{if } m > m_0.$$

Proof: By the uniform convergence assumption of A'.8,

$$1-\alpha_1 = \inf_{F_m \in \mathcal{D}_{1m}} \mathbb{P}\left[\frac{g_m(\lambda_m) - m(\theta)}{s_m(\theta, \xi)}\right] + \tilde{\epsilon}_m \tag{3.14}$$

and we may suppose that

$$|\tilde{\epsilon}_m| < \tfrac{1}{2}\alpha_1 \qquad \text{if } m > m_1. \tag{3.15}$$

By A'.9, A".9. and Lemma 3.2, (3.14) becomes

$$1 - \alpha_1 = \inf_{\theta \le \theta_1} \mathbb{P}\left[\ell^a_{m,c}\ \frac{g_m(\lambda_m) - m(\theta)}{s_c(\theta)}\right] + \tilde{\epsilon}_m \tag{3.16}$$

if $m_{\epsilon} > m_1$. By A".9, there exists a number $\sigma < \infty$ such that $s_c(\theta) \le \sigma$, $\theta \in \omega$, so that we may write, when $m > m_1$,

$$(3.17) \qquad 1 - \alpha_1 \geq \overline{\Phi}\left[\ell^a_{m,c} \frac{g_m(\lambda_m) - m(\theta_1)}{\sigma} \right] + \tilde{\epsilon}_m .$$

Now, by (3.1), (3.15), Lemma 3.2, and the Tchebycheff-Cantelli inequality, we obtain

$$g_m(\lambda_m) - m(\theta_1) \leq \frac{\sigma}{\ell^a_{m,c}} \left[\frac{1 - 1/2\ \alpha_1}{1/2\ \alpha_1} \right]^{1/2} ,$$

if $m > m_1$. The lemma follows immediately.

Let $\lambda(u)$ be defined by $1 - u = \overline{\Phi}(\lambda(u))$.

Lemma 3.4: If Assumptions A'.8, A'.9, A".9 are satisfied, and $\alpha_1 < 1/2$, then

$$\lim_{m \longrightarrow \infty} \ell^a_{m,c} \left[g_m(\lambda_m) - m(\theta_1) \right] = s_c(\theta_1)\lambda(\alpha_1) .$$

Proof: By Lemmas 3.2 and 3.3 and Assumption A".9,

$$\frac{g_m(\lambda_m) - m(\theta)}{s_c(\theta)}$$

is a non-increasing function of θ if m is sufficiently large. Thus, from (3.16), we obtain

$$1 - \alpha_1 = \overline{\Phi}\left[\ell^a_{m,c} \frac{g_m(\lambda_m) - m(\theta_1)}{s_c(\theta_1)} \right] + \tilde{\epsilon}_m$$

if m is sufficiently large. The lemma follows by continuity of $\lambda(u)$.

Lemma 3.5: If Assumptions A'.8, A'.9, A".9 are satisfied, and $\alpha_1 < 1/2$, then Assumptions A.8, A.9 are satisfied with d_0 any number greater than $s_c(\theta_1)\lambda(\alpha_1)/m'(\theta)$, and

$$H_c(d) = \overline{\Phi}\left[\lambda(\alpha_1) - \frac{m'(\theta_1)}{s_c(\theta_1)}\ d \right] .$$

Proof: We have noted above that A'.8 implies A.8, with

$$H_m(x,\theta,\underline{\xi}) = \overline{\emptyset}\left[\frac{g_m(x) - m(\theta)}{s_m(\theta,\underline{\xi})}\right] .$$

Now, if

$$g_m(\lambda_m) \leq m(\theta_1 + \ell_{m,c}^{-a} d), \qquad d > 0, \tag{3.18}$$

then by A'.9, A".9,

$$\sup_{F_m \in \mathcal{D}_m(\theta)} H_m(\lambda_m,\theta,\underline{\xi}) = \overline{\emptyset}\left[\ell_{m,c}^{a} \frac{g_m(\lambda_m) - m(\theta)}{s_c(\theta)}\right] \tag{3.19}$$

if $\theta \geq \theta_1 + \ell_{m,c}^{-a} d$. Furthermore, by A".9, $m(\theta_1+\ell_{m,c}^{-a} d) < m(\theta_1)+ \eta$ for m sufficiently large, so that by Lemma 3.2, the argument of $\overline{\emptyset}$ in (3.19) is a non-increasing function of θ when (3.18) holds and m is sufficiently large.

Now, it follows from A".9 and Lemma 3.4 that

$$\ell_{m,c}^{a}\left[m(\theta_1 + \ell_{m,c}^{-a} d) - g_m(\lambda_m)\right]$$

tends to the limit $dm'(\theta_1) - s_c(\theta_1)\lambda(\alpha_1)$ as $m \longrightarrow \infty$. Hence (3.18) will hold for m sufficiently large if we require $d \geq d_0$ and choose $d_0 > s(\theta_1)\lambda(\alpha_1)/m'(\theta_1)$.

We can thus assert that for m sufficiently large, and $d \geq d_0$,

$$\sup_{F_m \in \mathcal{D}_{m,c}(a,d)} H_m(\lambda_m,\theta,\underline{\xi}) = \overline{\emptyset}\left[\ell_{m,c}^{a} \frac{g_m(\lambda_m)-m(\theta_1+\ell_{m,c}^{-a} d)}{s_c(\theta_1+\ell_{m,c}^{-a} d)}\right] .$$

This, with A".9, completes the proof of Lemma 3.5.

We obtain (assuming A'.8, A'.9, A".9)

$$D_c(\alpha_1,\alpha_2) = \frac{s_c(\theta_1)}{m'(\theta_1)}\left[\lambda(\alpha_1) + \lambda(\alpha_2)\right] .$$

Furthermore, Assumption A.7 is satisfied provided

$$(3.20) \qquad \overline{\mathcal{Q}}\left[\lambda(\alpha_1) - \frac{m'(\theta_1)}{s_c(\theta_1)}\, d_0\right] > \alpha_2 .$$

Now, both (3.20) and the condition for Lemma 3.5 will hold if we set d_0 so that

$$\frac{s_c(\theta_1)\lambda(\alpha_1)}{m'(\theta_1)} < d_0 < \frac{s_c(\theta_1)}{m'(\theta_1)}\left[\lambda(\alpha_1) + \lambda(\alpha_2)\right]$$

which we may do if and only if $\alpha_2 < 1/2$.

Assumption A'.3: $\alpha_1 < 1/2$, $\alpha_2 < 1/2$.

Lemma 3.5 and the discussion above complete the proof of

Theorem 3.2: Let $\mathcal{J}$ be given by (3.4). If Assumptions A.1, A'.3, A.4, A.5, A'.8, A'.9, A".9 are satisfied, then for θ_1 and $c \in \mathcal{C}$ fixed and $\delta \longrightarrow 0$, we have

$$N_c(\mathcal{J}) \sim \left[\frac{s_c(\theta_1)}{m'(\theta_1)}\right]^{1/a}\left[\frac{\lambda(\alpha_1) + \lambda(\alpha_2)}{\delta}\right]^{1/a} .$$

We observe that inspection of the proofs shows that Assumption A'.9 may be replaced by

Assumption A*.9: There exists $a > 0$ and $s_c(\theta) > 0$ such that

$$\lim_{m \longrightarrow \infty} \sup_{\underline{\xi} \in \Omega(\theta)} \ell_{m,c}^{a}\, s_m(\theta, \underline{\xi}) = s_c(\theta)$$

uniformly for $\theta \in \omega$.

5. Asymptotic Comparison of Two Test Families.

In this section, we specialize the foregoing to certain classes of tests based on U-statistics, in the one- and two-sample cases.

Let $\phi(x_1, \ldots, x_r)$ be a (measurable) function of r variables. Let $\mathcal{D}$ be a class of distributions F(x) for which there is defined

the functional

$$\theta(F) = \int \cdots \int \phi(x_1, \ldots, x_r) dF(x_1) \cdots dF(x_r),$$

and for which the conditions of Theorem 2.1 are satisfied. Let

$$\mathcal{D}_n = \left\{ F_n : F_n(z_n) = \prod_{i=1}^{n} F(x_i), F \in \mathcal{D} \right\} ,$$

for $n \geq r$. Let $\theta(F_n) = \theta(F)$.

Consider the statistics

$$U_n = \binom{n}{r}^{-1} \sum_{1 \leq i_1 < \ldots < i_r \leq n} \phi(x_{i_1}, \ldots, x_{i_r}) \tag{3.21}$$

$$W_n = \left[\frac{n}{r}\right]^{-1} \sum_{i=0}^{[n/r]-1} \phi(x_{ri+1}, \ldots, x_{ri+r}) , \tag{3.22}$$

where $[n/r]$ denotes the largest integer contained in n/r. Let $\mathcal{J}_1$, $\mathcal{J}_2$ be families of tests of the form (3.4) based on U_n, W_n respectively, with $\mathcal{N}(\mathcal{J})$ equal to the set of integers $\geq r$. Consider the decision problem

$$(\{\mathcal{D}_{1n}\}, \{\mathcal{D}_{2n}\}, \alpha_1, \alpha_2)$$

where $\delta = \theta_2 - \theta_1$ is the "distance" between the classes $\mathcal{D}_{1n}$, $\mathcal{D}_{2n}$.

In order to compare the efficiencies of $\mathcal{J}_1$ and $\mathcal{J}_2$ for this problem asymptotically as $\delta \longrightarrow 0$, we apply the theorems of Chapters II and III to obtain $N(\mathcal{J}_1)$, $N(\mathcal{J}_2)$. We may note immediately that Assumption A.1 is satisfied for each of the two sequences of statistics.

By Theorem 2.1, Assumption A'.8 is satisfied for $\{U_n\}$ with

$$\underline{\zeta}(F_n) = (\zeta_1, \ldots, \zeta_r)$$

$$m(\theta) = \theta$$

$$s_n(\theta,\underline{\xi}) = \sigma(U_n)$$

$$g_n(U_n) = U_n$$

where $\xi_1,\ldots,\xi_r$ and $\sigma(U_n)$ are defined in Section 3 of Chapter II.

If $\phi(x_1,\ldots,x_r)$ takes the values zero or one only, then nW_n has binomial distribution with parameters $([n/r], \theta)$. We have in this case, as a consequence of Theorem 4.3 of [13],

$$N(\mathcal{J}_2) \sim r\theta_1(1-\theta_1)\left[\frac{\lambda(\alpha_1)+\lambda(\alpha_2)}{\delta}\right]^2$$

as $\delta \longrightarrow 0$.

If, on the other hand, ϕ is a more general function, we must further restrict the class $\mathcal{D}$ in order to have Assumption A'.8 satisfied. We might assume

$$E_F|\phi(X_1,\ldots,X_r)|^3 \leq M < \infty$$

for $F \in \mathcal{D}$ and establish the required uniform convergence by a proof using Berry's theorem [5]. Or we might make assumptions such as those given by Parzen [19], Chapter II. Assumption A'.8 will then be satisfied for $\{W_n\}$ with

$$\underline{\xi}(F_n) = \xi_r$$

$$m(\theta) = \theta$$

$$s_n^2(\theta_1, \underline{\xi}) = [n/r]^{-1}\,\xi_r$$

$$g_n(W_n) = W_n.$$

It is easy to show from (2.13) that

$$\lim_{n \longrightarrow \infty} \sup_{F \in \mathcal{D}(\theta)} n\sigma_F^2(U_n) = r^2 \sup_{F \in \mathcal{D}(\theta)} \zeta_1(F),$$

so that we have Assumption A*.9 satisfied for the families $\mathcal{J}_1$, $\mathcal{J}_2$ with a = 1/2 and, respectively,

$$s_1^2(\theta) = r^2 \sup_{F \in \mathcal{D}(\theta)} \zeta_1(F) \tag{3.23}$$

$$s_2^2(\theta) = r \sup_{F \in \mathcal{D}(\theta)} \zeta_r(F) . \tag{3.24}$$

We see that these functions $s(\theta)$ are bounded; it remains to assume the other parts of Assumption A".9.

Assumption A**.9: $s_c(\theta)$ is continuous at $\theta = \theta_1$ and there exists $\eta > 0$ such that $(u - \theta)/s_c(\theta)$ is a non-increasing function of θ for $\theta \in \omega$ if $\theta_1 < u < \theta_1 + \eta$.

We have established that under appropriate conditions

$$N(\mathcal{J}_1) \sim s_1^2(\theta_1) \left[\frac{\lambda(\alpha_1) + \lambda(\alpha_2)}{\delta}\right]^2$$

and

$$N(\mathcal{J}_2) \sim s_2^2(\theta_1) \left[\frac{\lambda(\alpha_1) + \lambda(\alpha_2)}{\delta}\right]^2$$

as $\delta \longrightarrow 0$. We may summarize the above discussion in :

Theorem 3.3: Let $\mathcal{D}$ be a class of distributions for which the functional $\theta(F)$ is defined and for which $\zeta_1 \geq b > 0$ and $E|\phi(X_1, \ldots, X_r)|^3 \leq M < \infty$. Let $\mathcal{J}_1$, $\mathcal{J}_2$ be the families of tests of the form (3.4) based on the sequences of statistics (3.21), (3.22), respectively. Then if $\mathcal{N}(\mathcal{J})$ is the set of integers $\geq r$ and Assumptions A'.3, A.4, A**.9 are satisfied for both $\mathcal{J}_1$ and $\mathcal{J}_2$, we have as $\delta \longrightarrow 0$

$$\text{eff}(\mathcal{T}_2/\mathcal{T}_1) \sim r \sup_{F\in\mathcal{D}(\theta_1)} \zeta_1(F)/\sup_{F\in\mathcal{D}(\theta_1)} \zeta_r(F)$$

$$\leq 1$$

for the problem $(\{\mathcal{D}_{1n}\}, \{\mathcal{D}_{2n}\}, \alpha_1, \alpha_2)$.

Corollary: Let $\theta(F)$ be a functional whose kernel ϕ takes on the values zero or one only, and let $\mathcal{D}$ be a class of distributions for which $\zeta_1 \geq b > 0$. Then if Assumption A'.3 is satisfied and if Assumptions A.4 and A**.9 hold for $\mathcal{T}_1$, we have as $\delta \longrightarrow 0$

$$\text{eff}(\mathcal{T}_2/\mathcal{T}_1) \sim \frac{r}{\theta_1(1-\theta_1)} \cdot \sup_{F\in\mathcal{D}(\theta_1)} \zeta_1(F).$$

For the two-sample case, using Theorem 2.2 and notation developed in Section 3 of Chapter II, we obtain an analogous theorem. Let

$$U_{mn} = \binom{n}{r}^{-1}\binom{m}{r}^{-1} \sum_{\substack{1\leq i_1<\ldots<i_r\leq n \\ 1\leq j_1<\ldots<j_r\leq n}} \cdot \phi(x_{i_1},\ldots,x_{i_r}; y_{j_1},\ldots,y_{j_r}) \tag{3.23}$$

$$W_{mn} = [t/r]^{-1} \sum_{i=0}^{[t/r]-1} \phi(x_{ri+1},\ldots,x_{ri+r};y_{ri+1},\ldots,y_{ri+r}) \tag{3.24}$$

where $t = \min(m,n)$.

It is found that, under the conditions stated below,

$$s^2_{c,1}(\theta_1) = r^2(c_1s_1 + c_2s_2) \sup_{(F,G)\in\mathcal{D}(\theta_1)} (\zeta_{10}/s_1 + \zeta_{01}/s_2),$$

$$s^2_{c,2}(\theta_1) = \frac{r(c_1s_1 + c_2s_2)}{\min(s_1,s_2)} \sup_{(F,G)\in\mathcal{D}(\theta_1)} \zeta_{rr} .$$

In this case, we see that the asymptotic relative efficiency of the two families is independent of (c_1, c_2).

Theorem 3.4: Let $\mathcal{D}$ be a class of pairs of distributions for which the functional $\theta(F,G)$ is defined and for which $\zeta_{10} \geq b > 0$, $\zeta_{01} \geq b > 0$, and $E|\phi_{0r}(X_1, \ldots, X_r)|^3 \leq M < \infty$, $E|\phi_{r0}(Y_1, \ldots, Y_r)|^3 \leq M < \infty$. Let $\mathcal{J}_1$, $\mathcal{J}_2$ be the families of tests of the form (3.4) based on the sequences of statistics (3.23), (3.24), respectively. Then if Assumptions A'.3, A.4, A.5, A**.9 are satisfied for both $\mathcal{J}_1$ and $\mathcal{J}_2$, we have for every $c \in \mathcal{C}$, as $\delta \longrightarrow 0$

$$\text{eff}^{(c)}(\mathcal{J}_2/\mathcal{J}_1) \sim \frac{r \min(s_1, s_2) \sup_{(F,G) \in \mathcal{D}(\theta_1)} \left[\frac{\zeta_{10}(F,G)}{s_1} + \frac{\zeta_{01}(F,G)}{s_2}\right]}{\sup_{(F,G) \in \mathcal{D}(\theta_1)} \zeta_{rr}(F,G)}$$

$$\leq 1,$$

for the problem $(\{\mathcal{D}_{1mn}\}, \{\mathcal{D}_{2mn}\}, \alpha_1, \alpha_2)$.

The analogous corollary may also be stated. In the statement of Theorem 3.4, we may suppress Assumption A.5 and consider the partition $\{\mathcal{J}_{i,s}\}$ of $\mathcal{J}_i$, where $\mathcal{N}(\mathcal{J}_i)$ contains vectors n with components $\geq r$, $i = 1, 2$. $\mathcal{J}_i$ is replaced by $\mathcal{J}_{i,s}$.

6. Examples.

The following are illustrations of the application of the corollaries of Theorems 3.4, 3.3, respectively.

Example I: $\theta(F,G) = P(X < Y)$; $\mathcal{D}$ consists of all pairs of continuous distributions, for which $\zeta_{10} + \zeta_{01} \geq b > 0$; $\alpha_1 < 1/2$, $\alpha_2 < 1/2$. By Theorem 3.4, for every s,

$$\text{eff}^{(c)}(\mathcal{J}_{2,s}/\mathcal{J}_{1,s}) \sim 1 \qquad \text{as } \delta \longrightarrow 0,$$

for the problem ($\{\mathcal{D}_{1n}\}$, $\{\mathcal{D}_{2n}\}$, α_1, α_2), where $\mathcal{D}_{in}$ are as given in Section 3 of this chapter (i = 1, 2).

We must verify: (i) A.4 is satisfied for $\mathcal{J}_1$. This is seen upon examination of the distribution of U_{mn} when (F,G) is given by (4.11), (4.10) in Chapter IV.

$$\text{(ii)} \quad \theta(1-\theta) = \frac{\sup(s_2\zeta_{10} + s_1\zeta_{01})}{\max(s_1,s_2)} \quad .$$

In addition, we may compute

$$N_c(\mathcal{J}_1) \sim N_c(\mathcal{J}_2) \sim \theta_1(1-\theta_1)\left[\frac{\lambda(\alpha_1) + \lambda(\alpha_2)}{\delta}\right]^2$$

whence for every $c \in \mathcal{C}$

$$\text{eff}^{(c)}(\mathcal{J}_2/\mathcal{J}_1) \sim \text{eff}^{(0)}(\mathcal{J}_2/\mathcal{J}_1) \sim 1.$$

Example II: $F = F(x,y)$, $\theta(F) = P[(X_1 - X_2)(Y_1 - Y_2) > 0]$; $\mathcal{F}$ consists of continuous bivariate distributions with $\zeta_1 \geq b > 0$; $\alpha_1 < 1/2$, $\alpha_2 < 1/2$.

$$\text{eff}(\mathcal{J}_2/\mathcal{J}_1) \sim \frac{2s^2(\theta_1)}{\theta_1(1-\theta_1)} < 1$$

where

$$s^2(\theta) = (\tfrac{1}{2} + |\theta - \tfrac{1}{2}|)^{3/2} - (\tfrac{1}{2} + |\theta - \tfrac{1}{2}|)^2,$$

subject to verification of the conjecture that $s^2(\theta)$ is the correct maximum for $\zeta_1(F)$.

We must verify A.4. Let $G_\theta(x,y)$ be given by uniform distribution of X on (0,1) and

$$Y = \begin{cases} 1 - \theta^{1/2} + X & \text{if } X \leq \theta^{1/2} \\ 1 - X & \text{if } X > \theta^{1/2}. \end{cases}$$

$\theta(G_{\theta'}) = \theta'$. Note that $\zeta_1(G_\theta) = s^2(\theta)$ if $\theta \geq 1/2$. The distribution of U_n is given by

$$P\left[U_n = \binom{n-k}{2}\right] = \binom{n}{k} \theta^{\frac{n-k}{2}} (1 - \theta^{1/2})^k$$

if $k = 2,\ldots,n$, and

$$P[U_n = 0] = n\theta^{1/2}(1 - \theta^{1/2})^{n-1} + (1 - \theta^{1/2})^n \quad .$$

The U-statistic on which the family $\mathcal{J}_1$ is based in this example is a linear function of Kendall's coefficient of difference-sign correlation [15].

The conjectured $s^2(\theta)$ is, as we have seen, not greater than the true value. It is attained for a distribution analogous to the discrete distribution for which Sundrum has shown [24] that ζ_1 attains its maximum in a class of discrete distributions.

7. Further Asymptotic Expressions for $N_c(\mathcal{J})$.

The results given in Sections 3 to 5 are not always applicable. We derive here an asymptotic expression for $N_c(\mathcal{J})$ which is applicable to certain tests of hypotheses of invariance.

Let $\theta(F_n)$ be a real-valued function defined for all $F_n \in \mathcal{D}_n$ and for every $n \in \mathcal{N}$. Assume that the function $\theta(F_n)$ takes on values in an interval ω (finite or infinite).

Let $\mathcal{J}$ be a family of tests of the form (3.4), defined for $n \in \mathcal{N}(\mathcal{J})$.

Let $\mathcal{D}_n(\theta)$ be the subset of $\mathcal{D}_n$ having $\theta(F_n) = \theta$ and suppose that for some $\theta_1 \in \omega$, $\mathcal{C}_{0n}$ is a subset of $\mathcal{D}_n(\theta_1)$ such that

(3.25) $$P(t_n \le x | F_n) = K(x,n) \qquad \text{if } F_n \in \mathcal{C}_{0n}.$$

Let $\theta_2 > \theta_1$ be numbers (independent of n) in ω and let

(3.26) $$\mathcal{D}_{1n} = \mathcal{C}_{0n}$$
$$\mathcal{D}_{2n} = \{F_n : \theta(F_n \ge \theta_2\}.$$

We will consider the asymptotic behavior of $N_c(\mathcal{J})$ as $\delta = \theta_2 - \theta_1 \longrightarrow 0$ with α_1, α_2 fixed, and θ_1 fixed.

Let λ_n be the smallest number which satisfies

(3.27) $$K(\lambda_n, n) \ge 1 - \alpha_1.$$

Then $\mathcal{N}(\mathcal{J})$ is the set of vectors n for which

$$\sup_{F_n \in \mathcal{D}_{2n}} P(t_n \le \lambda_n | F_n) \le \alpha_2.$$

$N_c(\mathcal{J})$ is determined from (3.2).

Assumption B.1: For every $n \in \mathcal{N}$ and every $\theta \in \omega$, there exists $H_n^*(x,\theta)$ and $\mathcal{D}_n^*(\theta) \subset \mathcal{D}_n(\theta)$, such that

$$P(t_n \le x | F_n) = H_n^*(x,\theta) \qquad \text{when } F_n \in \mathcal{D}_n^*(\theta),$$

where for every $x, H_n^*(x,\theta)$ is continuous on the right in θ at $\theta = \theta_1$.

Assumption B.2: $H_n^*(\lambda_n, \theta_1) > \alpha_2$.

If Assumptions A.1, B.1, B.2 are satisfied, and if for every $n \in \mathcal{N}(\mathcal{J})$ there exists $n' \in \mathcal{N}(\mathcal{J})$ with $n' > n$, then $N_c(\mathcal{J}) \longrightarrow \infty$ as $\delta \longrightarrow 0$.

The proof of Theorem 3.1 may be quoted (with appropriate re-interpretation of notation where necessary) to prove

Theorem 3.5: Let $\mathcal{J}$ be given by (3.4). If Assumptions B.1, B.2, A.1, A.5 - A.7 are satisfied, then as $\delta \longrightarrow 0$ we have

$$N_c(\mathcal{J}) \sim \left[\frac{D_c(\alpha_1,\alpha_2)}{\delta}\right]^{1/a} .$$

We have, in the notation of Section 4, Assumptions A.8, A.9 sufficient for A.6. We proceed now to consider a class of statistics having asymptotic normal distribution. We obtain sufficient conditions for Assumptions A.8, A.9. Among these are, as before, A'.8, A'.9, A".9. We have also

Assumption B.3: There exist numbers $b > 0$, $\sigma_c > 0$, a cdf $K_0(x)$ which is continuous and everywhere increasing (except possibly when it is equal to zero or one), and everywhere increasing functions $f_n(t_n)$, $n \in \mathcal{N}(\mathcal{J})$, such that for every real x

$$K(x,m) = K_0\left\{\frac{\ell_{m,c}^b}{\sigma_c}\left[f_m(x) - m(\theta_1)\right]\right\} + \epsilon_m(x) ,$$

where $\epsilon_m(x) \longrightarrow 0$ as $m \longrightarrow \infty$ and $m(\theta)$ is the function given in Assumption A'.8. Since we have Assumption A.5, the index n is replaced by m.

Let $\kappa(u)$ be defined by $1 - u = K_0(\kappa(u))$. Then by (3.27) and Assumption B.3,

$$1 - \alpha_1 = K_0\left\{\frac{\ell_{m,c}^b}{\sigma_c}\left[f_m(\lambda_m) - m(\theta_1)\right]\right\} + \tilde{\epsilon}_m$$

where $\tilde{\epsilon}_m \longrightarrow 0$ as $m \longrightarrow \infty$; and

$$\lim_{m \longrightarrow \infty} \ell_{m,c}^b\left[f_m(\lambda_m) - m(\theta_1)\right] = \sigma_c \kappa(\alpha_1) . \tag{3.28}$$

Assumption B.4: $g_m(x) \sim f_m(x)$ as $m \longrightarrow \infty$, for every x, $\{g_m(x)\}$ are the functions given in Assumption A'.8.

Now if for $d > 0$,

(3.29) $$g_m(\lambda_m) \le m(\theta_1 + \ell_{m,c}^{-a}\, d),$$

then

(3.30) $$\sup_{F_m \in \mathcal{D}(\theta)} H_m(\lambda_m,\theta,\underline{\xi}) = \overline{\mathcal{D}}\left[\ell_{m,c}^{a}\, \frac{g_m(\lambda_m) - m(\theta)}{s_c(\theta)}\right]$$

for $\theta \ge \theta_1 + \ell_{m,c}^{-a}\, d$.

By (3.27) and Assumptions A".9, B.4, $m(\theta_1) < g_m(\lambda_m) < m(\theta_1)+\eta$ for m sufficiently large, provided (3.29) holds and $\kappa(\alpha_1) > 0$. Hence for m sufficiently large and $\kappa(\alpha_1) > 0$,

$$\sup_{F_m \in \mathcal{D}_{m,c}(a,d)} H_m(\lambda_m,\theta,\underline{\xi}) = \overline{\mathcal{D}}\left[\ell_{m,c}^{a}\, \frac{g_m(\lambda_m) - m(\theta_1+\ell_{m,c}^{-a}\, d)}{s_c(\theta_1+\ell_{m,c}^{-a}\, d)}\right]$$

by Assumption A".9, if (3.29) holds.

Now, by Assumption B.4,

$$\ell_{m,c}^{a}\,[g_m(\lambda_m) - m(\theta_1)] \sim \ell_{m,c}^{a-b}\,\ell_{m,c}^{b}\,[f_m(\lambda_m) - m(\theta_1)]$$

and by (3.28),

$$\lim_{m \longrightarrow \infty} \ell_{m,c}^{a}\,[g_m(\lambda_m) - m(\theta_1)] = \begin{cases} 0 & \text{if } a < b \\ \sigma_c\, \kappa(\alpha_1) & a = b \\ \infty & a > b \end{cases} .$$

It follows that $\ell_{m,c}^{a}\,[g_m(\lambda_m) - m(\theta_1 + \ell_{m,c}^{-a}\, d)]$ tends to the limit

(3.31) $$\begin{cases} \sigma_c \kappa(\alpha_1) - dm'(\theta_1) & \text{if } a = b \\ -dm'(\theta_1) & a < b \end{cases} .$$

Hence (3.29) will hold for m sufficiently large if we choose

(3.32) $$\begin{cases} d_0 > \sigma_c \kappa(\alpha_1)/m'(\theta_1) & \text{when } a = b \\ d_0 > 0 & \text{when } a < b \end{cases} .$$

This completes the derivation of $H_c(d)$. We have, if $a \leq b$ and $\kappa(\alpha_1) > 0$,

$$H_c(d) = \overline{\emptyset}\left[\frac{\sigma_c \kappa(\alpha_1) - dm'(\theta_1)}{s_c(\theta_1)}\right] \qquad \text{if } a = b,$$

$$H_c(d) = \overline{\emptyset}\left[\frac{-dm'(\theta_1)}{s_c(\theta_1)}\right] \qquad \text{if } a < b,$$

for $d \geq d_0$, d_0 given by (3.32). We obtain

$$(3.33) \qquad D_c(\alpha_1,\alpha_2) = \begin{cases} [s_c(\theta_1)\lambda(\alpha_2) + \sigma_c \kappa(\alpha_1)]/m'(\theta_1) & \text{if } a = b \\ s_c(\theta_1)\lambda(\alpha_2)/m'(\theta_1) & \text{if } a < b \end{cases}.$$

Assumption A.6 will be satisfied provided we choose d_0 so that

$$m'(\theta_1)d_0 < \sigma_c\kappa(\alpha_1) + s_c(\theta_1)\lambda(\alpha_2) \qquad \text{when } a = b$$

$$m'(\theta_1)d_0 < s_c(\theta_1)\lambda(\alpha_2) \qquad \text{when } a < b,$$

which we may do if $\alpha_2 < 1/2$.

Assumption B.5: $\kappa(\alpha_1) > 0$ and $\alpha_2 < 1/2$.

Theorem 3.6: Let $\mathcal{J}$ be given by (3.4). If Assumptions B.1-B.5, A.1, A.5, A'.9, A".9 are satisfied, with $a \leq b$, then as $\delta \longrightarrow 0$

$$N_c(\mathcal{J}) \sim \left[\frac{D_c(\alpha_1,\alpha_2)}{\delta}\right]^{1/a}$$

where $D_c(\alpha_1,\alpha_2)$ is given by (3.33).

It should be noted that this theorem will not apply, for example, to the case of Lehmann's two-sample U-statistic corresponding to the functional $\Delta^2 = \int (F - G)^2 dF$, since the uniform convergence requirement of A'.8 is not satisfied. In general, this theorem may be

applicable in cases where the set $\mathcal{C}_{0n}$ is contained in $\mathcal{D}(\theta_1)$ for θ_1 an interior point of the interval ω.

8. Examples.

The examples given in Section 6 concern tests which may also be used for testing, respectively, the hypotheses $F = G$ and $F(x,y) = F_1(x)F_2(y)$. Both statistics have asymptotic normal distribution under such null hypotheses.

Example I: $F = G$ implies $\theta = 1/2$; hence, for $\theta_1 = 1/2$, the situation is the same as in Section 6, for the family $\mathcal{J}_{2,s}$ based on the binomially-distributed statistic. For $\mathcal{J}_{1,s}$, we obtain

$$\sigma_c^2 = (s_1 + s_2)(c_1 s_1 + c_2 s_2)/12\, s_1 s_2$$

and

$$D_c(\alpha_1,\alpha_2) = \left[\frac{c_1 s_1 + c_2 s_2}{12\, s_1 s_2}\right]^{1/2}\left[\sqrt{3\max(s_1,s_2)}\cdot\lambda(\alpha_2) + \sqrt{s_1 + s_2}\,\lambda(\alpha_1)\right].$$

Thus for every $c \in \mathcal{C}$,

$$\text{eff}^{(c)}(\mathcal{J}_{2,s}/\mathcal{J}_{1,s}) \sim \left[\frac{\lambda(\alpha_2) + \left\{\frac{s_1 + s_2}{3\max(s_1,s_2)}\right\}^{1/2}\lambda(\alpha_1)}{\lambda(\alpha_2) + \lambda(\alpha_1)}\right]^2 < 1,$$

for the problem $(\{\mathcal{C}_{0n}\}, \{\mathcal{D}_{2n}\}, \alpha_1, \alpha_2)$.

Example II: $F(x,y) = F_1(x)F_2(y)$ implies $\theta = 1/2$. For $\mathcal{J}_1$ we obtain $\sigma^2 = 1/9$, and

$$D(\alpha_1,\alpha_2) = \tfrac{1}{9}\lambda(\alpha_1) + \tfrac{1}{2}\lambda(\alpha_2).$$

Thus,

$$\text{eff}(\mathcal{J}_2/\mathcal{J}_1) \sim 1/4 \left[\frac{\lambda(\alpha_2) + \frac{2}{9}\lambda(\alpha_1)}{\lambda(\alpha_2) + \lambda(\alpha_1)}\right]^2$$

for the problem $(\{\mathcal{C}_{0n}\}, \{\mathcal{D}_{2n}\}, \alpha_1, \alpha_2)$.

CHAPTER IV

TWO-SAMPLE TESTS .

1. Introduction.

In this chapter, we consider the properties of various two-sample tests, for several types of two-sample decision problems.

We will be concerned chiefly with a class of pairs (F,G) of distributions for which the functional

$$(4.1) \qquad \theta(F,G) = P(X < Y) = \int\int \phi(x,y)\,dF(x)\,dG(y)$$

is defined, where

$$(4.2) \qquad \phi(x,y) = \begin{cases} 1 & \text{if} \quad x < y \\ 0 & \text{if} \quad x \geq y \end{cases} .$$

Two types of null hypothesis are of particular interest. The most usual null hypothesis for the two-sample problem is $F = G$. The Wilcoxon-Mann-Whitney statistic was proposed for tests of this hypothesis. When considering that statistic, however, one is led naturally to a partition of alternatives according to the value of $\theta(F,G)$. The next step is to consider a null hypothesis defined according to values of θ. We will see, for instance, that the Wilcoxon statistic can be shown to have a certain kind of weak optimum property for testing $H_1 : \theta \leq \theta_1$ against $H_2 : \theta \geq \theta_2$, $\theta_1 < \theta_2$, for large samples.

In this chapter, unless it is otherwise stated, we will suppose that we are given the $(n + m)$ mutually independent random variables $X_1,\ldots,X_n$ and $Y_1,\ldots,Y_m$, distributed according to $F(x)$, $G(y)$, respectively. (F,G) is supposed to be a pair of distributions

from a class $\mathcal{D}$ of pairs of distributions for which $0 < \theta(F,G) < 1$. Let $\mathcal{D}(\theta)$ be the subset of $\mathcal{D}$ in which $\theta(F,G) = \theta$.

The decision problem is to select one or the other of two hypotheses, each of which is given by a subset of the class $\mathcal{D}$, on the basis of information given by two observed samples. Thus, we have in general H: $(F,G)\varepsilon\ \mathcal{D}_0$ and H': $(F,G)\varepsilon\ \mathcal{D}_1$, where $\mathcal{D}_0$ and $\mathcal{D}_1$ are disjoint subsets of $\mathcal{D}$.

Let z_{mn} represent the vector $(x_1,\ldots,x_n;\ y_1,\ldots,y_m)$. Given $\alpha > 0$ and a statistic $t_{mn}(z_{mn})$, we will mean by a "test", a function $\phi(z_{mn} \mid t_{mn})$ which gives the probability of selecting H' (rejecting H) when z_{mn} has been observed, such that

$$\phi = \begin{cases} 1 & \text{if} \quad t_{mn} > c_{mn} \\ a_{mn} & \text{if} \quad t_{mn} = c_{mn} \\ 0 & \text{if} \quad t_{mn} < c_{mn} \end{cases} \tag{4.3}$$

where c_{mn} and a_{mn} $(0 \leq a_{mn} < 1)$ are chosen so that

$$\sup_{(F,G)\varepsilon\ \mathcal{D}_0} E_{FG}\ \phi(Z_{mn} \mid t_{mn}) = \alpha \quad . \tag{4.4}$$

Thus we consider a test which is randomized only to the extent necessary in order that the level of significance be exactly α. For simplicity, a_{mn} and c_{mn} will be written without their subscripts when it is not desired to emphasize their dependence on m,n.

A family $\mathcal{J}$ of tests will consist of tests of the form (4.3) based on the sequence $\{t_{mn}\}$, $m,n = 1, 2, \ldots$.

We will in particular be interested in the families of tests $\mathcal{U}$ and $\mathcal{W}$ based on the statistics (4.5) and (4.6), respectively. Suppose $n \leq m$ and let

$$(4.5) \qquad mnU_{mn} = \sum_{i=1}^{n}\sum_{j=1}^{m} \phi(x_i, y_i)$$

$$(4.6) \qquad nW_{mn} = \sum_{i=1}^{n} \phi(x_i, y_i),$$

where $\phi(x,y)$ is given by (4.2). $\mathcal{U}'$ and $\mathcal{W}'$ will denote the families of two-sided tests based on $|U_{mn} - 1/2|$, $|W_{mn} - 1/2|$, respectively.

2. Tests of $\theta \leq \theta_1$ against $\theta \geq \theta_2$; Small Samples.

Let us consider $H_1: \theta \leq \theta_1$ and $H_2: \theta \geq \theta_2$, $\theta_1 < \theta_2$. We show first that the family $\mathcal{W}$ provides a test which maximizes the minimum power with respect to H_2, among test of size α for testing H_1.

For testing H_1 against H_2, we have a test $\phi(z_{mn} \mid W_{mn})$ of the form (4.3) from the family $\mathcal{W}$. nW_{mn} has binomial distribution $B(n,\theta)$. Thus, for $(F,G) \in \mathcal{D}(\theta)$,

$$(4.7) \quad E_{FG}\phi(Z_{mn}|W_{mn}) = \sum_{j=c+1}^{n} \binom{n}{j}\theta^j(1-\theta)^{n-j} + a\binom{n}{c}\theta^c(1-\theta)^{n-c} = H(\theta;\, c,a),$$

which is an increasing function of θ, and we have a and c determined by

$$(4.8) \qquad \alpha = H(\theta_1;\, c,a).$$

Let μ be a measure on the real line R_1 having the following property. There exist disjoint intervals I_1, I_2, I_3 of R_1 such that

$$(4.9) \qquad \begin{aligned} &\text{(i)} \quad x_j \in I_j \quad (j = 1,2,3) \text{ implies } x_1 < x_2 < x_3 \\ &\text{(ii)} \quad 0 < \mu(I_j) < \infty, \qquad j = 1, 2, 3. \end{aligned}$$

Theorem 4.1: Let $\mathcal{D}_\mu$ be a class of pairs (F,G) of distributions having generalized density with respect to some fixed measure μ satisfying (4.9). Then a test from the family $\mathcal{W}$ of size α for test-

ing H_1, maximizes the minimum power with respect to H_2, provided $\mathcal{D}_\mu$ contains the pairs $(K(x,\theta),H(y))$, $0 < \theta < 1$, defined by (4.10),(4.11).

Proof: By Theorem 2 of [11], it will be sufficient to exhibit $(F_1,G_1)\varepsilon\ \mathcal{D}_\mu(\theta_1)$ and $(F_2,G_2)\varepsilon\ \mathcal{D}_\mu(\theta_2)$ such that $\phi(z_{mn}|W_{mn})$ is a most powerful test of size α for testing the simple hypothesis $\overline{H}_1$: $(F,G) = (F_1,G_1)$ against the simple alternative $\overline{H}_2$: $(F,G)=(F_2,G_2)$.

Let $H(y)$ be a distribution having the probability density

$$h(y) = \begin{cases} 1/\mu(I_2) & \text{if } y\varepsilon I_2 \\ 0 & \text{otherwise.} \end{cases} \tag{4.10}$$

Let $K(x,\theta)$ be a distribution with density

$$k(x,\theta) = \begin{cases} \theta/\mu(I_1) & \text{if } x \varepsilon I \\ (1-\theta)/\mu(I_3) & \text{if } x \varepsilon I_3 \\ 0 & \text{otherwise.} \end{cases} \tag{4.11}$$

Let $G_1(y) = G_2(y) = H(y)$ and $F_i(x) = K(x,\theta_i)$, $i = 1, 2$. The desired result follows from a familiar application of the Neyman-Pearson lemma.

We might broaden the statement of the theorem slightly, for the test from $\mathcal{W}$ actually maximizes the minimum power uniformly with respect to a partition according to values of θ of the subset of $\mathcal{D}_\mu$ with $\theta \geq \theta_2$.

It is also possible, for example, to prove Theorem 4.1 for a class $\mathcal{D}$ restricted to contain only distributions F,G which are continuous and everywhere increasing (unless equal to zero or one). The distributions K, H are replaced by sequences $\{K_{it}\}$, $\{H_{it}\}$ with $(K_{it},H_{it})\varepsilon\ \mathcal{D}(\theta_i)$, $(i = 1,2;\ t = 1,2,\ldots)$, such that the power of a most powerful test of level α for testing (K_{1t},H_{1t}) against (K_{2t},H_{2t}) tends as $t \longrightarrow \infty$, to the power of $\phi(z_{mn}|W_{mn})$ for testing

$(F,G)\varepsilon\ \mathcal{D}(\theta_1)$ at level α, against $(F,G)\varepsilon\ \mathcal{D}(\theta_2)$.

A similar result may be obtained for two-sided tests from the family $\mathcal{U}'$ for testing H_1': $\theta = 1/2$ against H_2': $|\theta - 1/2| \geq \delta > 0$. The proof follows as above. A m.p. test is obtained for testing the simple hypothesis that the joint distribution of $(X_1,\ldots,X_n)$ is $\prod_{i=1}^{n} K(x_i, 1/2)$ against the simple alternative that it is

$$1/2 \prod_{i=1}^{n} K(x_i, 1/2 - \delta) + 1/2 \prod_{i=1}^{n} K(x_i, 1/2 + \delta).$$

That a test from the family $\mathcal{U}$ does not in general maximize the minimum power may be seen by examination of the proof of Theorem 4.2.

Lemma 4.1: Let $\mathcal{D}_c$ be the class of pairs of continuous distributions. Then $\theta(F,G) = \int F dG$ and

$$\max_{(F,G)\varepsilon\,\mathcal{D}_c(\theta)} \int (1 - G)^m dF^n = \theta^s, \tag{4.12}$$

$$\max_{(F,G)\varepsilon\,\mathcal{D}_c(\theta)} \left\{ \int (1-G)^m dF^n + \int (1-F)^n dG^m \right\} = \theta^s + (1-\theta)^s, \tag{4.13}$$

where $s = \min(m,n)$.

Proof: We have three steps in the proof of (4.12).

(i) $\int (1 - G)^m dF^n \leq \int (1 - G)^s dF^s$. If $m > n$, then $(1 - G)^m \leq (1 - G)^n$. If $m < n$, integrate by parts to obtain

$$\int (1-G)^m dF^n = m \int (1-G)^{m-1} F^n dG \leq m \int (1-G)^{m-1} F^m dG = \int (1-G)^m dF^m.$$

(ii) $\int (1-G)^s dF^s \leq \theta^s$. The integral is equal to the probability of the event A: $\max(X_1,\ldots,X_s) < \min(Y_1,\ldots,Y_s)$, while θ^s is the probability of the event B: $X_1 < Y_1,\ldots,X_s < Y_s$; and $P(A) \leq P(B)$.

(iii) There exist $(F,G)\in \mathcal{D}_c(\theta)$, $0 < \theta < 1$, such that the upper bound θ^s is attained. Consider the distributions $K(x,\theta)$, $H(y)$ defined in (4.10), (4.11). Let μ be Lebesgue measure. If $n \leq m$, let $F(x) = K(x,\theta)$, $G(y) = H(y)$. Then $\int (1-G)^m dF^n = P(A) = P(B) = \theta^s$. If $n > m$, let $F(x) = H(x)$, $G(y) = K(y,1-\theta)$. Again, $P(A) = P(B)$.

To prove (4.13), we see that steps (i) and (ii) may be repeated with F and G interchanged and θ replaced by $1 - \theta = \int G dF$. That the upper bound $\theta^s + (1-\theta)^s$ may be attained for every θ is verified with the same distributions used in step (iii).

Theorem 4.2: If $(F,G)\in \mathcal{D}_c$, a test from the family $\mathcal{U}$ is inadmissible for testing H_1 against H_2 whenever $\alpha_1 < \theta_1^s$, where $s = \min(m,n) > 1$.

Proof: We show that a uniformly better test is available, namely a test from the family $\mathcal{W}$. For $m = n = 1$, of course, U_{mn} and W_{mn} are identical.

If $\alpha < \theta_1^s$, then the following tests are of size α with respect to H_1: $\theta \leq \theta_1$.

$$\phi(z_{mn}|W_{mn}) = \begin{cases} \theta_1^{-s}\alpha & \text{if } W_{mn} = 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\phi(z_{mn}|U_{mn}) = \begin{cases} \theta_1^{-s}\alpha & \text{if } U_{mn} = 1 \\ 0 & \text{otherwise,} \end{cases}$$

by Lemma 4.1. If $(F,G)\in \mathcal{D}_c(\theta)$,

$$E_{FG}\phi(Z_{mn}|W_{mn}) = \theta_1^{-s}\alpha\theta^s,$$

$$E_{FG}\ \phi(Z_{mn}|W_{mn}) \;=\; \theta_1^{-s}\,\alpha \int (1 - G)^m\, dF^n.$$

Thus, by Lemma 4.1,

$$E_{FG}\ \phi(Z_{mn}|W_{mn}) \geq E_{FG}\ \phi(Z_{mn}|U_{mn})$$

and it is easily seen that strict inequality holds for some (F,G).

Let $\mathcal{U}'$ be the family of (two-sided) tests of the form (4.3) based on the statistic $|U_{mn} - 1/2\ |$, for testing

$$H_1'\colon\ \theta = 1/2 \quad \text{against} \quad H_2'\colon\ |\theta - 1/2\ | \geq \delta > 0.$$

The second part of Lemma 4.1 may be used to show that a test from $\mathcal{U}'$ is inadmissible for testing H_1' against H_2' whenever $\alpha < 2^{-(s-1)}$, $s > 1$.

That the tests based on U_{mn} exhibit curious properties in small samples is strikingly illustrated by the following example. Suppose $(F,G) \in \mathcal{D}_c$. Let $m = n = 2$, $\alpha \leq \theta_1^2$. It is shown that for any π, $0 \leq 2\pi \leq \alpha\ \theta_1^{-2}$, the following test is of size α:

$$\bar{\phi}_\pi \;=\; \begin{cases} \alpha\ \theta_1^{-2} & \text{if } U_{22} = 1 \\ \pi & \text{if } U_{22} = 3/4 \\ 0 & \text{otherwise.} \end{cases}$$

$$E_{FG}\ \bar{\phi}_\pi = 2\pi\theta^2 + (\theta_1^{-2}\,\alpha - 2\pi) \int (1 - G)^2\, dF^2,$$

since

$$P(U_{22} = 3/4) = 2\theta^2 - 2 \int (1 - G)^2\, dF^2.$$

Now

$$\max_{(F,G)\in \mathcal{D}_c(\theta)} E_{FG}\ \bar{\phi}_\pi = \theta_1^{-2}\ \alpha\ \theta^2$$

by Lemma 4.1.

(1) The test $\bar{\phi}^*$ corresponding to $\pi = 1/2\ \alpha\ \theta_1^{-2}$ is uniformly better that $\bar{\phi}_\pi$ for smaller π.

(2) $\bar{\phi}^*$ has the same power function as the test $\phi\ (Z_{22}|W_{22})$ from the family $\mathcal{W}$. Hence $\bar{\phi}^*$ is not necessarily inadmissible.

(3) For $\alpha < \theta_1^2$, a test $\bar{\phi}_\pi$ is not of the form (4.3). Thus, by permitting greater randomization, we may obtain a test based on U_{22} which is no worse than $\phi(Z_{22}|W_{22})$. This weakens somewhat the effect of Theorem 4.2.

(4) For $\alpha = \theta_1^2$, $\bar{\phi}_\pi$ is a test of the form (4.3), and we see that it is possible that a test of the form (4.3) may not be uniquely determined by the fixing of the level of significance. Also, Theorem 4.2 cannot hold for $m = n = 2$, $\alpha = \theta_1^2$.

3. Rank Order Tests.

Let $u_j = u_j(m,n)$, $j = 1, \ldots, M$, denote an arrangement of n x's and m y's; let $\mathcal{Y}_{mn}$ denote the class of $M = \binom{m+n}{m}$ possible arrangements. For example,

$$\mathcal{Y}_{12} = \{(xxy), (xyx), (yxx)\}\ .$$

A rank order test (r. o. test) assigns to each u_j a number π_j $(0 \le \pi_j \le 1)$ which is the probability of rejecting the null hypothesis (usually H_0: F = G) if the (ordered) x's and y's of the sample z_{mn} have arrangement $u_j (z_{mn} \sim u_j)$. We will use the notation

$$\rho(z_{mn};\pi) = \pi_j \quad \text{if} \quad z_{mn} \sim u_j \quad (j = 1, \ldots, M)$$

where $\pi = (\pi_1, \ldots, \pi_M)$, for r.o. tests. For a test of size α, for testing H_0: $F = G$,

$$(4.14) \qquad \sum_{j=1}^{M} \pi_j = \alpha(m + n)! \,/\, m!\, n! \quad .$$

We will find it useful also to define a <u>symmetrical rank order test</u>, for testing H_0 against two-sided alternatives. Let u_j' denote the arrangement obtained from u_j by reversing the order; e.g.,

$$u_1(1,2) = (xxy), \qquad u_1'(1,2) = (yxx).$$

The set $\mathcal{Y}_{mn}$ may be divided into subsets, $\mathcal{Y}_{mn}^0$ in which $u_j' = u_j$, and $\mathcal{Y}_{mn}'$ in which $u_j' \neq u_j$. For a symmetrical rank order (s.r.o.) test, $\pi_k = \pi_j$ if $u_k = u_j'$. We will write

$$\rho_s(z_{mn};\, \pi) = \pi_j \text{ if } z_{mn} \sim u_j \quad (j = 1, \ldots, M)$$

for s.r.o. tests.

Thus, the arrangements $u_j \in \mathcal{Y}_{mn}'$ are considered in pairs in a s.r.o. test. We have for all $u_j \in \mathcal{Y}_{mn}$,

$$(4.15) \qquad U_{mn}(z_{mn}) = 1 - U_{mn}(z_{mn}') \text{ if } z_{mn} \sim u_j,\ z_{mn}' \sim u_j' \ .$$

Rank order tests and s.r.o. tests have as special cases tests based on the statistics U_{mn} and $|U_{mn} - 1/2\ |$, respectively, which assign the π_j according to the value of $U_{mn}(z_{mn})$ for $z_{mn} \sim u_j$.

Since more than one arrangement may be associated with a given value of U_{mn}, a test based on U_{mn} assigns the π_j to groups of arrangements. By (4.15), a test based on $|U_{mn} - 1/2|$ assigns the same probability to u_j and u_j'.

The power function of a r.o. test is given by

$$(4.16) \qquad E_{FG}\, \rho(Z_{mn};\, \pi) = \sum_{j=1}^{M} \pi_j \Pr(Z_{mn} \sim u_j).$$

We say that a test $\rho(z_{mn};\, \pi')$ maximizes the minimum power among r.o. tests of size α for testing H_0 against H': $(F,G) \in \mathcal{D}_1$ if for all π such that (4.14) holds

$$(4.17) \qquad \inf_{(F,G) \in \mathcal{D}_1} E_{FG}\, \rho(Z_{mn};\, \pi') \geq \inf_{(F,G) \in \mathcal{D}_1} E_{FG}\, \rho\,(Z_{mn};\, \pi).$$

And, of course, $\rho(z_{mn};\, \pi')$ is u.m.p. among r.o. tests if for all π satisfying (4.14)

$$(4.18) \qquad E_{FG}\rho(Z_{mn};\, \pi') \geq E_{FG}\rho(Z_{mn};\, \pi) \quad \text{for all } (F,G) \in \mathcal{D}_1.$$

For s.r.o. tests, replace ρ by ρ_s in (4.16) - (4.18). Consider now the use of r.o. tests for testing H_1: $\theta \leq \theta_1$ against H_2: $\theta \geq \theta_2$. In the example given at the end of Section 2, it is found that a test based on U_{mn} maximizes the minimum power. This does not happen in general. It is, however, always possible to construct a <u>rank order test</u> which maximizes the minimum power among tests of size α for testing H_1: $\theta \leq \theta_1$ against H_2: $\theta \geq \theta_2$. This is done by constructing a test whose power function is always equal to the power function of the test based on W_{mn}, as follows:

(1) A test of the form (4.3) based on W_{mn} may be written

$$\phi(z_{mn} \mid W_{mn}) = p_t \quad \text{if} \quad s\, W_{mn}(z_{mn}) = t$$

$$t = 0,1,\ldots,\ s = \min(m,n),$$

where $0 \leq p_t \leq 1$. We construct the symmetrized test $\phi^*(z_{mn} \mid W_{mn})$ given by

$$m!n!\ \ \phi^*(z_{mn} \mid W_{mn}) = \Sigma'\, \phi(z'_{mn} \mid W_{mn})$$

where Σ' denotes two-fold summation over the permutations

$$z'_{mn} = (x_{i_1},\ldots,x_{i_n};\ y_{j_1},\ldots,y_{j_m})$$

of the x's and y's of z_{mn}. Now

$$m!n!\ \phi^*(z_{mn} \mid W_{mn}) = \sum_{t=0}^{s} p_t \left[\text{no. } z'_{mn} \text{ with } s\, W_{mn}(z_{mn}) = t\right],$$

or

$$\phi^*(z_{mn} \mid W_{mn}) = \sum_{t=0}^{s} p_t \; P\left[{}_sW_{mn}(Z_{mn}) = t \;\middle|\; Z_{mn} \sim u(m,n) \right]$$

so that ϕ^* is clearly a r.o. test.

(2) Evidently, the power function of ϕ^* is the same as the power function of ϕ.

In the same way, we can show that the symmetrized form of $\phi(z_{mn} \mid |W_{mn} - 1/2|)$ is an s.r.o. test.

We may remark that the test $\overline{\phi}^*$ based on U_{22} which maximizes the minimum power in the example at the end of Section 2 happens to be identical with $\phi^*(z_{22} \mid W_{22})$

4. Tests of F = G; Small Samples.

Various results concerning the properties of small-sample rank order tests of the hypothesis H_0: F = G against certain classes of alternatives have been obtained by I. R. Savage [21]. Savage's approach is to examine the ordering induced on the set of possible arrangements of (ordered) x's and y's by the distributions (F, G) of the alternative hypothesis (all arrangements being equally likely under H_0). He considers alternatives which are all special cases of the slippage alternatives, H_s: F(x) > G(x) for all x.

Savage shows that it is impossible, for most sample sizes, to construct uniformly most powerful rank order tests for testing H_0 against H_s. His remarks apply a fortiori to the problem of constructing u.m.p. rank order tests against H_1: $\theta(F,G) \leq \theta_1$ or H_2: $\theta(F,G) \geq \theta_2$ $(\theta_1 < 1/2 < \theta_2)$.

The general conclusion reached by Savage is that the class of admissible tests for testing $F = G$ against H_s is very large. The same remark may be made about the classes of admissible tests for testing $F = G$ against H_1, H_2, or H_2': $|\theta(F,G) - 1/2| \geq \delta > 0$. We may, however, suggest that tests based on W_{mn} cannot be expected in general to have even the weak optimum properties that were shown for the problem of testing $\theta \leq \theta_1$ against $\theta \geq \theta_2$ $(\theta_1 < \theta_2)$.

Theorem 4.3. If $m = n = 2$, $\alpha < 1/2$, $(F,G) \in \mathfrak{D}_c$, the two-sided test of the form (4.3) based on $|W_{22} - 1/2|$ is inadmissible for testing H_0 against H_2'.

Proof: We show that a test based on $|U_{22} - 1/2|$ is uniformly better. Let

$$\phi(z_{22} \mid |W_{22} - 1/2|) = \begin{cases} 2\alpha & \text{if } |W_{22} - 1/2| = 1/2 \\ 0 & \text{otherwise.} \end{cases}$$

This test is of size α and

$$E_\theta\, \phi(z_{22} \mid |W_{22} - 1/2|) = 2\alpha \left[\theta^2 + (1 - \theta)^2\right].$$

Let

$$\phi_1(z_{22} \mid |U_{22} - 1/2|) = \begin{cases} 3\alpha & \text{if } |U_{22} - 1/2| = 1/2 \\ 0 & \text{otherwise,} \end{cases}$$

$$\phi_2(z_{22} \mid |U_{22} - 1/2|) = \begin{cases} 1 & \text{if } |U_{22} - 1/2| = 1/2 \\ 3\alpha - 1 & \text{if } |U_{22} - 1/2| = 1/4 \\ 0 & \text{otherwise.} \end{cases}$$

These tests are of size α for $0 < \alpha \leq 1/3$, $1/3 < \alpha < 1/2$, respectively.

$$E_{FG}\, \phi_1(Z_{22} \mid |U_{22} - 1/2|) = 3\alpha\,(1/3 + 2\Delta^2),$$

$$E_{FG}\, \phi_2(Z_{22} \mid |U_{22} - 1/2|) = (1/3 + 2\Delta^2) + 4(3\alpha - 1)\left[1/3 - \Delta^2 - \theta(1-\theta)\right],$$

where $\Delta^2 = \int (F - G)^2\, dG$. Now,

(4.19) $$\Delta^2 \geq (\theta - 1/2)^2.$$

With (4.19), we establish for $0 < \alpha < 1/3$ (with an obvious abbreviation of notation):

$$E_{FG}\, \phi_1(Z_{22}|U_{22}) - E_\theta\, \phi(Z_{22}|W_{22})$$

$$= 6\alpha\,\Delta^2 - 4\alpha(\theta - 1/2)^2 \geq 2\alpha(\theta - 1/2)^2 > 0, \text{ if } \theta \geq \delta.$$

and for $1/3 \leq \alpha < 1/2$

$$E_{FG}\, \phi_2(Z_{22}|U_{22}) - E_\theta\, \phi(Z_{22}|W_{22})$$

$$= 2(1 - 2\alpha)\left[3\Delta^2 - 2(\theta - 1/2)^2\right]$$

$$\geq 2(1 - 2\alpha)(\theta - 1/2)^2 > 0, \text{ if } \theta \geq \delta.$$

A corollary to Theorem 4.3 is that $\phi(z_{22} \mid |W_{22} - 1/2|)$ cannot maximize the minimum power for testing H_0 against H_2'.

We may ask whether, in the absence of "better" optimum properties, it is possible to find tests which maximize the minimum power among rank order tests of size α for testing H_0: $F = G$.

Table 4.1, at the end of this section, summarizes the results which have been obtained. It is clear that unless the alternative hypothesis is restricted, the class of admissible tests is too large to permit satisfactory results from this approach.

We consider one statistic not defined above, namely, the two-sample statistic proposed by Lehmann [16] for testing H_0 against alternatives H_3: $F \neq G$. Let

$$K(x_1, x_2, y_1, y_2) = \begin{cases} 1 & \text{if } \max(x_1, x_2) < \min(y_1, y_2) \\ & \text{or } \max(y_1, y_2) < \min(x_1, x_2) \\ 0 & \text{otherwise} \end{cases}$$

$$\binom{m}{2}\binom{n}{2} L_{mn}(z_{mn}) = \sum K(x_i, x_j, y_k, y_\ell)$$

where summation is extended over sets of integers (i, j, k, ℓ) with $1 \leq i < j \leq n$, $1 \leq k < \ell \leq m$.

$$E\, L_{mn}(Z_{mn}) = 1/3 + 2\Delta^2$$

where $\Delta^2 = \int (F - G)^2\, dG$.

The remainder of this section is devoted to the derivation of Table 4.1.

Conditions for Table 4.1: Let $\mathcal{D}$ be the class of pairs of continuous distributions (F, G). Let the one-sided alternatives $\mathcal{D}_2 = \{(F,G): \theta(F,G) > 1/2\}$ be partitioned according to values of

$\theta(F,G)$. Let the two-sided alternatives $\mathcal{D}_2' = \{(F,G): |\theta(F,G) - 1/2| > 0\}$ be partitioned according to values of $d = |\theta(F,G) - 1/2|$. Let the alternatives $\mathcal{D}_3 = \{(F,G): \Delta^2 > 0\}$ be partitioned according to values of Δ^2. The null hypothesis is H_0: $F = G$.

Tests denoted by $\phi(z_{mn}|t_{mn})$ are of the form (4.3); there is no ambiguity of definition. Tests denoted by $\bar{\phi}(z_{mn}|t_{mn})$ are tests based on t_{mn} but employing more extensive randomization. The following are the ones which appear in the table:

$$\bar{\phi}(z_{12}|U_{12}) = \begin{cases} 2\alpha & \text{if } U_{12} = 1 \\ \alpha & = 1/2 \\ 0 & = 0 \end{cases}$$

$$\bar{\phi}(z_{12}||U_{12} - 1/2|) = \alpha.$$

Derivation of Table 4.1: Theorem 4.3 gives an example of a proof of inadmissibility.

The set of r.o. tests (or of s.r.o. tests in the two-sided case) is reduced by use of necessary conditions for admissibility. Let $\mathcal{C}_1$ be a class of alternatives.

(i) If $P(Z_{mn} \sim u_j) \geq P(Z_{mn} \sim u_k)$ for all $(F,G) \in \mathcal{C}_1$, then $\rho(z_{mn};\pi)$ will be admissible only if $\pi_k > 0$ implies $\pi_j = 1$. (Savage [21], page 13).

(ii) If $P(Z_{mn} \sim u_j) + P(Z_{mn} \sim u_j') \geq P(Z_{mn} \sim u_k) + P(Z_{mn} \sim u_k')$ for all $(F,G) \in \mathcal{C}_1$, where $u_j, u_k \in \mathcal{Y}_{mn}'$, then $\rho_s(z_{mn}; \pi)$ will be admissible only if $\pi_k > 0$ implies $\pi_j = 1$.

(iii) If $P(Z_{mn} \sim u_j) + P(Z_{mn} \sim u'_j) \geq 2P(Z_{mn} \sim u_k)$ for all $(F,G) \in \mathcal{C}_1$, where $u_j \in \mathcal{Y}'_{mn}$, $u_k \in \mathcal{Y}^0_{mn}$, then $\rho_s(z_{mn}; \pi)$ will be admissible only if $\pi_k > 0$ implies $\pi_j = 1$.

When there is only one r.o. (s.r.o.) test which is not inadmissible, then that test is most powerful among r.o. tests.

It will be seen that the results are "complete" for $m = 1$, $n = 2$. For $m \geq 2$, $n \geq 2$, there are only certain significance levels and certain subsets of the alternatives for which results may be obtained without restrictions on the class of distributions considered. For larger sample sizes, it does not seem to be possible to obtain the power functions of the r.o. (s.r.o.) tests in manageable form.

TABLE 4.1

Sample sizes	Level of significance	Alternatives for which properties hold (uniformly)	Tests which maximize minimum power among r.o. (s.r.o.) tests	Inadmissible tests
$m = 1$	$0 < \alpha < \frac{1}{2}$	$\mathcal{D}_2(\theta)$	$\overline{\phi}(z_{12}\|U_{12})$	
$n = 2$		$\frac{1}{2} < \theta < \frac{2}{3}$	$\phi(z_{12}\|W_{12})$	
		$\mathcal{D}_2(\theta)$	$\phi(z_{12}\|U_{12})^*$	$\phi(z_{12}\|W_{12})$
		$\frac{2}{3} \leq \theta < 1$		$\overline{\phi}(z_{12}\|U_{12})$
		$\mathcal{D}_2'(d)$	$\overline{\phi}(z_{12}\|\|U_{12}-\frac{1}{2}\|)$	$\phi(z_{12}\|\|U_{12}-\frac{1}{2}\|)$
		$0 < d^2 < \frac{1}{12}$	$\phi(z_{12}\|\|W_{12}-\frac{1}{2}\|)$	
		$\mathcal{D}_2'(d)$	$\phi(z_{12}\|\|U_{12}-\frac{1}{2}\|)^*$	$\phi(z_{12}\|\|W_{12}-\frac{1}{2}\|)$
		$\frac{1}{12} \leq d^2 < \frac{1}{4}$		
$m = 2$	$0 < \alpha \leq \frac{1}{6}$	$\mathcal{D}_2(\theta)$	$\phi(z_{22}\|U_{22})^*$	$\phi(z_{22}\|W_{22})$
$n = 2$		$\frac{3}{4} \leq \theta < 1$**		
	$0 < \alpha < \frac{1}{4}$	$\mathcal{D}_2(\theta)$		$\phi(z_{22}\|W_{22})$
		$\frac{1}{2} \leq \theta^2 < 1$**		
	$0 < \alpha \leq \frac{1}{3}$	$\mathcal{D}_2'(d)$	$\phi(z_{22}\|\|U_{22}-\frac{1}{2}\|)^*$	$\phi(z_{22}\|\|W_{22}-\frac{1}{2}\|)$
		$\frac{1}{12} \leq d^2 < \frac{1}{4}$	$\phi(z_{22}\|L_{22})^*$	
	$0 < \alpha < \frac{1}{2}$	$\mathcal{D}_2'(d)$		$\phi(z_{22}\|\|W_{22}-\frac{1}{2}\|)$
		$0 < d^2 < \frac{1}{4}$		

Simple sizes	Level of significance	Alternatives for which properties hold (uniformly)	Tests which maximize minimum power among r.o. (s.r.o.) tests	Inadmissible tests
$m = 2$	$0 < \alpha \leq \frac{1}{3}$	$\mathfrak{D}_3(\Delta^2)$	$\phi(z_{22} \mid L_{22})^*$	$\phi(z_{22} \mid \lvert W_{22} - \frac{1}{2} \rvert)$
$n = 2$		$\frac{1}{6} \leq \Delta^2 < \frac{1}{3}$	$\phi(z_{22} \mid \lvert U_{22} - \frac{1}{2} \rvert)^*$	
	$0 < \alpha \leq \frac{3}{7}$	$\mathfrak{D}_3(\Delta^2)$		$\phi(z_{22} \mid \lvert W_{22} - \frac{1}{2} \rvert)$
		$0 < \Delta^2 < \frac{1}{3}$		
$m = 2$	$0 < \alpha \leq \frac{1}{5}$	$\mathfrak{D}_2'(d^2)$	$\phi(z_{23} \mid L_{23})^*$	
$n = 3$		$\frac{11}{52} \leq d^2 < \frac{1}{4}$**	$\phi(z_{23} \mid \lvert U_{23} - \frac{1}{2} \rvert)^*$	
	$0 < \alpha \leq \frac{3}{7}$**	$\mathfrak{D}_2'(d^2)$		$\phi(z_{23} \mid \lvert W_{23} - \frac{1}{2} \rvert)$
		$0 < d^2 < \frac{1}{4}$		
	$0 < \alpha \leq \frac{1}{5}$	$\mathfrak{D}_3(\Delta^2)$	$\phi(z_{23} \mid L_{23})^*$	
		$\frac{1}{6} \leq \Delta^2 < \frac{1}{3}$	$\phi(z_{23} \mid \lvert U_{23} - \frac{1}{2} \rvert)^*$	
	$0 < \alpha \leq \frac{3}{7}$**	$\mathfrak{D}_3(\Delta^2)$		$\phi(z_{23} \mid \lvert W_{23} - \frac{1}{2} \rvert)$
		$0 < \Delta^2 < \frac{1}{3}$		

* This test is most powerful among r.o. (s.r.o.) tests.

** This condition is sufficient but undoubtedly not necessary.

5. Asymptotic Properties.

The results of Chapter III were applied in Example I of Section 6, to show that the families $\mathcal{U}$ and $\mathcal{W}$ have asymptotic relative efficiency one for testing $\theta \leq \theta_1$ against $\theta \geq \theta_2$ when (F, G) is in the class of all pairs of continuous distributions (or certain subsets of that class).

For testing F = G against $\theta \geq \theta_2 > 1/2$, eff $(\mathcal{W}/\mathcal{U})$ is asymptotically less than one, as shown in Example I, Section 8 of Chapter III.

We have remarked that the asymptotic results of Chapter III will not apply to the problem of testing F = G against $\Delta^2 > 0$, using Lehmann's statistic. However, it might be reasonable in some experimental situations to suppose that the class $\mathcal{D}$ of pairs of continuous distributions (F,G) does not contain pairs (F,F). The assumption $\zeta_{10} + \zeta_{01} \geq a > 0$ (required for uniform convergence) implies that Δ^2 is bounded away from zero. We may, therefore, consider the problem of testing $a \leq \Delta^2 \leq \delta_1$ against $\Delta^2 \geq \delta_2 > \delta_1$. For this problem, Theorem 3.4 is applicable; the asymptotic upper bound for the variance must be found and shown to satisfy Assumption $A^{**}.9$.

In Section 7 below, we consider the behavior of tests based on the statistics U_{mn} and W_{mn} relative to the problem of testing $\theta \leq \theta_1$ against $\theta \geq \theta_2$, when the pair of distributions (F, G) is assumed to be in a certain class $\mathcal{D}_0$ which is restricted in such a way that tests based on W_{mn} do not maximize the minimum power. It is pointed out that under this restriction, the asymptotic relative efficiency of $\mathcal{U}$ with respect to $\mathcal{W}$ is greater than one.

6. Large-Sample Properties.

In this section, we derive various properties of the families $\mathcal{U}$ and $\mathcal{W}$ for testing $\theta \leq \theta_1$ against $\theta \geq \theta_2$, in large samples.

In the first place, in order to use tests from the family $\mathcal{U}$ for testing $\theta \leq \theta_1$, we must have an approximation for λ_{mn}, the smallest number such that

$$(4.20) \qquad \sup_{(F,G)\in\mathcal{D}_1} P_{FG}\,(U_{mn} > \lambda_{mn}) \leq \alpha, \qquad 0 < \alpha < 1/2,$$

where $\mathcal{D}_1$ is the subset of $\mathcal{D}$ having $\theta(F,G) \leq \theta_1$.

Suppose $\mathcal{D}$ contains all pairs of continuous distributions (F,G) having $0 < \theta < 1$. It is required at various points in the following discussion that for every (m,n) and every θ, $\mathcal{D}_{mn}(\theta)$ contain a pair of distributions such that $\sigma^2(U_{mn})$ attains its maximum $\theta(1-\theta)/\min(m,n)$; or at least that $\mathcal{D}_{mn}(\theta)$ contain a sequence of pairs of distributions for which $\sigma^2(U_{mn})$ tends to $\theta(1-\theta) / \min(m,n)$. The assumption that $\mathcal{D}$ contains all pairs of continuous distributions implies this but is unnecessarily strong; in certain lemmas, weaker assumptions are explicitly stated.

Now, the number λ_{mn} defined by (4.20) will always exist. In fact, since mnU_{mn} takes on integer values only, $mn\,\lambda_{mn}$ will always be an integer. We obtain from (4.20)

$$(4.21) \qquad \sup_{(F,G)\in\mathcal{D}_1} P_{FG}(U_{mn} \geq \lambda_{mn} + 1/mn) \leq \alpha < \sup_{(F,G)\in\mathcal{D}_1} P_{FG}(U_{mn} \geq \lambda_{mn}).$$

Consider the normal approximation to the distribution of U_{mn}. For u such that mnu is an integer,

$$(4.22)\quad P_{FG}(U_{mn} \geq u) = 1-P_{FG}(U_{mn} \leq u-1/mn) = \overline{\Phi}\left[\frac{\theta-u+1/mn}{\sigma(U_{mn})}\right] + \epsilon'_{mn}(u;F,G)$$

where, in the notation of Theorem 2.3,

$$(4.23)\qquad \epsilon'_{mn}(u;F,G) = \epsilon_{mn}\left(\frac{u - 1/mn - \theta}{\sigma(U_{mn})}\ ;\ F,G\right)\ ;$$

$\epsilon'_{mn} \longrightarrow 0$ as m, n $\longrightarrow \infty$, and is uniformly bounded for (F,G) having $\zeta_{10} + \zeta_{01}$ bounded away from zero. Now the bound (2.29) for $\epsilon_{mn}(u;F,G)$ is a function of u having its maximum at u = 0. Let $\bar{\epsilon}_{mn}(0;F,G)$ denote the bound (2.29) evaluated at u = 0. Then

$$|\epsilon'_{mn}(u;\ F,\ G)| \leq \bar{\epsilon}_{mn}(0;\ F,\ G).$$

Lemma 4.2: Let $\mathcal{D}$ be a class of pairs of continuous distributions (F,G). Let $\theta(F,G) = \int F dG$, $\zeta_{10} = \int(1 - G - \theta)^2\, dF$, $\zeta_{01} = \int(F-\theta)^2 dG$. If $\rho > 0$ and $\theta(1 - \theta) \geq 1/12 + \rho$, then $\zeta_{10} + \zeta_{01} \geq \rho$.

Proof: For $(F,G) \in \mathcal{D}$ such that $\theta(F,G) = \theta$, $\zeta_{10}+\zeta_{01} = \int(F-G)^2 dF$ $+ 2\theta(1 - \theta) - 1/3 \geq \theta(1- \theta) - 1/12$.

Lemma 4.3: If $\lambda_{mn} > \theta_1$ and $0 < \theta(F,G) < \rho < \min(\theta_1, 1/2)$, then for $n \leq m$

$$P_{FG}(U_{mn} > \lambda_{mn}) < \rho(1 - \rho)/(\theta_1 - \rho)^2 n.$$

Proof: If $\theta < \theta_1$, then

$$P(U_{mn} > \lambda_{mn}) = P\left[\frac{U_{mn} - \theta}{\sigma(U_{mn})} > \frac{\lambda_{mn} - \theta}{\sigma(U_{mn})}\right] \leq P\left[\frac{U_{mn} - \theta}{\sigma(U_{mn})} \geq \frac{\sqrt{n}\ (\theta_1 - \theta)}{\sqrt{\theta\ (1 - \theta)}}\right]$$

$$\leq \theta(1 - \theta) \ / \ (\theta_1 - \theta)^2\ n$$

by the Tchebycheff-Bienaymé inequality.

By Lemma 4.3, $0 < \rho < \min\ (\theta_1, 1/2)$, $\theta_1 < \lambda_{mn}$, and $\rho(1-\rho)/(\theta_1-\rho)^2 n < \alpha$ imply (if $n \leq m$)

$$(4.24) \qquad \sup_{(F,G)\in\mathcal{D}_1} P_{FG}\ (U_{mn} > \lambda_{mn}) = \sup_{\rho \leq \theta \leq \theta_1} \sup_{(F,G)\in\mathcal{D}(\theta)} P_{FG}(U_{mn} > \lambda_{mn}).$$

Furthermore, $\theta_1(1 - \theta_1) > 1/12$ implies the existence of ρ such that $\rho(1 - \rho > 1/12$, and $0 < \rho < \min\ (\theta_1, 1/2)$.

Now, if $u > \theta_1 \geq \theta(F,G)$, we have for $n \leq m$

$$(4.25) \qquad \sup_{(F,G)\in\mathcal{D}(\theta)} \overline{\mathcal{D}}\left(\frac{\theta - u}{\sigma(U_{mn})}\right) = \overline{\mathcal{D}}\left(\frac{\sqrt{n}(\theta-u)}{\sqrt{\theta(1-\theta}}\right) .$$

Thus, by Lemmas 4.2 and 4.3, we obtain from (4.21), (4.22), (4.24), (4.25)

$$(4.26) \qquad \alpha < \overline{\mathcal{D}}\left[\frac{\sqrt{n}\ (\theta_1 - \lambda_{mn} + 1/mn)}{\sqrt{\theta_1(1-\theta_1)}}\right] + \tilde{\epsilon}_{mn}$$

where

$$(4.27) \qquad |\tilde{\epsilon}_{mn}| \leq \sup_{\rho \leq \theta \leq \theta_1} \sup_{(F,G)\in\mathcal{D}(\theta)} |\overline{\epsilon}_{mn}(0;F,G)\ |$$

and $1/2 - 1/\sqrt{6} < \rho < \min(\theta_1, 1/2)$, provided $n \leq m$, $\lambda_{mn} - 1/mn > \theta_1$, $\theta_1(1 - \theta_1) > 1/12$, and $\rho(1 - \rho)/(\theta_1 - \rho)^2 n < \alpha$. We will write

$$(4.28) \qquad B(\rho, \theta_1, m, n) = \sup_{\rho \leq \theta \leq \theta_1} \sup_{(F,G) \in \mathcal{D}(\theta)} |\bar{\epsilon}_{mn}(0; F, G)| .$$

By Lemma 4.2 and Theorem 2.2, $B(\rho, \theta_1, m, n) \longrightarrow 0$, as $m, n \longrightarrow \infty$ in such a way that $m/(m+n) \longrightarrow p$ $(1/2 \leq p < 1)$.

But the last statement holds no matter how m, n tend to infinity. This follows from the promised improvement of Lemmas 2.6 and 2.8, which is included in the proof of Lemma 4.4. This lemma is proved only for the statistic U_{mn} defined by (4.5); generalizations are immediate. Let

$$\epsilon_{mn}(F, G) = \sup_{-\infty < u < \infty} \left\{ P\left(U_{mn} - \theta) / \sigma(U_{mn}) \leq u\right) - \Phi(u) \right\} .$$

Lemma 4.4: If $\zeta_{10} + \zeta_{01} > 0$, then

$$(4.29) \qquad |\epsilon_{mn}| \leq \frac{(4.8)(n\mu_3 + m\nu_3)}{(n\zeta_{10} + m\zeta_{01})^{3/2}} + \min_{\epsilon > 0} \left\{ \epsilon(2\pi)^{-1/2} + \frac{1}{\epsilon^2} \frac{\zeta_{11} - \zeta_{10} - \zeta_{01}}{m\zeta_{10} + n\zeta_{01}} \right\},$$

and $\epsilon_{mn} \longrightarrow 0$ as $\min(m,n) \longrightarrow \infty$, uniformly for (F, G) in a class with $\zeta_{10} + \zeta_{01}$ bounded away from zero.

Proof: Only the third term of the bound in (4.29) differs from the expression (2.29). In place of Lemma 2.6, we may write, from (2.22),

$$\sigma^2(U_{mn}) - \frac{\zeta_{10}}{n} - \frac{\zeta_{01}}{m} \leq \frac{\zeta_{11} - \zeta_{10} - \zeta_{01}}{mn} .$$

We then obtain, in place of Lemma 2.8,

$$P(|Y_{mn} - W_{mn}| > \epsilon) \leq \frac{1}{\epsilon^2} \frac{\zeta_{11} - \zeta_{10} - \zeta_{01}}{m\zeta_{10} + n\zeta_{01}} .$$

(If $r > 1$, the corresponding improvements in these lemmas lead to more complicated expressions.)

It remains to show that the first term of the bound in (4.29) is $O(\max(m^{-1/2}, n^{-1/2}))$. But $\mu_3 \leq \zeta_{10}$, $\nu_3 \leq \zeta_{01}$. This completes the proof.

Now, for any $(F_0, G_0) \epsilon \mathcal{D}_1$, $\alpha \geq P_{F_0G_0}(U_{mn} > \lambda_{mn})$. In particular let $F_0 = K(x,\Theta_1)$, $G_0 = H(y)$, where H,K are defined by (4.10), (4.11), with μ taken to be Lebesgue measure. If $(F,G) = (F_0,G_0)$, nU_{mn} has binomial distribution with parameters (n, Θ_1) and we have (see Uspensky [27], p. 129)

$$\alpha \geq 1 - P_{F_0G_0}(nU_{mn} \leq n\lambda_{mn}) = 1 - P_{F_0G_0}(nU_{mn} \leq \lfloor n\lambda_{mn} \rfloor)$$

where $\lfloor u \rfloor$ stands for the greatest integer less than or equal to u, whence

$$\alpha \geq 1 - \overline{\Phi}(\ell_{mn}) - D(\ell_{mn}, n, \Theta_1) \tag{4.30}$$

where

$$\ell_{mn} = \frac{\lfloor n\lambda_{mn} \rfloor + 1/2 - n\Theta_1}{\sqrt{n\Theta_1(1 - \Theta_1)}} \tag{4.31}$$

and

$$D(z,n,\Theta) = \frac{1 - 2\Theta}{6\sqrt{2\pi n\Theta(1-\Theta)}}(1-z^2)e^{-z^2/2} + \tag{4.32}$$

$$+ \frac{1 - 2\theta}{6\sqrt{2\pi n\theta(1-\theta)}}\left(\frac{1+4n(n+1)\theta^2}{4n\theta(1-\theta)}\right)\exp\left\{\frac{-(1+2n\theta)^2}{8n\theta(1-\theta)}\right\} + \overline{\phi}\left(\frac{1/2 + n\theta}{\sqrt{n\theta(1-\theta)}}\right) - 1 + \omega$$

with

$$(4.33) \qquad |\omega| < \frac{(.13) + (.18)|1-2\theta|}{n(\theta(1-\theta)} + \exp\left\{-\frac{3}{2}\sqrt{n\theta(1-\theta)}\right\}$$

provided $n\theta(1 - \theta) \geq 25$. Let

$$(4.34) \qquad b(\theta,n) = \sup_{-\infty < z < \infty} D(z,n,\theta);$$

$b(\theta,n) \longrightarrow 0$ as $n \longrightarrow \infty$.

We are now in a position to obtain upper and lower bounds for λ_{mn}. Let $\lambda(u)$ be defined by $1 - u = \overline{\phi}(\lambda(u))$.

Theorem 4.4: Let λ_{mn} be defined by (4.20), let $B(\rho, \theta_1, m,n)$ be given by (4.28) and $b(\theta_1,n)$ by (4.34). Suppose $\theta_1(1-\theta_1) > 1/12$ and $n\theta_1(1-\theta_1) \geq 25$. Let ρ be chosen so that

$$(4.35) \qquad 1/2 - 1/\sqrt{6} < \rho < \min(\theta_1, 1/2).$$

Then

$$(4.36) \qquad \lambda_{mn} \geq \theta_1 + \left(\frac{\theta_1(1-\theta_1)}{n}\right)^{1/2} \cdot \lambda(\alpha + b(\theta_1, n)) - 1/2n \text{ if } n > N$$

and

$$(4.37) \qquad \lambda_{mn} < \theta_1 + \left(\frac{\theta_1(1-\theta_1)}{n}\right)^{1/2} \cdot \lambda(\alpha - B(\rho,\theta_1,m,n)) + 1/mn \text{ if } m \geq n > N,$$

where $N = N(\alpha,\theta_1,\rho)$ is an integer such that

(i) $-\alpha < b(\theta_1,n) < 1/2 - \alpha$ if $n > N$,

(ii) $B(\rho,\theta_1,m,n) < \alpha$ if $m \geq n > N$,

(iii) $n > \rho(1-\rho)/(\theta_1-\rho)^2\alpha$ if $n > N$,

(iv) $n > 3(1.02)^2/\lambda^2(\alpha + b(\theta_1,n))$ if $n > N$.

Proof: We verify first the existence of an integer N satisfying (i)-(iv): (i) and (iii) need no comment; (ii) is a consequence of Lemmas 4.3 and 4.4, given (4.35); (iv) will be possible since by (i) the right-hand side of the inequality is positive, and decreases with n.

From (4.30), (4.34), and (i),

$$\underline{\beta}(\ell_{mn}) \geq 1 - \alpha - b(\theta_1,n),$$

$$\ell_{mn} \geq \lambda(\alpha + b(\theta_1,n)) .$$

Now, from (4.31),

$$\sqrt{n\theta_1(1-\theta_1)}\ \ell_{mn} \leq n\lambda_{mn} + 1/2 - n\theta_1,$$

so that (i) implies

$$(4.38)\quad n\lambda_{mn} + 1/2 - n\theta_1 \geq \sqrt{n\theta_1(1-\theta_1)}\ \lambda(\alpha + b(\theta_1,n)) > 0,$$

from which we obtain (4.36).

Now, from (4.38) and (i), $\lambda_{mn} - 1/mn > \theta_1$ if $N < n \leq m$ and

$$(4.39)\qquad n^{-1/2} + 2n^{-3/2} \leq 2\sqrt{\theta_1(1-\theta_1)}\ \lambda\ (\alpha + b(\theta_1,n)).$$

Thus, from (4.26) and (4.27), $\theta_1(1-\theta_1) > 1/12$ with (4.35), (i), (4.39), and (iii) implies

$$\underline{\beta}\left[\frac{\sqrt{n}\ (\lambda_{mn} - 1/mn - \theta_1)}{\sqrt{\theta_1(1-\theta_1)}}\right] < 1 - \alpha + B(\rho,\theta_1,m,n)$$

when $m \geq n > N$. With (ii), we obtain (4.37).

Finally, we show that (iv) is sufficient for (4.39). We require

$$n^{-1/2}\left(\frac{n+2}{n}\right) \leq 2\sqrt{\theta_1(1-\theta_1)}\ \lambda(\alpha + b(\theta_1, n))$$

if $n > N$. We have assumed $\theta_1(1 - \theta_1) > 1/12$; $n\theta_1(1-\theta_1) \geq 25$ implies $n \geq 100$; hence this inequality surely holds if (i) holds and

$$n^{1/2} \geq (1.02)\sqrt{3} \,/\, \lambda(\alpha + b(\theta_1, n)).$$

This completes the proof.

For large m and n, we may on the basis of Theorem 4.4 use the approximation $\lambda_{mn} \cong \mu_{mn}$, where

$$\mu_{mn} = \theta_1 + \left(\frac{\theta_1(1-\theta_1)}{\min(m,n)}\right)^{1/2} \cdot \lambda(\alpha)\ . \tag{4.40}$$

Remarks.

(1) Trial computations show that we must have N close to one million in order to satisfy (ii) for $\alpha = .05$, even in the favorable case that $\theta_1 = 1/2$, $m = n$.

(2) For the values of α which are usually of interest, condition (iv) will not be a significant restriction.

(3) $B(\rho,\theta_1,m,n)$ will not be very sensitive to changes in ρ as $\rho \longrightarrow \theta_1$. Thus, (i) and (ii), particularly the latter, are in practice the conditions which must be satisfied before the theorem can be applied.

(4) By symmetry, we may of course interchange m and n in Theorem 4.4.

The next problem is to obtain some information as to the sample sizes m and n which would be required to solve the problem $(\{\mathcal{D}_{1,mn}\}, \{\mathcal{D}_{2,mn}\}, \alpha, \beta)$ with a test based on the statistic U_{mn}.

Let $\hat{\mathcal{U}}$ be a family of tests of the form

$$\hat{\phi}(z_{mn}|U_{mn}) = \begin{cases} 1 & \text{if } U_{mn} > \lambda_{mn} \\ 0 & \text{if } U_{mn} \leq \lambda_{mn}, \end{cases} \tag{4.41}$$

where λ_{mn} is determined by

$$\sup_{(F,G)\in \mathcal{D}_1} E_{FG}\, \hat{\phi}(Z_{mn}|U_{mn}) \leq \alpha, \tag{4.42}$$

and $\mathcal{N}(\hat{\mathcal{U}})$ contains all pairs of positive integers. We have defined $\mathcal{M}(\hat{\mathcal{U}})$ as that subset of $\mathcal{N}(\hat{\mathcal{U}})$ in which

$$\inf_{(F,G)\in \mathcal{D}_2} E_{FG}\, \hat{\phi}(Z_{mn}|U_{mn}) \geq 1 - \beta. \tag{4.43}$$

Lemma 4.5: Suppose the pair of (continuous) distributions $(F(x), G(y))$ is in $\mathcal{D}$ if and only if the pair $(1 - G(-x), 1 - F(-y))$ is in $\mathcal{D}$. Then, for the family of tests $\hat{\mathcal{U}}$, $\lambda_{mn} = \lambda_{nm}$, and $(m,n)\in\mathcal{M}(\hat{\mathcal{U}})$ if and only if $(n,m)\in\mathcal{M}(\hat{\mathcal{U}})$.

Proof: In the notation developed in Section 3 of this chapter, let u(m,n) represent an arrangement of n x's and m y's. Let u'(m,n) be the arrangement obtained from u(m,n) by reversing the order, and let $\bar{u}(m,n)$ be the arrangement of m x's and n y's obtained from u(m,n) by replacing the x's by y's and the y's by x's. We have seen that if $z_{mn} \sim u(m,n)$ and $z'_{mn} \sim u'(m,n)$, then $U_{mn}(z_{mn}) = 1 - U_{mn}(z'_{mn})$. Now, also, if $\bar{z}_{nm} \sim \bar{u}(m,n)$, $U_{mn}(z_{mn}) = 1 - U_{nm}(\bar{z}_{nm})$. Thus, $U_{mn}(z_{mn})$

$= U_{nm}(\bar{z}'_{nm})$, where $\bar{z}'_{nm} \sim \bar{u}'(m,n)$, which is obtained from $\bar{u}(m,n)$ by reversing the order.

Let $F'(x) = 1 - G(-x)$, $G'(y) = 1 - F(-y)$. If F,G are the distributions of X, Y, then F', G' are the distributions of -Y, -X. Thus $\theta(F,G) = P(X < Y) = \theta(F',G')$. Furthermore,

$$P_{FG}(Z_{mn} \sim u(m,n)) = P_{F'G'}(Z_{nm} \sim \bar{u}'(m,n)).$$

It follows that, for any constant c,

$$\sup_{(F,G)\in\mathcal{D}_1} P_{FG}(U_{mn} > c) = \sup_{(F,G)\in\mathcal{D}_1} P_{FG}(U_{nm} > c)$$

and

$$\inf_{(F,G)\in\mathcal{D}_2} P_{FG}(U_{mn} > c) = \inf_{(F,G)\in\mathcal{D}_2} P_{FG}(U_{nm} > c).$$

This completes the proof.

By Lemma 4.5, we may restrict our attention to tests based on samples of sizes m and n, with $n \leq m$. Suppose also that n is so large that (i)-(iv) of Theorem 4.4 hold. Consider the test

$$\phi^*(z_{mn} \mid U_{mn},\rho) = \begin{cases} 1 & U_{mn} > \lambda^*_{mn} \\ 0 & U_{mn} \leq \lambda^*_{mn} \end{cases} \tag{4.44}$$

where ρ, θ_1 satisfy the conditions of Theorem 4.4, and

$$\lambda^*_{mn} = \theta_1 + \left(\frac{\theta_1(1-\theta_1)}{n}\right)^{1/2} \cdot \lambda(\alpha - B(\rho,\theta_1,m,n)) + 1/mn. \tag{4.45}$$

By this choice of λ^*_{mn}, we have

(4.46) $\quad E_{FG}\,\phi^*(Z_{mn}|U_{mn},\,\rho) \le \alpha \quad \text{for all } (F,G) \in \mathcal{D}_1.$

Let $\mathcal{U}^*$ denote the family of tests of the form (4.44) with $\mathcal{N}(\mathcal{U}^*) = \{(m,n):\, m \ge n > N\}$, where N is given by Theorem 4.4. We will obtain some information about $\mathcal{M}(\mathcal{U}^*)$ for $0 < \beta < 1/2$.

Lemma 4.6: If $u < \theta_2$ and $\max(\theta_2, 1/2) < \rho' < \theta(F,G) < 1$, then if $n \le m$

$$P_{FG}(U_{mn} > u) > 1 - \rho'(1-\rho')/(\rho' - \theta_2)^2\, n.$$

The proof is analogous to the proof of Lemma 4.3.

Now $\theta_2 > \lambda^*_{mn}$ if and only if

(4.47) $$\theta_2 - \theta_1 > \left(\frac{\theta_1(1-\theta_1)}{n}\right)^{1/2} \cdot \lambda(\alpha - B(\rho,\theta_1,m,n)) + 1/mn.$$

By Lemmas 4.2 and 4.6, if (4.47) holds, $\theta_2(1-\theta_2) > \rho'(1-\rho') > 1/12$, $\rho' > \max(\theta_2, 1/2)$, and $n > \rho'(1-\rho')/(\rho'-\theta_2)^2\beta$, $\mathcal{M}(\mathcal{U}^*)$ is given by the set of integers (m,n) such that

(4.48) $$Q_{mn} = \inf_{\theta_2 \le \theta \le \rho'} \; \inf_{(F,G)\in\mathcal{D}(\theta)} E_{FG}\,\phi^*(Z_{mn}\,|\,U_{mn},\rho) \ge 1 - \beta.$$

We will derive upper and lower bounds for the quantity Q_{mn} in (4.48), and hence sets $\mathcal{M}_0(\mathcal{U}^*) \subset \mathcal{M}(\mathcal{U}^*) \subset \mathcal{M}_1(\mathcal{U}^*)$.

Recall the distributions (F_0, G_0) for which nU_{mn} has binomial distribution. If we replace θ_1 by θ_2, we obtain

$$Q_{mn} \le P_{F_0G_0}(U_{mn} > \lambda^*_{mn}) \le 1 - P_{F_0G_0}(nU_{mn} \le \lceil n\lambda^*_{mn} \rceil) < 1 - \underline{\beta}(\ell'_{mn}) - D(\ell'_{mn}, n, \theta_2)$$

where

$$(4.49) \qquad \ell'_{mn} = \frac{\lfloor n\lambda^*_{mn} \rfloor + 1/2 - n\theta_2}{\sqrt{n\theta_2(1-\theta_2)}}$$

and $D(z,n,\theta)$ is defined by (4.32). Let

$$(4.50) \qquad b'(\theta,n) = \inf_{-\infty < z < \infty} D(z,n,\theta) .$$

Then

$$(4.51) \qquad Q_{mn} \leq Q^{(1)}_{mn} = \overline{\Phi}(-\ell'_{mn}) - b'(\theta_2, n).$$

Now, from (4.22), we have

$$(4.52) \quad P(U_{mn} > \lambda^*_{mn}) = P(U_{mn} \geq \frac{1}{mn}\lceil mn\lambda^*_{mn} \rceil + 1/mn)$$

$$= \overline{\Phi}\left[\frac{\theta - \lceil mn\, \lambda^*_{mn} \rceil / mn}{\sigma(U_{mn})} \right] + \epsilon'_{mn} .$$

If $u < \theta$, $n \leq m$, then

$$(4.53) \qquad \inf_{(F,G)\in\mathcal{D}(\theta)} \overline{\Phi}\left(\frac{\theta - u}{\sigma(U_{mn})}\right) = \overline{\Phi}\left(\frac{\sqrt{n}\,(\theta-u)}{\sqrt{\theta(1-\theta)}}\right) ;$$

and (4.47) implies $\theta_2 > \lceil mn\, \lambda^*_{mn} \rceil / mn$. Hence, given the conditions under which (4.48) holds, we have from (4.52), (4.53)

$$(4.54) \qquad Q_{mn} \geq Q^{(0)}_{mn} = \overline{\Phi}\left[\frac{\sqrt{n}\,(\theta_2 - \lambda^*_{mn})}{\sqrt{\theta_2(1-\theta_2)}}\right] - B'(\rho',\theta_2,m,n)$$

where from (4.23)

$$(4.55)\qquad B'(\rho',\theta_2,m,n) = \sup_{\theta_2 \le \theta \le \rho'} \sup_{(F,G)\in\mathcal{D}(\theta)} \bar{\epsilon}_{mn}(0;F,G).$$

We now state and complete the proof of

Theorem 4.5: Let $\theta_j(1-\theta_j) \ge 1/12 + \eta$, $\eta > 0$, $n\,\theta_j(1-\theta_j) \ge 25$ $(j = 1, 2)$, and choose ρ, ρ' so that $\rho(1-\rho) > 1/12$, $\rho'(1-\rho') > 1/12$, and $\rho < \min(\theta_1, 1/2)$, $\rho' > \max(\theta_2, 1/2)$. Let $b(\theta_1,n)$, $B(\rho,\theta_1,m,n)$, $b'(\theta_2,n)$, $B'(\rho',\theta_2,m,n)$ be given by (4.34), (4.28), (4.50), (4.55), respectively. Let N be the integer given by Theorem 4.4 and let N' be an integer such that $N' \ge N$ and

(i') $-(1/2 - \beta) < b'(\theta_2, n) < \beta$ if $n > N'$,

(ii') $B'(\rho',\theta_2, m, n) < \beta$ if $m \ge n > N'$,

(iii') $n > \rho'(1-\rho') \,/\, (\rho' - \theta_2)^2\beta$ if $n > N'$,

(iv') $n > 3(1.02)^2 \,/\, \lambda^2(\beta - b')$ if $n > N'$,

(v) $n > \frac{1}{(\theta_2-\theta_1)^2}\left(\sqrt{\theta_1(1-\theta_1)}\,\lambda(\alpha - B) + 1/m\sqrt{n}\right)^2$ if $m \ge n > N'$.

Let M_0, M_1 be integers defined by

$$(4.56)\qquad n \ge \frac{1}{(\theta_2-\theta_1)^2}\left(\sqrt{\theta_2(1-\theta_2)}\,\lambda\,(\beta - B') + \sqrt{\theta_1(1-\theta_1)}\,\lambda(\alpha - B) + 1/m\sqrt{n}\right)^2 \quad \text{if } m \ge n \ge M_0,$$

$$(4.57)\qquad n \ge \frac{1}{(\theta_2-\theta_1)^2}\left(\sqrt{\theta_2(1-\theta_2)}\,\lambda(\beta-b') + \sqrt{\theta_1(1-\theta_1)}\,\lambda(\alpha-B) + 1/m\sqrt{n} - 1/2\sqrt{n}\right)^2 \text{ if } m \ge n \ge M_1.$$

Let $\theta_2 - \theta_1$ be so small that $M_0 \ge M_1 > N'$. Then

$\mathcal{M}_0(\mathcal{U}^*) \subset \mathcal{M}(\mathcal{U}^*) \subset \mathcal{M}_1(\mathcal{U}^*)$, where $\mathcal{M}_j(\mathcal{U}^*) = \{(m,n): m \geq n \geq M_j\}$ $(j = 0,1)$.

Proof: From (4.51), (4.54) we have $Q_{mn}^{(1)} \geq Q_{mn} \geq Q_{mn}^{(0)}$, and we obtain the sets $\mathcal{M}_j(\mathcal{U}^*)$ from the relations $Q_{mn}^{(j)} \geq 1 - \beta$ $(j = 0, 1)$.

Thus, from (4.51), (4.49)

$$\overline{\Phi}(-\ell'_{mn}) \geq 1 - \beta + b',$$

$$-\ell'_{mn} \geq \lambda(\beta - b'),$$

$$n\theta_2 - 1/2 - \lfloor n\lambda^*_{mn} \rfloor \geq \sqrt{n\theta_2(1 - \theta_2)}\,\lambda(\beta - b'),$$

and the set in which $Q_{mn}^{(1)} \geq 1 - \beta$ will be contained in the set where

$$(4.58) \qquad n\theta_2 + 1/2 - n\lambda^*_{mn} \geq \sqrt{n\theta_2(1 - \theta_2)}\;\lambda(\beta - b').$$

The inequality (4.58) defines M_1.

From (4.54), it is sufficient for $Q_{mn}^{(0)} \geq 1 - \beta$ that

$$\overline{\Phi}\left[\frac{\sqrt{n}\,(\theta_2 - \lambda^*_{mn})}{\sqrt{\theta_2(1 - \theta_2)}}\right] \geq 1 - \beta + B',$$

$$(4.59) \quad \sqrt{n}(\theta_2 - \theta_1) \geq \sqrt{\theta_2(1 - \theta_2)}\;\lambda(\beta - B') + \sqrt{\theta_1(1-\theta_1)}\lambda(\alpha - B) + 1/m\sqrt{n}.$$

The inequality (4.59) defines M_0.

We may use Q_{mn} to obtain $\mathcal{M}(\mathcal{U}^*)$ since all the conditions of (4.48) have been assumed, and (4.47) follows from (v).

It remains to verify that all of the conditions of Theorem 4.5 can be satisfied.

The quantities $|b(\theta_1,n)|$, $|b'(\theta_2,n)|$, $B(\rho,\theta_1,m,n)$, $B'(\rho',\theta_2,m,n)$ can be uniformly bounded for $\theta_j(1 - \theta_j) \geq 1/12 + \eta$, $j = 1, 2$. Hence (i), (i'), (ii), (ii'), (iv), and (iv') can be satisfied with an integer N" independent of θ_1, θ_2.

Since $\theta_j(1-\theta_j)$ is bounded away from 1/12 ($j = 1, 2$), (iii) and (iii') can be satisfied with N" independent of θ_1, θ_2.

Now $M_0 \geq M_1$ by construction , and we can choose $(\theta_2 - \theta_1)$ so small that $M_1 > N'$ provided M_1 is greater than the quantity on the right-hand side of (v). This will be so if

$$\sqrt{\theta_2(1-\theta_2)}\ \lambda(\beta - b') - 1/(2\sqrt{n}) > 1/(m\sqrt{n}) \tag{4.60}$$

when $m \geq n > N'$, and (iv') is sufficient for (4.60). This completes the proof.

Theorem 4.5 may be described in the following summary: First, a set of conditions on sample sizes is necessitated by the crudeness of our bounds on the deviation from the normal approximation (i.e., all except condition (v)). Second, we choose $(\theta_2 - \theta_1)$ so small that condition (v) is sufficient for the others. Then, for the family $\mathcal{U}^*$ with $\mathcal{N}(\mathcal{U}^*)$ defined by (v), we have bounds for $\mathcal{M}(\mathcal{U}^*)$.

Now, for $c = (c_1, c_2)$, $c_1 \geq 0$, $c_2 \geq 0$, $c_1 + c_2 = 1$, we have defined

$$N_c(\mathcal{U}^*) = \min_{(m,n)\in\mathcal{M}(\mathcal{U}^*)} (c_1 n + c_2 m). \tag{4.61}$$

It follows from Theorem 4.5 that (under the conditions stipulated there) for every c

$$M_1 \leq N_c(\mathcal{U}^*) \leq M_0. \tag{4.62}$$

In other words, an experimenter using a test from the family $\mathcal{U}^*$ would need to take no more than M_0 observations in order to solve the problem $(\{\mathcal{D}_{1,mn}\}, \{\mathcal{D}_{2,mn}\}, \alpha, \beta)$.

By the symmetry shown in Lemma 4.5, (4.62) holds for the family $\mathcal{U}^{**}$ defined as $\mathcal{U}^*$ with n and m interchanged, and hence for the union of the two families.

On the basis of Theorem 4.5, one might use the approximation $N_c(\mathcal{U}^*) \cong \bar{N}(\mathcal{U}^*)$, where

$$\bar{N}(\mathcal{U}^*) = \frac{1}{(\theta_2-\theta_1)^2}\left[\sqrt{\theta_2(1-\theta_2)}\,\lambda(\beta) + \sqrt{\theta_1(1-\theta_1)}\,\lambda(\alpha)\right]^2. \tag{4.63}$$

This approximation, while more symmetric, would in practice by very little different from the asymptotic

$$N_c(\mathcal{U}^*) \sim \frac{\theta_1(1-\theta_1)}{(\theta_2-\theta_1)^2}\,(\lambda(\alpha) + \lambda(\beta))^2.$$

In order to make certain comparisons, we perform a similar set of calculations for the family $\mathcal{W}$. Only the definitions and results are given here.

Suppose $n \leq m$. Let $n\lambda_n$ be the integer defined by

$$P_{\theta_1}(W_{mn} > \lambda_n) \leq \alpha < P_{\theta_1}(W_{mn} > \lambda_n - 1/n). \tag{4.64}$$

If

$$-\alpha < b'(\theta_1, n) < 1 - \alpha, \tag{4.65}$$

then

$$(4.66) \qquad \lambda_n < \lambda_n^* = \theta_1 + \left(\frac{\theta_1(1-\theta_1)}{n}\right)^{1/2} \cdot \lambda(\alpha + b'(\theta_1,n)) + 1/2n \ ,$$

where $b'(\theta,n)$ is defined by (4.50).

Let $\mathcal{W}^*$ be the family of tests given by

$$(4.67) \qquad \phi^*(z_{mn}|W_{mn}) = \begin{cases} 1 & \text{if} \quad W_{mn} > \lambda_n^* \\ 0 & \text{if} \quad W_{mn} \leq \lambda_n^* \end{cases} ,$$

with $\mathcal{N}(\mathcal{W}^*)$ containing pairs of positive integers (m,n) such that $n \leq m$ and (4.65) holds. Now $\mathcal{M}(\mathcal{W}^*)$ is the subset of $\mathcal{N}(\mathcal{W}^*)$ in which

$$(4.68) \qquad P_{\theta_2}(W_{mn} > \lambda_n^*) \geq 1 - \beta.$$

But, since the distribution of W_{mn} depends only on the smaller of the two sample sizes, (4.68) will also define the integer $N_c(\mathcal{W}^*) = N(\mathcal{W}^*)$.

We obtain $M_1' \leq N_c(\mathcal{W}^*) \leq M_0'$, where M_1', M_0' are defined as follows:

$$(4.69) \quad n \geq \frac{1}{(\theta_2-\theta_1)^2}\left[\sqrt{\theta_1(1-\theta_1)}\,\lambda(\alpha+b'(\theta_1,n)) + \sqrt{\theta_2(1-\theta_2)}\,\lambda(\beta-b'(\theta_2,n))\right]^2$$
$$\text{if } n \geq M_1' \ ,$$

provided (4.65) holds and

$$(4.70) \qquad -(1-\beta) < b'(\theta_2,n) < \beta;$$

$$(4.71) \quad n \geq \frac{1}{(\theta_2-\theta_1)^2}\left[\sqrt{\theta_1(1-\theta_1)}\,\lambda(\alpha+b'(\theta_1,n)) + \sqrt{\theta_2(1-\theta_2)}\,\lambda(\beta-b(\theta_2,n)) + n^{-1/2}\right]^2 \quad \text{if } n \geq M_0' \ ,$$

provided (4.65) holds and

$$-(1-\beta) < b(\theta_2,n) < \beta, \tag{4.72}$$

where $b(\theta,n)$ is defined by (4.34).

7. Large-Sample Comparison of $\mathcal{U}$ and $\mathcal{W}$.

For testing $\theta \leq \theta_1$ against $\theta \geq \theta_2$, we have seen that tests from $\mathcal{W}$ are preferable in very small samples; also that the asymptotic relative efficiency of the families $\mathcal{U}$ and $\mathcal{W}$ is one. Intuitively, nevertheless, one would expect to find that tests from $\mathcal{U}$, which appear to utilize more of the "information" in the samples, would be better in some sense than tests from $\mathcal{W}$. The following discussion tends to confirm this expectation. It is valid only under restrictions which imply that the sample sizes are very large indeed (one million or more, say). Presumably, however, a more precise approximation to the distribution of U_{mn} would permit us to demonstrate the same propositions for more reasonable sample sizes.

We say "J_1 is more efficient that J_2" if the index of efficiency of the former is smaller than that of the latter, with respect to a specified problem. For a given class $\mathcal{D}$ of pairs of distributions (F,G), we write $(\mathcal{D}, \theta_1, \theta_2, \alpha,\beta)$ to represent the problem $(\{\mathcal{D}_{1,mn}\}, \{\mathcal{D}_{2,mn}\}, \alpha, \beta)$, since $\mathcal{D}_{mn}$ contains distributions F_{mn} of the form

$$F_{mn}(\underline{z}) = \prod_{i=1}^{n} \prod_{j=1}^{m} F(x_i)\, G(y_j), \quad (F,G)\in\mathcal{D},$$

and $\mathcal{D}_{1,mn}$, $\mathcal{D}_{2,mn}$ are the subsets of $\mathcal{D}_{mn}$ with $\theta(F,G) \leq \theta_1$ and $\theta(F,G) \geq \theta_2$, respectively.

(I) $\mathcal{W}^*$ is more efficient than $\mathcal{U}^*$ for the problem $(\mathcal{D}, \theta_1, \theta_2, \alpha, \beta)$ if $\mathcal{D}$ contains all pairs of continuous distributions, provided $(\theta_2 - \theta_1)$ is chosen so small that the sample sizes must be large enough to satisfy the conditions of Theorem 4.5, as well as certain conditions implied below.

Consider M_1, M_0', the lower and upper bounds, respectively, for $N(\mathcal{U}^*)$, $N(\mathcal{W}^*)$. (We drop the subscript c in $N_c(\mathcal{T})$ since the bounds are the same for every c.) We show that if m and n are sufficiently large, $M_1 > M_0'$. From (4.57), (4.71), it is sufficient to show that

$$(4.73)\quad \sqrt{\theta_2(1-\theta_2)}\left[\lambda(\beta - B'(\rho', \theta_2, m, n)) - \lambda(\beta - b(\theta_2, n))\right] + \sqrt{\theta_1(1-\theta_1)}\left[\lambda(\alpha - B(\rho, \theta_1, m, n)) - \lambda(\alpha + b'(\theta_1, n))\right] > n^{-1/2}.$$

Now $\lambda(u)$ is a decreasing function of u and B, $B' = O(n^{-1/3})$ while $b, b' = O(n^{-1/2})$, so that the quantities in square brackets are positive and in fact have positive lower bounds of order $n^{-1/3}$. Hence (4.73) will hold if m,n are sufficiently large ($n \leq m$).

Proposition (I) may be interpreted to mean that, if we consider test families $\mathcal{U}^*$, $\mathcal{W}^*$ (in order to be certain that the level of significance is at most α) we may solve the problem $(\mathcal{D}, \theta_1, \theta_2, \alpha, \beta)$ "more economically" (i.e., with smaller samples) by using a test from $\mathcal{W}^*$.

This result is to be expected, since we know that a test based on W_{mn} maximizes the minimum power for testing $\theta \leq \theta_1$ against $\theta \geq \theta_2$, while tests based on U_{mn} do not in general have this property.

At this point, therefore, it is of interest to compare the behavior of tests from the two families with respect to slightly modified problems, for which tests from $\mathcal{W}$ do not maximize the minimum power.

In the remainder of this section, we consider only families of tests based on samples of equal size. With this restriction, we have the variance of U_{mn} depending on only one parameter besides θ, namely,

$$(4.74) \qquad n\,\sigma^2(U_{nn}) = 2\theta(1-\theta) + \Delta^2 - 1/3 - 1/n\,[\theta(1-\theta)+\Delta^2 - 1/3],$$

where $\Delta^2 = \Delta^2(F,G) = \int (F - G)^2\, dF$.

Now $n\,\sigma^2(U_{nn})$ attains its maximum $\theta(1-\theta)$, for $(F,G)\in\mathcal{D}(\theta)$, when $\Delta^2(F,G) = 1/3 - \theta(1-\theta)$. But the pairs of distributions for which this maximum is attained are rather unusual: in a typical case, the random variable distributed according to F must take on values in two intervals separated by a third interval in which must be found the entire range of the random variable distributed according to G.

Suppose we impose the condition that, for all $(F,G)\in\mathcal{D}(\theta)$,

$$(4.75) \qquad \Delta^2(F,G) \leq 1/3 - 4/3\,\theta(1 - \theta)\ .$$

First, under this condition, the proof of Theorem 4.1 fails, as do also the proofs of the modifications and extensions of that theorem mentioned in Section 2 of this chapter. Second, under this condition, we retain in $\mathcal{D}(\theta)$ pairs of distributions of several common types, among which are the following:

(1) Normal. Let $N(\mu,\sigma)$ denote the normal distribution with mean μ and variance σ^2. Let $F = N(0,\sigma)$, $G = N(\mu\sigma,\sigma)$. Then θ,Δ^2 depend only on μ (cf. Sundrum [25]), and (4.75) holds.

(2) Rectangular. Let R(a,b) denote the uniform distribution on the interval (a,b). (4.75) holds if we let F = R(0,1), G = R(δ,1+δ), $\delta \geq 0$.

3. Exponential. Let E(a,b) denote the distribution whose cdf is

$$E(x;a,b) = \begin{cases} 0 & x < a \\ 1-\exp\left\{\frac{-(x-a)}{b}\right\} & x \geq a \end{cases}.$$

Then (4.75) holds if we let F = E(0,b), G = E(ab,b); or if we let F = E(a,1), G = E(ab,b).

(4) "Lehmann's alternatives," see [17]. (4.75) holds if (F,G) = (F,F^a), a > 0.

Finally, under the restriction (4.75), we have

$$\max_{(F,G)\in\mathcal{D}(\theta)} n\sigma^2(U_{nn}) = \frac{n+1/2}{n}\,\frac{2}{3}\,\theta(1-\theta).$$

Let $\hat{\mathcal{D}}_2$ denote the subset of $\mathcal{D}$ containing all pairs of continuous distributions (F,G) with $0 < \theta(F,G) < 1$ and $\Delta^2(F,G) \leq 1/3 - 4\theta(1-\theta)/3$ if $\theta(F,G) = \theta \geq \theta_2$. In the problem $(\hat{\mathcal{D}}_2, \theta_1, \theta_2, \alpha, \beta)$, then, the null hypothesis is unchanged while the "unusual" distributions are eliminated from the set of alternatives. This is a relatively conservative modification of the problem.

As in Section 6, we find an integer $\hat{M}_0$ which is an upper bound for $N(\psi^*)$ relative to the problem $(\hat{\mathcal{D}}_2, \theta_1, \theta_2, \alpha, \beta)$. Under the conditions of Theorem 4.5, $\hat{M}_0$ is defined by

$$(4.76)\quad n \geq \frac{1}{(\theta_2-\theta_1)^2}\left(\frac{n+1/2}{n}\right)\left[\sqrt{\theta_1(1-\theta_1)}\,\lambda(\alpha-B(\rho,\theta_1,n,n)) + \right.$$

$$+\sqrt{2/3\ \theta_2(1-\theta_2)}\ \lambda(\beta - B'(\rho', \theta_2, n, n)) + n^{-3/2}\Big]^2 \text{ if } n \geq \hat{M}_0.$$

(II) $\mathcal{U}^*$ is more efficient than $\mathcal{W}^*$ for the problem $(\hat{\mathcal{D}}_2, \theta_1, \theta_2, \alpha, \beta)$, provided $(\theta_2 - \theta_1)$ is so small that n is sufficiently large to satisfy the conditions of Theorem 4.5 as well as certain conditions implied below.

The index of efficiency of $\mathcal{W}^*$ is unchanged for this problem, since it depends only on θ_1, θ_2, α, β. We show that $\hat{M}_0$ is smaller than the lower bound M_1' for $N(\mathcal{W}^*)$. By (4.76), (4.69), it is sufficient to show that

$$(4.77)\quad \sqrt{\theta_1(1-\theta_1)}\left[\lambda(\alpha + b'(\theta_1, n)) - \left(\frac{n+1/2}{n}\right)^{1/2}\lambda(\alpha - B(\rho, \theta_1, n, n))\right]$$

$$+\sqrt{\theta_2(1-\theta_2)}\left[\lambda(\beta - b'(\theta_2, n)) - \left(\frac{n+1/2}{n}\right)^{1/2}\left(\frac{2}{3}\right)^{1/2}\lambda(\beta - B'(\rho', \theta_2, n, n))\right]$$

$$\geq 2n^{-3/2}.$$

But the second term of (4.77) has a positive lower bound of order one while the first has a negative lower bound of order $n^{-1/3}$.

Now, if we further restrict $\mathcal{D}$ to a subset $\mathcal{D}_0$ in which $\Delta^2(F,G) \leq 1/3 - 4/3\ \theta(1-\theta)$ for every θ, we may obtain a new (and smaller) upper bound for λ_{mn} and hence a new family of tests $\mathcal{U}_0$ having level of significance α for the problem $(\mathcal{D}_0, \theta_1, \theta_2, \alpha, \beta)$. Proposition (II) will hold a fortiori with $\mathcal{U}^*$, $\hat{\mathcal{D}}_2$ replaced by $\mathcal{U}_0$, $\mathcal{D}_0$.

Furthermore, it is evident that the asymptotic relative efficiency of $\mathcal{U}$ with respect to $\mathcal{W}$ is greater than one for the problem $(\mathcal{D}_0, \theta_1, \theta_2, \alpha, \beta)$.

8. Discontinuous Distributions and Ties.

In this chapter we have assumed (except in Theorem 4.1) that the two samples are drawn from continuous distributions F and G. This assumption may be relaxed in certain ways.

The small-sample exact distributions of U_{mn} and W_{mn}, considered in Sections 2 to 4, were calculated for continuous distributions. Similar calculations might be made in the more general case, but this has not been done.

Some of the large-sample and asymptotic results may, however, be modified without too much difficulty.

Let $\mathcal{D}'$ be the class of all pairs of distributions (F,G) such that F and G have no common point of discontinuity. Nothing needs to be changed in the foregoing discussion if the class $\mathcal{D}$ of all pairs of continuous distributions is replaced by the union of $\mathcal{D}$ and $\mathcal{D}'$. Stoker [22] has pointed this out in connection with the bounds for deviation from normality of the distribution of U_{mn}. That there is no difficulty with uniform convergence will be clear in the discussion of a more general case below.

We note that for $(F,G) \in \mathcal{D}+\mathcal{D}'$, we retain the properties of the functional $\theta(F,G) = P(X < Y)$ which recommend it as a measure of the difference between the two distributions, namely: If $(F,G) \in \mathcal{D}+\mathcal{D}'$ and $F(x) = G(x)$, $-\infty < x < \infty$, then $\theta(F,G) = 1/2$; if $F(x) \geq G(x)$, $-\infty < x < \infty$, then $\theta(F,G) \geq 1/2$. These properties are retained because $P(X = Y) = 0$.

The situation is changed, however, if we admit a wider class of pairs of discontinuous distributions. Theorem 4.1 will still hold,

but $\theta(F,G) = P(X < Y)$ is no longer a reasonable parameter. In an extreme case, we may even have $\theta(F,F) = 0$. In the remainder of this section, we let $\mathcal{D}$ represent an arbitrary class of pairs of distributions (F,G) having $0 < \theta(F,G) < 1$.

Since, in practice, it is frequently unrealistic to suppose that $P(X = Y) = 0$, and that there will be no tied observations, the Wilcoxon-Mann-Whitney is usually (cf. [9]) defined by

$$mnU_{mn} = \sum_{i=1}^{n} \sum_{j=1}^{m} \phi'(x_i, y_j)$$

where

$$\phi'(x,y) = \begin{cases} 1 & x < y \\ 1/2 & x = y \\ 0 & x < y \end{cases} .$$

The corresponding functional is

$$\theta'(F,G) = P(X < Y) + 1/2\, P(X = Y) = 1/2 \int [F(x+0) + F(x-0)]\, dG(x).$$

The functionals ζ'_{10}, ζ'_{01}, ζ'_{11} may also be defined with ϕ replaced by ϕ', as is clear from Section 3, Chapter II. Nothing is affected in Chapters II and III, except that the examples in Sections 6 and 8 of Chapter III are valid only under the condition that ϕ takes values zero or one only. The argument in Sections 5,6, and 7 of this chapter, however, fails in several respects. We will discuss first those points which depend on the fact that $\phi(x,y)$ takes values zero or one only; and second, those points which depend on continuity of the two distributions.

Where we formerly had $\zeta_{11} = \theta(1-\theta)$, we now have

$$\zeta'_{11} = \theta'(1 - \theta') - 1/4\ P(X = Y) .$$

Evidently, however, we have (provided $\mathcal{D}(\theta')$ is sufficiently large, cf. Theorem 4.1)

$$\max_{(F,G)\in\mathcal{D}(\theta')} \zeta'_{11}(F,G) = \theta'(1 - \theta') .$$

It is easy to verify that we still have (if $n \leq m$)

$$\max_{(F,G)\in\mathcal{D}(\theta')} n\sigma^2(U_{mn}) = \theta'(1-\theta') .$$

Thus, except for the question of uniform convergence, the treatment of the distribution of U_{mn} is unchanged, and analogs of Theorems 4.4 and 4.5 might be proved. The conditions for uniform convergence are considered below. (Certain trivial modifications are required at points where we made use of the fact that mnU_{mn} takes integer values only when the distributions are continuous.)

The statistic W_{mn}, when defined with $\phi'(x,y)$ instead of $\phi(x,y)$, no longer has binomial distribution. While its distribution can be computed exactly for small samples, it depends on $P(X = Y)$ as well as on θ'. The distribution is not convenient to handle in general. But W_{mn} is still a sum of independent, identically distributed random variables having asymptotic normal distribution. Uniform convergence may be established under the condition that ζ'_{11} is bounded away from zero, and the error incurred by using the normal approximation is (by Berry's theorem, for example) of order $n^{-1/2}$. An approximation to the index of efficiency for the family $\mathcal{W}$ relative to the problem

$(\mathcal{D}, \theta_1', \theta_2', \alpha, \beta)$ may then be obtained by the methods used in Section 6 of this chapter.

The continuity of F and G is used in establishing the relation

$$\zeta_{10} + \zeta_{01} = \Delta^2 - 1/3 + 2\theta(1 - \theta),$$

where $2\Delta^2 = \int (F - G)^2 \, d(F + G)$. In the present case, we have

$$\zeta_{10}' + \zeta_{01}' = \Delta'^2 - 1/3 + 2\theta(1-\theta) + p,$$

where $2\Delta'^2 = \int (\overline{F} - \overline{G})^2 \, d(F + G)$,

$$2\overline{F} = F(x + 0) + F(x - 0), \quad 2\overline{G} = G(x + 0) + G(x - 0),$$

and p is a certain function of the probabilities of ties, which may be positive or negative (but is non-negative when F,G have no common points of discontinuity). Conditions sufficient to bound $\zeta_{10}' + \zeta_{01}'$ away from zero are sufficient for uniform convergence to normality of U_{mn}. A condition on θ' analogous to the condition $\theta(1-\theta) > 1/12$ used in Section 6 will not suffice, unless F,G have no common points of discontinuity.

The details of the comparison of $\mathcal{U}$ and $\mathcal{W}$ for testing $\theta' \leq \theta_1'$ against $\theta' \geq \theta_2'$ have not been carried out. It is evident, however, that $\text{eff}(\mathcal{U}/\mathcal{W})$ is asymptotically one under conditions sufficient for the required uniform convergences. While we may expect the first proposition of Section 7 to be retained, although perhaps with added conditions, the second proposition would have to be reconsidered extensively.

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