ON THE MINIMALITY OF A BOUNDEDLY COMPLETE
SUFFICIENT SUB FIELD

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SUMMARY

This note, written for its pedagogical interest, attempts at a simplification of a proof due to R. R. Bahadur (1957) of the minimality of a boundedly complete sufficient sub-field.

NOTATIONS AND DEFINITIONS

Let \((\mathcal{X}, \mathcal{A}, \mathcal{P})\) be our probability structure. That is, \(\mathcal{P}\) is a family \(\{P\}\) of probability measures on a \(\sigma\)-field \(\mathcal{A}\), sub-sets of a sample space \(\mathcal{X}\).

Definition 1: The set \(N \in \mathcal{A}\) is said to be \(\mathcal{P}\)-null if
\[
P(N) = 0 \quad \text{for all } P \in \mathcal{P}.
\]

Definition 2: (a) The two sets \(A\) and \(B\) belonging to \(\mathcal{A}\) are said to be \(\mathcal{P}\)-equivalent if their symmetric difference \(A\Delta B\) is \(\mathcal{P}\)-null.

(b) The two \(\mathcal{A}\)-mble functions \(f\) and \(g\) are said to be \(\mathcal{P}\)-equivalent if the set \(\{x \mid f(x) \neq g(x)\}\) is \(\mathcal{P}\)-null.

Definition 3: An \(\mathcal{A}\)-mble function \(f\) is said to be \(\mathcal{P}\)-integrable if
\[
\int |f| \, dP < \infty \quad \text{for all } P \in \mathcal{P}.
\]

\(^1\)This research was supported by the Mathematics Division of the Air Force Office of Scientific Research.

\(^\dagger\)As usual we use the term sub-field to mean a sub-\(\sigma\)-field.
Definition 4: A sub-field $\mathcal{A}_*$ of $\mathcal{A}$ is said to be sufficient if corresponding to each $\mathcal{P}$-integrable, $\mathcal{A}$-mble $f$ there exists an $\mathcal{A}_*$-mble $f_*$ such that, for all $B \in \mathcal{A}_*$ and $P \in \mathcal{P}$,

$$\int_B f \, dP = \int_B f_* \, dP.$$ 

The function $f_*$ is then called the conditional expectation of $f$ given $\mathcal{A}_*$ and is determined upto a $\mathcal{P}$-equivalence.

Definition 5: The sub-field $\mathcal{A}_0$ is said to be boundedly complete if the only bounded $\mathcal{A}_0$-mble functions satisfying the identity

$$\int f \, dP = 0$$

for all $P \in \mathcal{P}$ are those that are $\mathcal{P}$-equivalent to zero.

Definition 6: $\mathcal{A}_0$ is said to be a minimal sufficient sub-field if each member of $\mathcal{A}_0$ is $\mathcal{P}$-equivalent to some member of every alternative sufficient sub-field $\mathcal{A}_*$.

Now if $\mathcal{A}_*$ be sufficient then for each $\mathcal{A}$-mble and square $\mathcal{P}$-integrable $f$ the conditional expectation $f_*$ is also square $\mathcal{P}$-integrable and we have in addition

$$\int f^2 \, dP = \int f_*^2 \, dP + \int (f-f_*)^2 \, dP$$

for all $P \in \mathcal{P}$.

In other words,

$$\int f^2 \, dP \geq \int f_*^2 \, dP$$

for all $P \in \mathcal{P}$.

the sign of equality holding for all $P \in \mathcal{P}$ if and only if $f$ and $f_*$ are $\mathcal{P}$-equivalent.

**THEOREM**

**Theorem:** If $\mathcal{A}_0$ be a boundedly complete sufficient sub-field then $\mathcal{A}_0$ is a minimal sufficient sub-field.
Proof: Let \( \mathcal{A}_* \) be any alternative sufficient sub-field and let \( A \) be an arbitrary member of \( \mathcal{A}_o \). We have to prove the existence of a set \( B \in \mathcal{A}_* \) such that \( A \) and \( B \) are \( \mathcal{P} \)-equivalent.

Let \( f \) be the indicator of \( A \) and let \( f_* \) be the conditional expectation of \( f \) given \( \mathcal{A}_* \) and \( f_{*o} \) the conditional expectation of \( f_* \) given \( \mathcal{A}_o \).

Since \( f \) is bounded we can, without any loss of generality, assume that both \( f_* \) and \( f_{*o} \) are bounded.

Now, from definition 4 we have, for each \( P \in \mathcal{P} \),

\[
\int f \, dP = \int f_* \, dP = \int f_{*o} \, dP
\]

Thus,

\[
\int (f - f_{*o}) \, dP = 0 \text{ for all } P \in \mathcal{P}
\]

and \( f - f_{*o} \) is a bounded \( \mathcal{A}_o \)-measurable function. From the bounded completeness of \( \mathcal{A}_o \) it then follows that \( f \) and \( f_{*o} \) are \( \mathcal{P} \)-equivalent and hence

\[
\int f^2 \, dP = \int f_{*o}^2 \, dP \text{ for all } P \in \mathcal{P}
\]

But we know that for all \( P \in \mathcal{P} \)

\[
\int f^2 \, dP \geq \int f_*^2 \, dP \geq \int f_{*o}^2 \, dP
\]

Therefore,

\[
\int f^2 \, dP = \int f_{*o}^2 \, dP \text{ for all } P \in \mathcal{P}
\]

and hence \( f \) and \( f_* \) are \( \mathcal{P} \)-equivalent.

Thus, the set

\[
A = \{ x \mid f(x) = 1 \}
\]

is \( \mathcal{P} \)-equivalent to the set

\[
B = \{ x \mid f_*(x) = 1 \}
\]

and so \( B \) is the \( \mathcal{A}_* \)-measurable set we are searching after.
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