ON THE STRUCTURE OF INCOMPLETE BLOCK DESIGNS

by

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INTRODUCTION

The object of these investigations is to study the structural properties of incomplete block designs. The investigation of structural properties began with the work of R.A. Fisher, who proved the inequality $b \geq v$ for balanced incomplete block designs. Fisher also noticed that every two blocks of a symmetric balanced incomplete block design contain $\lambda$ treatments in common.

Hussain and Nandi were the first to prove that certain balanced incomplete block designs are impossible, by using the method of complete enumeration. Bruck and Ryser introduced analytical methods for the proof of the impossibility of some symmetrical balanced incomplete block designs. Further work for symmetrical designs was done by Shrikhande and by Chowla and Ryser.

In this study there are developed for the first time analytical methods for the investigation of unsymmetrical incomplete block designs. Two unsymmetrical balanced incomplete block designs are proved to be impossible, and for
such designs in general, inequalities are found for the number of treatments common to two blocks.

The methods are generalized to the class of intra-inter group balanced incomplete block designs. An inequality analogous to Fisher's inequality is found for this class, and several other theorems about structure are given.

For the important class of group divisible designs, which is a sub-class of the intra-inter group designs, the symmetric case has been investigated. An interesting theorem is found on the relations among the blocks, and several theorems useful in proving impossibilities are given.
CHAPTER I

THE IMPOSSIBILITY OF CERTAIN UNSYMMETRICAL
BALANCED INCOMPLETE BLOCK DESIGNS

1. The History of the Problem

1.1 The interest of mathematicians in combinatorial problems, involving the arrangement of a finite number of things in sets or patterns, satisfying given conditions, can be traced back to at least as far as Euler, who in 1782 interested himself in the construction of Latin Squares and Graeco Latin Squares. It was, however, only at the beginning of the second quarter of the present century that the importance of combinatorial problems for the proper designing of biological experiments began to be understood, mainly through the work of R. A. Fisher and his associates. Yates in 1936 first introduced into experimental studies a new type of design called the Balanced Incomplete Block Design, which has since that time been widely used. In this case v varieties or treatments are

\[1\] Numbers in square brackets refer to the references listed at the end of the chapter.
compared in such a manner that each treatment is assigned to $r$ experimental units. The units themselves are arranged into $b$ more or less homogeneous blocks, each containing $k$ experimental units. Any two treatments are required to occur together in the same block $\lambda$ times, the treatments occurring in a given block being all different. Hence the design depends on the five parameters $v, b, r, k, \lambda$. Clearly the following conditions are necessary:

\[ b k = vr \]

and

\[ r(k-1) = (v-1)\lambda. \]

Fisher also showed that

\[ b \geq v, \text{ or } k \geq r. \]

For a given per plot error variance and a given $r$, the comparisons between treatments are made with less accuracy in balanced incomplete block designs than in randomized block designs. However, the reduction in the per plot error variance which results from the smaller block size in the balanced incomplete block designs usually is
sufficient to more than offset the theoretical loss of accuracy, and therefore to justify their use.

A number of balanced incomplete block designs were found by Yates $[8]$, and by Fisher and Yates $[10]$. They restricted themselves to the designs with $r \leq 10$, since these are the practically useful cases. In the 1938 edition of Fisher and Yates' tables, the parameters were listed for 16 designs for which no solutions were known.

Bose $[11]$ gave general methods of solution for certain series of designs, which included as special cases the designs for which solutions were already known. By use of these general methods, he also gave solutions for four previously unsolved cases. By following the methods of Bose, Battacharya $[12-14]$ gave solutions for four additional designs.

The work of Fisher, Yates, Bose, and Battacharya provided solutions for all of the designs with $r \leq 10$, except the following:
Hussain \(\left[15,16\right]\) proved the non-existence of the designs (10) and (14). His method of proof consisted essentially of enumeration. Wandi \(\left[17\right]\) showed the impossibility of (8), by showing that the design cannot exist unless (10) exists.

Shrikhande \(\left[18\right]\) proved the non-existence of (30) by considering the matrix \(N\bar{N}'\), where \(N\) is the incidence matrix of the design. He showed that if \(v\) is even, then \((r - \lambda)\) must be a perfect square, from which it follows that the design cannot exist. Shrikhande gave an alternative proof of the im-
possibility of (10) and (14). (10) is ruled out by the perfect square condition, and (14) by consideration of the Hasse invariant $c_p(NN')$, where $p$ is an odd prime. These methods will be discussed in detail below.

Chowla and Ryser [19], in a sequel to a paper by Bruck and Ryser [20], employed similar methods to prove the impossibility of these designs. In fact, they gave a general result, of which (14) is a special case. They showed that if $v = b = 4t + 1$, and $(\lambda : p) = -1$, where $t$ is an arbitrary integer and $p$ is an odd prime which is a factor of the square-free part of $(r - \lambda)$, then the design does not exist. Also, for $t$ and $p$ so defined, if $v = b = 4t + 3$, and $(-\lambda : p) = -1$, then the design does not exist. The symbol $(a \mid b)$ is the Legendre residue symbol.

1.2 Of the designs listed in 1.1.13 there remain to be examined (12), (26), (24), and (31). It is the object of this chapter to show that (12) and (28) are impossible, and to give a proof alternative to Nandi's of the impossibility of (8). The investigations will incidentally throw
much light on the structure of balanced incomplete block designs in general.

Before proceeding further, it is desirable to establish firmly that proofs of the impossibility of (12) and (23) are really needed. Designs which have \( v = b \) and \( r = k \) are called "symmetrical" designs. Associated with every symmetrical design is a "derived" design, which has the following relation to the symmetrical design. If the parameters of the symmetrical design are \( v, b, r, k, \) and \( \lambda \), then the parameters of the derived design, which are indicated by asterisks, are

\[
1.1.21 \quad v^* = v - r, \quad b^* = b - 1, \quad r^* = r, \quad k^* = k - \lambda, \quad \lambda^* = \lambda.
\]

If a solution of a symmetrical design exists, then a solution of the derived design may be obtained by deleting a block and all of the treatments in the block from the symmetrical design. Such a solution of the derived design is said to be "adjoinable", since the symmetrical design can be built up from it by suitably adjoining \( k \) new treatments, \( \lambda \) to each block, and a block consisting of the new treatments. There do exist, however, in certain cases non-adjoinable solutions for the
class of designs given by 1.1.21.

An instructive example is due to Battacharya \( \text{[12]} \). Associated with the symmetrical design \( v=b=25, r=k=9, \lambda =3 \) is the derived Design \( v=16, b=24, r=9, k=6, \lambda =3 \). In this case solutions exist for the symmetrical design, and hence there exists an adjoinable solution for the derived design. Since it is known that every two blocks of a symmetrical design have \( \lambda \) treatments in common, it follows that no two blocks of an adjoinable derived design can have more than \( \lambda \) treatments in common. If a solution exists for the derived design which contains two blocks which have more than \( \lambda \) treatments in common, then clearly the solution is non-adjoinable. Battacharya gave a solution of the derived design for the above case which contains two blocks which have 4 treatments in common. This non-adjoinable solution is reproduced below, and the two blocks with four treatments in common are starred.
1:1.22

(1,2,7,8,14,15), (3,5,7,8,11,13), (2,3,8,9,13,16),
(3,5,8,9,12,14), (1,6,7,9,12,13)*, (2,5,7,10,13,15),
(3,4,7,10,12,16), (3,4,6,13,14,15), (4,5,7,9,12,15),
(2,4,9,10,11,13), (3,6,7,10,11,14), (1,2,3,4,5,6),
(1,4,7,8,11,16), (2,4,8,10,12,14), (5,6,8,10,15,16),
(1,6,8,10,12,13)*, (1,2,3,11,12,15), (2,6,7,9,14,16),
(1,4,5,13,14,16), (2,5,6,11,12,16), (1,3,9,10,15,16),
(4,6,8,9,11,15), (1,5,9,10,11,14), (11,12,13,14,15,16).

The above considerations show that the existence of a symmetrical design implies the existence of the corresponding derived design. Also the non-existence of a derived design implies the non-existence of the corresponding symmetrical design. But the non-existence of a symmetrical design does not imply the non-existence of the corresponding derived design, since a non-adjoinable solution may nevertheless exist. In particular the non-existence of designs (14) and (30) of Fisher's tables does not rule out the possible existence of non-adjoinable solutions for (12) and (28). In the next section there will be established a fundamental theorem which besides being useful for
establishing the impossibility of the two last mentioned designs, has intrinsic interest in as much as it gives a helpful insight into the structural nature of balanced incomplete block designs.

2. A Fundamental Theorem

2.1 Before consideration of the theorem, it is convenient to evaluate the following determinant:

\[
|A| = \begin{vmatrix}
\alpha & \beta & \cdots & \beta & e_{1,v+1} & \cdots & e_{1,v+t} \\
\beta & \alpha & \cdots & \beta & e_{2,v+1} & \cdots & e_{2,v+t} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\beta & \beta & \cdots & \alpha & e_{v,v+1} & \cdots & e_{v,v+t} \\
e_{v+1,1} e_{v+1,2} & \cdots & e_{v+1,v} & e_{v+1,v+1} & \cdots & e_{v+1,v+t} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
e_{v+t,1} e_{v+t,2} & \cdots & e_{v+t,v} & e_{v+t,v+1} & \cdots & e_{v+t,v+t} \\
\end{vmatrix}
\]

Let the following operations be carried out on the rows and columns of \( A \):

(i) Multiply the last \( t \) columns by \( \sum \alpha + (v-1)\beta \) \( \sum \alpha - \beta \),
and write an offsetting factor outside.

(ii) Add rows 1,2,...,v-1 to row v.

(iii) Take the factor $\sqrt{\alpha + (v-1)\beta}$ out of row v.

(iv) Multiply row v by $\beta$ and subtract this product from rows 1,2,...,v-1.

(v) Take the factor $(\alpha - \beta)$ out of rows 1,2,...,v-1.

(vi) Subtract rows 1,2,...,v-1 from row v.

(vii) Subtract suitable multiples of columns 1,2,...,v from columns v+1,v+2,...,v+t so as to make the elements which are both in the first v rows and also in the last t columns equal to zero.

This method of evaluation leads to

**Lemma 1.2.1**: 

$$A = \sqrt{\alpha + (v-1)\beta} \int_{-t+1}^{t}(\alpha - \beta)^{v-t-1} B_t,$$

where $B_t$ is of order $t \times t$, and the elements of $B_t$ are

$$b_{jk} = (\alpha + (v-1)\beta)(\alpha - \beta)e_{v+j,v+k}$$

$$- (\alpha + (v-1)\beta) \sum_{i=1}^{v} e_{i,v+k} e_{v+j,i} + \beta \sum_{i=1}^{v} e_{i,v+k} \sum_{i=1}^{v} e_{v+j,i}.$$
2.2 Consider the "incidence" matrix $N_o$ of the design, i.e.,

$$N_o = \begin{pmatrix} n_{11} & \cdots & n_{1b} \\ \vdots & \ddots & \vdots \\ n_{v1} & \cdots & n_{vb} \end{pmatrix},$$

where the rows represent treatments, the columns represent blocks, and $n_{ij} = 1$ or 0 according as the $i$-th treatment does or does not occur in the $j$-th block. Since every treatment is replicated $r$ times,

$$\sum_{j=1}^{b} n_{ij} = r, \quad (i = 1, \ldots, v),$$

and since every treatment must occur $\lambda$ times with every other treatment,

$$\sum_{j=1}^{b} n_{ij} n_{kj} = \lambda, \quad (i, k = 1, \ldots, v).$$

Hence,

$$N_o N_o' = \begin{pmatrix} r & \lambda & \cdots & \lambda \\ \lambda & r & \cdots & \lambda \\ \vdots & \vdots & \ddots & \vdots \\ \lambda & \lambda & \cdots & r \end{pmatrix},$$
where \( N_o^t \) denotes the transpose of \( N_o \).

Clearly

\[
1.2.25 \quad N_o N_o^t = rk(r - \lambda)^{v-1}.
\]

Let \( \alpha_{ju} \) be the number of treatments common to the \( j \)-th and \( u \)-th blocks \((j,u = 1,2, \ldots, b)\). Consistently with this we can put \( \alpha_{jj} = k \) \((j=1,2, \ldots, b)\). Then the \( b \times b \) matrix

\[
1.2.26 \quad S_b = (\alpha_{ju}) = N_o^t N_o
\]

is defined to be the **structural matrix of the design**. \( S_b \) is a symmetric matrix, in which the \( j \)-th row or column corresponds to the \( j \)-th block and the successive elements of the \( j \)-th row or column give the number of treatments which this block has in common with the 1st, 2nd, \ldots, \( b \)-th blocks.

If we select any \( t \) blocks from the design \((t \leq b)\), and if \( N_{oo} \) is the submatrix of the incidence matrix \( N_o \) formed by choosing the columns corresponding to these blocks, then \( N_{oo} \) may be defined to be the incidence matrix of the chosen
set of blocks. \( S_t = N_t^t N_t \) is then a submatrix of the structural matrix \( S_b \), obtained by selecting the rows and columns corresponding to the chosen blocks. Its diagonal elements are all \( k \) and any other element specifies the number of treatments common to the blocks which correspond to the row and column in which the element lies.

2.3 Let the incidence matrix be extended by adjoining \( b \) new rows, so that the \( j \)-th adjoined row consists of zero elements except the \( j \)-th, which is unity. We thus get

\[
N_2 = \begin{pmatrix}
N_t \\
\vdots \\
10 \ldots 0 \\
01 \ldots 0 \\
\vdots \\
00 \ldots 1
\end{pmatrix},
\]

which is of order \((v+b) \times b\). Then

\[
N_2 N_2^t = \begin{pmatrix}
N_t N_t^t & N_t \\
N_t^t & I
\end{pmatrix},
\]
where $I$ is the identity matrix of order $b \times b$.

By application of Lemma 1.2.1, obtain

$$1.2.33 \quad |N_2N_2^T| = r^{-b+1}k(r - \lambda)^{v-b-1} |C_b|,$$

where

$$1.2.34 \quad c_{jj} = (r-k)(r-\lambda), \quad \text{and} \quad c_{ju} = \lambda k - r\alpha_j u,$$

and where $(j,u = 1, \ldots, b, j \neq u)$ and $\alpha_j u$ is the number of treatments which blocks $j$ and $u$ have in common. $1.1.11$ has been used in replacing $(r + (v-1) \lambda)$ by $rk$.

The matrix $C_b$ given by $1.2.33$ is of considerable importance in the succeeding theory. It is a symmetric matrix whose elements are in (1.1) correspondence with the elements of the structural matrix $S_b$. In fact we can write

$$1.2.35 \quad C_b = \lambda k E_b + r(r - \lambda)I_b - rS_b ,$$

where $I_b$ is the unit matrix and $E_b$ is the singular $b \times b$ matrix all of whose elements are unity.

The matrix $C_b$ is defined as the characteristic matrix of the design. The $j$-th row or the $j$-th
column of \( C_b \) corresponds to the \( j \)-th block of the design.

If we choose any set of \( t \) blocks from the design, then the characteristic matrix of this set of \( t \) blocks is defined to be the sub-matrix \( C_t \) of \( C_b \) obtained by selecting the rows and columns corresponding to this set of blocks. Let \( N_{20} \) be the sub-matrix of \( N_2 \) which is obtained by keeping only those of the adjoined rows which have unity in the columns corresponding to the chosen blocks. Then an analogous argument shows that

\[
\text{1.2.36} \quad |N_{20}N'_{20}| = r^{-t+1}k(r - \lambda)^{v-t-1}|C_t|.
\]

2.4 When \( P \) is a matrix of order \( s \times t \), \( t \geq s \), a well-known algebraic theorem states that \( |PP'| \) may be expressed as a sum of squares of \( s \times s \) determinants. Clearly \( |PP'| \geq 0 \).

Hence if \( b > v + t \), then \( |N_{20}N'_{20}| \geq 0 \). Further, since the elements of \( N_{20} \) are integers if \( b = v + t \), then \( |N_{20}N'_{20}| \) is a perfect integral square.

Finally if \( b \leq v + t \), then \( |N_{20}N'_{20}| = 0 \). Hence
from 1.2.36 we get the following fundamental theorem.

**Theorem 1.2.1:** If $C_t$ is the characteristic matrix of any set of $t$ blocks chosen from a balanced incomplete block design with parameters $v, b, r, k, \lambda$ then

1.2.41 (i) $|C_t| \geq 0$ if $t < b-v$,

1.2.42 (ii) $|C_t| = 0$ if $t \geq b-v$, and

1.2.43 (iii) $(rk)^{-b+v+1}(r - \lambda )^{2v-b-1} |C_t|$ is a perfect integral square if $t = b-v$.

N.B. We recall that the diagonal elements of $C_t$ are $(r-k)(r-\lambda)$, and the element in the $j$-th row and the $u$-th column is $\lambda k - r \alpha_{ju}$, where $\alpha_{ju}$ is the number of treatments common to the $j$-th and the $u$-th of the chosen blocks, i.e., $C_t = \lambda k E_t + r(r-\lambda )I_t - rS_t$ where $S_t$ is the structural matrix of the chosen blocks, $E_t$ is a singular $t \times t$ matrix with each element unity, and $I_t$ is the $t \times t$ unit matrix.
Corollary 1.2.1: The characteristic matrix $C_b$ of the design is non-negative, so that the quadratic form

$$1.2.44 \quad \sum C_{ju} x_j x_u$$

where the $c$'s are given by 1.2.34 is non-negative.

2.5 To illustrate the kind of information which is contained in this theorem, consider the design with parameters $r=7$, $k=5$, $b=21$, $v=15$, and $\lambda =2$. Let the treatments be denoted by letters, and consider whether it is possible to fill up four blocks in such a way that each block will have two treatments in common with each of the other three blocks. One way in which this can be done is as follows:

$$1.2.51 \quad (ABCDE), (ABFGH), (ACFIJ), (BCFKL)$$

But can these four blocks form part of the completed design? To answer this question, apply Theorem 1. Now,

$$c_{jj} = (r - \lambda)(r - k) = 10,$$

and

$$c_{ju} = \lambda k - r \alpha_j u = -4, \quad (u \neq j).$$
Hence,
\[
|C_4| = \begin{vmatrix}
10 & -4 & -4 & -4 \\
-4 & 10 & -4 & -4 \\
-4 & -4 & 10 & -4 \\
-4 & -4 & -4 & 10 \\
\end{vmatrix} = (14)^3(-2) \ll 0,
\]
and by (i) of Theorem 1 it follows that 1.2.51 is impossible, and in fact that any set of four blocks of the type considered cannot form a part of the completed design.

Now we shall derive some simple consequences of Theorem 1. Assume that \( b < v \). By use of Theorem 1 this assumption will be contradicted. Let \( t=1 \), so that

\[
1.2.53 \quad |C_1| = (r - k)(r - \lambda),
\]
which by (ii) of Theorem 1.2.1 is equal to zero. Hence, either \( r = \lambda \) or \( r = k \). If \( r = \lambda \), then the design is a randomized block design, which is a degenerate case of a balanced incomplete block design. Let this possibility be excluded. Then it is necessary that \( r = k \). But from the first of the parametric conditions of 1.1.11, it follows that \( b = v \), which contradicts the assumption that \( b < v \).
Hence, the

**Corollary 1.2.2:** For a balanced incomplete block design it is necessary that \( b \geq v \), unless \( r = \lambda \), in which case the design degenerates into a randomized block design.

This is Fisher's Inequality, 1.1.12.

Again, let \( t=2 \). Then

\[
1.2.54 \quad |C_2| = \begin{vmatrix} (r-\lambda)(r-k) & \lambda k - \alpha_{12} \cdot r \\ \lambda k - \alpha_{12} \cdot r & (r-\lambda)(r-k) \end{vmatrix}
\]

\[
= (r-\lambda)^2(r-k)^2 - (\lambda k - \alpha_{12} \cdot r)^2.
\]

If the design is symmetric, then \( r = k \), and 1.2.54 reduces to

\[
1.2.55 \quad |C_2| = -k^2(\lambda - \alpha_{12}).
\]

By (ii) of Theorem 1, \( |C_2| = 0 \). Hence, \( \lambda = \alpha_{12} \).

This result is stated in

**Corollary 1.2.3:** For a symmetric design every block has \( \lambda \) treatments in common with every other block.

This was first noticed by Fisher.
Consider 1.2.55 without requiring that \( r = k \).

Then by (i) of Theorem 1 it is necessary that

\[
1.2.56 \quad (r - \lambda)^2 (r - k)^2 - (\lambda k - \alpha_{2,r})^2 \geq 0.
\]

This relation provides bounds for \( \alpha_{2,r} \), which are stated in

Corollary 1.2.56:

\[
\sqrt{2\lambda k + r(r - \lambda - k)} \geq \alpha_{jk} \geq -(r - \lambda - k),
\]

where \( \alpha_{jk} \) is the number of intersections of blocks \( j \) and \( k \).

In some cases the bounds are very good. For example, consider the design with parameters \( r = 7, k = 5, v = 15, b = 21 \), and \( \lambda = 2 \). In this case the bounds are

\[
1.2.57 \quad 20/7 \geq \alpha_{jk} \geq 0,
\]

and it is shown below that \( \alpha_{jk} = 1 \) or \( 2 \).

Also consider the design with parameters \( r = 9, k = 6, v = 16, b = 24 \), and \( \lambda = 3 \). The bounds are

\[
1.2.58 \quad 4 \geq \alpha_{jk} \geq 0,
\]

and Bhattacharya's example cited before, 1.1.22 shows that there actually exists a solution in which two blocks intersect in four treatments.
3. The Structure of Balanced Incomplete Block Designs of the Series \( v=\frac{k(k+1)}{2} \), \( b=(k+1)(k+2)/2 \), \( r=k+2 \), \( k \), \( \lambda =2 \)

3.1 It is the object of this section to develop several lemmas about the relations between blocks of any design belonging to this series. The first two lemmas do not depend on Theorem 1.2.1, but subsequent lemmas are based on it.

Consider an initial block \( B_1 \), which contains the \( k \) treatments \( a_1, \ldots, a_k \). It is desired to know how the \( a_j \) are distributed among the remaining \( (b-1) \) blocks. Let there be \( i_1 \) blocks which contain \( i \) of the treatments \( a_j \). Then the following relations are necessary:

1.3.11 (i) \[ \sum_{i=0}^{k} n_i = b-1 = \frac{k(k+1)}{2} , \]

(ii) \[ \sum_{i=0}^{k} i n_i = k(r-1) = k(k+1) , \text{ and} \]

(iii) \[ \sum_{i=0}^{k} i(i-1)n_i = k(k-1) . \]

Now (i) is clear. To prove (ii) consider the
replications of the $k$ treatments in $B_1$. Each of the $a_j$ has been replicated once in $B_1$ and therefore must be replicated $(r-1)$ additional times among the remaining blocks. Hence $k(r-1)$ is the number of plots allotted to the $a_j$ in the remaining blocks. But this number is also given by $\sum_{i=0}^{k} i n_i$. This proves (ii).

To prove (iii), consider the number of pairs to be formed among the $a_j$. Every pair must occur 2 times altogether, but every pair has already occurred once in $B_1$. Hence, the required number of pairs still to be formed is $k(k-1)/2$. But this number is also given by $\frac{1}{2} \sum_{i=1}^{k} i(i-1)n_i$.

Consider

$$1.3.12 \quad Q = \sum_{i=0}^{k} (i-1)(i-2)n_i,$$

where $n_i$, $(i=0, \ldots, k)$, is a positive or zero integer. Now
1.3.13 \[ \psi = \sum_{i=0}^{k} i(i-1)n_i - 2 \sum_{i=0}^{k} in_i + \sum_{i=0}^{k} n_i \]

\[ = (k)(k-1) - 2k(k+1) + k(k+3) \]

\[ = 0. \]

Since \( n_i \geq 0 \) and \( n_i \equiv 0 \), it follows from 1.3.13 that each term of \( \psi \) is zero. Hence

1.3.14 \[ n_i = 0 \quad \text{for} \quad i = 0 \quad \text{and} \quad k \geq i > 2. \]

From (i) and (ii) of 1.3.11 we obtain

1.3.15 \[ n_1 + n_2 = \frac{k(k+3)}{2} \]

and

\[ n_1 + 2n_2 = k(k+1), \]

whence

1.3.16 \[ n_2 = \frac{k(k-1)}{2} \]

and

\[ n_1 = 2k. \]

These results are contained in

**Lemma 1.3.1**: Any block of the design has two treatments in common with \( \frac{k(k-1)}{2} \) other blocks, and one treatment in common with \( 2k \) other blocks.
3.2 Next consider two initial blocks, $B_1$ and $B_2$, which contain treatments as follows:

\[ B_1: \theta_1 \ldots \theta_\gamma a_1 \ldots a_{k-\gamma} \]
\[ B_2: \theta_1 \ldots \theta_\gamma b_1 \ldots b_{k-\gamma} \]

The treatments $\theta_i$ ($i=1, \ldots, \gamma; \gamma=1,2$) are the $\gamma$ treatments which $B_1$ and $B_2$ have in common. It is desired to determine the ways in which these treatments may be distributed among the remaining $(b-2)$ blocks.

The remaining $(b-2)$ blocks are of several types depending on how the treatments of $B_1$ and $B_2$ occur in them. If $\gamma=2$, then $\theta_1$ and $\theta_2$ occur together twice in $B_1$ and $B_2$ and cannot occur together again in any other block. The types of blocks are defined in

**Definition 1.3.1**: Type 1. The block contains two treatments from each of $B_1$ and $B_2$. It is of sub-type $1_1$ or $1_2$ according as one $\theta_i$ does not, or does occur as one of the two treatments.
Type 2. The block contains two treatments from one of $B_1$ and $B_2$, but only one treatment from the other. It is of sub-type $2_1$ or $2_2$ according as one $\theta_i$ does not, or does occur as one of the treatments.

Type 3. The block contains one treatment from each of $B_1$ and $B_2$. It is of sub-type $3_1$ or $3_2$ according as one $\theta_i$ is not, or is the treatment.

These types are illustrated below:

<table>
<thead>
<tr>
<th>Type of block</th>
<th>Treatments from $B_1$ and $B_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1_1$</td>
<td>$a_i a_j b_k \theta_j'$</td>
</tr>
<tr>
<td>$1_2$</td>
<td>$a_i b_j \theta_k$</td>
</tr>
<tr>
<td>$2_1$</td>
<td>$a_i a_j b_k$</td>
</tr>
<tr>
<td>$2_2$</td>
<td>$a_i \theta_j$</td>
</tr>
<tr>
<td>$3_1$</td>
<td>$a_i b_j$</td>
</tr>
<tr>
<td>$3_2$</td>
<td>$\theta_i$</td>
</tr>
</tbody>
</table>

Denote the number of blocks of type $\kappa_i$, $(\kappa=1,2,3; i=1,2)$, by $x_{\kappa i}$.

Consider the pairs which must be formed among
the treatments of $B_1$ and $B_2$. Certain pairs occur in $B_1$ and $B_2$, leaving the following pairs to occur in the remaining ($b-2$) blocks:

1.3.23

<table>
<thead>
<tr>
<th>Type of pair</th>
<th>Number of pairs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_i b_j$</td>
<td>$n_1 = 2(k-\gamma)^2$</td>
</tr>
<tr>
<td>$a_i a_j$ or $b_i b_j$</td>
<td>$n_2 = (k-\gamma)(k-\gamma-1)$</td>
</tr>
<tr>
<td>$e_i a_j$ or $e_k b_j$</td>
<td>$n_3 = 2\gamma(k-\gamma)$</td>
</tr>
</tbody>
</table>

From 1.3.22 and 1.3.23 we obtain the following equations:

1.3.24

(a) $4x_{11} + x_{12} + 2x_{21} + x_{31} = n_1$,
(b) $2x_{11} + x_{21} = n_2$,
(c) $2x_{12} + x_{22} = n_3$,
(d) $x_{12} + x_{22} + x_{32} = \gamma k$,
(e) $4x_{11} + 3x_{21} + x_{22} + 2x_{31} + 2x_{32} = 2(k+1)(k-\gamma)$,
(f) $x_{11} + x_{12} + x_{21} + x_{22} + x_{31} + x_{32} = k(k+3)/2 - 1$.

Equations (a), (b), and (c) arise from consideration of the number of pairs of different types which are contained in the ($b-2$) blocks. Thus, (a) is based on the pairs of the type $a_i b_j$. 
(1.3.23). Blocks of type $l_1$ contain 4 such pairs; blocks of type $2_1$ contain 2 such pairs, etc. The sum of these pairs in the different types of blocks is $n_1$. Similarly, (b) is based on (ii) of 1.3.23, and (c) is based on (iii) of 1.3.23.

Equations (d) and (e) of 1.3.24 arise from consideration of the number of replications of the treatments. The $\theta_i$ occur both in block $B_1$ and in block $B_2$. Hence, each $\theta_i$ must occur $k$ times in the remaining $(b-2)$ blocks, or all $\theta_i$ must occur $k\gamma$ times in those blocks. The blocks of types $l_2$, $2_2$, and $3_2$ each contain one $\theta_i$. Hence, the equation (d). Similarly, (e) is based on the replications of the $a_i$ and $b_j$.

Equation (f) is clear.

The equations of 1.3.24 may be solved to determine the number of blocks of types 1, 2, and 3. The result of this solution is contained in

Lemmas 1.3.2: With respect to 2 initial blocks which have $\gamma$, $(\gamma=1,2)$ treatments in common, there exist
\[(k-\gamma)(k-\gamma+1) + k \cdot \gamma - k(k+3)/2 -1\]

blocks of type 1,

\[2\gamma(k-\gamma)\]

blocks of type 2, and

\[k(2-\gamma) - 2\gamma(1-\gamma)\]

blocks of type 3.

3.3 Now consider several structural matrices for 5 blocks. The first structural matrix to be considered is

\[
S_5^{(1)} = \begin{pmatrix}
k & 1 & 1 & 2 & 2 \\
k & 1 & 2 & 2 \\
k & 1 & 1 \\
k & \alpha_{45}
\end{pmatrix},
\]

which is a symmetric matrix. The element \(\alpha_{45}\) is unknown, and it is desired to know what values are admissible for \(\alpha_{45}\), if the five blocks which have \(S_5^{(1)}\) for their structural matrix are to form a part of the completed design. Of course, the admissible value is 1 or 2, or both.
Associated with $S_{5}^{(1)}$ is the characteristic matrix $C_{5}^{(1)}$, and the characteristic determinant, $|C_{5}^{(1)}|$. Consider the elements of $C_{5}^{(1)}$. For the series of designs under consideration, $r-k = 2$ and $r-\lambda = k$.

Hence,

$$c_{jj} = 2k$$

and

$$c_{ju} = k - 2 \text{ or } -4,$$

according as $\alpha_{ju} = 1$ or $2$, where $j$ and $u$ refer to the $j$-th and $u$-th blocks of the set of 5 blocks being considered. Hence,

\[
\begin{vmatrix}
2k & (k-2) & (k-2) & -4 & -4 \\
2k & (k-2) & -4 & -4 \\
2k & (k-2) & (k-2) & & \\
2k & \alpha_{45}^* & \alpha_{45}^* & \\
2k & & &
\end{vmatrix}
\]

where the elements below the diagonal have been omitted since the matrix is symmetric. This convention will always be followed. Also $\alpha_{45}^* = (k-2)$ or $-4$ according as $\alpha_{45}^* = 1$ or $2$. This determinant
is of the same form as $|A|$ of 1.2.11. Hence its value is given by Lemma 1.2.1, if the proper substitutions are made. The matrix $B_2^{(1)}$ which will now be introduced is analogous to $B_t$ of that lemma.

We can obtain as elements of $|B_2^{(1)}|$:

\[ b_{11} = b_{22} = 4(k-1)(k+2)(2k) - 4(k-1)(32+(k-2)^2) + (k-2)(k-10)^2, \]

and

\[ b_{12} = b_{21} = 4(k-1)(k+2) \alpha_{45}^* - 4(k-1)(32+(k-2)^2) + (k-2)(k-10)^2. \]

Subtract the second row of $|B_2^{(1)}|$ from the first row, and then add the first column to the second column. We obtain

\[ |B_2^{(1)}| = 4(k-1)(k+2)(2k - \alpha_{45}^*) \sqrt{(k-1)(k+2)} (\alpha_{45}^* + 2k) - 8(k-1)(32+(k-2)^2) + 2(k-2)(k-10)^2. \]

In the following argument exclude the degenerate case $k = 1$ from consideration. Now according to Theorem 1.2.1 it is necessary that $|C_5^{(1)}| \geq 0$, and hence that $|B_2^{(1)}| \geq 0$. Clearly $(2k - \alpha_{45}^*) > 0$. 
for both $\alpha_{45}^* = (k-2)$ and $\alpha_{45}^* = -4$. But consider the last factor of 1.3.335, which reduces to

$$2(k+2)/\alpha(k-1)\alpha_{45}^* + (k^2 - 2k) \geq 0.$$ 

Now either $\alpha_{45}^* = -4$ or $\alpha_{45}^* = (k-2)$. If $\alpha_{45}^* = -4$ we obtain

$$1.3.34 \quad 2(k+2)^2(k-10) \geq 0,$$

whence it follows that $\alpha_{45}^*$ cannot be $-4$ and hence $\alpha_{45}$ cannot be 2 unless $k \geq 19$. If $\alpha_{45}^* = (k-2)$ we obtain

$$1.3.345 \quad 6(k+2)^2(k-4) \geq 0,$$

whence it follows that $\alpha_{45}^*$ cannot be $(k-2)$ and hence $\alpha_{45}$ cannot be 1 unless $k \geq 4$. Those results are contained in

Lemma 1.3.3:

(i) If $k < 4$, then there cannot exist 5 blocks with $S_5^{(1)}$ as structural matrix.

(ii) If $4 \leq k \leq 9$, then in $S_5^{(1)}$, $\alpha_{45} = 1$. 

(iii) If \( k \geq 10 \), then both values of \( \alpha_{45} \) are admissible in \( S_5^{(1)} \).

Let the second structural matrix to be considered be

\[
S_5^{(1)} = \begin{pmatrix}
    k & 1 & 1 & 2 & 2 \\
    k & l & 1 & 1 & 1 \\
    k & 1 & 1 & \alpha_{45} \\
    k & 1 & 1 & \alpha_{45} \\
    k & \alpha_{45} & \alpha_{45} & \alpha_{45} & \alpha_{45}
\end{pmatrix}
\]

The corresponding characteristic determinant is

\[
|C_5^{(2)}| = \begin{vmatrix}
    2k & (k-2) & (k-2) & -4 & -4 \\
    2k & (k-2) & (k-2) & (k-2) & (k-2) \\
    2k & (k-2) & (k-2) & (k-2) & (k-2) \\
    2k & \alpha_{45} & \alpha_{45} & \alpha_{45} & \alpha_{45} \\
    2k & \alpha_{45} & \alpha_{45} & \alpha_{45} & \alpha_{45}
\end{vmatrix}
\]

Again using Lemma 1.2.1 we obtain as elements of \( B_2^{(2)} \): 

\[
b_{11} = b_{22} = 4(k-1)(k+2)(2k-4(k-1)(16+2(k-2)^2) + (k-2)(-4 + 2(k-2))^2,
\]
and

\[ b_{12} = b_{21} = 4(k-1)(k+2) \alpha_{45}^* - 4(k-1)(16+2(k-2)^2) + (k-2)(-1+2(k-2))^2. \]

Further reduction yields

1.3.36

\[ |B_2^{(2)}| = 16(k-1)(k+2)^2(2k - \alpha_{45}^*) \sqrt{k-1} \alpha_{45}^* + 2(k-4). \]

Exclude \( k = 1 \), which is a degenerate case. Now according to Theorem 1.2.1, \( |C_5^{(2)}| \geq 0 \), and hence

\[ |B_2^{(1)}| = 0. \]

We see that \( (2k - \alpha_{45}^*) \geq 0 \) for both \( \alpha_{45}^* = (k-2) \) and \( \alpha_{45}^* = -4 \). But consider the last factor of 1.3.36. If \( \alpha_{45}^* = -4 \) we obtain

1.3.37

\[ -2k - 4 \geq 0, \]

which is impossible for all \( k \), since \( k > 1 \). Hence \( \alpha_{45} = 2 \) is impossible for all \( k \). If \( \alpha_{45}^* = (k-2) \) we obtain

1.3.375

\[ (k+2)(k-3) \geq 0, \]

whence it follows that \( \alpha_{45}^* = (k-2) \) and hence that \( \alpha_{45} = 1 \) are not possible unless \( k \geq 3 \).
These results are contained in

**Lemma 1.3.4:**

(i) If $k < 3$, then there cannot exist 5
blocks with $S_{5}^{(2)}$ as structural matrix.

(ii) If $k \geq 3$, then in $S_{5}^{(2)}$, $a_{45} = 1$.

Consider a third structural matrix

\[ S_{5}^{(3)} = \begin{pmatrix}
  k & 1 & 1 & 1 & 2 \\
  k & 1 & 2 & 1 \\
  k & 2 & 1 \\
  k & a_{45} \\
  k & / 
\end{pmatrix} \]

The corresponding characteristic determinant is

\[ C_{5}^{(3)} = \begin{vmatrix}
  2k & (k-2) & (k-2) & (k-2) & -4 \\
  2k & (k-2) & -4 & (k-2) \\
  2k & -4 & (k-2) \\
  2k & a_{45} \\
  2k 
\end{vmatrix} \]

Apply Lemma 1.2.1, and then the elements of $B_{2}^{(3)}$ are
\[ b_{11} = 4(k-1)(k+2)(2k) - 4(k-1)(k-2)^2 + 32 \\
+ (k-2)(k-10)^2, \]

\[ b_{22} = 4(k-1)(k+2)(2k) - 4(k-1)(2(k-2)^2 + 16) \\
+ (k-2)(2k-8)^2, \]

and

\[ b_{12} = b_{21} = 4(k-1)(k+2) \alpha_{45}^* - 4(k-1)(-12(k-2)) \\
+ (k-2)(k-10)(2k-8). \]

Further reduction yields

1.3.39

\[
\begin{vmatrix}
B_2^{(3)} \\
4(k+2) \end{vmatrix}^2 = (5k-14)(k+2) + 2(k-1) \alpha_{45}^* + (k-2)(k+8) \\
2(k-1) \alpha_{45}^* + (k-2)(k+8) \\
+ (k-2)(k+2) \\
\]

\[ = 16(k+2)^2(k-1)(\alpha_{45}^*)^2 - (k-2)(k+8) \alpha_{45}^* \\
+ (k-2)(k^2 - k - 19). \]

Exclude \( k = 1 \). By Theorem 1.2.1, \( c_3^{(3)} \geq 0 \), and hence \( B_2^{(3)} \geq 0 \). Let \( \alpha_{45}^* = -4 \). We obtain from the last factor of 1.3.39,

\[ (k+2)^2(k-3) \geq 0, \]

whence \( \alpha_{45}^* \neq -4 \) unless \( k > 2 \) and hence \( \alpha_{45}^* \neq 2 \).
unless \( k > 2 \). Let \( \alpha_{45} = (k-2) \) and obtain from the last factor of 1.3.39,

\[
-(k-2)(k+2)^2 \geq 0,
\]

which is not satisfied except for \( k=2 \). By placement of the treatments in the blocks it is clear that the design with \( k=2 \) cannot contain five blocks with the structural matrix \( S_5(3) \). Hence, \( \alpha_{45} = 1 \) is impossible for all \( k \). These results are contained in

**Lemma 1.3.5:**

(i) If \( k < 3 \), then there cannot exist 5 blocks with \( S_5(3) \) as structural matrix.

(ii) If \( k \geq 3 \), then in \( S_5(3) \), \( \alpha_{45} = 2 \).

4. The Hasse Invariant \( c_p(A) \)

4.1 In order to prove the impossibility of design number (12) of Fisher and Yates' table, 1.1.13, recourse is made to certain invariants of a quadratic form. In this section these invariants are defined and discussed, and especially attention
is given to the remarkable Hasse invariant, \( c_p(A) \),
which is defined below.

As a preliminary to this subject, consider several definitions. If \( f \) and \( g \) are two quadratic forms with rational coefficients, then we define their rational congruence in the following manner.

**Definition 1.4.1:** If there is a non-singular linear transformation with rational coefficients which takes \( f \) into \( g \), then \( f \) and \( g \) are "rationally congruent" forms.

Let \( f = X'AX \), where \( A \) is the coefficient matrix of the form, and \( X \) is the column matrix of the variables \( x_i \). If the transformation is \( X = TY \), where \( T \) is a coefficient matrix and \( Y \) is a column matrix of variables \( y_i \), then the transformation takes \( f \) into

\[
1.4.11 \quad g = Y'T'ATY = Y'(T'AT)Y,
\]

which shows that the coefficient matrix of \( g \) is equal to \( T'AT \). Hence, corresponding to the transformation of \( f \) into \( g \) is the transformation of \( A \) into \( T'AT \), and it is clear that Definition 1
could equally well have been phrased in terms of the coefficient matrices. It is important to know the conditions which are necessary and sufficient for $f$ and $g$ to be rationally congruent. Paul 21 has summarized the relevant theory in a convenient manner which is followed here. Let $p$ be a given prime, $a$ and $b$ be non-zero rational numbers, and $r$ be a positive integer. Then by the equation

$$1.4.12 \quad ax^2 \equiv b \pmod{p^r}$$

is meant that

$$1.4.125 \quad (ax^2 - b)/p^r$$

is a rational number which, after cancelling out factors common to the numerator and denominator, has a denominator which is prime to $p$, i.e., an integer modulo $p$. If $1.4.12$ is satisfied for all $r$, then $a$ and $b$ are said to be in the same $p$-adic class.

For given $p$ and for every $r$, the relation $1.4.12$ may or may not be satisfied. As an example let $a=2$ and $b= -\frac{7}{2}$. Then $1.4.12$ becomes
2x^2 \equiv -\frac{7}{2} \pmod{p^r}.

If p = 2 and r = 1, then setting x = \frac{1}{2} we obtain

\[ 2\left(\frac{1}{4}\right) + \frac{7}{2} = \frac{8}{2} = 4, \]

which when divided by $p^r = 2^1$, so that the denominator is 1, an integer modulo 2. And for any r there exist values for x which will solve 1.4.12.

Also if p = 11, solutions exist for all r. Hence for p = 2 and for p = 11, 2 and $-\frac{7}{2}$ are in the same p-adic class. However, if p = 3, 5, or 7 solutions do not exist for all r, and 2 and $-\frac{7}{2}$ are in different p-adic classes.

It can be proved that for p odd there are four p-adic classes, containing the respective numbers

1, p, v, pv,

where v denotes any given quadratic non-residue mod p; and for p = 2, there are exactly eight p-adic classes, containing the respective numbers

1, 3, -1, -3, 2, 6, -2, -6.

To find the p-adic class of any non-zero rational
number write it in the form $s^2 \ p^\alpha k$, where $s$ is a rational number, $\alpha = 0$ or $1$, and $k$ is an integer prime to $p$; and replace $k$ by any number of the same quadratic character, $1$ or $\nu$ if $p > 2$, $\pm 1$ or $\pm 3$ if $p = 2$.

It is useful to introduce a conventional "prime" $p$, called the prime $\infty$; and to understand that the solvability of 1.4.12 for every $r$ means the solvability of

$$ax^2 = b$$

with $x$ real. For the prime $\infty$ it is evident that there are two $p$-adic classes, one consisting of all positive and the other of all negative rationals.

Consider a generalization of 1.4.12.

**Definition 1.4.2:** If $a$ and $b$ are any non-zero rational numbers, $p$ is any prime, and $r$ is a positive integer, then the symbol

$$(a,b)_p$$

is defined to have the value $+1$ or $-1$ according as the congruence
\[ ax^2 + by^2 \equiv 1 \pmod{p^n} \]

has or has not for each \( r \), rational solutions \( x_r \) and \( y_r \).

The symbol \( (a,b)_p \) has the same value as Hilbert's norm-residue symbol in those cases in which Hilbert's symbol is defined. Certain properties of \( (a,b)_p \) will be enumerated below, but first consider a product of such symbols,

\[
1.4.13 \quad c_p = c_p(f) = (-1,-D_n)_p \prod_{i=1}^{n-1} (D_i,-D_i+1)_p,
\]

where \( f \) is a quadratic form with rational coefficients and \( D_i \) is the leading principal minor determinant of order \( i \) in the coefficient matrix of the form. This quantity is known as Hasse's invariant for the quadratic form \( f \), since if no \( D_1 = 0 \), it is invariant under rational transformations of the form \( f \). In view of the discussion below Definition 1.4.1, the invariant \( c_p \) might as well refer to the coefficient matrix \( A \) of the quadratic form \( f \). Hence we define

\[
1.4.135 \quad c_p(A) = c_p(f).
\]
Let \( n \) = the number of variables in a quadratic form, \( i \) = the index of the form, and \( d \) = the square-free integer part of the determinant of the form. Then an answer is provided to the question of when two forms are rationally congruent by the following

**Theorem L.4.1:** Two forms of \( f \) and \( g \) are rationally congruent if and only if they have the same values for their invariants

\[ n, i, d, \text{ and } c_p \text{ for every } p. \]

These invariants are not independent of each other but satisfy certain relations which will not be stated here. For a proof of this important theorem consult the book by Jones [22].

The important properties of the symbol \((a, b)_p\) are

\[ (a, b)_\infty = -1 \text{ if and only if } a \text{ and } b \]

are negative;

\[ (p^\alpha m, p^\alpha m')_p = (-1 | p)^{\alpha} \ (m | p)^{\alpha'} (m' | p)^{\alpha} \]

if \( p > 2 \); and

\[ (2^\alpha m, 2^\alpha m')_2 = (2 | m)^{\alpha} (2 | m')^{\alpha} (-1)^{m-1} (m'-1)/4 \]

if \( p = 2 \).
Here \( m \) and \( m' \) denote integers prime to \( p \); \( \alpha \) and \( \alpha' \) are 0 or 1. For example, \( (2, -\frac{5}{2}, p) \) has the value -1 if \( p = 5 \); \((+1)(-1)(+1) = -1 \) if \( p = 2 \); +1 for all other primes \( p \).

Let \( \alpha, \beta, \gamma, \rho, \) and \( \sigma \) be non-zero rational numbers. Then the following properties follow from the properties of 1.4.14:

1.4.15 \[
1. \quad (\alpha, \beta)_p = (\beta, \alpha)_p. \\
2. \quad (\alpha\sigma^2, \beta\sigma^2)_p = (\alpha, \beta)_p. \\
3. \quad (\alpha, -\alpha)_p = 1. \\
4. \quad (\alpha, \beta, \gamma)_p = (\alpha, \beta)_p (\alpha, \gamma)_p. \\
5. \quad (\alpha, \alpha)_p = (\alpha, -1)_p. \\
6. \quad (-1, -1)_p = 1 \text{ for } p \text{ odd; } \\
\quad \quad \quad = -1 \text{ for } p = 2 \text{ and } \infty. \\
7. \quad (\alpha, \beta)_p = 1 \text{ if } \alpha, \beta, \text{ or } (\alpha + \beta) \\
\quad \quad \quad \text{ is in the } p\text{-adic class of } 1. \\
8. \quad \text{If } \alpha \text{ is an integer and if } p \text{ is an odd } \\
\quad \quad \quad \text{ prime, then } (p, \alpha) = (\alpha, p) = (\alpha | p), \text{ which} \\
\quad \quad \quad \text{ is the Legendre residue symbol.}
9. There are only a finite even number of primes (including \( p = \infty \)) for which the symbol \( (\alpha, \beta)_p = -1 \). This is usually written as

\[
\prod_p (\alpha, \beta)_p = -1.
\]

10. \( (\alpha, \alpha + 1)_p = (-1, \alpha + 1)_p \), for \( p \) odd,

where

\[
\prod_{j=1}^{m} (j, j+1)_p = ((m+1)!, -1)_p \text{ for } p \text{ odd.}
\]

Bruck and Ryser gave 10. and 11. in \( \text{[20]} \).

4.2 Next consider the invariant \( c_p(N_0N') \),

where \( N_0 \) is the incidence matrix of a balanced incomplete block design. The elements of \( N_0 \) are 0 and 1, so that the elements of \( N_0N' \) are rational numbers. Further, \( N_0N' \) is symmetric and therefore may be regarded as the coefficient matrix of a quadratic form.

Recall that
1.4.21 \[
N_{\circ}^\circ \cdot N_{\circ}^\circ' = \begin{pmatrix}
\lambda & r & \cdots & \lambda \\
r & \lambda & \cdots & \lambda \\
\vdots & \vdots & \ddots & \vdots \\
\lambda & \lambda & \cdots & r
\end{pmatrix}.
\]

Now there exists an elementary transformation matrix \( L \) with rational coefficients such that

1.4.22
\[
L' (N_{\circ}^\circ N_{\circ}^\circ') L = Q = \begin{pmatrix}
2(r-\lambda) & (r-\lambda) & \cdots & (r-\lambda) & -(r-\lambda) \\
(r-\lambda) & 2(r-\lambda) & \cdots & (r-\lambda) & -(r-\lambda) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
(r-\lambda) & (r-\lambda) & \cdots & 2(r-\lambda) & -(r-\lambda) \\
-(r-\lambda) & -(r-\lambda) & \cdots & -(r-\lambda) & r
\end{pmatrix}.
\]

This obviously is true since \( Q \) is obtained from \( N_{\circ}^\circ N_{\circ}^\circ' \) by subtracting the last row of \( N_{\circ}^\circ N_{\circ}^\circ' \) from the other rows, and then by subtracting the last column from the other columns. Clearly \( Q \) is rationally congruent to \( N_{\circ}^\circ N_{\circ}^\circ' \), so that

1.4.23 \[
c_p(Q) = c_p(N_{\circ}^\circ N_{\circ}^\circ').
\]

We shall now compute \( c_p(Q) \). If \( D_1 \) is the leading principal minor determinant of order 1,
then

\[ \text{1.4.24} \quad D_i = (r - \lambda)^i (i+1), \quad i=1,2, \ldots, v-1, \text{ and} \]

\[ D_v = |2| = |N_o N'| = \text{rk}(r - \lambda)^{v-1}. \]

Omit p for convenience and write

\[ \text{1.4.25} \quad c(N_o N') = (-1, -D_v)(D_{v-1}, -D_v)^{v-2} \prod_{i=1}^{v-2} (D_1, -D_{i+1}). \]

Consider

\[ \text{1.4.251} \]

\[ (-1, -D_v)(D_{v-1}, -D_v) = (-1, -\text{rk}(r - \lambda)^{v-1}((r - \lambda)^{v-1}_v, \]

\[ - \text{rk}(r - \lambda)^{v-1}), \]

which by 4. of 1.4.15,

\[ = (-1, -1)(-1, r)(-1, k)(-1, (r - \lambda)^{v-1})((r - \lambda)^{v-1}, r) \]

\[ ((r - \lambda)^{v-1}, k)((r - \lambda)^{v-1}, -(r - \lambda)^{v-1}(v, r)(v, k) \]

\[ (v, -(r - \lambda)^{v-1}). \]

Now from 3. of 1.4.15, \((r - \lambda)^{v-1}, -(r - \lambda)^{v-1}) = 1.\]

Also it is clear that

\[ (a, b^n) = (a, b)^n. \]

Hence, 1.4.251 becomes
1.4.252 \((-1, r)(-1, k)(-1, r- \lambda)^{v-1}(r- \lambda, r)^{v-1}\)

\((r- \lambda, k)^{v-1}(v, r)(v, k)(v, -1)(v, r- \lambda)^{v-1}\).

Also,

1.4.253

\[\prod_{i=1}^{v-2} (D_i, -D_{i+1}) = \prod_{i=1}^{v-2} ((r- \lambda)^{i+1}(i+1), -(r- \lambda)^{i+1}(i+2)),\]

which by 4. of 1.4.15,

\[= \left\{ \prod_{i=1}^{v-2} \left( (r- \lambda)^{i+1}(i+1), -(r- \lambda)^{i+1}(i+2) \right) \right\} S,\]

where

1.4.254 \(S = \prod_{i=1}^{v-2} \left( ((r- \lambda)^{i+1}, i+2), ((r- \lambda)^{i+1}, i+1) \right) \).

By 4. of 1.4.15, 1.4.253 becomes

1.4.255 \[\prod_{i=1}^{v-2} \{(r- \lambda, r- \lambda)^{i+1}(i+2, -1)(i+1, -1)\}\]

which by 5. of 1.4.15

\[= \prod_{i=1}^{v-2} \{(r- \lambda, -1)^{i+1}(i+2, -1)(i+1, -1)\}\]

\[= (r- \lambda, -1)^{(v-1)(v-2)/2)((v-1)!,-1)(v!,-1),\]
since \( \Pi_{i=1}^{v-2} (r-\lambda,-1)^i = (r-\lambda,-1)^{(v-1)(v-2)/2} \),

\( \Pi_{i=1}^{v-2} (i+2,-1) = (v!,v-1)(2,-1)(1,-1) = (v!,v-1) \),

and \( \Pi_{i=1}^{v-2} (i+1,-1) = ((v-1)!,v-1)(1,-1) = ((v-1)!,v-1) \).

Finally, 1.4.255 becomes

1.4.256 \( (r-\lambda,-1)^{(v-1)(v-2)/2} (v,-1) \).

Now consider \( S \) from 1.4.273.

1.4.257

\[
S = \Pi_{i=1}^{v-2} \left\{ ((r-\lambda)^i,i+2)((r-\lambda)^{i+1},i+1) \right\}
\]

\[
= \Pi_{i=1}^{v-2} \left\{ ((r-\lambda)^i,i+2)((r-\lambda)^{i-1},i+1) \right\},
\]

since by 2. of 1.4.15,

\( ((r-\lambda)^2,i+1) = 1 \).

Then

1.4.258 \( S = \prod_{i=1}^{v-2} (r-\lambda)^i \prod_{i=0}^{v-3} (r-\lambda)^i,i+2) = (r-\lambda,v)^{v-2} \).
Since all terms occur twice except \((r-\lambda,v)^{v-2}\) and \((1,2)\) which has the value +1 for p odd. Therefore from 1.4.25, 1.4.252, 1.4.256, and 1.4.258,

\begin{equation}
\begin{aligned}
c_p(N_o N_o') &= (-1,-1)_p (-1,r)_p (-1,k)_p (-1,r-\lambda)_p \binom{v}{v-1}/2 \\
&\quad (r-\lambda,r)_p^{v-1} (r-\lambda,k)_p^{v-1} (v,r)_p \\
&\quad (v,k)_p (v,r-\lambda)_p.
\end{aligned}
\end{equation}

This result is contained in

**Lemma 1.4.1:** The value of $c_p(N_o N_o')$ for $p$ is given by 1.4.26.

It is of interest to note that if $r=k$, then 1.4.26 will reduce to

\begin{equation}
1.4.27 \quad c_p(N_o N_o') = (-1,r-\lambda)_p^{v(v-1)/2} (v,r-\lambda)_p,
\end{equation}

for $p$ odd, which was given by Shrikhande \(\sqrt{13}\).

4.3 It is necessary to evaluate $c_p$ for $N_{20} N_{20}'$, where $N_{20}$ is obtained from $N_2$ by keeping only those of the adjoined row vectors which have
unity in the b-v columns corresponding to a chosen set of b-v blocks. Clearly \( N_{20} \) is of order \( b \times b \), so that \( c_p(N_{20}^N) \) exists. It will be seen that this computation can be based on the value just found for \( c_p(N_o^N) \). Clearly \( D_i, 1 = 1, \ldots, v, \) is the same for \( N_{20}N_{20}^i \) as for \( N_oN_o^i \). Hence, for the leading principal minor determinants of order \( \leq v \) in \( c_p(N_{20}N_{20}^i) \) we obtain

\[
1.4.31 \quad (-1, -D_v) c_p(N_o^N) .
\]

Clearly any principal minor determinant of 
\( |N_{20}N_{20}^i| \) may be evaluated according to Lemma 1.2.1, and in fact we have seen that

\[
1.4.32 \quad D_{v+t} = r^{-t+1} k(r-\lambda)^{v-t-1} c_t ,
\]

where \( t = 1, \ldots, b-v, \)

\[
c_{jj} = (r-k)(r-\lambda) ,
\]

and 

\[
c_{ju} = \lambda k - r \alpha_{ju}, (j \neq u; j, u = 1, \ldots, b-v) .
\]

Now consider the matrix \( \sum r(r-\lambda) T^{-1}_{m+c} = \mathbb{E}_t \).
t=1, \ldots, b-v, where \( I_t \) is the identity matrix of order \( t \times t \) and \( \langle r(r-\lambda) \rangle_t^{-1} \) is a scalar. Then consider
\[
P = \begin{pmatrix} N_0 N_0' & 0 \\ 0 & 0 & C_{b-v} E_{b-v} \end{pmatrix},
\]
where \( C_{b-v} \) is the characteristic matrix for the \((b-v)\) blocks of \( N_0 \) which have 1's in the last \((b-v)\) rows of \( N_{20} \), and \( E_{b-v} = E_t \) with \( t = b-v \).

Let \( D_{i}^{(\mathcal{P})} \) be the leading principal minor determinant of order \( i \) in \( P \). Then clearly
\[
D_{i}^{(\mathcal{P})} = D_{i}, \quad i = 1, \ldots, v .
\]

Also,
\[
D_{v+t}^{(\mathcal{P})} = |N_0 N_0' | |C_t E_t |
= \langle \text{rk}(r-\lambda) \rangle_{v-1} \langle r^{-t}(r-\lambda)^{-t} |C_t | \rangle
= r^{-t+1} k(r-\lambda)^{v-t-1} |C_t | ,
\]
t=1, \ldots, b-v. Hence
\[
D_{v+t}^{(\mathcal{P})} = D_{v+t} .
\]
From 1.4.34 and 1.4.35 it follows that

1.4.36 \[ c_p(p) = c_p(N_{20}N_{20}'). \]

If f and g are two forms with no variables in common, then we know \( \langle 22 \rangle \) that

1.4.37 \[ c_p(f+g) = c_p(f)c_p(g)(-1,-1)_p(|f|,|g|)_p. \]

Since \( N_{o}N_{o}' \) and \( C_{b-v}E_{b-v} \) have no variables in common, we have from 1.4.36 and 1.4.37 the

**Theorem 1.4.2:**

\[ c_p(N_{20}N_{20}') = c_p(N_{o}N_{o}')c_p(C_{b-v}E_{b-v})(-1,-1)_p \]

\[ (|N_{o}N_{o}'|,|C_{b-v}E_{b-v}|)_p, \]

where \( N_{o} \) is the incidence matrix,

\[ N_{20} = \begin{pmatrix} N_{o} \\ I_{b-v} \\ 0 \end{pmatrix} \]

\( C_{b-v} \) is the characteristic matrix for the \((b-v)\) columns (blocks) of \( N_{o} \) which pass through \( I_{b-v} \),

\( E_{b-v} = \overline{r(r-\lambda)} \overline{r-1}I_{b-v} \), and \( I_{b-v} \) is the identity matrix of order \((b-v) \times (b-v)\).
5. The Impossibility of Balanced Incomplete Block Designs (8), (12), (28) of Fisher and Yates' Table

5.1 In this section the proof of the impossibility of the designs (8), (12), and (28) of 1.1.13 is completed.

From Lemmas 1.3.1 and 1.3.2 it follows that there exist two rows of the structural matrix of the design $S_b = N'N_o$ which are as follows:

\[
\begin{array}{cccccccc}
1.5.11 & k & 1 & l & 1 & \ldots & 1 & 1 & 2 & \ldots & 2 & 2 & \ldots & 2 \\
 & k & 1 & l & 1 & \ldots & 1 & 2 & 2 & \ldots & 2 & 2 & \ldots & 2 \\
 & k & & & & & & & & & & & & 1 & \ldots & 1 \\
\end{array}
\]

where the partitions break up the matrix $S_b$ into submatrices $A, B, C, D$, and $E$, in left to right order. According to Lemma 1.3.1, rows 1 and 2 both contain $k(k-1)/2$, 2's and $2k$, 1's. Now since blocks 1 and 2 intersect in 1 treatment, it follows from Lemma 1.3.2 that there exist $(k-1)(k-2)/2$ blocks of type 1, $2(k-1)$ blocks of type 2, and $k$ blocks of type 3. Hence, it is necessary that $A$ contain 3 columns, that $B$ contain $(k-1)$ columns, that $C$ contain $(k-1)$ columns, that $D$ contain $(k-1)(k-2)/2$
columns, and that $E$ contain $(k-1)$ columns.

Consider how the third row of $S_b$ may be filled up. By Lemma 1.3.1 it must contain $k(k-1)/2$, 2's and $2k$, 1's. Since block 3 intersects block 1 in 1 treatment, it follows by considering blocks 1 and 3 as initial blocks that the number of blocks of types 1, 2, and 3 must be as given in the preceding paragraph. Also block 3 intersects block 2 in 1 treatment, so the same result holds for blocks 2 and 3 as initial blocks. Unfortunately these conditions are met by numerous arrangements of the 1's and 2's in row 3. In fact, it follows from Lemmas 1.3.1 and 1.3.2 that if there are $(k-j-1)$ 2's in row 3 of B, then there are $j$ 2's in row 3 of C, $\sqrt{(k-1)(k-2)/2} - j$, 2's in row 3 of D, and $j$, 2's in row 3 of E, ($j=1, \ldots, k-1$).

5.2 Consider $S_{k+2}$, the structural matrix for the following $(k+2)$ blocks: the blocks of $A$, the $j$ blocks of $B$ which have 2 in row 3, and the $(k-j-1)$ blocks of $E$ which have 1 in row 3. I.e.,
1.5.21 \[
S_{k+2} = \begin{pmatrix}
 k & 1 & 1 & \cdots & 1 & 2 & \cdots & 2 \\
 k & 1 & 2 & \cdots & 2 & 1 & \cdots & 1 \\
 k & 2 & \cdots & 2 & 1 & \cdots & 1 \\
 F & & & & & G & & \\
 & & & & & H & &
\end{pmatrix}
\]

where \( F \) and \( H \) have \( k \) in the main diagonal, and are symmetric matrices. The other elements of \( F \), \( G \), and \( H \) are so far unknown and will be determined below. Of course \( S_{k+2} \) is symmetric. Comparison of the structure of \( S_{k+2} \) with the structures of \( S_5^{(1)} \) of 1.3.31, \( S_5^{(2)} \) of 1.3.35, and \( S_5^{(3)} \) of 1.3.38 shows that Lemmas 1.3.3, 1.3.4, and 1.3.5 apply. Hence the elements of \( F \) are 1, the elements of \( G \) are 2, and the elements of \( H \) are 1, for \( k < 10 \).

Corresponding to \( S_{k+2} \) is the characteristic matrix \( C_{k+2} \). It is useful to compute
1.5.22

\[ |c_{k+2}| = \]

\[
\begin{array}{cccc}
(2k)(k-2)(k-2) & (k-2) & \ldots & (k-2) \\
(2k)(k-2) & (-4) & \ldots & (-4) \\
(2k) & (-4) & \ldots & (-4) \\
\end{array}
\]

\[
\begin{array}{cccc}
(2k)(k-2) & \ldots & (k-2) & (-4) \\
(k-2)(2k) & \ldots & (k-2) & (-4) \\
\vdots & \vdots & \vdots & \vdots \\
(k-2)(k-2) & \ldots & (2k) & (-4) \\
\end{array}
\]

Clearly \( c_{k+2} \) is of the same form as \( A \) of 1.2.11 and therefore Lemma 1.2.1 applies. We obtain

1.5.221

\[ |c_{k+2}| = \sqrt{4(k-1)} \sqrt[3]{4} \left( k^2 + (k+2) - k^3 \right) |B_{k-1}|, \]

where

\[ b_{mn} = 4(k-1)(k+2)c_{m+3,n+3} - \zeta(k-1) \sum_{i=1}^{3} c_{im} c_{in} \]

\[ + (k-2) \sum_{i=1}^{3} c_{im} \sum_{i=1}^{3} c_{in}, \]

\((m \neq n; \ m, n = 1, \ldots, k-1)\).

Explicitly, the typical element in the first j
rows and columns of $B_{k-1}$ is

$$1.5.222 \quad b_{mn} = 4(k-1)(k+2)c_{m+3,n+3} - 4(k-1)((k-2)^2 + 32) + (k-2)(k-10)^2,$$

whence

$$b_{mm} = (k+2)^2(5k-14),$$

and

$$b_{mn} = (k+2)^2(k-10), \quad (m \neq n; \ m, n = 1, \ldots, j).$$

The typical element in the first $j$ rows and the last $(k-j-1)$ columns of $B_{k-1}$ is

$$1.5.223$$

$$b_{mn} = 4(k-1)(k+2)c_{m+3,n+3} + 48(k-1)(k-2) + (k-2)(k-10)(2k-8),$$

$$(m=1, \ldots, j; \ n=j+1, \ldots, k-1), \text{ whence since }$$

$$c_{m+3,n+3} = (-4),$$

$$b_{mn} = 2(k+2)^2(k-4).$$

The typical element in the last $(k-j-1)$ rows and columns is
\[ b_{mn} = 4(k-1)(k+2)c_{m+3,n+3} - 4(k-1)(16+2(k-2)^2) + (k-2)(2k-8)^2, \]

whence

\[ b_{mn} = 4(k+2)^2(k-2), \]

and

\[ b_{mn} = -4(k+2)^2, (m \neq n; m, n = j+1, \ldots, k-1). \]

From 1.5.22, ..., 1.5.224,

1.5.23

\[ |B_{k-1}| = (k+2)^2(k-1). \]

\[
\begin{vmatrix}
(5k-14)(k-10) & \cdots & (k-10) \\
(5k-14) & \cdots & (k-10) \\
\vdots & & \vdots \\
(5k-14) & \cdots & (5k-14) \\
\end{vmatrix}
\begin{vmatrix}
2(k-4) & \cdots & 2(k-4) \\
\vdots & & \vdots \\
2(k-4) & \cdots & 2(k-4) \\
4(k-2)(-4) & \cdots & (-4) \\
4(k-2) & \cdots & (-4) \\
4(k-2) & & \\
\end{vmatrix}
\]

\[ = (k+2)^2(k-1) |B'_{k-1}|, \text{ say.} \]

Now \( B'_{k-1} \) is of the same form as \( A \) of 1.2.11.
Hence, by Lemma 1.2.1,

1.5.235

\[ |B_{k-1}^j| = \sum (j+4)^{k-(10j+4)} \sum^{k+j+2} \sum 4(k-1) \sum^{j-k} |D_{k-j-1}|, \]

where the typical element of \( D_{k-j-1} \) is

\[ d_{su} = 4(k-1)(j(k-10)+4(k-1))b_{s+j,u+j}^1 b_{s+j,u+j}^2, \]

\( (s,u=1, \ldots, k-j-1). \)

Clearly,

1.5.24 \( |D_{k-j-1}| = (d_{ss}-d_{su})^{k-j-2} (d_{ss}+(k-j-2)d_{su}), \)

which by 1.5.235

\[ = \sum 16(k-1)^2 (4(k-1)-j(10-k)) \sum^{k-j-2} \]

\[ \cdot \sum 16j(k-1)^2 (k-6) (j-(k-2)) \sum, \]

\( (j=0,1, \ldots, k-1). \)

Now according to (ii) of Theorem 1.2.1,

\[ |C_{k+2}| = 0. \]

This requirement will be used to rule out certain possibilities for \( j \). From 1.5.221, 1.5.235, and 1.5.24, it is clear that \( |C_{k+2}| = 0 \) when and only when \( k=1 \) or \( k=6 \), and \( j=0 \), \((k-2) \) or
4(k-1)/(10-k). Of course j has to be an integer.

5.3 Let k=8. Then from 1.5.25 j=14, but as j \leq 7 so j = 14 is not permissible. Hence for 
\[ |C_9| = 0, \] either j=0 or j=6. If j=0, then consider 
\[ S_9^{(1)} \] for blocks 1,2,3, and any 6 blocks of C of 
1.5.11. Then from 1.2.36, 1.5.22, ..., 1.5.24,

1.5.31 \[ \left| \left( N_{20}^{(1)} \right) \left( N_{20}^{(1)} \right)' \right| = 2^{33}, \]

where \( N_{20}^{(1)} \) is obtained from \( N_2 \) by keeping only 
those 9 of the adjoined row vectors which have 
unities in the columns (blocks) for which \( S_9^{(1)} \) is 
the structural matrix.

If j=6, then consider \( S_9^{(2)} \) for blocks 1,2,3, 
and the six blocks of C of 1.5.11 for which the 
third row contains 2. Then

1.5.32 \[ \left| \left( N_{20}^{(2)} \right) \left( N_{20}^{(2)} \right)' \right| = 2^{85}, \]

where \( N_{20}^{(2)} \) is obtained from \( N_2 \) by keeping only 
those 9 of the adjoined row vectors which have
unities in the columns (blocks) for which $S^{(2)}_9$ is the structural matrix.

The determinant $\left| (N_{20}^{(1)})(N_{20}^{(1)}) \right|$, $i=1,2$, is not a perfect integral square. But from (iii) of Theorem 1.2.1, it must be a perfect integral square. Hence the

**Theorem 1.5.1:** The balanced incomplete block design with parameters $r=10$, $k=8$, $b=45$, $v=36$, and $\lambda=2$ is impossible.

5.4 Although a similar argument might be given for $k=5$, an easy proof is as follows. Consider

\begin{equation}
S_4 = \begin{pmatrix}
5 & 1 & 1 & 1 \\
5 & 2 & 2 & 5 \\
5 & & \alpha_{34} & 5 \\
5 & & & 5
\end{pmatrix},
\end{equation}

where $\alpha_{34}$ is unknown but is either 1 or 2. The corresponding characteristic determinant is
\[ |C_4| = \begin{vmatrix} 10 & 3 & 3 & 3 \\ 10 & -4 & -4 \\ 10 & \alpha_{34} \\ 10 \end{vmatrix} = (13)(7^2)(10 - \alpha_{34})(13 \alpha_{34} + 3\theta), \]

By Theorem 1.2.1, it is necessary that \(|C_4| \geq 0\).

Clearly \((10 - \alpha_{34}) \geq 0\) for both \(\alpha_{34} = -4\) and \(\alpha_{34} = 3\). But consider

\[ 1.5.43 \quad 13 \alpha_{34} + 38 \geq 0, \]

or \(\alpha_{34} \geq -38/13 = -3\).\]

From 1.5.43 it follows that \(\alpha_{34} = k-2=3\). Hence, \(\alpha_{34} = 1\) and blocks 1,3, and the four blocks of C of 1.5.11 have the structural matrix

\[ S_6 = \begin{pmatrix} 5 & 1 & \cdots & 1 \\ 1 & 5 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 5 \end{pmatrix}. \]

The corresponding characteristic determinant is
\[ \begin{vmatrix} 10 & 3 & \ldots & 3 \\ 3 & 10 & \ldots & 3 \\ \vdots & \vdots & & \vdots \\ 3 & 3 & \ldots & 10 \end{vmatrix} = 7^5 \cdot 5^2 , \]

and from 1.2.36,

\[ |N_{20}^{\prime}N_{20}'| = 5^5 , \]

where \( N_{20} \) is obtained from \( N_2 \) by keeping only the 6 of the adjoined row vectors which have unities in those columns (blocks) for which \( S_6 \) is the structural matrix. Now \( |N_{20}^{\prime}N_{20}'| \) is not a perfect integral square, which contradicts (iii) of Theorem 1.2.1. Hence, the

**Theorem 1.5.2:** The balanced incomplete block design with parameters \( r=7, k=5, b=21, v=15 \), and \( \lambda =2 \) is impossible.

This result was obtained by Nandi \( \left(17\right) \) by a different method.
5.5 Next consider the design which has \( k=6 \). From 1.5.24 it is seen that \((k-6)\) is a factor of \( |D_{k-j-1}| \). Hence the argument used for \( k=8 \) will not apply for \( k=6 \).

Consider 1.5.11 in which two rows of \( S_b \) are given. Assume that there do not exist five blocks having for their structural matrix

\[
S_5^{(4)} = \begin{pmatrix}
6 & 1 & \ldots & 1 \\
1 & 6 & \ldots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \ldots & 6
\end{pmatrix}
\]

This assumption will be contradicted. For the assumption to be true it is necessary for row 3 of \( C \) to contain exactly three 2's. For if it contains less than three 2's then it contains at least three 1's, which we may for definiteness take to be in columns 1, 2, and 3 of \( C \). But then blocks 1 and 3 of \( A \), and 1, 2, and 3 of \( C \) form \( S_5^{(4)} \), by Lemma 1.3.4. If row 3 of \( C \) contains more than three 2's, then by Lemma 1.3.3, blocks 1 of \( A \) and any 4 blocks of \( C \) which have 2 in row 3 form
$S_5^{(4)}$. Hence, there exist three blocks such that the first three rows of $S_b$ are as shown below.

\[
\begin{array}{c|cccccccc}
1.5.52 & 6 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
       & 6 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\
       & 6 & 1 & 1 & 2 & 2 & 2 & 2 & 1 & 1 & 1 \\
\end{array}
\]

Denote the element in row $i$ and column $j$ of sub-matrix $\mathcal{F}$ by $(i,j)$. Then for $S_5^{(4)}$ not to exist, it is necessary in $B$ that

\[
1.5.53 \quad (4,2) = (4,3) = (5,3) = 2.
\]

But then blocks 1, 2, 4, 5, and 6 of $S_b$ form $S_5^{(2)}$ with $\alpha_{45} = 2$, which contradicts Lemma 1.3.4. Hence, the

**Lemma 1.5.1**: There exist five blocks having $S_5^{(4)}$ of 1.5.51 for their structural matrix.

Without loss of generality, let $S_5^{(4)}$ be the
leading principal minor matrix of order 5 in $S_b$. Let $S_b$ be partitioned as in 1.5.11. Then row 3 of B contains at least two 1's and cannot contain more than three 2's. Hence, row 3 of C cannot contain fewer than two 2's. If row 3 of C contains $u$, 2's, then by Lemma 1.3.2, row 3 of E contains $(5-u)$, 1's, $(u=2, \ldots, 5)$.

Case 1. Row 3 of C contains either two or three 2's. Then row 3 of E contains at least two 1's. Let $S_7^{(1)}$ be the structural matrix for the three blocks of A, any two blocks from C which have 2 in row 3, and two blocks from E which have 1 in row 3. Then

$$
S_7^{(1)} = \begin{pmatrix}
6 & 1 & 1 & 1 & 1 & 2 & 2 \\
6 & 1 & 2 & 2 & 1 & 1 \\
6 & 2 & 2 & 1 & 1 \\
6 & 1 & 2 & 2 \\
6 & 2 & 2 \\
6 & 1 \\
6
\end{pmatrix},
$$

where the partitions separate the blocks from A, C, and E, in that order. The elements in rows 4, 5, and 6 and not in the main diagonal of $S_7^{(1)}$ are
uniquely determined by Lemmas 1.3.3, 1.3.4, and 1.3.5.

Case 2. Row 3 of C contains either four or five 2's. Let $S_7^{(2)}$ be the structural matrix for the three blocks of A and any four blocks of C which contain 2 in row 3. Then

$$S_7^{(2)} = \begin{pmatrix}
6 & 1 & 1 & 1 & 1 \\
6 & 1 & 2 & 2 & 2 \\
6 & 2 & 2 & 2 & 2 \\
6 & 1 & 1 & 1 \\
6 & 1 & 1 \\
6 & 1 \\
6 \\
\end{pmatrix},$$

where the partitions separate the blocks from A and C.

An easy computation shows that (iii) of Theorem 1.2.1, i.e., the perfect integral square condition, does not rule out either Case 1 or Case 2. However, recall the invariant $c_p$, which will now be used.

The characteristic matrix which corresponds to $S_7^{(1)}$ is
1.5.56
\[
C_7^{(1)} = \begin{pmatrix}
12 & 3 & 3 & 3 & 3 & -3 & -3 \\
12 & 3 & -3 & -3 & 3 & 3 \\
12 & -3 & -3 & 3 & 3 \\
12 & 3 & -3 & -3 \\
12 & -3 & -3 \\
12 & 3 \\
12
\end{pmatrix},
\]

and to \(S_7^{(2)}\) is
\[
C_7^{(2)} = \begin{pmatrix}
12 & 3 & 3 & 3 & 3 & 3 & 3 \\
12 & 3 & -3 & -3 & -3 & -3 & -3 \\
12 & -3 & -3 & -3 & -3 \\
12 & 3 & 3 & 3 \\
12 & 3 & 3 \\
12 & 3 \\
12
\end{pmatrix}.
\]

Multiply the last two rows and columns of \(C_7^{(2)}\) by -1. The result is \(C_7^{(1)}\). Hence \(C_7^{(2)}\) is rationally congruent to \(C_7^{(1)}\), and it follows that consideration of Case 1 only is sufficient.

Define \(N_{20}\) as the matrix which is obtained from \(N_2\) by keeping only the 7 adjoined row vectors which contain unities in the 7 columns (blocks) for which \(S_7^{(1)}\) is the structural matrix. Then we
wish to compute $c_p(N^o_{20}N^o_{20})$. By Theorem 1.4.2,

$$1.5.57 \quad c_p(N^o_{20}N^o_{20}) = c_p(N^o_{0}N^o_{0})c_p(C^{(1)} E_7)(-1,-1)_p$$

$$= (|N^o_{0}N^o_{0}|, |C^{(1)} E_7|).$$

From 1.4.26, for p odd,

$$1.5.571 \quad c_p(N^o_{0}N^o_{0}) = (-1,2^3)(-1,2\cdot 3)(-1,2\cdot 3)^{210}(2,3,2^3)^{20}$$

$$= (-1,3)(3,2)(7,2),$$

where p is omitted for convenience.

Consider $c_p(C^{(1)} E_7)$. From 1.4.345, if $D_t$ is a leading principal minor determinant of order $t (t=1, \ldots, 7$) in $C^{(1)} E_7$, then

$$1.5.572 \quad D_t = \sqrt[r-r-\lambda]{r}^{-t} |C^{(1)}_t|$$

$$= \sqrt[k(k+2)]{k}^{-t} |C^{(1)}_t|$$

$$= 2^{-dt} 3^{-t} |C^{(1)}_t|. $$

Clearly $C^{(1)}_7$ is of the same form as $C^{(1)}_{k+2}$ of 1.5.22.
Hence from 1.5.221, ..., 1.5.24 and 1.5.572, if 
t=1,2,3 then

1.5.573 \[ D_t = 2^{-t-1} 3^{-t} (2+t); \]

if \( t = 3, \ldots, 7 \) then

\[ D_t = 2^{-t-1} 3^{-t} (8-t). \]

Hence,

1.5.574

\[ c_p(C_7^{(1)}E_7) = (2^{-2}, 2^{-13^{-2}}) (2^{-13^{-2}}, 2^{-43^{-3}3^{-5}}) \]
\[ \cdot (2^{-43^{-3}3^{-5}}, 2^{-33^{-4}}) (2^{-33^{-4}}, 2^{-63^{-4}}) \]
\[ \cdot (2^{-63^{-4}}, 2^{-63^{-6}}) (2^{-63^{-6}}, 2^{-83^{-7}}) \]
\[ \cdot (-1, 2^{-83^{-7}}) \]
\[ = (5, -1) \]

for \( p \) odd.

Also \((-1, -1) = 1 \) for \( p \) odd, and

1.5.575 \[ (|N_0^*N_0^*|, |C_7^{(1)}E_7|) = (2^{243^{-21}}, 2^{-83^{-7}}) \]
\[ = (3, -1) \]

for \( p \) odd.
From 1.5.57, ..., 1.5.575, it follows that

\[ c_p(N_{20}N_{20}^t) = (3, -1)^2(3, 2)(7, 2)(5, -1), \]

for \( p \) an odd prime, and for \( p=3 \),

\[ 1.5.58 \quad c_3(N_{20}N_{20}^t) = -1. \]

But consider

\[ 1.5.59 \quad N_{20}^{-1}(N_{20}N_{20}^t)(N_{20}^{-1})^t = I, \]

the identity matrix. Clearly \( c_p(I) = +1 \) for all odd primes \( p \). Also from 1.5.59, \( N_{20}N_{20}^t \) and \( I \) are rationally congruent, and it follows from Theorem 1.4.1 that \( N_{20}N_{20}^t \) and \( I \) have the same value for \( c_p \), for all primes \( p \). However, since from 1.5.58 \( c_3(N_{20}N_{20}^t) = -1 \), Theorem 1.4.1 is contradicted, and hence we obtain

**Theorem 1.5.3**: The balanced incomplete block design with parameters \( r=8, k=6, b=22, v=21, \) and \( \lambda=2 \) is impossible.
REFERENCES


CHAPTER II

ON THE STRUCTURE OF INTRA-INTER GROUP BALANCED INCOMPLETE BLOCK DESIGNS

1. Some Theorems on the Structure of Intra-Inter Group Designs

1.1 In this chapter we shall consider a generalization of balanced incomplete block designs. Let the treatments be divided into $m$ groups of $v_k$ ($k = 1, \ldots, m$) treatments each. Let any two treatments within group $k$ occur together in $\lambda_{kk}$ blocks, and let a treatment from group $k$ occur with a treatment from group $s$ in $\lambda_{ks}$ blocks. Each treatment of group $k$ is replicated $r_k$ times. There are $b$ blocks and $k$ plots per block. The designs of this class are called intra-inter group balanced incomplete block designs. This class of designs was introduced by Nair and Rao\(^1\) in 1942.

The following conditions are necessary:

\(^1\)Numbers in square brackets refer to references at the end of the chapter.
2.1.11 \( \lambda_{s}^{k'} = \lambda_{s}^{k'} \), \((s=1, \ldots, m)\),

2.1.12 \( b_{k} = \sum_{k'=1}^{m} r_{k'} v_{k'} \), and

2.1.13 \( r_{k}(k-1) = \sum_{s=1}^{m} v_{s} \lambda_{s}^{k'} - \lambda_{k'}^{k'} \), \((k'=1, \ldots, m)\).

We also define

2.1.14 \( v = \sum_{k'=1}^{m} v_{k'} \).

A sub-class of especial interest, the group divisible designs, was introduced by Bose [1] in 1951. Let \( r_{k'} = r \), \( v_{k'} = n \), \( \lambda_{k'}^{k'} = \lambda_{1} \), and \( \lambda_{s}^{k'} = \lambda_{2} \).

It is seen that 2.1.13 becomes

2.1.15 \( r(k-1) = (n-1) \lambda_{1} + (m-1)n \lambda_{2} \),

whence

2.1.155 \( r+(n-1)\lambda_{1} - n\lambda_{2} = r_{k} - v\lambda_{2} \).

It is easy to see that the group divisible designs may be regarded also as a sub-class of the class of partially balanced designs. In fact, the parameters of the second kind are
2.1.16 \[ p_{11}^1 = n-2 , \]
\[ p_{12}^1 = p_{21}^1 = 0 , \]
\[ p_{22} = v-n , \]
\[ p_{22}^2 = v-2n , \]
\[ p_{12}^2 = p_{21}^2 = n-1 , \]
and \[ p_{11}^2 = 0 , \]

which satisfy the necessary conditions for such parameters.

In this chapter we shall generalize the methods of Chapter I so as to accommodate the intra-inter group designs, and shall deduce a number of interesting results.

1.2 In this section we shall define the structural matrices of the general intra-inter group balanced incomplete block designs.

Let \( \alpha_{\mathcal{I}j_u}^\mathcal{I} \) = the number of treatments from group \( \mathcal{I} \) which blocks \( j \) and \( u \) have in common. Then

\[
\sum_{\mathcal{I}=1}^{m} \alpha_{\mathcal{I}j_u}^\mathcal{I} = \alpha_{j_u}
\]

is the number of treatments which blocks \( j \) and \( u \)
have in common. The \( b \times b \) symmetric matrix

\[
S_{bm} = (\alpha_{ju}) = N_o'N_o
\]

is defined to be the **structural matrix of the design.** In \( S_{bm} \) the \( j \)-th row or column corresponds to the \( j \)-th block and the successive elements of the \( j \)-th row or column give the number of treatments which this block has in common with the 1st, 2nd, ..., \( b \)-th blocks.

Now select any \( t \) blocks \((t \leq b)\) and \( p \) groups \((p \leq m)\) from the design. Let \( N_{oo} \) be the submatrix of \( N_o \) formed by keeping only the \( t \) chosen columns and the rows corresponding to those treatments which belong to the \( p \) chosen groups. Then the \( b \times b \) symmetric matrix

\[
S_{tp} = N_{oo}'N_{oo}
\]

is defined to be the **structural matrix of the \( t \) chosen blocks and the \( p \) chosen groups of the design.** In \( S_{tp} \) the \( j \)-th row or column corresponds to the \( j \)-th among the chosen blocks, and the successive elements of the \( j \)-th row or column give the number of treatments belonging to the \( p \) selected groups which this
block has in common with the 1st, 2nd, ..., t-th blocks.

1.3 In this section we shall determine the characteristic matrices for the general class of the intra-inter group balanced incomplete block designs.

Let \( N_0 \) denote the incidence matrix of the design. Then

\[
N_0 \cdot N_o' = \begin{bmatrix}
\lambda_{ll} \cdots \lambda_{ll} & \cdots & \lambda_{lm} \lambda_{lm} & \cdots & \lambda_{lm} \\
\lambda_{ll} \lambda_{ll} & \cdots & \lambda_{ll} & \cdots & \lambda_{lm} \lambda_{lm} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\lambda_{ll} \cdots r_1 & \cdots & \lambda_{lm} \lambda_{lm} & \cdots & \lambda_{lm} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\lambda_{lm} \cdots \cdots \lambda_{lm} & \cdots & \lambda_{mm} r_m & \cdots & \lambda_{mm} \\
\lambda_{lm} \cdots \cdots \lambda_{lm} & \cdots & \lambda_{mm} \lambda_{mm} & \cdots & \lambda_{mm} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\lambda_{lm} \lambda_{lm} & \cdots \cdots \lambda_{lm} & \cdots & \lambda_{mm} \lambda_{mm} & \cdots & r_m
\end{bmatrix}
\]

Adjoin to \( N_0 \) the matrix \( I_b \), which is the identity matrix of order \( b \times b \) to form the matrix \( N_2 \), i.e.,
2.1.32
\[ N_2 = \begin{pmatrix} N_o \\ I_b \end{pmatrix} , \]

which is of order \((b+v) \times b\).

Clearly

2.1.33
\[ |N_2N'_2| = \begin{vmatrix} N_oN'_o & N_o \\ N'_o & I_b \end{vmatrix} . \]

It is convenient to write

2.1.331
\[ |N_2N'_2| = \begin{vmatrix} N_oN'_o & N_o & 0_1 \\ N'_o & I_b & 0_2 \\ J & 0_3 & I_m \end{vmatrix} , \]

where \(0_1\) is of order \(v \times m\), with every element 0; \(0_2\) is of order \(b \times m\), with every element 0; \(0_3 = 0'_2\); \(I_m\) is the identity matrix of order \(m \times m\); and

2.1.332
\[ J = \begin{pmatrix} 1 & \ldots & 1 & 0 & \ldots & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 1 & \ldots & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & 0 & \ldots & 1 \end{pmatrix} , \]
which is of order \( m \times v' \), \( k = 1, \ldots, m \), given by the partitions.

\[ |N'_o N'_2| \] will be evaluated by a sequence of operations on its rows and columns. At the end of each operation let the submatrix \( K \) bear the superscript \( i \), i.e., write \( K^i \), where \( i = \) the number of the operation. This will be the convention regardless of whether \( K \) has been transformed by the operation \( i \).

The operations are as follows:

1. Multiply the columns through \( N_o \) and \( 0_1 \) by

\[
d = \prod_{k=1}^{m} (r^{-\lambda} \ldots) \ldots
\]

and write \( d^{-(b+m)} \) outside.

2. Subtract suitable multiples of the last \( m \) rows from the rows which pass through \( (N_o N'_o)^1 \) so that the diagonal elements of \( (N_o N'_o)^1 \) become \( r^{-\lambda} \ldots \), \( k = 1, \ldots, m \), and the non-diagonal elements become 0. Since \( 0^{1}_{1} = 0 \), these operations have no effect on \( (N_o)^1 \). However \( 0^{1}_{1} \) is affected and becomes
2.1.334 \[ C_1^2 = \begin{pmatrix} -\lambda_{11}^d & \cdots & -\lambda_{1m}^d \\ \vdots & \ddots & \vdots \\ -\lambda_{1m}^d & \cdots & -\lambda_{mm}^d \end{pmatrix} \]

3. From rows \( v_{k+1}, \ldots, v_{k+1}, (k=0, \ldots, m-1) \), take outside the factor

2.1.335 \[ r_{k+1} - \lambda_{k+1, k+1} \]

4. Subtract from the columns which pass through \( N_0^3 \) and \( O_1^3 \) suitable multiples of the columns through \( (N_0^3 N_0^3)^3 \) so that \( N_0^4 = O_1^4 = (0) \).

We now have

2.1.336 \[ |N_2^4 N_2^4| = \prod_{k=1}^{m} (r_{k} - \lambda_{k}^*) v_{k+1}^{(b+m)} \begin{vmatrix} I_b^4 & 0^4 \\ 0^4 & I_m^4 \end{vmatrix}, \]

where the last determinant is the determinant of the characteristic matrix \( C_{bm} \) of the design. I.e.,

2.1.337 \[ C_{bm} = \begin{pmatrix} I_b^4 & 0^4 \\ 0^4 & I_m^4 \end{pmatrix} \]
is the characteristic matrix of the design. The elements of $C_{bm}$ will now be considered.

Denote the elements of $C_{bm}$ by $c_{ju}$ if the element is in $I^4_b$, by $c_{us}$ if the element is in $O^4_2$, by $c_{su}$ if the element is in $O^4_3$, and by $c_{ks}$ if the element is in $I^4_m$. Let $\alpha_{ju}^\kappa$ = the number of treatments from group $\kappa$ which blocks $j$ and $u$ have in common, and let

$$2.1.338 \quad \Pi^\kappa = (r^\kappa - \lambda^\kappa)^{-1}d,$$

Then

$$2.1.339 \quad c_{ju} = d \cdot \delta_{ju} - \sum_{\kappa=1}^{m} \Pi^\kappa \alpha_{ju}^\kappa, \quad (j,u=1,\ldots,b),$$

where $\delta_{ju}$ is the Kronecker Delta;

$$c_{us} = \sum_{\kappa=1}^{m} \alpha_{jj}^\kappa \lambda_{ks}, (u=1,\ldots,b; \ s=1,\ldots,m),$$

$$c_{su} = -\Pi_{s} \alpha_{uu}^s, \quad (s=1,\ldots,m; u=1,\ldots,b);$$

and

$$c_{ks} = d \cdot \delta_{ks} + \Pi_{\kappa^v} \lambda_{ks}, \quad (\kappa,s=1,\ldots,m),$$
where $\delta_{ks}$ is the Kronecker Delta.

Choose any $t$ blocks ($t \leq b$) and any $p$ groups ($p \leq m$) of the design. There are $t$ columns of $N_2$ which correspond to these $t$ blocks. Also, if we order the groups so that the first $p$ groups are chosen, then there are $\sum_{k=1}^{p} v_k$ rows of $N_2$ which correspond to these $p$ groups. If in $N_2$ we keep only the $t$ adjoined row vectors which have unities in the chosen columns, and the $\sum_{k=1}^{p} v_k$ original rows of the $p$ groups, we form the matrix $N_{20}$. Repeat the argument just given to obtain the value of $|N_{20} N_{20}^t|$. Then

$$2.1.34 \quad |N_{20} N_{20}^t| = \prod_{k=1}^{p} (r_k^{-1} \lambda_k^{(t+p)})^{v_k} |C_{tp}|,$$

where

$$2.1.35 \quad C_{tp} = \begin{pmatrix} I_t^4 & 0_2^4 \\ 0_3^4 & I_p^4 \end{pmatrix}.$$

In 2.1.35, $I_t^4$ is of order $t \times t$, $0_2^4$ is of order
t \times p, \ C_3^t is of order p \times t, and I_p^t is of order p \times p. The elements of C_{tp} are given by 2.1.339 if in 2.1.339 we replace b by t and m by p. We define C_{tp} to be the characteristic matrix of the t chosen blocks and the p chosen groups.

1.4 We shall state without proof an analogue to Theorem 1.2.1. The proof is similar to the argument of paragraph 2.4 of Chapter I.

Theorem 2.1.1: If C_{tp} is the characteristic matrix of a set of t blocks and p groups chosen from an intra-inter group balanced incomplete block design with parameters b,k,v_\kappa,r_\kappa,\lambda_\kappa,s, (\kappa,s=1,\ldots,p), then

2.1.41 (i) \quad |C_{tp}| \geq 0 \quad if \quad t < b - \sum_{\kappa=1}^{p} v_\kappa,

2.1.42 (ii) \quad |C_{tp}| = 0 \quad if \quad t > b - \sum_{\kappa=1}^{p} v_\kappa, \quad and

2.1.43 (iii) \quad \prod_{\kappa=1}^{p} (r_\kappa - \lambda_\kappa) \cdot v_\kappa^{-(t+p)} \quad |C_{tp}| \quad is \quad a

perfect integral square if \quad t = b - \sum_{\kappa=1}^{p} v_\kappa.
1.5 In this section we shall consider the special cases of the characteristic matrix for \( m \) groups in which \( t=0 \).

Let \( t=0 \). Then from 2.1.339 the elements of \( C_{om} \) are given by

\[
2.1.51 \quad c_{\kappa s} = \delta_{\kappa s} \cdot \prod_{i=1}^{m} (r_i - \lambda_{ii}) + \prod_{\kappa} v_{\kappa} \lambda_{kk} \kappa_{s}
\]

\[
= \prod_{\kappa} \left( \delta_{\kappa s} (r_{kk} - \lambda_{kk}) + v_{\kappa} \lambda_{kk} \kappa_{s} \right),
\]

\((\kappa, s=1, \ldots, m)\). Hence

\[
2.1.515 \quad N_0 N_0^\prime = \prod_{\kappa=1}^{m} (r_{kk} - \lambda_{kk}) v_{\kappa}^{-1}.
\]

\[
\begin{vmatrix}
(r_1 + (v_1-1) \lambda_{11}) & v_1 \lambda_{12} & \cdots & v_1 \lambda_{1m} \\
v_2 \lambda_{21} & (r_2 + (v_2-1) \lambda_{22}) & \cdots & v_2 \lambda_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
v_m \lambda_{m1} & v_m \lambda_{m2} & \cdots & (r_m + (v_m-1) \lambda_{mm})
\end{vmatrix}
\]

\[
= \prod_{\kappa=1}^{m} (r_{kk} - \lambda_{kk}) v_{\kappa}^{-1} \left| C_{om} \right|, \text{ say.}
\]

If \( r_{kk} = \lambda_{kk}, (\kappa=1, \ldots, m) \), the design is
degenerate. If \( r_{\kappa} = \lambda_{\kappa\kappa} \) for some \( \kappa \)'s but not for all \( \kappa \)'s, the design is singular. If \( |C_{om}| > 0 \) and \( r_{\kappa} \neq \lambda_{\kappa\kappa} \), \( (\kappa=1,\ldots,m) \), the design is regular. If \( |C_{om}| = 0 \) and \( r_{\kappa} \neq \lambda_{\kappa\kappa} \), \( (\kappa=1,\ldots,m) \), the design is semi-regular. These definitions were given by Bose \( \sim 2 \) for the group divisible designs, which are a special case of our class, as well as a special case of the partially balanced designs.

Now consider the matrix

\[
2.1.52 \quad A = \begin{pmatrix}
\alpha_1 \\
\vdots \\
\alpha_m
\end{pmatrix},
\]

where \( \alpha_{\kappa} = (\alpha_{\kappa_{11}}, \ldots, \alpha_{\kappa_{bb}}) \). Clearly \( \alpha \) is of order \( m \times b \). From the necessary parametric conditions \( (2.1.11, \ldots, 2.1.13) \), it follows at once that

\[
2.1.521 \quad (\alpha_{\kappa} \cdot \alpha_{\kappa}) = v_{\kappa}(r + (v_{\kappa-1}) \lambda_{\kappa\kappa})
\]

and

\[
(\alpha_{\kappa} \cdot \alpha_{s}) = v_{\kappa} v_{s} \lambda_{\kappa s},
\]

\( (\kappa \neq s; \kappa, s=1,\ldots,m) \), where \( (a \cdot b) \) is the scalar
product of vectors $a$ and $b$. Hence

$$AA' = 
\begin{pmatrix}
v_1(r_1+(v_1-1)\lambda_{11}) & v_1v_2 \lambda_{12} & \cdots & v_1v_m \lambda_{1m} \\
v_1v_2 \lambda_{12} & v_2(r_2+(v_2-1)\lambda_{22}) & \cdots & v_2v_m \lambda_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
v_1v_m \lambda_{1m} & v_2v_m \lambda_{2m} & \cdots & v_m(r_m+(v_m-1)\lambda_{mm})
\end{pmatrix}.
$$

If we define $(\alpha^k \cdot \alpha^s)$ as the second moment about $0$ for groups $k$ and $s$, then we may define $AA'$ as the second moment matrix for groups.

2.1.523 Let

$$V = 
\begin{pmatrix}
v_1 & 0 & \cdots & 0 \\
0 & v_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & v_m
\end{pmatrix}.
$$

Then since

$$AA' = 
\begin{pmatrix}
v_1 & 0 & \cdots & 0 \\
0 & v_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & v_m
\end{pmatrix}_{\text{com}},
$$

we have the
Theorem 2.1.2: For intra-inter group balanced incomplete block designs,

$$AA' = V \cdot C_{om},$$

where $$AA'$$ is the second moment matrix for groups,

$$V = \begin{pmatrix} 0 & \cdots & 0 \\ 0 & v_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & v_m \end{pmatrix},$$

and $$C_{om}$$ is the characteristic matrix for $$t=0$$.

Since $$V$$ is non-singular, we have the

Corollary 2.1.1: $$AA'$$ and $$C_{om}$$ have the same rank.

Let $$AA'$$ be of rank $$p$$, $$0 < p \leq m$$. We shall show that at least one principal minor determinant of order $$p$$ in $$AA'$$ is not zero. Since $$AA'$$ is of rank $$p$$, there exists at least one $$p \times p$$ submatrix $$M$$ of $$A$$ which is of rank $$p$$. Let the rows which pass through $$M$$ form the $$p \times b$$ matrix $$A_1$$. From a well known theorem in algebra, $$|A_1 A_1'|$$ may be expressed
as a sum of squares of determinants of order \( p \), and hence cannot be negative. But one of these squares is \( |M|^2 \), which is positive, so that \( |A_1A'_1| \) is positive, not zero. It is evident that \( |A_1A'_1| \) is a principal minor determinant of order \( p \) in \( AA' \). Hence, the

**Lemma 2.1.1:** If the rank of \( AA' \) is \( p \), then there exists at least one principal minor determinant of order \( p \) in \( AA' \) which is positive.

**Corollary 2.1.2:** If the rank of \( C_{om} \) is \( p \), then there exists at least one principal minor determinant of order \( p \) in \( C_{om} \) which is positive.

We now shall develop an inequality for the semi-regular designs. Let the rank of \( C_{om} \) be \( p \), \( 0 < p < m \). Then there are \( (m-p) \) linearly independent linear relations among the row vectors of \( C_{om} \). Now consider \( N_oN'_o \) which is given by 2.1.31. Let \( \beta_1^p \) denote the \( i \)-th row vector of group \( \lambda \) in \( N_oN'_o \) after carrying out operation \( p \), and make the same convention for the column vector \( \gamma_1^p \).
The operations are

2.1.53 1. Add $\beta_1^{(0)}$ to $\beta_v^{(0)}$,

2. Subtract $\gamma_{1}^{1}$ from $\gamma_{1}^{1}$, and

3. Subtract suitable multiples of $\gamma_{1}^{1}$
   from $\gamma_{v_{1}}^{12}, \ldots, \gamma_{v_{m}}^{1m}$ so as to make $\gamma_{v_{k}}^{12}$
   null except for its $v_{k}$-th elements,

   $(i=1, \ldots, v_{k}-1; \ k=1, \ldots, m)$. 

Then $N_0 \cdot N_0'$ has been transformed into

2.1.54 $Q = \begin{pmatrix}
\hat{A}_{11} & \hat{A}_{12} & \cdots & \hat{A}_{1m} \\
\hat{A}_{21} & \hat{A}_{22} & \cdots & \hat{A}_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
\hat{A}_{m1} & \hat{A}_{m2} & \cdots & \hat{A}_{mm}
\end{pmatrix},$

where

2.1.55 $\hat{A}_{\kappa\kappa'} = \begin{pmatrix}
(r_{\kappa} - \lambda_{\kappa\kappa'}) & 0 & \cdots & 0 \\
0 & (r_{\kappa} - \lambda_{\kappa\kappa'}) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & (r_{\kappa} + (v_{\kappa}-1) \lambda_{\kappa\kappa'})
\end{pmatrix},$
and
\[ \hat{\mathbf{x}}_s = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & v^s \lambda \mathbf{x}_s \end{pmatrix}. \]

Consider the elements in the last row and column of each sub-matrix of \( Q \). These elements form the matrix \( C_{om} \). Since the other elements in the last row of each group are zero, we may use the \((m-p)\) relations among the rows of \( C_{om} \) to transform \((m-p)\) of the row vectors of \( Q \) into null vectors. It follows that the rank of \( Q \) is at most \( v-m+p \).

We shall show that the rank of \( Q \) cannot be less than \( v-m+p \). For definiteness let the leading principal minor determinant of order \( p \) in \( C_{om} \), say \( D_p \), be non-zero. That such a determinant exists follows from Lemma 2.1.2. Use the \((m-p)\) linear relations to transform the vectors \( \beta^v_{\mathbf{x}^3} \) \((k=p+1, \ldots, m)\) into null row vectors. Then delete the null row vectors and the corresponding column vectors to form \( Q' \). It is easy to see that
2.1.56 \(\psi' = D_p \prod_{\lambda=1}^{m} (r_{\lambda} - \lambda) \lambda^{v-1}\),

which is positive. Hence the rank of \(N_0 N_0'\) is \(v-m+p\).

Clearly the rank of \(N_0 \leq b\). Now from a well-known theorem in algebra, the rank of a product matrix cannot exceed the rank of either factor matrix. Hence, the

2.1.57 \(\text{Rank } N_0 N_0' \leq \text{Rank } N_0 \leq b\),

or \(b \geq v-m+p\).

This result is summarized in the following

\textbf{Theorem 2.1.3:} If for the semi-regular case of the intra-inter group balanced incomplete block design the rank of the characteristic matrix \(C_{om}\) is \(p\) where \(1 \leq p < m\), then

\(b \geq v-m+p\).

Consider a special case. Let \(t=0\) and \(m=2\).

Then
2.1.58 \[ |c_{02}| = \begin{vmatrix} v_1 \lambda_{12} & v_2 \lambda_{12} \\ v_2 \lambda_{12} & (r_2+(v_2-1)\lambda_{22}) \end{vmatrix} = k^2(r_1r_2 - b \lambda_{12}). \]

If the design is semi-regular, then the rank of \(C_{02}\) is 1 and

\[ r_1r_2 = b \lambda_{12}. \]

Hence, the

Corollary 2.1.3: If \(m=2\), then for the semi-regular case it is necessary that

\[ b = \frac{r_1r_2}{\lambda_{12}}. \]

For the group divisible designs, \(p=1\). Hence, the

Corollary 2.1.4: For the semi-regular group divisible designs, \(b \geq v-m+1\).

This was proved by Bose \(\mathcal{S}_2\).

Again consider \(C_{0m}\). Let the rank of \(C_{0m}\) and
hence of $\mathbf{A}'$ be $p$, $0 \leq p \leq m$. Then there are $(m-p)$ linearly independent linear relations among the row vectors $\alpha^{k'}(k'=1,\ldots,m)$, of $\mathbf{A}$. Hence, the

**Lemma 2.1.2:** If for an intra-inter group balanced incomplete block design the rank of the characteristic matrix $C_{om}$ is $p$, $0 \leq p \leq m$, then only $p$ of the $\alpha^{k'}$ are linearly independent, where $\alpha^{k'}$ is the vector whose $j$-th element is the number of treatments from group $k'$ which occur in block $j$.

Clearly, the $\alpha^{k'}$s satisfy the relation

$$2.1.59 \quad \sum_{k'=1}^{m} \alpha^{k'} = (k, \ldots, k),$$

whence it follows that the vector $\gamma=(1,\ldots,1)$ lies in the vector space of the $\alpha^{k'}$s. Let us take $\gamma$ and any $p-1$ of the $\alpha^{k'}$s which are independent and independent of $\gamma$ as the basis of the vector space. Then each of the other $(m-p)$ $\alpha^{k'}$s is expressible as a linear combination of the vectors of the basis. In particular, if $p=1$, the other $m-p$ vectors are dependent on $\gamma$. 
Hence, the

**Corollary 2.1.5**: If \( p=1 \), then \( \alpha^\lambda=(c_\lambda, \ldots, c_\lambda) \),

\( (\lambda=1, \ldots, m) \), where \( c_\lambda = \frac{\lambda r^\lambda}{b} \).

For the semi-regular group divisible designs,

\( c_\lambda = \frac{r}{m} \), as was shown by Bose \( \text{C-2-J} \).

1.6 Now consider some special cases of \( G_{tm} \).

Let \( m=1 \). Then from 2.3.239,

\[
C_{tl} = \begin{pmatrix}
(r-\lambda-k) & -\alpha_{12} & \cdots & -\alpha_{1t} & | & \lambda k \\
& -\alpha_{12} & (r-\lambda-k) & \cdots & -\alpha_{2t} & | & \lambda k \\
& & \vdots & \ddots & \vdots & \vdots & : \\
& & -\alpha_{1t} & -\alpha_{2t} & \cdots (r-\lambda-k) & | & \lambda k \\
& & -k & -k & \cdots & -k & | & rk
\end{pmatrix},
\]

and it is easy to see that \( C_{tl} \) can be reduced to a determinant of order \( t \times t \) which has the elements

2.1.62 \( c_{jj} = (r-\lambda)(r-k) \), and

\( c_{ju} = \lambda k - r \alpha_{ju} \), \( (j,u=1, \ldots, t) \),

which is in agreement with 1.2.33 and 1.2.34.
Let \( \lambda_{\mathcal{K}'} = \lambda_1 \), and \( \lambda_{\mathcal{K}''} = \lambda_2 \), \( r_{\mathcal{K}} = r \), and \( v_{\mathcal{K}} = n \). This is the group divisible case. Then

\[
2.1.63 \quad C_{tm} = \begin{pmatrix}
B_1 & B_2 \\
B_3 & B_4
\end{pmatrix},
\]

where

\[
2.1.631 \quad B_1 = \begin{pmatrix}
(r - \lambda_1^{-k}) & -\alpha_{12} & \cdots & -\alpha_{1t} \\
-\alpha_{12} & (r - \lambda_1^{-k}) & \cdots & -\alpha_{2t} \\
\vdots & \vdots & \ddots & \vdots \\
-\alpha_{1t} & -\alpha_{2t} & \cdots & (r - \lambda_1^{-k})
\end{pmatrix},
\]

\[
2.1.632 \quad B_2 = \begin{pmatrix}
((\lambda_1 - \lambda_2) \alpha_{11} + \lambda_2^{-k}) & \cdots & \gamma((\lambda_1 - \lambda_2) \alpha_{1t}^{-m} + \lambda_2^{-k}) \\
\vdots & \ddots & \vdots \\
((\lambda_1 - \lambda_2) \alpha_{tt}^{-1} + \lambda_2^{-k}) & \cdots & ((\lambda_1 - \lambda_2) \alpha_{tt}^{-m} + \lambda_2^{-k})
\end{pmatrix},
\]

\[
2.1.633 \quad B_3 = \begin{pmatrix}
-\alpha_{11}^{t} & \cdots & -\alpha_{tt}^{t} \\
\vdots & \ddots & \vdots \\
-\alpha_{11}^{m} & \cdots & -\alpha_{tt}^{m}
\end{pmatrix},
\]

and
2.1.634  

\[ B_4 = \begin{pmatrix} 
(r+(n-1) \lambda_1) & n \lambda_2 & \ldots & n \lambda_2 \\
 n \lambda_2 & (r+(n-1) \lambda_1) & \ldots & n \lambda_2 \\
 \vdots & \vdots & \ddots & \vdots \\
 n \lambda_2 & n \lambda_2 & \ldots & (r+(n-1) \lambda_1) 
\end{pmatrix} \]

Now \( C_{tm} \) is of the same form as \( A \) of Lemma 1.2.1.

Hence, apply that lemma to obtain \( |C_{tm}| \), where the typical element is

2.1.635  

\[ c_{ju} = \text{rk}(rk-n\lambda_2)\delta_{ju} \]

\[ + \sum_{k=1}^{m} (rk\alpha_{uu}^f - n\lambda_2^{jk}) \cdot \]

\[ \cdot (\lambda_1 - \lambda_2) \alpha_{jj}^f + \lambda_2^{jk} \]

Specifically

2.1.636  

\[ |C_{t+m}| = (r+(n-1) \lambda_1 + (m-1)n \lambda_2)^{m-1} \cdot (r+(n-1) \lambda_1 - n \lambda_2) \cdot |C_t| \cdot \]

Further use will be made of 2.1.636 in section 2 of this chapter.
"2. Theorems on the Structure of Symmetrical Group Divisible Designs"

2.1 In this section we shall consider the special case of the group divisible designs in which the number of blocks is equal to the number of treatments. It will be shown under certain conditions that the blocks fall into groups just as the treatments do, and that the relations among the blocks are the same as the relations among the treatments.

2.2 Recall from 2.1.15 the necessary parametric condition

\begin{align*}
2.2.21 \quad r(k-1) &= (n-1)\lambda_1 + (m-1)n\lambda_2, \text{ or} \\
&= r^k - v\lambda_2 \\
&= r^2 - v\lambda_2,
\end{align*}

since the design is symmetrical. Assume that the design is regular. Hence

\begin{align*}
2.2.22 \quad r^2 &> v\lambda_2, \quad r > \lambda_1.
\end{align*}

Let us compute \(|N_{20}N_{20}'|\), which is defined by 2.3.24.
Now since \( r = k \), 2.1.635 becomes

2.2.23

\[
c_{jj} = r^2 \prod (r^2 - v \lambda_2)(\lambda_2 - \lambda_1) \left( \sum_{\ell=1}^{m} (\alpha_{jj}^{\ell})^2 - n \lambda_2 \right)
\]

\[
= r^2 (\lambda_2 - \lambda_1) \prod (r^2 - v \lambda_2) \left( \sum_{\ell=1}^{m} (\alpha_{jj}^{\ell})^2 - n \lambda_2 \right)
\]

and

\[
c_{ju} = r^2 \prod (r^2 - v \lambda_2)(\lambda_2 - \alpha_{ju}) - (\lambda_2 - \lambda_1) \left( \sum_{\ell=1}^{m} (\alpha_{ju}^{\ell})^2 - n \lambda_2 \right)
\]

\[(j \neq u; j, u = 1, \ldots, t). \text{ Hence}
\]

2.2.235

\[
|N_{20} N_{20}| = (rk)^{-t+1} (rk - v \lambda_2)^{m-t-1} (r - \lambda_1)^{m(n-1)-t} |C_t|,
\]

where the elements of \( C_t \) are given by 2.2.23.

Now let \( t = 1 \). Then since the factor outside \(|C_1|\) in 2.2.235 is positive, it follows from

Theorem 2.1.1 that \(|C_1| = 0\). Hence, the only element of \( C_1 \), i.e.,

2.2.24

\[
c_{11} = r^2 (\lambda_2 - \lambda_1) \prod (r^2 - v \lambda_2) \left( \sum_{\ell=1}^{m} (\alpha_{11}^{\ell})^2 - n \lambda_2 \right)
\]
is zero. Since $r^2(\lambda_2 - \lambda_1) \neq 0$, it is necessary that

$$2.2.245 \quad (r^2 - v\lambda_2) - \left( \sum_{k=1}^{m} \left( \alpha_{11}^k \right)^2 - n\lambda_2 \right) = 0,$$

whence

$$\sum_{k=1}^{m} \left( \alpha_{11}^k \right)^2 = r^2 - v\lambda_2 + n\lambda_2.$$  

Now let $t=2$. Since $c_{11}=c_{22}=0$, it is necessary that $c_{12}=c_{21}=0$. Hence from 2.2.23,

$$2.2.25 \quad \alpha_{12} = \lambda_2 + \frac{e}{(r^2 - v\lambda_2)} (\lambda_1 - \lambda_2),$$

where

$$e = \sum_{k=1}^{m} \alpha_{11}^k \alpha_{22}^k - n\lambda_2.$$  

From 2.2.245 and the observation that

$$\alpha_{jj}^k = 0, \quad (j=1,2; \quad k=1, \ldots, m),$$

it follows that

$$2.2.26 \quad -n\lambda_2 \leq e \leq r^2 - v\lambda_2.$$  

Hence, if $e = r^2 - v\lambda_2$, $\alpha_{12} = \lambda_1$. This is one bound on $\alpha_{12}$, being the upper bound when $\lambda_1 - \lambda_2$ is positive and the lower bound when $\lambda_1 - \lambda_2$ is negative. Likewise when $e = -n\lambda_2$. 
2.2.265 \quad \alpha_{12} = \lambda_2 - \frac{n \lambda_2}{r^2 - v \lambda_2} (\lambda_1 - \lambda_2),

which gives the other bound on \( \alpha_{12} \), viz., the lower bound when \( \lambda_1 - \lambda_2 \) is positive, and the upper bound when \( \lambda_1 - \lambda_2 \) is negative. Hence we get the following

**Theorem 2.2.1**: For a regular symmetrical group divisible design (i.e., with \( r^2 > v \lambda_2 \), \( r > \lambda_1 \)), the number of treatments common to two blocks satisfies the inequalities

2.2.27 \quad \lambda_2 - \frac{n \lambda_2}{r^2 - v \lambda_2} (\lambda_1 - \lambda_2) \leq \alpha_{12} \leq \lambda_1,

when \( \lambda_1 > \lambda_2 \).

The inequalities are reversed when \( \lambda_1 < \lambda_2 \).

Also the following lemma is obvious from 2.2.25 and 2.2.26.

**Lemma 2.2.1**: If \( h \) is the H.C.F. of \( r^2 - v \lambda_2 \) and \( \lambda_1 - \lambda_2 \), then
\[ \alpha_{12} = \lambda_2 + f \left( \frac{\lambda_1 - \lambda_2}{n} \right), \]

where \( f \) is an integer obeying the inequalities

\[ -\frac{nh \lambda_2}{r^2 - v \lambda_2} \leq f \leq h. \]

2.3 Consider an initial block \( B_1 \). Let \( n_j \) be the number of blocks among the remaining \((b-1)\) blocks which has \( j \) treatments in common with \( B_1 \). Then from the definition of the design we obtain

\[
\sum_{j=0}^{k} n_j = b-1 = v-1, \quad \text{and} \\
\sum_{j=0}^{k} jn_j = r(k-1) = r(r-1).
\]

Also consider \( \sum_{j=0}^{k} j(j-1)n_j \), which is twice the number of pairs of treatments of \( B_1 \) which lie among the other \((v-1)\) blocks. This number is given by

\[
N = \sum_{\ell=1}^{m} \alpha_{11}^\ell (\alpha_{11}^\ell - 1)(\lambda_{1-1}) + \sum_{s = 1}^{m} \alpha_{11}^s \alpha_{11}^s (\lambda_{2-1}).
\]
From 2.2.245

\[ 2.2.321 \quad \sum_{k=1}^{m} (\alpha_{ll}^k)^2 = r + (n-1) \lambda_1. \]

Also it is necessary that

\[ 2.2.322 \quad \sum_{k=1}^{m} \alpha_{ll}^k = r = k. \]

Hence,

\[ 2.2.323 \quad \sum_{k=1}^{m} (\alpha_{ll}^k - 1) = (n-1) \lambda_1. \]

Further, since

\[ 2.2.324 \quad r^2 = (\sum_{k=1}^{m} \alpha_{ll}^k)^2 = \sum_{k=1}^{m} (\alpha_{ll}^k)^2 + \sum_{s,k=1}^{m} \alpha_{ll}^s \alpha_{ll}^k, \]

it follows from 2.2.321 that

\[ 2.2.325 \quad \sum_{s,k=1}^{m} \alpha_{ll}^s \alpha_{ll}^k = r^2 - r - (n-1) \lambda_1 \]

\[ = (m-1)n \lambda_2. \]

Hence from 2.2.32, 2.2.323, and 2.2.325 we obtain
2.2.326 \[ N = (n-1)(\lambda_1)(\lambda_1-1) + (m-1)(n)(\lambda_2)(\lambda_2-1) \]
\[ = \Sigma_{j=0}^{k} j(j-1)n_j. \]

Now consider

2.2.33 \[ B = \Sigma_{j=0}^{k} (j-\lambda_1)(j-\lambda_2)n_j \]
\[ = \Sigma_{j=0}^{k} j^2n_j - (\lambda_1 + \lambda_2) \Sigma_{j=0}^{k} jn_j + \lambda_1 \lambda_2 \Sigma_{j=0}^{k} n_j. \]

From 2.2.33, 2.2.31, and 2.2.326,

2.2.331 \[ B = \left( (n-1)(\lambda_1)(\lambda_1-1) + (m-1)(n)(\lambda_2)(\lambda_2-1) \right) \]
\[ + r(r-1) \] - \left( (\lambda_1 + \lambda_2)(r)(r-1) \right) \]
\[ + \lambda_1 \lambda_2 (v-1). \]

We can show that \( B = 0 \) by expressing it entirely in terms of \( m, n, \lambda_1, \) and \( \lambda_2, \) as follows:

2.2.332 \[ B = (n-1)(\lambda_1)(\lambda_1-1) + (m-1)(n)(\lambda_2)(\lambda_2-1) \]
\[ + (n-1)\lambda_1 + (m-1)n \lambda_2 - (\lambda_1 + \lambda_2)(n-1)\lambda_1 \]
\[ - (\lambda_1 + \lambda_2)(m-1)n \lambda_2 + \lambda_1 \lambda_2 (mn-1) \]
\[ = 0. \]
Hence the following

**Theorem 2.2.2:** For a regular/group divisible design if \( n_j \) denotes the number of blocks having \( j \) treatments in common with a given initial block then

\[
2.2.34 \quad B = \sum_j n_j(j-\lambda_1)(j-\lambda_2) = 0.
\]

2.4 Let \( \lambda_1 - \lambda_2 \) and \( r^2 - v \lambda_2 \) be relatively prime. Then \( h=1 \). Hence from Lemma 2.2.1, \( \alpha_{12} \) cannot lie in the open interval \((\lambda_1, \lambda_2)\). Then every term of \( B \) is positive or zero. But since \( B=0 \), every term must be zero. We thus get the

**Lemma 2.2.2:** If \( \lambda_1 - \lambda_2 \) and \( r^2 - v \lambda_2 \) are relatively prime, then any two blocks of the regular symmetrical group divisible design intersect in either \( \lambda_1 \) or \( \lambda_2 \) treatments.

Consider the matrix \( A' \) which is the transpose of the matrix \( A \) (2.1.52):
2.2.41 \[ A' = \begin{pmatrix} a_{11}^1 & a_{11}^2 & \cdots & a_{11}^m \\ a_{22}^1 & a_{22}^2 & \cdots & a_{22}^m \\ \vdots & \vdots & \ddots & \vdots \\ a_{bb}^1 & a_{bb}^2 & \cdots & a_{bb}^m \end{pmatrix} \]

The elements in the rows of $A'$ add up to $k$ so that

2.2.42 \[ \sum_{j=1}^{m} a_{jj}^k = k, \]

and also satisfy

2.2.43 \[ \sum_{j=1}^{m} (a_{jj}^k)^2 = r(n-1) \lambda_1. \]

The number of intersections of any two blocks, say $B_1$ and $B_2$, is determined by the scalar product of the corresponding rows of $A'$, through the formula 2.2.25. Even if $h \neq 1$, it may not be possible to choose the elements of $A'$, which must be positive integers satisfying 2.2.42 and 2.2.43, so that $12$ lies in the open interval $(\lambda_1, \lambda_2)$. In fact, in all particular cases which have been examined it has been found impossible to do so. Hence we can state...
Lemma 2.2.3: If it is impossible to determine elements of \( A' \) of 2.2.41, so as to satisfy 2.2.42 and 2.2.43 and giving \( \alpha_{1j} \) lying in the open interval \((\lambda_1, \lambda_2)\), then any two blocks of the regular symmetric group divisible design intersect in \( \lambda_1 \) or \( \lambda_2 \) treatments.

Consider the following example: \( r = k = 9 \), \( v=b=45 \), \( m=3 \), \( n=15 \), \( \lambda_1=3 \), and \( \lambda_2=1 \). Here \( h=2 \), so that Lemma 2.2.2 does not apply. It is easy to verify that the only non-negative integers which will satisfy 2.2.42 and 2.2.43 are 1,1,7. However, it is impossible to form \( \sum_{\ell=1}^{m} \alpha_{jj} \alpha_{uu}^\ell \), using only 1,1,7 as the elements in the \( j \)-th and \( u \)-th rows of \( A' \), so that \( \alpha_{ju} \) lies in the open interval \((\lambda_1, \lambda_2)\). Hence Lemma 2.2.3 applies.

Later on we shall make use of the following lemma which follows directly from 2.2.245 and 2.2.25.

Lemma 2.2.4: If \( \alpha_{12} = \lambda_1 \), then
\[ \sum_{k=1}^{\lambda_1} \alpha_{11}^k \alpha_{22}^k = \sum_{k=1}^{\lambda_1} (\alpha_{11}^k)^2 \]

and conversely.

2.5 Suppose that the conditions of Lemma 2.2.2 or 2.2.3 are satisfied, so that any two blocks intersect in either \(\lambda_1\) or \(\lambda_2\) treatments. Then denoting as before by \(n_{\lambda_1}\) and \(n_{\lambda_2}\) the numbers of blocks with which a given initial block has \(\lambda_1\) or \(\lambda_2\) treatments in common, we have from 2.2.31

2.2.31 \[ n_{\lambda_1} + n_{\lambda_2} = v-1, \quad \text{and} \]

\[ \lambda_1 n_{\lambda_1} + \lambda_2 n_{\lambda_2} = r(r-1), \]

whence

2.2.52 \[ n_{\lambda_1} = n-1, \quad \text{and} \]

\[ n_{\lambda_2} = (m-1)n. \]

Thus with respect to any initial block \(B_1\)
there are \((n-1)\) other blocks which have \(\lambda_1\) treatments in common with it. But from Lemma
2.2.4 any one of the blocks, say \(B_j\), is such that

\[
2.2.53\quad \sum_{k=1}^{m} \alpha_{jj}^{k} = \frac{m}{\sum_{k=1}^{m} (\alpha_{11}^{k})^2}.
\]

Clearly \(2.2.53\) implies that \(\alpha_{jj}^{k} = \alpha_{11}^{k}\) for every \(j\). Then by the converse part of Lemma 2.2.4, every two of these \(n\) blocks have \(\lambda_1\) treatments in common. Thus if the blocks \(B_1\) and \(B_2\) have

\(\lambda_1\) treatments in common, and the blocks \(B_1\) and \(B_3\) have \(\lambda_1\) treatments in common, then \(B_2\) and \(B_3\) also have \(\lambda_1\) treatments in common. Thus the blocks are divided into groups with the following characteristics:

1. Two blocks from the same group contain \(\lambda_1\) treatments in common.

2. Two blocks from different groups contain \(\lambda_2\) treatments in common.

Hence we have the following theorem,
Theorem 2.2.3: For the regular symmetrical group divisible designs, if the conditions of Lemma 2.2.2 or Lemma 2.2.3 are met, then the blocks fall into m groups of n treatments each, which are such that any two blocks from the same group contain $\lambda_1$ treatments in common and any two blocks from different groups contain $\lambda_2$ treatments in common.

Corollary 2.2.1: Under the same conditions as Theorem 2.2.1 if $N_0$ is the incidence matrix of a symmetrical group divisible design with $r^2 \neq v\lambda_2$, then

$$NN' = N'N.$$

3. On the Impossibility of Some Group Divisible Designs

3.1 We shall consider the regular symmetrical group divisible designs, and shall develop three theorems for them which are useful in demonstrating that certain designs are impossible.

3.2 Consider the incidence matrix $N_0$ of the
design. From 2.2.235 it follows for \( t=0 \) that

\[
2.3.21 \quad |NN'_0| = |N_{20}N'_0| = r^2 (r^2 - v \lambda_2)^{m-1} (r - \lambda_1)^m (n-1).
\]

Since \( N_0 \) is of order \( v \times v \), it is necessary that \( |NN'_0| \) be a perfect square. Hence if \( m \) is even, \( r^2 - v \lambda_2 \) must be a perfect square. Also if \( m \) is odd and \( n \) is even, \( r - \lambda_1 \) must be a perfect square. We may state this result in

**Theorem 2.3.1:** For the regular symmetrical group divisible designs, (i) if \( m \) is even, then \( r^2 - v \lambda_2 \) must be a perfect square; (ii) if \( m \) is odd and \( n \) is even, then \( r - \lambda_1 \) must be a perfect square.

It is clear that this theorem is a corollary to Theorem 2.1.1, but is of sufficient importance to be stated explicitly here.

We give below a table of regular symmetrical group divisible designs which can be proved to be impossible in this manner. The table is illustrative and not exhaustive.
\[
\begin{array}{cccccccc}
v & b & r & k & m & n & \lambda_1 & \lambda_2 \\
22 & 22 & 5 & 5 & 11 & 2 & 0 & 1 \\
44 & 44 & 7 & 7 & 22 & 2 & 0 & 1 \\
92 & 92 & 10 & 10 & 46 & 2 & 0 & 1 \\
42 & 42 & 7 & 7 & 21 & 2 & 2 & 1 \\
56 & 56 & 8 & 8 & 28 & 2 & 2 & 1 \\
90 & 90 & 10 & 10 & 45 & 2 & 2 & 1 \\
88 & 88 & 10 & 10 & 22 & 4 & 2 & 1 \\
30 & 30 & 8 & 8 & 15 & 2 & 0 & 2 \\
\end{array}
\]

3.3 We shall now compute the Hasse invariant \( c_p(\text{NN'}) \) for the regular symmetric group divisible designs, using the properties of the Hilbert symbol discussed in section 4 of Chapter I.

Subtract the in-th column and row of \( \text{NN'} \circ \circ \circ \) from the preceding \((n-1)\) columns and rows, \((i=1,2,...,m)\) to form the matrix \( R \). Clearly \( \text{NN'} \) and \( R \) are rationally congruent, so that

\[
2.3.31 \quad c_p(\text{NN'}) = c_p(R).
\]

Let \( D_h \) be the leading principal minor determinant of order \( h \) in \( R \). Then
2.3.32 \[ D_j = (r - \lambda_1)^j (j + 1), \quad (j=1, 2, \ldots, n-1), \]

and \[ D_{in+j} = (r - \lambda_1)^i (n-1) + j (rk-v \lambda_2)^i \]

\[ \cdot \left\{ \frac{r+(n-1) \lambda_1 + (i-1)n \lambda_2}{n} \right\} (j+1), \]

\((i=1, \ldots, m-1; j=0, \ldots, n-1).\)

Hence

2.3.321 \[ D_{in+j} = D_{in} D_j, \quad (i=1, \ldots, m-1; j=0, \ldots, n-1), \]

where \[ D_{in} = (r - \lambda_1)^i (n-1) (rk-v \lambda_2)^i \]

\[ \cdot \left\{ \frac{r+(n-1) \lambda_1 + (i-1)n \lambda_2}{n} \right\}. \]

Let us set

2.3.322 \[ Q = r - \lambda_1 \quad \text{and} \]

\[ P_i = r+(n-1) \lambda_1 + (i-1)n \lambda_2, \]

so that

2.3.323 \[ P_0 = r+(n-1) \lambda_1 - n \lambda_2 \]

\[ = rk-v \lambda_2, \quad \text{and} \]

\[ P_m = r+(n-1) \lambda_1 + (m-1)n \lambda_2 \]

\[ = rk. \]
Honcc we have

$$D_j = (j+1)\varphi^j, \ (j=1,2,\ldots,n-1), \text{ and}$$

$$D_{in} = Q^{i(n-1)}p^{i-1}p_i, \ (i=1,\ldots,m).$$

Recall from 1.4.13 that

$$c_p(R) = (-1,-D_{mn}) \prod_{i=0}^{m-1} \prod_{j=0}^{n-1} (D_{in+j}, -D_{in+j+1}),$$

where $p$ is omitted for convenience.

Let us display the product

$$\prod_{i=0}^{m-1} \prod_{j=1}^{n-1} \left(D_{ln+j}, -D_{ln+j+1}\right) = X, \text{ say},$$

in the following way:

$$X =
\begin{pmatrix}
(D_1,-D_2) & \cdots & (D_{n-1},-D_n) \\
(D_{n},-D_{n+1}) & \cdots & (D_{n+n-1},-D_{2n}) \\
\vdots & \ddots & \vdots \\
(D_{in},-D_{in+1}) & \cdots & (D_{in+n-1},-D_{i(n+1)n}) \\
\vdots & \cdots & \vdots \\
(D_{m-ln},-D_{m-ln+1}) & (D_{m-ln+2},-D_{m-ln+3}) & \cdots & (D_{m-ln+n-1},-D_{mn})
\end{pmatrix}.$$
so that

2.3.332 \[ c_p(R) = X(-1, -D_{mn}). \]

We shall separate out the last column in \( X \) and calculate out the remaining product by rows. Let us first calculate the \( i \)-th row, i.e.,

2.3.333

\[ (D_{1n}, -D_{in+1}) (D_{in+1}, -D_{in+2}) \ldots (D_{in+n-2}, -D_{in+n-1}) \]

\[ = (D_{in}, -D_{1n}) (D_{1n}, -D_{2n}) \ldots (D_{n-2n}, -D_{n-1n}) \]

\[ = (D_{in}, D_{1n}) \cdot (D_{1n}, D_{2n}) \ldots (D_{n-2n}, D_{n-1n}) \]

\[ = (D_{in}, D_{n-1n}) \prod_{j=1}^{n-2} (D_{jn} - D_{j+n}) \]

since by 3. of 1.4.15, \((\alpha, -\alpha)_p = 1\). Hence the value of \( X \), except for the last column, is

2.3.334 \[ \prod_{i=1}^{m-1} (D_{in}, D_{n-1}) \left\{ \prod_{j=1}^{n-2} (D_{jn}, -D_{j+n}) \right\}^m. \]

Let us now calculate the last column of \( X \):
2.3.335

\[(D_{n+1}, -D_n) (D_{n+2}, -D_{2n}) \ldots (D_{m-1}, n+1, -D_{mn}) \]

\[= (D_{n+1}, -D_n) (D_{n+2}, -D_{2n}) \ldots (D_{m-1}, n+1, -D_{mn}) \]

\[= (D_{n+1}, -D_n) \cdot (D_{n+2}, -D_{2n}) (D_{n+1}, -D_{2n}) \ldots \]

\[= (D_{m-1}, n+1, -D_{mn}) (D_{n+1}, -D_{mn}) \]

\[= (D_{n+1}, -1)^m (D_{n+1}, -D_{mn}) \prod_{i=1}^{m-1} (D_{in}, D_{n+1}) \prod_{i=1}^{m-1} (D_{in}, -D_{(i+1)n}) \cdot \]

From 2.3.33, 2.3.334, and 2.3.335, it follows that

2.3.336

\[c_p(R) = (-1, -D_{mn}) (D_{n+1}, -1)^m (D_{n+1}, -D_{mn}) \]

\[\left\{ \prod_{j=1}^{n-2} (D_{j}, -D_{j+1}) \right\} \prod_{i=1}^{m-1} (D_{in}, -D_{(i+1)n}) \cdot \]

Now from 2.3.324,

2.3.337 \((-1, -D_{mn}) = (-1, -Q^{n-1}) + P^{m-1}p_m \)

\[= (Q, -1)^m (P, -1)^{m-1} (P_m, -1), \]

and \((D_{n+1}, -1)^m = (nQ^{n-1}, -1)^m \)

\[= (n, -1)^m (Q, -1)^{m-1}, \]
and \( (D_{n-1}, D_{mn}) = (n, Q^{m(n-1)} p_{o}^{m-1} p_{m}) \)
\[ = (n, Q)^{m(n-1)} (n, P_{o})^{m-1} (n, P_{m}) \]
\[ (Q, -1)^{m(n-1)} (Q, P_{o})^{m-1} (n, P_{m})^{n-1}. \]

Also from 2.3.324,

2.3.338
\[
\left\{ \frac{n-2}{\prod_{j=1}^{2} (D_{j}, -D_{j+1})} \right\}^{m} = \left\{ \prod_{j=1}^{n-2} ((j+1)Q_{j}^{j} - (j+2)Q_{j+1}^{j+1}) \right\}^{m}
\]
\[ = (Q, -1)^{m(n-1)} (n-2)^{2} (Q, n)^{m-2} (n, -1)^{m}. \]

We have yet to consider

2.3.34
\[ Y = \prod_{i=1}^{m-1} (D_{in}, -D_{i+1}n). \]

Now from 2.3.324, if we set \( S = Q^{n-1} \),

2.3.341
\[ (D_{in}, -D_{(i+1)n}) = (S_{i}^{i-1} p_{o}^{i-1} p_{i}, -S_{i+1}^{i+1} p_{o}^{i} p_{i+1}) \]
\[ = (S_{i}^{i}, -S_{i+1}^{i+1}) (S_{i}^{i}, p_{o}^{i}) (S_{i}^{i}, p_{i+1}) \]
\[ (p_{i-1}^{i-1}, S_{i+1}^{i+1}) (p_{i-1}^{i-1}, -p_{o}^{i}) (p_{i-1}^{i-1}, p_{i+1}) \]
\[ (p_{i}, S_{i+1}^{i+1}) (p_{i}, p_{o}^{i}) (p_{i}, -p_{i+1}) \]
\[= (S,-1)^i (S,P_0)^i (S,P_{i+1})^i \]
\[(S,P_0)^{i-1} (P_0,-1)^{i-1} (P_0,P_{i+1})^{i-1} \]
\[(P_i,S)^{i+1} (P_i,P_0)^i (P_i,-P_{i+1}).\]

Now let

2.3.342 \[A = \prod_{i=1}^{m-1} (P_0,P_{i+1})^{i-1} (P_0,P_1)^i.\]

Multiply \(A\) by \(\prod_{i=1}^{m-1} (P_0,P_{i+1})^2 = 1\), so that

2.3.343 \[A = (P_0,P_1)(P_0,P_m)^m.\]

Let

2.3.344 \[B = \prod_{i=1}^{m-1} (S,P_{i+1})^i (S,P_1)^{i+1}.\]

Multiply \(B\) by \(\prod_{i=1}^{m-1} (S,P_{i+1})^2\), so that

2.3.345 \[B = (S,P_1)^2 (S,P_m)^{m+1}\]
\[= (S,P_m)^{m+1}.\]

Hence, from 2.3.34, 2.3.341, ..., 2.3.345 obtain
2.3.346 \[ Y = (S,-1)^{(m-1)}(m)/2(S,P_o)^{m-1}(P_o,-1)^{(m-1)(m-2)/2} \]

\[(P_o,P_1)(P_o,P_m)^{m}(S,P_m)^{m+1}m-1 \prod_{i=1}^{m-1} (P_i,-P_{i+1}),\]

which since \( S = q^{n-1} \) becomes

2.3.347

\[ Y = (Q,-1)^{(n-1)(m-1)(m)/2} (Q,P_o)^{(n-1)(m-1)} \]

\[(P_o,-1)^{(m-1)(m-2)/2}(P_o,P_1)(P_o,P_m)^{m} \]

\[(Q,P_m)^{(n-1)(m+1)}m-1 \prod_{i=1}^{m-1} (P_i,-P_{i+1}).\]

From 2.3.31, 2.3.336, 2.3.337, 2.3.338, and 2.3.347,

2.3.35

\[ c_p(N_0,N_1) = (P_m,-1)(Q,n)^{m}(P_o,P_m)^{m}(P_1,P_o) \]

\[(Q,-1)^{(m(n-1)(m+n-1))/2}(Q,P_m)^{m(n-1)} \]

\[(P_m,n)(P_o,n)^{m-1}(P_o,-1)^{m(m-1)}/2 \]

\[ m-1 \prod_{i=1}^{m-1} (P_i,-P_{i+1}),\]

where \( \omega = r - \lambda_1, P_o = r k - v \lambda_2, P_1 = r + (n-1) \lambda_1, \) and

\( P_m = r k. \) This form of \( c_p(N_0,N_1) \) is useful in computing
\[ c_p(N_{20}, N'_{20}) \], for clearly there is a theorem analogous to Theorem 1.4.2.

However, for the purpose at hand, \( r = k \), so that \( c_p(N_{0}, N'_{0}) \) reduces to

\[
2.3.36 \quad c_p(N_{0}, N'_{0}) = (Q, n)^m p(q, -1)^m(m-1)(m+n-1)/2
\]

\[
(P_0, n)^m p^{-1}(P_0, -1)^m p_0^m - 1/m p_0^{m-1} \prod_{i=0}^{m-1} (P_i, -P_{i+1}).
\]

The product

\[
2.3.37 \quad U = \prod_{i=0}^{m-1} (P_i, -P_{i+1}) p
\]

may be reduced in the following way. Let us first assume that \( P_m = r^2 \) and \( n \lambda_2 \) are prime to each other. Then no two consecutive \( P_i \) contain a common factor. Consider some particular \( P_j \), \((j=1, \ldots, m-1)\). Now \( P_j \) occurs in two consecutive symbols, and nowhere else. Thus, we have

\[
2.3.371 \quad T = (P_{j-1}, -P_j) p(P_j, -P_{j+1}).
\]
Let $p$ be a factor of $P_j$. Then it does not occur as a factor of $P_{j-1}$, or $P_{j+1}$. Hence

$$2.3.372 \quad T = (P_{j-1} \mid p)^{\delta_j} (P_{j+1} \mid p)^{\delta_j},$$

where $\delta_j$ is the power to which $p$ occurs in $P_j$.

Now

$$2.3.373 \quad P_{j-1} = P_j - n\lambda_2, \quad \text{and} \quad P_{j+1} = P_j + n\lambda_2.$$

Hence,

$$2.3.374 \quad T = (P_j - n\lambda_2 \mid p)^{\delta_j} (P_j - n\lambda_2 \mid p)^{\delta_j}.$$

If $\delta_j$ is even, then $T = +1$. If $\delta_j$ is odd, then

$$2.3.375 \quad T = (-n\lambda_2 \mid p)(n\lambda_2 \mid p)$$

$$= (n\lambda_2 \mid p)^2 = +1.$$

We are left to consider the first and last symbols. They give

$$2.3.376 \quad W = (P_0, -P_1 \mid p)^{\delta_0} (P_{m-1}, -P_m \mid p)^{\delta_m},$$
where \( \delta_1 \) is the power to which the prime \( p \) occurs in \( P_i \), \( i=0,m \). Since \( P_m = r^2 \),

\[
2.3.377 \quad W = (P_o, -P_1)^{\delta_0}.
\]

Hence the

**Lemma 2.3.1:** If \( n \lambda_2 \) and \( r^2 \) are relatively prime, then

\[
\prod_{i=0}^{m-1} (P_i, -P_{i+1})^p = (P_o, -P_1)^{\delta_0},
\]

where \( \delta_0 \) is the power to which \( p \) occurs as a factor of \( P_o \).

Now assume that \( h \) is the H.C.F. of \( P_m = r^2 \) and \( n \lambda_2 \). Then \( h \) is the H.C.F. of any two consecutive \( P_i \)’s.

Again consider the product

\[
2.3.38 \quad U = \prod_{i=0}^{m-1} (P_i, -P_{i+1})^p = \prod_{i=0}^{m-1} (hP^*_i, -hP^*_i)^p = \prod_{i=0}^{m-1} (h, P^*_i)(h, P^*_{i+1})(P^*_i, -P^*_{i+1})
\]
\[ = (h, P^*_0)(h, P^*_m) \prod_{i=0}^{m-1} (P^*_i, -P^*_i+1) \].

Let \[ V = \prod_{i=0}^{m-1} (P^*_i, -P^*_i+1) \]. Then since no two consecutive \( P^*_i \)'s contain a common factor, \( V \) reduces to \( W \) of 2.3.377, if we replace the \( P^*_i \)'s in \( W \) by \( P^*_i \)'s and the \( \delta^*_i \)'s by \( \delta^*_i \)'s. I.e.,

\[
2.3.38 \quad V = (P^*_0, -P^*_1)^{\delta^*_0} (P^*_1, -P^*_m)^{\delta^*_m},
\]

where \( \delta^*_i \) is the power in which \( p \) occurs in \( P^*_i \).

Consider \( (P^*_0, -P^*_1)^{\delta^*_0} \), where \( p \) occurs to the odd power \( \alpha \) in \( P^*_0 \). (A prime to an even power need not be considered since then \( \delta^*_0 \) is even.) Then

\[
2.3.382 \quad (P^*_0, -P^*_1)^{\delta^*_0} = (c^\alpha, -c^\alpha - L^*)^{\delta^*_0},
\]

where \( c \) is an integer, and

\[
2.3.383 \quad L^* = n \lambda_2 / h.
\]
Further

\[ 2.3.384 \quad (P^*_0,-P^*_1)_p^{\delta^*_p} = (c\alpha\gamma,-L\gamma)^{\delta^*_p}_p \]

\[ = (c,-L\gamma)^{\delta^*_p}_p (\alpha\gamma,-L\gamma)^{\delta^*_p}_p \]

\[ = (-L\gamma^*\gamma \mid p)^{\delta^*_p}_p. \]

Consider \((P^*_m,-P^*_m)_{m-1}^{\delta^*_p}_m\), where \(p\) occurs to the odd power \(\beta\) in \(P^*_m\). Observe that the square free part of \(P^*_m\) and \(h\) must be the same.

Hence

\[ 2.3.385 \quad (P^*_m,-P^*_m)_{m-1}^{\delta^*_p}_m = (P^*_m,-h)_{m-1}^{\delta^*_h}_p, \]

where \(\delta^*_h\) is the odd power \(\beta\) to which \(p\) occurs in \(h\). Hence, by an argument similar to that at 2.3.382 and following,

\[ 2.3.386 \quad (P^*_m,-h)_{m-1}^{\delta^*_h}_p = (-L\gamma^*\gamma \mid p)^{\delta^*_h}_p, \]

and from 2.3.381, 2.3.384, and 2.3.386,

\[ V = (-L\gamma^*\gamma \mid p)^{\delta^*_p+\delta^*_h}_p. \]
Since \( P_o = P_o^\text{wh} \), \( \delta_o = \delta_o^\text{w} + \delta_h^\text{h} \), and

\[ V = (-L^\text{w} \mid p)^\delta_o, \]

which by 2.3.383

\[ = (- \frac{n \lambda_2}{h} \mid p)^\delta_o. \]

Clearly \( (h, P_m^\text{w}) = (h, h) \), so that from 2.3.38 and 2.3.387,

\[ 2.3.39 \quad U = (h, P_o) (\frac{n \lambda_2}{h} \mid p)^\delta_o, \]

and by 2.3.36 and 2.3.39,

\[ 2.3.391 \quad c_p(N_o N_o^\text{t}) = (Q, n)^m_{p} (Q, -1)^m_{p} (m-1) (m+n-1)/2 \]

\[ \quad \times (P_o, n)^{m-1}_{p} (P_o, -1)^{m-1}_{p} \]

\[ \quad \times (h, P_o) (\frac{n \lambda_2}{h} \mid p)^\delta_o, \]

where \( h \) is the highest common factor of \( r^2 \) and \( n \lambda_2 \), \( \delta_o \) is the power of \( p \) in \( P_o \), \( \omega = r - \lambda_1 \) and

\( P_1 = r + (n-1) \lambda_1 + (1-1) n \lambda_2 \). We therefore have the

**Lemma 2.3.2:** For the regular, symmetric group
divisible designs, the Hasse invariant $c_p(N_oN_o')$ for $p$ an odd prime is given by 2.3.391.

By an argument analogous to that following 1.5.59 we obtain the

**Theorem 2.3.2:** If a regular, symmetric group divisible design exists, then

$$c_p(N_oN_o') = +1$$

for all odd primes $p$.

We give below a table of regular symmetrical group divisible designs which can be shown to be impossible by using the above theorem. The table is only illustrative and not exhaustive.

<table>
<thead>
<tr>
<th>$v$</th>
<th>$b$</th>
<th>$r$</th>
<th>$k$</th>
<th>$m$</th>
<th>$n$</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
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<td>6</td>
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<td>3</td>
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<td>1</td>
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<td>10</td>
<td>31</td>
<td>3</td>
<td>0</td>
<td>1</td>
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<td>95</td>
<td>10</td>
<td>10</td>
<td>19</td>
<td>5</td>
<td>0</td>
<td>1</td>
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<td>13</td>
<td>3</td>
<td>3</td>
<td>1</td>
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<td>7</td>
<td>3</td>
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<td>23</td>
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<td>11</td>
<td>3</td>
<td>0</td>
<td>3</td>
</tr>
</tbody>
</table>
The first of these is worked out in full as an illustration. We find

\[ Q = r - \lambda_1 = 6, \quad P_0 = r^2 - v \lambda_2 = 3, \quad h = 3, \]

so that

\[ c_p(N_oN'_o) = (6,3)_p^{ll}(6,-1)_p^{(11)}(11)(2)(13)/2(3,3)^{10}_p \]

\[ \frac{(11)(10)+1}{(3,-1)_p^2} \cdot (3,3)_p(-1|p)^5 \]

and for \( p = 3 \),

\[ c_3(N_oN'_o) = (-1|3)^1 = -1. \]

Hence the design is impossible.

3.4 Recall the restrictions on the \( \alpha_j^j \)'s for block \( B_j \):

\[ 2.3.41 \quad \sum_{\ell=1}^{m} \alpha_j^\ell = r, \quad \text{and} \]

\[ \sum_{\ell=1}^{m} (\alpha_j^\ell)^2 = r + (n-1)\lambda_1. \]

Also, under the conditions of Lemma 2.2.2 or of Lemma 2.2.3 it must be possible to choose the elements of \( A' \) of 2.2.41 so that there exist
two rows of $A'$ whose scalar product is $n \lambda_2$. I.e., for some $B_j$ and $B_u$,

$$2.3.42 \quad \sum_{\kappa=1}^{m} \alpha_{jj}^{\kappa} \alpha_{uu}^{\kappa} = n \lambda_2.$$ 

Hence, we have

**Theorem 2.3.3:** For a regular symmetric group divisible design to exist, it is necessary that there exist non-negative integers $\alpha_{jj}^{\kappa}$ ($\kappa=1, \ldots, m$) which satisfy 2.3.41; and moreover, if the conditions of Lemmas 2.2.2 or 2.2.3 are met, then there exist two sets of non-negative integers $\alpha_{tt}^{\kappa}$ ($t=j, u; \kappa=1, \ldots, m$) which satisfy 2.3.41 and 2.3.42.

Consider the design with parameters $r=k=9$, $n=11$, $m=3$, $b=v=33$, $\lambda_1=5$, and $\lambda_2=1$. Since $m$ and $n$ both are odd, this design is not shown to be impossible by Theorem 2.3.1. Also, let us compute the invariant $c_p(N_0 N_0')$. We find that $\omega=r-\lambda_1=4$, $P_0=r^2-v \lambda_2=48$, and $h=1$. Hence

$$c_p(N_0 N_0') = (-11 \mid p)^5_0.$$
The only odd prime which is a factor of $P_0$ is 3 and

$$c_3(N_0N'_0) = (-11 \mid 3)^1$$

$$= (-2 \mid 3)$$

$$= (1 \mid 3)$$

$$= +1 .$$

Hence Theorem 2.3.2 does not rule out the design.

But consider the first part of Theorem 2.3.3. It obviously is impossible to choose $\alpha_j^\ell$'s which satisfy

$$\sum_{\ell=1}^{3} \alpha_j^\ell = 9 \quad \text{and}$$

$$\sum_{\ell=1}^{3} (\alpha_j^\ell)^2 = 59 .$$

Hence, by Theorem 2.3.3, the design is impossible.

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