ON THE GENERAL RENEWAL PROCESS

by

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INTRODUCTION

This dissertation considers some problems that arise in Smith's expository article "On Renewal theory and its ramifications" \cite{217} and the discussion thereon. In his article Smith deals very comprehensively with the subject, from its historical background right up to the present stage. He discusses also several applied aspects of renewal theory. He does not consider the inferential problems since this aspect was discussed at some length by Cox \cite{27} and Cox and Smith \cite{4, 57}. It was Feller who recognized and championed the methods of renewal theory in his papers \cite{7, 87} where he deals with what we call discrete renewal processes. Later notable contributions have been made, among others, by Blackwell \cite{1, 27}, Takacs \cite{247}, Kendall, D.G. \cite{13, 147}, Karlin \cite{11, 127}, Cox \cite{3, 4, 57}, Hammersley \cite{9, 107} and Smith \cite{17, 18, 19, 20, 227}.

This dissertation is divided into three parts. In the first part, the results of Smith \cite{207} on the cumulants of a renewal process are extended to the case of a general renewal process. We first establish the asymptotic representation theorems for the $\phi$-moments and $\psi$-cumulants and thereby for the cumulants of $N_t$ for a general renewal process. The table of the first eight cumulants of a renewal process is then extended to the case of a General Renewal process. We finally prove in this section a theorem which leads to a check on the calculations. As a particular
application of our general results we derive the cumulants of the "Equilibrium process".

In the second part of this dissertation an estimate is proposed for the estimation of the variance-time curve of a renewal process. Assuming "equilibrium" it is shown that this estimator is asymptotically unbiased for any underlying renewal process. Under null hypothesis that the underlying renewal process is random (Poisson process) the variance and autocovariance function of the estimator are computed, and the consistency and asymptotic normality of the estimator are established. The later result provides a large sample test of significance for randomness of the underlying process, other tests based on the likelihood ratio criterion having been discussed earlier by Sukhatme. Finally, several difficulties which one encounters for obtaining useful non-null distributions of the estimator are discussed.

In the third part of this paper we have shown that a certain "generalization" of renewal theory proposed by Hammersley, is essentially included in Smith's theory of cumulative processes. We have shown that a separate treatment for the multidimensional version of Hammersley's random variable is unnecessary and that the one dimensional case of his random variable differs from the r.v. considered by Smith in his theory of cumulative processes in that Smith's r.v. contains just one more additional term of the renewal sequence. This simple difference takes the degree of complication a step further in Hammersley's case. Besides showing
how it is necessary to make a certain "global" assumption of Smith, we have finally demonstrated that apart from the minor difference in the one dimensional version of the generalizations considered by both, which is a matter of definition, the formulae for the asymptotic mean and variance of Smith, which have been carried to a further degree of accuracy in this part, will meet the requirements of Hammersley's multidimensional version completely.
NOTATION

<table>
<thead>
<tr>
<th>Standard Symbol</th>
<th>Meaning of the Symbol</th>
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<tr>
<td>r.v.</td>
<td>random variable</td>
</tr>
<tr>
<td>R.P.</td>
<td>Renewal Process</td>
</tr>
<tr>
<td>G.R.P.</td>
<td>General Renewal Process</td>
</tr>
<tr>
<td>L.T.</td>
<td>Laplace Transform</td>
</tr>
<tr>
<td>L-S.T.</td>
<td>Laplace-Stieltjes Transform</td>
</tr>
<tr>
<td>o(T)</td>
<td>A quantity which, when divided by T tends to zero as T tends to its limit</td>
</tr>
<tr>
<td>O(T)</td>
<td>A quantity of the order of magnitude of T</td>
</tr>
<tr>
<td>i.e.</td>
<td>that is</td>
</tr>
<tr>
<td>. . .</td>
<td>therefore</td>
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<tr>
<td>∈</td>
<td>belongs to</td>
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<tr>
<td>→</td>
<td>tends to</td>
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<tr>
<td>D →</td>
<td>converges in distribution to</td>
</tr>
<tr>
<td>B.V.</td>
<td>Bounded variation</td>
</tr>
<tr>
<td>w.r.t.</td>
<td>with respect to</td>
</tr>
<tr>
<td>R.H.S.</td>
<td>Right hand side</td>
</tr>
<tr>
<td>L.H.S.</td>
<td>Left hand side</td>
</tr>
<tr>
<td>N(0, σ²)</td>
<td>Normal distribution with mean 0 and variance σ²</td>
</tr>
<tr>
<td>~</td>
<td>is asymptotically equal to</td>
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PART I

ON THE CUMULANTS OF A GENERAL RENEWAL PROCESS

1.0. Summary

In this part the results of Smith[20] on the cumulants of a Renewal Process are extended to the case of a General Renewal Process. After establishing the asymptotic representation theorems for the \( \Phi \)-moments and \( \psi \)-cumulants of a General Renewal Process, the table of the first eight cumulants of a Renewal Process has been extended to the case of a General Renewal Process. A theorem is proved leading to a check on the calculations. As a particular case of the General Renewal Process, the cumulants of the "Equilibrium Process" are obtained.

1.1. Some notation and Preliminary Lemmas:

Let \( \{ x_i \}, i = 1, \ldots, n, \ldots \) be an infinite sequence of independent, non-negative, identically distributed random variables which are not all zero with probability one and let their distribution function be \( F(x) \). Unless otherwise stated it is assumed that \( F(x) \) is such that for some \( K \) its \( K \)th convolution has an absolutely continuous component. Let \( \mu_r = E(x_1^r) \), if this moment exists. Such a sequence of r.v.'s is called a renewal process (R.p.). Further, let \( x_0 \) be a non-negative, r.v independent of \( x_i (i \geq 1) \) with distribution function \( K(x) \), \( K(x) \) not necessarily being identical with \( F(x) \) and let \( \nu_r = E(x_0^r) \), if this moment exists.

Then the augmented sequence \( \{ x_i \}, i = 0, 1, \ldots \) is called a General Renewal process. (G.R.P.)

Let \( S_k = x_1 + \cdots + x_k \). \( N_t \) is defined as the maximum suffix \( k \)
for which \( S_k \leq t \), subject to the condition \( N_t = 0 \), if \( x_1 > t \). In the obvious renewal application the \( x_1 \) represent the lifetimes of articles being renewed and \( N_t \) is the number of renewals made by time \( t \) subject to the original object having been installed at time \( t = 0 \). On the other hand, one considers a G.R.P., when the initial installation was not known to be at \( t = 0 \), but only to be at some point chosen on the positive semiaxis of the time scale in accordance with a certain probability distribution. In particular, \( x_0 \) of the G.R.P., might be the "residual lifetime" of the article in use at the arbitrary chosen origin \( t = 0 \).

For a G.R.P., let \( \tilde{S}_{-1} = 0; \tilde{S}_k = x_0 + x_1 + \ldots + x_k \), \( (k = 0, 1, \ldots) \) and for all \( t \geq 0 \), define the random variable \( \tilde{N}_t \) as the greatest integer \( k \) such that \( \tilde{S}_{k-1} \leq t \).

In this chapter the random variable \( \tilde{N}_t \) associated with a G.R.P. is studied following the lines of study by Smith[20] of the random variable \( N_t \) associated with a R.P. We deal only with a continuous G.R.P., that is, we suppose there is no \( \tilde{\omega} > 0 \) such that with probability one every \( X_n \) is divisible by \( \tilde{\omega} \). It will be convenient to introduce another random variable \( \tilde{N}_t' \) defined as the maximum suffix \( k \), such that \( \tilde{S}_k \leq t \), for \( t \geq 0 \). It will be seen in lemma 2, that the study of \( \tilde{N}_t \) and \( \tilde{N}_t' \) are equivalent and we therefore concentrate on \( \tilde{N}_t' \) rather than \( \tilde{N}_t \) as its study is less complicated. Let us introduce some moment formulae w.r.t. the r.v. \( N_t \); when these moments involve either \( \tilde{N}_t \) or \( N_t' \) instead of \( N_t \), the corresponding symbol \( \tilde{N}_t \) or \( N_t' \) is put on the moments. Let \( m_r(t) \) and \( \kappa_r(t) \) denote respectively
the \( r \)th conventional moment and cumulant of \( N_t \). Let
\[
\phi_k(t) = E(N_t + 1)(N_t + 2) \ldots (N_t + k)
\]
be the \( \phi \)-moment of order \( k \) with generating function
\[
\phi_t(\xi) = E\left(\frac{1}{1 - \xi}\right)N_t + 1
\]
The corresponding \( \psi \)-cumulant of order \( k \) is obtained by a formal power series expansion of the cumulant generating function
\[
\psi_t(\xi) = \log \phi_t(\xi).
\]
Let \( M_t(z) = E e^{zn} \) and let \( K_t(z) = \log M_t(z) \).
Then it can be shown (Smith [20]) that
\[
M_t(z) = E e^{zn} = e^{-z} \phi_t(1 - e^{-z}),
\]
and hence that
\[
K_t(z) = -z + \psi_t(1 - e^{-z}),
\]
and
\[
\psi_t(\xi) = \log(1 - \xi) + K_t(-\log(1 - \xi)).
\]
By these formulae, the \( \psi \)-cumulants of \( N_t \) can be expressed in terms of the conventional cumulants and vice versa.

In what follows we will assume that the functions we are dealing with vanish identically for negative values of their argument. If \( \Lambda(t) \) is any function of bounded variation \( (B.V.) \) in every finite interval, then we write
\[
\Lambda^*(s) = \int \exp(-st) \Lambda(t) \, dt,
\]
for the Laplace-Stieltjes transform \( (L-S.T.) \) of \( \Lambda(t) \). If \( \Lambda(t) \) is any function which is integrable in every finite interval then we write
\[
\Lambda^0(s) = \int \exp(st) \Lambda(t) \, dt,
\]
for the Laplace transform (L.T.) of $\mathcal{L}(t)$. All the transforms that arise in this chapter will exist for $\Re(s) > 0$. Also, when $\mathcal{L}(t)$ has both an L.S.T and an L.T., then

$$\mathcal{L}^*(s) = s \mathcal{L}^0(s)$$

We will make repeated use of the following lemma of Smith[20],

**Lemma A** A necessary and sufficient condition for the distribution function $F(x)$ of a non-negative r.v. to have its first $k$ moments finite is that there exists another distribution function $F_k(x)$ such that

$$F^*(s) = 1 - \mu_1 s + \frac{\mu_2 s^2}{2} - \cdots - \frac{\mu_{k-1} (-s)^{k-1}}{(k-1)!}$$

$$+ \frac{\mu_k (-s)^k}{k!} F_k(s).$$

Let $\gamma_n$ and $\gamma_n'$ stand for finite rational functions of $\mu_1$, $\mu_2$, ..., $\mu_n$ and $\nu_1$, $\nu_2$, ..., $\nu_n$ respectively, and let $\gamma_{1j}$ denote a rational function in $\mu_1$, $\mu_2$, ..., $\mu_i$ and $\nu_1$, $\nu_2$, ..., $\nu_j$. If $\mu_n < \infty$, and $K^*_n(s) = 1 - F^*(n)(s)$, $n \geq 1$, then

$$K_n(x)$$

is a function of B.V. (Smith[20]); Let $M^*_w$ stand for any multinomial expression in $K^*_1, K^*_2, \ldots K^*_w$, with coefficients $\gamma_w$ and with maximum degree of any term not greater than $d$.

We shall now prove

**Lemma 1.**

$$\phi_n^*(s) = K^*(s) \phi_n(s)$$

**Proof** By definition

$$\phi_n^*(t) = E(N_t^i + 1)(N_t^i + 2) \ldots (N_t^i + n),$$

where $N_t^i$ is the maximum $k$ such that $\bar{s}_k \leq t$. 
\[
\Pr \left[ N_t^* = k \right] = \Pr \left[ \xi_k < t < \xi_{k+1} \right],
= K(t) \star \mathcal{F}_k(t) - K(t) \star \mathcal{F}_{k+1}(t),
\]

where \( \mathcal{F}_j(t) \) denotes the \( j \)th convolution of \( \mathcal{F}(x) \), and \( \star \) represents the Stieltjes convolution.

\[
\phi_n^*(t) = \sum_{k=0}^{\infty} (k+1)(k+2) \cdots (k+n) \left[ K(t) \star \mathcal{F}_k(t) - K(t) \star \mathcal{F}_{k+1}(t) \right]
\]

Taking \( L - S. T \) on both sides we get

\[
\phi_n^*(s) = \sum_{k=0}^{\infty} (k+1)(k+2) \cdots (k+n) \left[ F^*(s) \right]^k \left[ 1 - F^*(s) \right]
\]

= \( K^*(s) \sum_{k=0}^{\infty} (k+1) \cdots (k+n) \left[ F^*(s) \right]^k \left[ 1 - F^*(s) \right] \).

But Smith [10] has shown that

\[
\phi_n^*(s) = \sum_{k=0}^{\infty} (k+1) \cdots (k+n) \left[ F^*(s) \right]^k \left[ 1 - F^*(s) \right] = \frac{n!}{\left[ 1 - F^*(s) \right]^n}
\]

Hence the lemma is proved.

Lemma 2.

\[
\tilde{\phi}_n^*(t) = 1 + \frac{n}{n!} \sum_{j=1}^{n} \phi_j^*(t)
\]

Proof:

By definition

\[
\tilde{\phi}_n^*(t) = E \left( \tilde{N}_t + 1 \right) \left( \tilde{N}_t + 2 \right) \cdots \left( \tilde{N}_t + n \right)
= n! \left[ 1 - K(t) \right]
\]

\[
+ \sum_{k=1}^{\infty} (k+1) \cdots (k+n) \left[ K(t) \star \mathcal{F}_k(t) - K(t) \star \mathcal{F}_{k+1}(t) \right]
\]

Taking \( L - S. T \) on both sides, we get
\[ \phi_n^*(s) = n! \left[ 1 - K^*(s) \right] + \sum_{k=1}^{\infty} \frac{(k+1)(k+2)\ldots}{k!} \left( k+n \right) K^*(s) \left[ \frac{F^*(s)}{l - F^*(s)} \right]^{k-1} \left[ 1 - F^*(s) \right] \]

\[ = n! \left[ 1 - K^*(s) \right] + K^*(s) \left[ 1 - F^*(s) \right] \]

\[ \times \left[ \frac{(n+1)!}{2!} F^* + \frac{(n+2)!}{3!} F^*^2 + \ldots \right] \]

\[ = n! \left[ 1 - K^*(s) \right] + \frac{(n+1)!}{(n+1) F^*(s)} K^*(s) \left[ 1 - F^*(s) \right] \]

\[ \times \left[ \frac{(n+1) F^* + (n+1)(n+2) F^*^2 + \ldots}{2} \right] \]

\[ = n! \left[ 1 - K^*(s) \right] + n! K^*(s) \frac{1 - F^*(s)}{F^*(s)} \]

\[ \times \left[ \frac{1 - F^*(s)}{F^*(s)} \right]^{(n+1)} \cdot \]

\[ \phi_n^*(s) = 1 - K^*(s) + \frac{K^*(s)}{F^*(s)} \left[ \frac{1 - n}{1 - F^*(s)} \right] - \left( 1 - F^*(s) \right) \]

\[ = 1 - K^*(s) + \left[ 1 + (1 - F^*(s)) + \left( 1 - F^*(s) \right)^2 + \ldots + (1 - F^*(s))^n \right] \frac{K^*(s)}{1 - F^*(s)} \]

\[ = 1 - K^*(s) + K^*(s) \left[ 1 + \phi_1^*(s) + \frac{\phi_2^*(s)}{2!} + \ldots + \frac{\phi_n^*(s)}{n!} \right] . \]
\[ \sum_{j=1}^{\infty} \frac{\phi^*(s)K^*(s)}{j!} \]

\[ = 1 + \sum_{j=1}^{\infty} \frac{\phi^*(s)}{j!}, \text{ by Lemma 1.} \]

\[ \frac{\phi_n(t)}{n!} = \sum_{j=1}^{n} \frac{\phi_j(t)}{j!} \]

The importance of the above result is that it shows that the asymptotic behavior of the moments \( \frac{\phi_n(t)}{n!} \) can easily be inferred from the asymptotic behavior of the moments \( \phi_j(t) \).

1.2. The Asymptotic form of \( \frac{\phi_n(t)}{n!} \).

We shall first obtain the asymptotic form of \( \phi_n(t) \) and then, by using Lemma 2, the asymptotic form of \( \frac{\phi_n(t)}{n!} \) can be written down immediately.

From lemma 1, we have

\[ (1.2.1) \quad \phi_n(t) = \frac{\phi_n'(s)}{n!} = K^*(s) \phi^*(s), \]

and we have therefore,

\[ (1.2.2) \quad \phi_n'(t) = \int_0^t \phi_n(t-x) \, dK(x). \]

If \( \mu_{n+1} < \infty \), we have from theorem 1 of Smith[20], that

\[ (1.2.3) \quad \phi_n(t) = \gamma_1 t^n + \gamma_2 t^{n-1} + \ldots + \gamma_n t + M_n(t), \]

where \( M_n(t) = \gamma_{n+1} t^n \in B \), \( B \) being the class of functions \( \lambda(t) \) which are of bounded variation, tend to zero as \( t \to +\infty \) and are such that
\( \lambda(t) - \lambda(t-\alpha) = o\left(\frac{1}{t}\right) \) as \( t \to \infty \), for every \( \alpha > 0 \).

Substituting from (1.2.3) in (1.2.2) we get

\[
(1.2.4) \quad \phi'_n(t) = \int_0^t \sum_{j=1}^n \gamma_j (t-x)^{n+1-j} d\iota(x) + \int_0^t M^n_n(t-x) d\eta(x).
\]

Taking \( \ell - s \cdot t \) on both sides of (1.2.4), we get

\[
(1.2.5) \quad \phi'_n(s) = \sum_{j=1}^n \frac{\gamma_j}{s^{n+1-j}} K^*(s) + M^n_n(s) K^*(s).
\]

From (1.2.5), it is evident that we should assume \( \gamma_n < \infty \) in order to account for the entire polynomial part of \( \phi'_n(t) \).

But, from Lemma A, for \( \gamma_n < \infty \), it is both necessary and sufficient that

\[
(1.2.6) \quad K^*(s) = 1 + \gamma_1 s + \frac{\gamma_2}{2} s^2 + \ldots + \frac{\gamma_{n-1}}{n-1} s^{n-1} + \gamma_n K^*(n)(s) s^n
\]

where \( K^*_n(t) \) is a distribution function.

Now

\[
\phi'_n(s) = \frac{\gamma_n}{s} K^*(s) + \frac{\gamma_{n-1}}{s^2} K^*(s) + \ldots + \frac{\gamma_1}{s^n} K^*(s) + M^n_n(s) K^*(s).
\]

\[
= \frac{\gamma_n}{s} \left[ 1 + \gamma_1 s K^*_1(s) \right] + \frac{\gamma_{n-1}}{s^2} \left[ 1 + \gamma_2 s K^*_2(s) \right] + \ldots + \frac{\gamma_1}{s^n} \left[ 1 + \gamma_1 s \right] + \frac{\gamma_{n-1}}{s} \left[ 1 + \gamma_2 s \right] + \ldots + \frac{\gamma_1}{s^n} \left[ 1 + \gamma_1 s \right]
\]
\[ \gamma_j s^{j-1} + \gamma_j s^{K^*(j)}(s) \] + \ldots + \\
\frac{1}{s} \left[ \sum_{j=1}^{n} \gamma_j \right] + \frac{1}{s^2} \left[ \sum_{j=1}^{n-1} \gamma_j \right] + \\
\ldots + \frac{1}{s^k} \left[ \sum_{j=1}^{n+1-k} \gamma_j \right] + \frac{1}{s^{n-1}} \left[ \sum_{j=1}^{2} \gamma_j \right] + \frac{1}{s^n} \left[ \sum_{j=1}^{n} \gamma_j \right] + \\
\gamma_j s^{n+1-j} K^*(j)(s) + M_n(s) K^*(s), \\
(1.2.7) = \frac{\gamma_{n,n-1}}{s} + \frac{\gamma_{n-1,n-2}}{s^2} + \ldots + \frac{\gamma_{n+1-k}}{s^k} + n-k + \ldots + \frac{\gamma_{2,1}}{s^{n-1}} + \frac{\gamma_{1}}{s^n} + \frac{\gamma_{n,1} K^*(1)}{s} + \frac{\gamma_{n-1,2} K^*(2)}{s} + \\
\ldots + \frac{\gamma_{1,n} K^*(n)}{s} + M_n(s) K^*(s). \\

Hence
\[ (1.2.8) \phi_n(t) = \gamma_1 t^n + \gamma_{2,1} t^{n-1} + \ldots + \gamma_{n-1,n-2} t^2 + \]
\[ \gamma_{n,n-1} t + \omega(t), \]
where

\[(1.2.9) \quad \omega'(t) = \frac{1}{n} \sum_{i=1}^{n} K_i(t) + \sum_{i=1}^{n} \gamma_{n-1,2} K_2(t) + \cdots
\]

\[\gamma_{1,n} K_n(t) + \int_0^t M_n(t-x) \, dK(x).\]

We will now prove that \(\omega'(t) - \gamma_{n+1,n} \) belongs to the class \(B\) if we further assume \(\gamma_{n+1,n} < \infty\).

**Lemma 1.2.1**

If the first moment of a distribution function \(F(x)\) is finite then \((F(x) - 1)\) belongs to the class \(B\).

**Proof:**

Clearly \(F(x) - 1\) is of bounded variation and tends to zero as \(x \to \infty\).

To prove \(x \left[ F(x) - F(x-\alpha) \right] \) tends to zero as \(x \to +\infty\) for every \(\alpha > 0\), we notice that

\[x \left[ F(x) - F(x-\alpha) \right] < x \left[ 1 - F(x-\alpha) \right]
\]

\[= x \int_{x-\alpha}^{\infty} dF(x)
\]

\[= (x-\alpha) \int_{x-\alpha}^{\infty} dF(x) + \int_{x-\alpha}^{\infty} t \, dF(t) + \int_{x-\alpha}^{\infty} \alpha \, dF(t).
\]

The right hand side tends to zero as \(x \to \infty\) and the left hand side is always non-negative. Hence

\[\lim_{x \to \infty} x \left[ F(x) - F(x-\alpha) \right] = 0.
\]
and our lemma is proved.

In fact if \( \mu_n < \infty \), \( \lim_{x \to \infty} x^n \left[ F(x) - F(x - \alpha) \right] = 0 \)

for \( \alpha > 0 \);

this result is a straightforward generalization of the above lemma.

Since \( \nu_n < \infty \) the means of \( K_1(t), \ldots, K_{n-1}(t) \) are all finite and if we further assume \( \nu_{n+1} < \infty \) the mean of \( K_n(t) \) is finite and applying lemma 1.2.1 we have

\[
(1.2.10) \quad \gamma_{n,1}K_1(t) + \gamma_{n-1,2}K_2(t) + \ldots
\]

\[
= \gamma_{1,n}K_n(t) - \gamma_{n,n} \in B
\]

Smith [2,0] has proved that if \( \mu_{n+1} < \infty \)

\[
(1.2.11) \quad M_n^n(t) - \gamma_{n+1} = \lambda(t), \text{ where } \lambda(t) \in B.
\]

Using (1.2.11) we discover that if

\[
\phi(t) = \int_0^t M_n^n(t-x) \, dK(x)
\]

\[
= \gamma_{n+1}K(t) + \int_0^t \lambda(t-x) \, dK(x),
\]

then

\[
(1.2.12) \quad t \left[ \phi(t) - \phi(t - \alpha) \right] = t \gamma_{n+1} \left[ K(t) - K(t - \alpha) \right]
\]

\[
+ \int_0^t t \left[ \lambda(t-x) - \lambda(t-x - \alpha) \right] \, dK(x)
\]

\[
+ t \int_{t-\alpha}^t \lambda(t-x) \, dK(x).
\]
It is not difficult to show that every term on the right hand side of (1.2.12) tends to zero as \( t \to \infty \).

(1.2.13) \( \phi(t) - \gamma_{n+1} \in B \).

Combining (1.2.10) and (1.2.13) we obtain that if \( \mu_{n+1} < \infty \), \( \nu_{n+1} < \infty \), then

\[
\omega'(t) - \gamma_{n,n} - \gamma_{n+1} = \omega'(t) - \gamma_{n+1,n} \in B
\]

and we have therefore proved the following theorem.

**Theorem 1.1**

If \( \mu_{n+1} < \infty \), \( \nu_{n+1} < \infty \), then as \( t \to \infty \)

\[
\phi'(t) = \gamma_{1,n} t + \gamma_{2,n} t^{n-1} + \ldots + \gamma_{n-1,n-2} t^2 + \gamma_{n,n-1} t + \gamma_{n+1,n} + \omega(t),
\]

where \( \omega(t) \) belongs to the class \( B \).

The corresponding theorem for \( \tilde{\phi}_n(t) \) of a G.R.P. follows from Lemma 2 and the closure properly of \( B \) under linear combinations. Notice that in all three of the moments \( \phi_n(t) \), \( \phi'_n(t) \) and \( \tilde{\phi}_n(t) \) the coefficient of \( t^n \) is the same \( \gamma_1 \), it is in fact \( n! \mu_1^n \mu_1 \) (where \( \mu_1 \) is the first moment of the distribution).

Now for a discussion of the asymptotic behavior of the cumulants of a G.R.P., we need additional information concerning the remainder term in theorem 1. We will achieve this by putting additional restrictions on the \( \mu \)'s and \( \nu \)'s.

We have from (1.2.7) that

(1.2.14) \( \phi_n^*(s) = \frac{\gamma_1}{s^n} + \frac{\gamma_{2,1}}{s^{n-1}} + \ldots + \frac{\gamma_{n-1,n-2}}{s^2} + \frac{\gamma_{n,n-1}}{s} \)
+ \gamma_n 1^{K^*}(1)(s) + \gamma_{n-1} 2^{K^*}(2)(s) \\
+ \ldots \gamma_{1,n} K^*(n)(s) + M_{n}^{n}(s) K^*(s).

Therefore we have for the Laplace transform of \phi_n'(t),

\begin{equation}
\phi_n'(t) = \frac{\gamma_1}{s+1} + \frac{\gamma_2}{s^2} + \ldots + \frac{\gamma_{n,n-1}}{s^n}
\end{equation}

\begin{equation}
+ \frac{\gamma_{n,1}^{K^*}(1)(s)}{s} + \frac{\gamma_{n-1,2}^{K^*}(2)(s)}{s} + \ldots
\end{equation}

\begin{equation}
+ \frac{\gamma_{1,n} K^*(n)(s)}{s} + \frac{M_{n}^{n}(s) K^*(s)}{s}
\end{equation}

We will now prove the following basic lemma.

**Lemma 1.2.2.**

If \mu_{n+p+1}<\infty, \nu_{n+p}<\infty, and p>0, then

\begin{equation}
(-1)^p \phi_n(p)(s) = \frac{\gamma_1}{s^{n+p+1}} + \frac{\gamma_2}{s^{n+p}} + \ldots \frac{\gamma_{n,n-1}}{s^{n+2}} + \frac{\gamma_{n+1,n}}{s^{n+1}}
\end{equation}

+ o\left(\frac{1}{s}\right), \text{ as } s \to \infty.

**Proof!**

Differentiating (1.2.15) p times on both sides, we get

\begin{equation}
(-\frac{d}{ds})^p \phi_n'(s) = \frac{\gamma_1}{s^{n+p+1}} + \frac{\gamma_2}{s^{n+p}} + \ldots + \frac{\gamma_{n,n-1}}{s^{n+2}}
\end{equation}

\begin{equation}
+ \gamma_{n,1} (-\frac{d}{ds})^p \left[\frac{K^*(1)(s)}{s}\right]
\end{equation}

\begin{equation}
+ \gamma_{n-1,2} (-\frac{d}{ds})^p \left[\frac{K^*(2)}{s}\right] + \ldots
\end{equation}

\begin{equation}
+ \gamma_{1,n} (-\frac{d}{ds})^p \left[\frac{K^*(n)}{s}\right]
\end{equation}
We show that if the first \( p \) moments of any distribution function \( F(x) \) of a non-negative random variable are finite then
\[
(1.2.17) \quad \left( -\frac{d}{ds} \right)^P \left[ \frac{F^*(s)}{s} \right] = \frac{1}{s^{p+1}} + o \left( \frac{1}{s} \right).
\]

To begin with we remark that since the first \( p \) moments of the distribution function \( r(x) \) are finite, we have from Lemma A,
\[
\frac{F^*(s)}{s} = \frac{1}{s} + \gamma_1 + \gamma_2 s + \gamma_3 s^2 + \ldots \gamma_{p-1} s^{p-2} + \\
+ \gamma_p s^{p-1} F^*(p)(s),
\]
where \( F^*(p)(t) \) is a distribution function.

Differentiation \( p \) times on both sides of this equation yields
\[
(1.2.18) \quad \left( -\frac{d}{ds} \right)^P \frac{F^*(p)}{s} = \frac{1}{s^{p+1}} + \gamma_p \left( -\frac{d}{ds} \right)^P \left[ s^{p-1} F^*(p)(s) \right].
\]

But for any function \( \mathcal{L}(t) \) of bounded variation, it follows from Smith\[20\] that
\[
(1.2.19) \quad \begin{cases} 
(a) \quad \mathcal{L}^*(p)(s) = o \left( \frac{1}{s^p} \right) \text{ as } s \to 0 +, \\
(b) \quad \left( -\frac{d}{ds} \right)^P [s^{p-1} \mathcal{L}^*(p)(s)] = o \left( \frac{1}{s} \right).
\end{cases}
\]

From (1.2.18) and (b) above, (1.2.17) follows.

If we now apply (1.2.17) to all the terms \( \frac{K^*(1)}{s}, \frac{K^*(2)}{s}, \ldots, \frac{K^*(n)}{s} \) and use the fact that \( K^*(2) \) and other \( K^* \)'s have their first \( p \) moments finite (in view of \( \nu_{np} < \infty \)), then we find
\( (1.2.20) \quad \gamma_{n,1} \left( \frac{-d}{ds} \right)^p \left[ \frac{K^*(1)(s)}{s} \right] + \ldots + \gamma_{1,n} \left( \frac{-d}{ds} \right)^p \left[ \frac{K^*(n)(s)}{s} \right] \)

\[ = \gamma_{n,n} + o\left( \frac{1}{s} \right). \]

But if \( \mu_{n+p+1} < \infty \), then \( M^*_n \) can be written as

\[ M^*_n(s) = \gamma_{n+1} + \gamma_{n+2}s + \gamma_{n+p}s^2 + \ldots + \gamma_{n+p}s^{p-1} + s^{p}M^*_n, \]

where \( M^*_n \) is B.V., and hence

\[ (1.2.21) \quad M^*_n(s)K^*(s) = \frac{\gamma_{n+1}K^*}{s} + \gamma_{n+2}K^* + \gamma_{n+p}sK^*(s) \]

\[ + \ldots + \gamma_{n+p}s^{p-2}K^* + s^{p-1}K^*M^*_n. \]

Upon differentiating (1.2.21) w.r.t \( s \), \( p \) times, and using (1.2.17) and (1.2.19), we find

\[ (1.2.22) \quad \left( \frac{-d}{ds} \right)^p \left[ \frac{M^*_nK^*}{s} \right] = \frac{\gamma_{n+1}}{s^{p+1}} + o\left( \frac{1}{s} \right), \]

and on combining (1.2.20) and (1.2.22) we discover that

\[ (1.2.23) \quad \gamma_{n,1} \left( \frac{-d}{ds} \right)^p \left[ \frac{K^*(1)(s)}{s} \right] + \ldots + \gamma_{1,n} \left( \frac{-d}{ds} \right)^p \left[ \frac{K^*(n)(s)}{s} \right] \]

\[ + \left( \frac{-d}{ds} \right)^p \left[ \frac{M^*_nK^*}{s} \right] = \frac{\gamma_{n+1,n}}{s^{p+1}} + o\left( \frac{1}{s} \right). \]

The lemma is therefore proved.

From Lemma 1 we have that
\[ (1.2.24) \quad \phi_n^*(s) = K^*(s) \phi_n^*(s) \]

If we divide both sides by \( s \) we find that

\[ (1.2.25) \quad \phi_n^0(s) = K^*(s) \phi_n^0(s), \]

and hence that

\[ (1.2.26) \quad \phi_n^0(p)(s) = \phi_n^0(p)(s) K^*(s) + \left( \begin{array}{c} p \\ 1 \end{array} \right) \phi_n^0(p-1)(s) K^*(1)(s) \]

\[ + \ldots \left( \begin{array}{c} p \\ r \end{array} \right) \phi_n^0(p-r)(s) K^*(r)(s) \]

\[ + \ldots \phi_n^0(s) K^*(p)(s). \]

Let us write

\[ (1.2.27) \quad K^*[p](s) = (-1)^p K^*(p)(s) = \gamma_p^0 K^*(p). \]

Then, if \( K(t) \) has its first \( n+p \) moments finite, the first \( n \) moments of \( K^*[p] \) are finite, and upon using (1.2.27), (1.2.26) can be written as

\[ (1.2.28) \quad \phi_n^0(p) = \phi_n^0(p) K^* + \gamma_1^0 \phi_n^0(p-1) K^*[1] \]

\[ + \ldots \gamma_r^0 \phi_n^0(p-r) K^*[r] + \gamma_p^0 \phi_n^0 K^*[p], \]

where \( K^*[r], r=0,1,\ldots, p \) has its first \( n+p-r \) moments finite.

Now, we have from Smith[20], that if \( n+p+1 < \infty, n>0, p>0, \)

then

\[ (1.2.29) \quad (-1)^p \phi_n^0(p) = \frac{\gamma_1}{s^{n+p+1}} + \frac{\gamma_2}{s^{n+p}} + \ldots + \frac{\gamma_{n+1}}{s^{p+1}}. \]

Where \( \gamma_n^0 \) is the L.T. of a function which belongs both to the
class B and to the class L (L is the class of functions \( \lambda(t) \) such that \( \lambda(t) \) belongs to \( L_1(0, \infty) \)). We denote the intersection \( (L+T) \) by R.

Applying (1.2.29) repeatedly in (1.2.28), and using the fact that

\[
\phi_1^n = \frac{\gamma_1}{s+1} + \frac{\gamma_2}{s^2} + \cdots \frac{\gamma_n}{s^n} + M_0^n,
\]

we obtain

\[
(1.2.30) \quad (-1)^p \phi_1^n = \left[\frac{\gamma_1}{s+p+1} + \frac{\gamma_2}{s+p^2} + \cdots + \frac{\gamma_n}{s^n} + M_0^n \right]^{K*}
\]

Comparing (1.2.30) with lemma 1.2.2 and using Lemma A, we get

\[
(1.2.31) \quad (-1)^p \phi_1^n = \frac{\gamma_1}{s+p+1} + \frac{\gamma_2}{s+p^2} + \cdots + \frac{\gamma_n}{s^{n+2}}
\]
\[ \mathcal{L}^0 = \sum_{l=0}^{p-1} \sum_{j=1}^{n+1} \gamma_{j, n+p+1-j-l} K^0_{(n+p+1-j-l)} \]

\[ + \sum_{j=1}^{n} \gamma_j K^0_P(n+1-j)+\mathcal{L}^0_{K^*} + \mathcal{L}^0_{K^*[p]} \]

\[ + \ldots + \mathcal{L}^0_P K^*[p-1] + M_{n} Q K^*[p] \]

It is clear from (1.2.32) that \( \mathcal{L}^0 \) is the Laplace transform of a function of B.V.

But lemma 1.2.2 leads to

\[ \mathcal{L}^0 = o\left(\frac{1}{s}\right) \]

(1.2.33) i.e.

\[ \text{Lt}_{s \to 0} \mathcal{L}^0 = \text{Lt}_{t \to \infty} \mathcal{L}(t) = \mathcal{L}(\infty) = o. \]

Noticing the fact that \((-1)^p \phi^{(p)}(t)\) is the L.T. of \( t^p \phi^{(p)}(t) \),

we get from (1.2.31) that

\[ \phi_n(t) = \gamma_1 t^n + \gamma_{2,1} t^{n-1} + \ldots + \gamma_{n,n-1} t + \gamma_{n+1,n} + \frac{\mathcal{L}(t)}{t^p} \]

where \( \mathcal{L}(t) \) is of bounded variation and tends to zero at \( \infty \).

Also, in view of theorem 1, since \( \frac{\mathcal{L}(t)}{t^p} \) must be bounded, we can equivalently write \( \mathcal{L}(t) \) as \( \frac{\lambda'(t)}{(1+t)^p} \) where \( \lambda'(t) \) is B.V. and tends to zero at \( + \infty \).
From (1.2.32) we get

\begin{align*}
(1.2.35) \tilde{\mathcal{L}}(t) &= \sum_{l=0}^{p-1} \sum_{j=1}^{n+1} \gamma_j, n+p+1-j-l \left[ K_{(n+p+1-j-l)}^{(l)}(t) - 1 \right] \\
&+ \sum_{j=1}^{n+1} \gamma_j K^{(j)}(t)_{(n+1-j)} + \int_0^t \mathcal{L}(t-x) \, dK(x) \bigg|_{x=x-x} \\
&+ \ldots \quad \int_0^t \int_{\mathcal{L}}(t-x) \, dK^{[p-1]}(x) + \gamma_{n+1} K^{[p]}(t) \\
&+ \int_0^t \lambda(t-x) \, dK^{[p]}(x), \text{ where}
\end{align*}

\begin{align*}
M_n^p(t) - \gamma_{n+1} = \lambda(t) \in B.
\end{align*}

Since \( \tilde{\mathcal{L}}(t) \to 0 \) as \( t \to \infty \), the right hand side of (1.2.35) must also tend to zero and hence (1.2.35) can also be written as

\begin{align*}
(1.2.36) \tilde{\mathcal{L}}(t) &= \sum_{l=0}^{p-1} \sum_{j=1}^{n+1} \gamma_j, n+p+1-j-l \left[ K_{(n+p+1-j-l)}^{(l)}(t) - 1 \right] \\
&+ \sum_{j=1}^{n+1} \gamma_j \left[ K^{[j]}(t) - 1 \right]_{(n+1-j)} \\
&+ \int_0^t \mathcal{L}(t-x) \, dK(x) + \ldots \quad \int_0^t \int_{\mathcal{L}}(t-x) \, dK^{[p-1]}(x) + \gamma_{n+1} \left[ K^{[p]}(t) - 1 \right] \\
&+ \int_0^t \lambda(t-x) \, dK^{[p]}(x)
\end{align*}

The terms in (1.2.36) whose first moments are not finite in accordance with the assumptions made so far are \( K_{(n+p)}^{[p-1]}, K_{(n+1)}^{[p-1]} \) and \( K_{(n)}^{[p]} \) and all these have their means finite if we further...
assume that \( \gamma_{n+p+1} < \infty \). Also, if \( F(x) \) is a distribution function with a finite mean \( \mu_1 \), then \( F(x)-1 \in L_1(0, \infty) \) as is seen from the fact that

\[
(1.2.34) \quad \int_0^\infty |F(x)-1| \, dx = \mu_1.
\]

In view of lemma 1.2.1,

\[
(1.2.35) \quad \sum_{i=0}^{p-1} \sum_{j=1}^{n+1} \gamma_{j, n+p+1-j-1} \left[ K^{[i]}(t) - 1 \right] + \sum_{j=1}^{n} \gamma_j \left[ K^{[P]}(t) - 1 \right] + \gamma_{n+1} \left[ K^{[P]}(t) - 1 \right]
\]

belongs both to \( B \) and to \( L_1(0, \infty) \).

Now, \( \Omega_1^0, \Omega_2^0, \ldots, \Omega_p^0 \) are L.T.'s of functions which belong to \( B \). Also \( \Omega_2(t), \ldots, \Omega_p(t) \) belong to the class \( L_1(0, \infty) \), and \( \Omega_1(t) \) belongs to \( L_1(0, \infty) \), as shown by Smith [20], if in addition \( \mu_{n+p+2} < \infty \); the function \( \lambda(t) = M_n^\gamma(t) - \gamma_{n+1} \) belongs to \( B \) and it can also be seen to belong to \( L_1(0, \infty) \) by noticing that \( \lambda = M_n^\gamma - \gamma_{n+1} \) and so \( \lambda^0 = M_n^\gamma_{n+1} \), where \( M_n^\gamma \) is B.V. since \( \mu_{n+1} < \infty \).

If \( \lambda(t) \in B \), we have shown in (1.2.12) that

\[
(1.2.36) \quad \mu(t) = \int_0^t \lambda(t-x) \, dK(x) \in B.
\]

Further, if \( \lambda(t) \in L_1(0, \infty) \), then \( \mu(t) \) also belongs to \( L_1(0, \infty) \).

We have thus proved
(A) If \( \mu_{n+p+1} < \infty \), \( \nu_{n+p+1} < \infty \), \( n > 0 \), \( p > 0 \) then

\[
(1.2.37) \quad \phi'_n(t) = \gamma_1 t^n + \gamma_{2,1} t^{n-1} + \cdots + \gamma_{n,n-1} t^{n-1} + \gamma_{n+1,n} \frac{\bar{\lambda}(t)}{t^p},
\]

where \( \bar{\lambda}(t) \in B \) and if

(B) \( \mu_{n+p+1} < \infty \), \( \nu_{n+p+1} < \infty \), \( n > 0 \), \( p > 0 \) then

\[
(1.2.38) \quad \phi'_n(t) = \gamma_1 t^n + \gamma_{2,1} t^{n-1} + \cdots + \gamma_{n,n-1} t^{n-1} + \gamma_{n+1,n} \frac{\bar{\lambda}(t)}{t^{p-1}}
\]

where \( \bar{\lambda}(t) \in B \) and also to \( L_1(0, \infty) \).

Comparing (A) and (B) we must have

\[
\frac{\bar{\lambda}(t)}{t^{p-1}} = \bar{\lambda}(t) \in L_1(0, \infty)
\]

(1.2.39) \( \frac{\bar{\lambda}(t)}{t^{p-1}} \) or equivalently as \( \frac{\lambda'(t)}{1+t} \in L_1(0, \infty) \).

Combining (1.2.37) and (1.2.39) we have proved

Theorem 1.2.

If \( \mu_{n+p+1} < \infty \), \( \nu_{n+p+1} < \infty \), \( p > 0 \), \( n > 0 \), then

\[
\phi'_n(t) = \gamma_1 t^n + \gamma_{2,1} t^{n-1} + \cdots + \gamma_{n,n-1} t^{n-1} + \gamma_{n+1,n} \frac{\lambda'(t)}{(1+t)^p}, \quad \text{where} \quad \lambda'(t) \in \mathbb{R}.
\]

Let us now recall the relationship (Lemma 2)

\[
(1.2.40) \quad \phi_n(t) = 1 + \sum_{j=1}^{n} \frac{t}{j!} \phi'_j(t),
\]

between \( \phi'_n(t) \) and \( \tilde{\phi}_n(t) \) of the G.R.P.
In view of the additive property of the class $R$ and (1.2.40), the result corresponding to Theorem 1.2 for a G.R.P. may be stated as

**Theorem 1.2'**

If $\mu_{n+p+1} < \infty$, $\nu_{n+p+1} < \infty$, $p > 0$, $n > 0$, then

$$\phi_n(t) = \gamma_1 t^n + \gamma_{2,1} t^{n-1} + \ldots + \gamma_{n,n-1} t$$

$$+ \gamma_{n+1,n} \frac{\lambda(t)}{(1+t)^p},$$

where $\lambda(t) \in R$.

1.3 The Asymptotic behavior of the $\psi$-cumulants:

$\psi_n(t)$ is a linear combination with numerical weights of a finite number of terms like

$$\phi_{p_1}(t) \phi_{p_2}(t) \ldots \phi_{p_r}(t),$$

where $p_1 + p_2 + \ldots + p_r = n$.

Using (1.2.41) in the individual terms of (1.3.1) and also the property that if $\lambda_1(t) \in R$, $\lambda_2(t) \in R$ then $\lambda_1(t) \lambda_2(t) \in R$, we get

$$\psi_n(t) = \gamma_1 t^n + \gamma_{2,1} t^{n-1} + \ldots + \gamma_{n,n-1} t$$

$$+ \gamma_{n+1,n} \frac{\lambda(t)}{(1+t)^p},$$

where $\lambda(t) \in R$.

In the corresponding representation for $\psi_n(t)$ of the Renewal Process, Smith [21] has shown that all the coefficients of the non-linear terms in $t$ vanish identically. Before establishing that the same thing is true for $\psi_n(t)$ of the General
Renewal Process, we will demonstrate the truth by computing the first of three $\Psi$'s of a G.R.P.

In what follows the finiteness of the appropriate moments is assumed, the remainder term omitted and the sign " ~ " is used to represent the "approximation" sign.

$\Psi_1(t)$: Assumption: $\mu_2 < \infty, \nu_1 < \infty$.

We have by definition,

$$\Psi_1(t) = \eta_1(t) = 1 + \phi'_1(t).$$

Taking L - S. T on both sides,

$$\tilde{\phi}'_1 = 1 + \phi'_1.$$  

But

$$\phi'_1 = K^* \phi'_1 = \frac{K^*}{1-F^*}$$

$$\frac{1}{\mu_1 s} \left[ 1 - \frac{\mu_2}{2! \mu_1} s + \ldots \right]^{-1} \left[ 1 - \nu_1 s + \frac{\nu_2 s^2}{2!} \ldots \right]$$

$$\frac{1}{\mu_1 s} \left[ 1 + \frac{\mu_2}{2! \mu_1} s + \ldots \right] \left[ 1 - \nu_1 s + \ldots \right]$$

$$\frac{1}{\mu_1 s} + \left( \frac{\mu_2}{2 \mu_1^2} - \frac{\nu_1}{\mu_1} \right).$$

$$\phi'_1(t) \sim \frac{t}{\mu_1} + \left( \frac{\mu_2}{2 \mu_1^2} - \frac{\nu_1}{\mu_1} \right).$$

$$\eta'_1(t) \sim \frac{t}{\mu_1} + \left( \frac{\mu_2}{2 \mu_1^2} + 1 - \frac{\nu_1}{\mu_1} \right).$$

If $\mu_1 = \nu_1$ we get the corresponding result of a Renewal Process.
Also \( \tilde{m}_1(t) = \psi_1(t) - 1 = \bar{\psi}_1(t) - 1 - \frac{t}{\mu_1} + \left( \frac{\mu_2}{2\mu_1^2} - \frac{\nu_1}{\mu_1} \right) \).

\( \bar{\psi}_2(t) \): Assumption \( \mu_2 < \infty \), \( \nu_2 < \infty \).

By definition

\[ \bar{\psi}_2(t) = \psi_2(t) - \psi_1(t). \]

But

\[ \psi_2(t) = 2 + 2 \phi_1(t) + \phi_2(t). \]

Now

\[ \phi_2^* = K^* \phi_2^* = 2! \frac{K^*}{(1-K^*)^2} = \frac{2!}{(1-K^*)^2} \left[ \frac{\mu_1 s - \mu_2 s^2 + \frac{\mu_3}{2} s^3 \ldots}{2!} \right] \]

\[ = \frac{2!}{\mu_1^2 s^2} \left[ \frac{\mu_2}{2\mu_1} + \frac{\mu_3}{3\mu_1} \right]^{-2} \left[ 1 - \frac{\nu_1 s + \nu_2 s^2}{2!} \ldots \right] \]

\[ = 2! \left[ \frac{1}{\mu_1^2 s^2} + \left( \frac{\mu_2}{\mu_1^3} - \frac{\nu_1}{2\mu_1} \right) \frac{1}{s} + \left( \frac{3\mu_2^2}{4\mu_1^4} - \frac{\mu_3}{3\mu_1^3} - \frac{\nu_1 \mu_2}{\mu_1^3} + \frac{\nu_2}{2! \mu_1^2} \right) \right] \]

\[ + \ldots \]

\[ \phi_2'(t) = e^{-t^2/\mu_1^2} + 2 \left( \frac{\mu_2}{\mu_1^3} - \frac{\nu_1}{2\mu_1} \right) t \]

\[ + \left( \frac{3\mu_2^2}{2\mu_1^4} - \frac{2\mu_3}{3\mu_1^3} - \frac{2\nu_1 \mu_2}{\mu_1^3} + \frac{\nu_2}{2! \mu_1^2} \right) \].
Substituting for \( \hat{\phi}_2(t) \) and \( \hat{\phi}_1(t) \) in \( \tilde{\phi}_2(t) \), we get

\[
\tilde{\phi}_2(t) = \frac{t^2}{\mu_1} + \left( \frac{2}{\mu_1} + \frac{2\mu_2}{\mu_1^2} - \frac{2\nu_1}{\mu_1^2} \right) t
+ \left( 2 + \frac{\mu_2}{\mu_1^2} + \frac{3\mu_2^2}{2\mu_1^4} - \frac{2\mu_3}{3\mu_1^3} - \frac{2\nu_1}{\mu_1} - \frac{2\nu_1\mu_2}{\mu_1^3} + \nu_2 \right).
\]

\[
\therefore \tilde{\psi}_2(t) = \tilde{\phi}_2(t) - \tilde{\phi}_1(t) - \frac{\mu_2}{\mu_1^3} t
+ \left( 1 + \frac{5\mu_2^2}{4\mu_1^4} - \frac{2\mu_3}{3\mu_1^3} - \frac{\nu_1\mu_2}{\mu_1^3} - \frac{\nu_1^2}{\mu_1^2} + \frac{\nu_2}{\mu_1} \right).
\]

Also

\[
\text{Var} \left[ \tilde{N}_t \right] = \tilde{\phi}_2(t) - 3 \tilde{\phi}_1(t) + 1 - \tilde{m}_1(t)
- \left( \frac{\mu_2}{\mu_1^3} - \frac{1}{\mu_1} \right) t + \left( 3 - \frac{\mu_2}{2\mu_1^2} + \frac{5\mu_2^2}{4\mu_1^4} - \frac{2\mu_3}{3\mu_1^3}
- \frac{\nu_1^2}{\mu_1^2} - \frac{2\nu_1}{\mu_1} - \frac{\nu_1\mu_2}{\mu_1^3} + \frac{\nu_2}{\mu_1} \right).
\]

If we put \( \nu_1 = \mu_1 \), \( \nu_2 = \mu_2 \) in the above we get the variance-time relation of the renewal process obtained earlier by Smith [2, 0],

\[
\text{Var} \left[ N_t \right] \sim \left( \frac{\mu_2}{\mu_1^3} - \frac{1}{\mu_1} \right) t + \left( \frac{5\mu_2^2}{4\mu_1^4} - \frac{2\mu_3}{3\mu_1^3} - \frac{\mu_2}{2\mu_1^2} \right).
\]
\[ \tilde{\psi}_3(t) : \text{Assumption } \mu_4 < \infty, \nu_3 < \infty. \]

By definition
\[ \tilde{\psi}_3(t) = \tilde{\varphi}_3(t) - 3 \tilde{\varphi}_2(t) \tilde{\varphi}_1(t) + 2 \tilde{\varphi}_1^3(t). \]

But
\[ \frac{\tilde{\varphi}_3(t)}{3!} = 1 + \frac{\varphi_1'(t)}{2!} + \frac{\varphi_2'(t)}{3!} + \frac{\varphi_3'(t)}{3!}. \]

\[ \therefore \frac{\tilde{\varphi}_3(t)}{\mu_1^3} = \left( \frac{9\mu_2 + 6\mu_1^2 - 2\mu_1 \nu_1}{2\mu_1^4} \right) t^2 \]
\[ + \frac{t}{\mu_1^5} \left[ 6\mu_1^4 + 6\mu_1^2 \nu_2 + 9\nu_2^2 - 3\mu_1 \nu_1 \right] \]
\[ - 9\nu_1 \mu_1 \nu_2 + 3\mu_1^2 \nu_2 - 6\mu_1^3 \nu_1 \]
\[ + \frac{1}{10\mu_1^6} \left[ 60\mu_1^6 + 30\mu_1^4 \nu_2 + 45\mu_1^2 \nu_2^2 - 20\mu_1^3 \nu_3 \right. \]
\[ + 72\mu_1 \mu_2 \nu_3 - 108\mu_1^2 \nu_1 + 36\nu_1 \nu_3 \nu_1 \]
\[ + 54 \mu_1^2 \nu_2 \nu_2 \]
\[ - 12 \mu_1^3 \nu_3 - 60\mu_1^5 \nu_1 - 60\mu_1^3 \nu_2 \nu_1 \]
\[ + 36 \mu_1^4 \nu_2 \nu_2 \right]. \]

We find in \( \tilde{\psi}_3(t) \):

Coefficient of \( t^3 = \frac{1}{\mu_1^3} (1 - 3 + 2) = 0, \)

Coefficient of \( t^2 = \frac{-6\mu_2 + 6\mu_1^2 - 6\nu_1 \mu_1}{\mu_1^4} \).
\[- \frac{6 \mu_1^2 + 3 \mu_2 - 6 \nu_1 \mu_1}{2 \mu_1^4} \]
\[+ \frac{6 \mu_1^2 + 3 \mu_2 - 6 \nu_1 \mu_1}{2 \mu_1^4} \]
\[+ \frac{9 \mu_2 + 6 \mu_2 - 6 \nu_1 \mu_1}{2 \mu_1^4} \equiv 0. \]

Also as in the case of \( \tilde{\psi}_1(t) \) and \( \tilde{\psi}_2(t) \) we find for the contribution due to \( K(x) \) to the coefficient of \( t \) in \( \tilde{\psi}_3(t) \) to be

\[\frac{3 \mu_1^2 \nu_2 - 6 \mu_1^3 \nu_1 - 9 \nu_1 \mu_1 \nu_2}{\mu_1^5} - \frac{3 \mu_1^2 \nu_2 - 6 \mu_1^3 \nu_1 - 6 \mu_1 \mu_2 \nu_1}{\mu_1^5} \]
\[- \frac{3 [2 \mu_1^2 + \mu_2 - 2 \nu_1 \mu_1] ^2}{2 \mu_1^5} - 6 \nu_1 \mu_1 \nu_2 \]
\[+ \frac{3 [2 \mu_1^2 + \mu_2 - 2 \nu_1 \mu_1] ^2}{2 \mu_1^5} \equiv 0. \]

We will now prove that the above observations are true for \( \tilde{\psi}_n(t) \) of a General Renewal Process. We will also show that the coefficient of \( t \) in the asymptotic representation of \( \tilde{\psi}_n(t) \) does not involve the moments of \( K(x) \) and finally that the contribution to the constant term due to \( K(x) \) is by way of an additive function of \( \mu \)'s and \( \nu \)'s.

We have proved in lemma 2 that

(1.3.3) \[ \frac{\phi^*}{r !} = 1 + \frac{\phi^*}{2} + \frac{\phi^*}{2 !} + \frac{\phi^*}{r !} \]
\[ (1.3.4) \quad \frac{\tilde{\phi}^*}{r+1!} - \frac{\phi^*_r}{r!} = \frac{\phi^{'*}}{r+1!} \]

Multiplying both sides of (1.3.4) by \( z^{r+1} \) and summing w.r.t. \( r \) from 0 to \( \infty \), we get

\[ (1 - z) \quad \tilde{\phi}_s(z)^* = \phi^*_s(z). \]

But

\[ \phi^*_s(z) = 1 + \phi^*_1(s)z + \phi^*_2(s) \frac{z^2}{2!} + \ldots \]

\[ = 1 + K^* \phi^*_1 z + K^* \phi^*_2 \frac{z^2}{2!} + \ldots \]

\[ = 1 - K^* + K^* \left[ 1 + \phi^*_1(s)z + \phi^*_2(s) \frac{z^2}{2!} + \ldots \right] \]

\[ (1.3.5) \quad = 1 - K^* + K^* \phi^*_s(z). \]

Substituting from (1.3.5) in (1.3.4), we obtain

\[ (1.3.6) \quad \tilde{\phi}_s(z) = \frac{1 - K^*(s)}{1 - z} + \frac{K^*(s) \phi^*_s(z)}{1 - z}. \]

Furthermore from (1.3.3) we find

\[ (1.3.7) \quad \tilde{\phi}^*_r = K^* \left[ 1 + \phi^*_1 + \phi^*_2 \frac{z}{2!} + \ldots \frac{\phi^*_j}{j!} \right] + 1 - K^*. \]

If we now let \( Q = 1 - F^* \),

and express \( K^* \) in the form

\[ (1.3.8) \quad K^* = 1 + \lambda_1 Q + \lambda_2 Q^2 + \ldots \]

then on substituting from (1.3.8) into (1.3.7) we discover that
\[ (1.3.9) \quad \frac{\hat{\phi}}{n!} = (1 + \lambda_1 + \lambda_2 + \ldots + \lambda_n) \\
+ (1 + \lambda_1 + \ldots \lambda_{n-1}) \frac{1}{\lambda} + (1 + \lambda_1 + \ldots \lambda_{n-2}) \frac{1}{\lambda^2} \\
+ \ldots + \frac{1}{\lambda^n} + R^*(s) \]

where

\[ (1.3.10) \quad R^*(s) = \sum_{j=1}^{\infty} \left( \lambda_j \sum_{k=1}^{j} Q^k \right) + \sum_{j=1}^{\infty} \lambda_j Q^j. \]

Now

\[ \sum_{k=1}^{j} Q^k = \frac{Q}{1 - Q} \]

and substituting in \( R^*(s) \)

we get

\[ R^*(s) = \frac{Q}{1 - Q} \sum_{j=1}^{\infty} \lambda_j \left[ 1 - \frac{Q^j}{1 - Q} \right] + \sum_{j=1}^{\infty} \lambda_j Q^j \]

\[ = \frac{Q}{1 - Q} \sum_{j=1}^{\infty} \lambda_j + \sum_{j=1}^{\infty} \lambda_j Q^j \left[ \frac{1 - 2Q}{1 - Q} \right]. \]

Thus, since \( \sum \lambda_j = -1 \),

\[ R^*(s) = \frac{1}{1 - Q} \sum_{j=0}^{\infty} \lambda_j Q^j - \frac{1 - 2Q}{1 - Q} - \frac{Q}{1 - Q} \]

\[ = K^* \frac{1 - 2Q}{1 - Q} - 1 \]

\[ (1.3.11) \quad = \frac{K^*}{F^*} (2F^* - 1) \cdot 1 \]

Thus, we have

\[ (1.3.12) \quad \frac{\hat{\phi}}{n!} = (1 + \lambda_1 + \ldots + \lambda_n) + (1 + \lambda_1 + \ldots \lambda_{n-1}) \phi_1(t) \\
+ (1 + \lambda_1 + \ldots \lambda_{n-2}) \frac{\phi_2(t)}{2!} \\
+ \ldots (1 + \lambda_1) \frac{\phi_{n-1}(t)}{(n-1)!} + \frac{\phi_n(t)}{n!} + R(t), \]
where in view of (1.3.11) \( \lim_{s \to 0} R^*(s) = \lim_{t \to \infty} R(t) = 0. \)

From (1.3.9) we derive

\[
(1.3.13) \quad \frac{\phi^*_n}{(n+1)!} - \frac{n}{n!} = \lambda_{n+1} + \lambda_n \phi^*_1 + \lambda_{n-1} \frac{\phi^*_2}{2!} + \cdots \lambda_1 \frac{\phi^*_n}{n!} + \frac{\phi^*_{n+1}}{(n+1)!}.
\]

Multiplying both sides of (1.3.13) by \( z^{n+1} \) and summing for \( n \) from \( 0 \) to \( \infty \), leads to

\[
(1.3.14) \quad (1 - z) \phi^*_s(z) = \bigwedge_s(z) \phi^*_s(z),
\]

so that

\[
(1.3.15) \quad (1 - z) \Phi_t(z) = \bigwedge(z) \Phi_t(z),
\]

where

\[
(1.3.16) \quad \bigwedge(z) = 1 + \lambda_1 z + \lambda_2 z^2 + \cdots.
\]

Upon taking logarithms on both sides of (1.3.15), we find

\[
(1.3.17) \quad \Psi_t(z) = \Psi_t(z) + \log \bigwedge(z) - \log (1 - z),
\]

where

\( \Psi_t(z) \) is the \( \tilde{\psi} \)-cumulant generating function of the G.R.P.

and \( \Psi_t(z) \) is the \( \psi \)-cumulant generating function of the R.P.

Let \( \gamma_{j,j} \) denote the coefficient of \( \frac{z^j}{j!} \) in the power series expansion of \( \log \bigwedge(z) - \log (1 - z) \).

Then from (1.3.17) we obtain

\[
(1.3.18) \quad \tilde{\psi}_n(t) = \psi_n(t) + \gamma_{n,n}
\]
Smith has shown that if \( \mu_{n+p+1} < \infty, \ p > 0 \), then

\[
\psi_n(t) = \gamma_n t + \gamma_{n+1} + \frac{\lambda(t)}{(1+t)^p}, \quad \text{where}
\]

\( \lambda(t) \in \mathbb{R} \)

Substituting (1.3.19) in (1.3.18) we find

\[
\tilde{\psi}_n(t) = \gamma_n t + \gamma_{n+1} + \frac{\gamma_{n,n} + \lambda(t)}{(1+t)^p},
\]

\( \lambda(t) \in \mathbb{R} \).

Comparing (1.3.20) and (1.3.2) we see that the rational functions \( \gamma_1, \gamma_2, \ldots, \gamma_{n-1,n-2} \) vanish identically and that \( \gamma_{n,n-1} \) is a \( \gamma_n \), which means that the coefficient of \( t \) in \( \tilde{\psi}_n(t) \) of the G.R.P. does not involve any moments of \( K(x) \), and finally that

\[
\gamma_{n+1,n} = \gamma_{n+1} + \gamma_{n,n}.
\]

This shows that the contribution due to being a G.R.P. is by way of an addition of a \( \gamma_{n,n} \) to the constant term in the asymptotic representation of the \( \psi_n \) - cumulant. We have thus proved that

**Theorem 1.3.1**

If \( \mu_{n+p+1} < \infty, \ \nu_{n+p+1} < \infty, \ n > 0, \ p > 0 \), then

\[
\tilde{\psi}_n(t) = \gamma_n t + \gamma_{n+1} + \frac{\gamma_{n,n} + \lambda(t)}{(1+t)^p}, \quad \text{where}
\]

\( \lambda(t) \in \mathbb{R} \).

1.4 The formal calculations:-

Assuming \( \mu_1 = 1 \), Smith has tabulated the first eight \( \gamma_n \).
and $\gamma_{n+1}$ of (1.3.22). To complete Smith's table for a G.R.P, in other words to extend the table of the $\psi$-cumulants to the G.R.P, we will now tabulate the first eight $\gamma_{n, n}$, $n = 1, 2, \ldots, 8$ under the assumption $\mu_1 = 1$.

$\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ are the coefficients of $(1 - F^*)$, $(1 - F^*)^2$, \ldots $(1 - F^*)^n$ \ldots in the following power series representation of $K^*$.

(1.4.1) $K^* = 1 + \lambda_1 (1 - F^*) + \lambda_2 (1 - F^*)^2 + \ldots$ \ldots

Denoting the moments about the origin of the distribution $K(x)$ by $\nu$'s and the corresponding moments of $F(x)$ by $\mu$'s, setting $\mu_1 = 1$, we obtain, by equating the coefficients of the first powers of $s$ on either side of the relation (1.4.1), the following:

\[
\begin{align*}
- \nu_1 &= \lambda_1 \\
- \frac{\nu_2}{2} &= \frac{-\mu_2}{2} \lambda_1 + \lambda_2 \\
- \frac{\nu_3}{6} &= \frac{\mu_3}{6} \lambda_1 - \frac{\mu_2}{2} \lambda_2 + \lambda_3 \\
- \frac{\nu_4}{24} &= \frac{-\mu_4}{24} \lambda_1 + \left(\frac{\mu_2}{4} + \frac{\mu_3}{3}\right) \lambda_2 - \frac{3\mu_2}{2} \lambda_3 + \lambda_4 \\
- \frac{\nu_5}{120} &= \frac{\mu_2}{120} \lambda_1 - \left(\frac{\mu_4}{12} + \frac{\mu_2 \mu_3}{6}\right) \lambda_2 + \left(\frac{3\mu_2}{4} + \frac{\mu_3}{2}\right) \lambda_3 - 2\mu_2 \lambda_4 + \lambda_5 \\
- \frac{\nu_6}{720} &= \frac{-\mu_6}{720} \lambda_1 + \left(\frac{\mu_3}{36} + \frac{\mu_2}{60} + \frac{\mu_2 \mu_4}{24}\right) \lambda_2
\end{align*}
\]
\[- \left( \frac{\mu_4}{8} + \frac{\mu_2 \mu_5}{2} + \frac{\mu_2^3}{8} \right) \lambda_3 \]

\[+ \left( \frac{3\mu_2^2}{2} + \frac{2\mu_5}{3} \right) \lambda_4 - \frac{5\mu_2}{2} \lambda_5 + \lambda_6 \]

\[- \frac{\nu_7}{5040} = \frac{\mu_7}{5040} \lambda_1 - \left( \frac{\mu_6}{360} + \frac{\mu_2 \mu_5}{120} + \frac{\mu_3 \mu_4}{72} \right) \lambda_2 \]

\[+ \left( \frac{\mu_2}{12} + \frac{\mu_5}{40} + \frac{\mu_2 \mu_4}{8} + \frac{\mu_3 \mu_2}{8} \right) \lambda_3 \]

\[- \left( \frac{\mu_4}{6} + \frac{\mu_2^2}{2} + \mu_2 \mu_3 \right) \lambda_4 \]

\[+ \left( \frac{5\mu_2^2}{2} + \frac{5\mu_3}{6} \right) \lambda_5 - 3\mu_2 \lambda_6 + \lambda_7 \]

\[\left( \frac{\mu_8}{40320} \right) = \frac{-\mu_8}{40320} \lambda_1 + \left( \frac{\mu_4^2}{576} + \frac{\mu_7}{2520} + \frac{\mu_2 \mu_6}{720} + \frac{\mu_3 \mu_5}{360} \right) \lambda_2 \]

\[- \left( \frac{\mu_6}{240} + \frac{\mu_2 \mu_5}{40} + \frac{\mu_2 \mu_4}{24} + \frac{\mu_3 \mu_2}{24} + \frac{\mu_2 \mu_4}{32} \right) \lambda_3 \]

\[+ \left( \frac{\mu_2}{16} + \frac{\mu_5^2}{6} + \frac{\mu_2 \mu_3}{2} + \frac{\mu_2 \mu_4}{4} + \frac{\mu_5}{30} \right) \lambda_4 \]

\[- \left( \frac{5\mu_4}{24} + \frac{5\mu_2 \mu_3}{3} + \frac{5\mu_2}{4} \right) \lambda_5 + \left( \frac{15\mu_2^2}{4} + \mu_3 \right) \lambda_6 \]

\[- \frac{7\mu_2}{2} \lambda_7 + \lambda_8 \]

Solving (1.4.2) for the \( \lambda \)'s we get
\[ \lambda_1 = -\nu_1 \]

\[ \lambda_2 = \frac{\nu_2}{2} - \frac{\nu_1 \mu_3}{2} \]

\[ \lambda_3 = \frac{\nu_1 \mu_3}{6} - \frac{\nu_3}{6} + \frac{\nu_2 \mu_2}{2} - \frac{\nu_1 \mu_2^2}{2} \]

\[ \lambda_4 = \frac{\nu_4}{24} - \frac{\nu_3 \mu_4}{24} + \frac{5\nu_2 \mu_2^2}{8} - \frac{5\nu_1 \mu_2^3}{8} - \frac{\nu_2 \mu_2^2}{6} \]

\[ \frac{-\nu_3 \mu_2}{4} + \frac{5\nu_4 \mu_2 \mu_3}{12} \]

\[ \lambda_5 = \frac{\nu_1 \mu_3}{120} - \frac{\nu_5}{120} + \frac{\nu_2 \mu_4}{24} - \frac{\nu_1 \mu_2 \mu_3}{8} - \frac{\nu_2 \mu_2 \mu_3}{2} \]

\[ + \frac{7\nu_2 \mu_2^2 \mu_3}{8} - \frac{3\nu_3 \mu_2^2}{8} + \frac{7\nu_2 \mu_2^3}{8} - \frac{7\nu_1 \mu_2^4}{8} \]

\[ - \frac{\nu_1 \mu_3^2}{12} + \frac{\nu_3 \mu_3}{12} + \frac{\nu_4 \mu_2}{12} \]

(1.4.3)

\[ \lambda_6 = \frac{\nu_6}{720} - \frac{\nu_1 \mu_6}{720} + \frac{7\nu_2 \mu_2^2}{72} - \frac{\nu_2 \mu_3}{120} + \frac{7\nu_2 \mu_2 \mu_4}{48} \]

\[ - \frac{7\nu_1 \mu_2 \mu_3^2}{18} + \frac{7\nu_1 \mu_2 \mu_5}{240} - \frac{7\nu_1 \mu_2^2 \mu_4}{24} + \frac{7\nu_1 \mu_3 \mu_4}{144} \]

\[ - \frac{\nu_3 \mu_4}{48} + \frac{7\nu_3 \mu_2 \mu_5}{24} - \frac{7\nu_2 \mu_2^2 \mu_3}{6} + \frac{7\nu_1 \mu_2 \mu_3}{4} \]

\[ - \frac{7\nu_3 \mu_2^3}{12} + \frac{21\nu_2 \mu_2^4}{16} - \frac{21\nu_1 \mu_2^5}{16} + \frac{7\nu_4 \mu_2^2}{48} \]

\[ - \frac{\nu_4 \mu_2}{36} - \frac{\nu_2 \mu_2}{48} \]
\[ \lambda_7 = \frac{v_1 \mu_7}{5040} - \frac{v_7}{5040} + \frac{v_2 \mu_6}{720} - \frac{v_2 \mu_7 \mu_3}{30} - \frac{v_2 \mu_3 \mu_4}{18} \\
- \frac{v_1 \mu_2 \mu_6}{180} + \frac{3v_1 \mu_2 \mu_5}{40} + \frac{v_1 \mu_2 \mu_7 \mu_3}{4} + \frac{v_3 \mu_5^3}{18} \\
- \frac{v_1 \mu_3 \mu_3}{90} - \frac{5v_1 \mu_2 \mu_3^2}{18} - \frac{v_3 \mu_2^2}{240} + \frac{v_3 \mu_2}{18} \\
- \frac{v_3 \mu_2 \mu_4}{12} + \frac{3v_3 \mu_2^2 \mu_4}{4} + \frac{v_2 \mu_2 \mu_3^2}{2} + \frac{3v_2 \mu_2 \mu_4^2}{8} \\
- \frac{5v_2 \mu_2^3 \mu_3}{2} - \frac{5v_1 \mu_2 \mu_4^3}{8} + \frac{55v_1 \mu_2 \mu_4}{16} + \frac{v_4 \mu_4}{144} - \frac{v_1 \mu_4^2}{144} + \frac{v_4 \mu_2^3}{4} + \frac{33v_2 \mu_2^5}{16} \\
- \frac{33v_1 \mu_2^6}{16} - \frac{15v_3 \mu_2^4}{16} - \frac{v_4 \mu_2 \mu_3}{9} - \frac{v_5 \mu_2^2}{24} + \frac{v_5 \mu_3}{144} + \frac{v_6 \mu_2}{240} \]
\[ + \frac{\nu_3 \mu_3 \mu_4}{32} - \frac{5\nu_3 \mu_2 \mu_3^2}{16} - \frac{15\nu_3 \mu_2 \mu_4}{64} \]

\[ - \frac{3\nu_2 \mu_2 \mu_4}{32} - \frac{5\nu_2 \mu_2 \mu_4}{16} + \frac{55\nu_2 \mu_2 \mu_3^2}{32} \]

\[ + \frac{55\nu_2 \mu_3 \mu_4}{64} + \frac{11\nu_3 \mu_3 \mu_5}{64} - \frac{55\nu_1 \mu_3 \mu_3^2}{16} \]

\[ - \frac{165\nu_4 \mu_2 \mu_4}{128} + \frac{55\nu_4 \mu_2 \mu_3}{128} + \frac{\nu_4 \mu_3^2}{48} \]

\[ - \frac{5\nu_4 \mu_2 \mu_3}{16} + \frac{\nu_4 \mu_2 \mu_4}{32} - \frac{\nu_4 \mu_3}{720} + \frac{\nu_4 \mu_4 \mu_5}{320} \]

\[ + \frac{429\nu_2 \mu_2 \mu_6}{128} - \frac{165\nu_2 \mu_2 \mu_3}{32} - \frac{429\nu_1 \mu_2}{128} \]

\[ + \frac{429\nu_1 \mu_2 \mu_5}{64} - \frac{5\nu_2 \mu_3^3}{72} - \frac{99\nu_3 \mu_2^3}{64} \]

\[ + \frac{55\nu_2 \mu_3 \mu_4}{32} - \frac{\nu_2 \mu_4}{576} + \frac{\nu_2 \mu_2 \mu_3^2}{32} - \frac{5\nu_2 \mu_2^3}{64} \]

\[ + \frac{3\nu_6 \mu_2^2}{320} - \frac{\nu_6 \mu_3}{720} - \frac{\nu_7 \mu_2}{1440} \]

The first eight \( a_n \) (n=1,2,..8), where \( a_n \) is the coefficient of \( \frac{z^n}{n!} \) in the power series expansion of \( \log \Lambda(z) \), are given by

\[ a_1 = \lambda_1 \]

\[ a_\frac{2}{2!} = \lambda_2 - \lambda_1^{1/2} \]
\[
\begin{align*}
\frac{a_3}{3!} &= \lambda_3 - \lambda_1 \lambda_2 + \lambda_1^3 \\
\frac{a_4}{4!} &= \lambda_4 - \left(\frac{\lambda_2^2}{2} + \lambda_1 \lambda_3\right) + \lambda_1^2 \lambda_2 - \lambda_1^{1/4} \\
\frac{a_5}{5!} &= \lambda_5 - \left(\lambda_1 \lambda_4 + \lambda_2 \lambda_3\right) + \left(\lambda_1 \lambda_2^2 + \lambda_1^2 \lambda_3\right) - \lambda_1^3 \lambda_2 + \lambda_1^5 \\
\frac{a_6}{6!} &= \lambda_6 - \left(\frac{\lambda_3^2}{2} + \lambda_1 \lambda_5 + \lambda_2 \lambda_4\right) + \left(\lambda_1 \lambda_4^2 + 2 \lambda_1 \lambda_2 \lambda_3 + \lambda_2^3 \lambda_3\right) - \left(\frac{3 \lambda_1^2 \lambda_2}{2} + \lambda_1^3 \lambda_3\right) + \lambda_1^4 \lambda_2 - \lambda_1^6/6. \\
\frac{a_7}{7!} &= \lambda_7 - \left(\lambda_1 \lambda_5 + \lambda_2 \lambda_3 + \lambda_3 \lambda_4\right) + \left(\lambda_1 \lambda_3^2 + \lambda_1 \lambda_2^2 + 2 \lambda_1 \lambda_2 \lambda_3 + \lambda_2^2 \lambda_3\right) - \left(\lambda_1^3 \lambda_2^3 + \lambda_1 \lambda_3^3\right) - \lambda_1^5 \lambda_2 + \lambda_1^7/7. \\
\frac{a_8}{8!} &= \lambda_8 - \lambda_1^2 + \frac{\lambda_1^2 \lambda_2}{2} + \left(\lambda_1 \lambda_7 + \lambda_2 \lambda_6 + \lambda_3 \lambda_5\right) + \left(\lambda_1 \lambda_6 + 2 \lambda_1 \lambda_2 \lambda_5 + 2 \lambda_1 \lambda_3 \lambda_4 + \lambda_2 \lambda_3^2 + \lambda_2^2 \lambda_4\right) - \left(\frac{\lambda_2^2}{4} + \frac{3 \lambda_1^2 \lambda_3}{2} + 3 \lambda_1 \lambda_2 \lambda_3 + \lambda_1^3 \lambda_5 + 3 \lambda_1 \lambda_2 \lambda_4\right) + \left(\lambda_1^4 \lambda_2^3 + 4 \lambda_1^3 \lambda_2^2 \lambda_3 + 2 \lambda_1^2 \lambda_2^3\right) - \left(\frac{3 \lambda_1^4 \lambda_2}{2} + \lambda_1^5 \lambda_3\right) + \lambda_1^6 \lambda_2 - \lambda_1^8/8.
\end{align*}
\]
Now

\[ \gamma_{n,n} = a_n + (n-1)! \]

Substituting for the $\lambda$'s from (1.4.3) in (1.4.4), we have

finally from (1.4.5),

\[
\begin{align*}
\gamma_{1,1} &= -v_1 + 1 \\
\gamma_{2,2} &= v_2 - v_1 \mu_2 - v_1^2 + 1 \\
\gamma_{3,3} &= v_1 \mu_3 - v_3 + 3v_2 \mu_2 - 3v_1 \mu_2^2 + 3v_1 v_2 - 3v_1^2 \mu_2 - 2v_1^3 + 2 \\
\gamma_{4,4} &= v_4 - v_1 \mu_4 + 15v_2 \mu_2^2 - 15v_1 \mu_2^3 - 4v_2 \mu_3 + 6v_1 \mu_2 \\
&+ 10v_1 \mu_2^2 - 3v_2^2 - 15v_1 \mu_2^2 + 18v_1 v_2 \mu_2 + 4v_1 \mu_3 \\
&- 4v_1 v_3 + 12v_2^2 v_2 - 12v_1 \mu_2 - 6v_1^4 + 6 \\
\gamma_{5,5} &= v_5 - v_5 + 5v_2 \mu_4 - 15v_1 \mu_2 \mu_4 - 60v_2 \mu_2 \mu_3 - 105v_1 \mu_2^2 \mu_3 \\
&- 45v_3 \mu_2^2 + 105v_2 \mu_2^3 - 105v_1 \mu_2^4 - 10v_1 \mu_2^2 + 10v_3 \mu_2 \\
&+ 10v_4 \mu_2 + 5v_1 v_4 - 5v_1 \mu_4 + 135v_1 \mu_2 \mu_3 - 105v_1 \mu_2^2 \\
&- 30v_1 \mu_3 - 40v_1 v_3 \mu_2 + 60v_2 \mu_2 \mu_3 + 10v_2 v_3 - 30v_2 \mu_2 \\
&- 30v_1 \mu_3 - 90v_1 \mu_2^2 + 120v_1 v_2 \mu_2 + 20v_1 \mu_3 \\
&- 20v_1 v_3 + 60v_2 v_2 - 60v_1 \mu_2 - 24v_1^5 + 24 \\
\gamma_{6,6} &= v_6 - v_1 \mu_6 + 70v_2 \mu_2^3 - 6v_2 \mu_3 + 105v_2 \mu_2 \mu_4 - 280v_1 \mu_3^2 + 21v_1 \mu_2 \mu_5
\end{align*}
\]
\[-210 \nu_1 \mu_2 \mu_4^2 + 35 \nu_1 \mu_2 \mu_4 - 15 \nu_3 \mu_4 + 210 \nu_3 \mu_2 \mu_3 - 840 \nu_2 \mu_2 \mu_3^2\]

\[+ 1260 \nu_1 \mu_2 \mu_3^3 - 420 \nu_3 \mu_2^3 + 945 \nu_2 \mu_2 - 945 \nu_1 \mu_2^5 + 105 \nu_4 \mu_2^2\]

\[-20 \nu_1 \mu_3 - 15 \nu_2 \mu_2 - 70 \nu_1 \mu_3^2 - 10 \nu_2^2 - 315 \nu_2 \mu_2^2 - 945 \nu_1 \mu_2 + 80 \nu_1 \nu_3 \mu_3\]

\[-630 \nu_1 \nu_2 \mu_2 \mu_3 + 840 \nu_1 \mu_2 \mu_3^2 + 150 \nu_2 \nu_3 \mu_2 - 420 \nu_1 \nu_3 \mu_2\]

\[+ 1260 \nu_1 \nu_3 \mu_2 + 6 \nu_1 \nu_3 - 6 \nu_1 \nu_3 \nu_2 - 45 \nu_1 \nu_2 \mu_4 - 105 \nu_1 \mu_2 \mu_4 + 75 \nu_1 \nu_4 \mu_2\]

\[-15 \nu_2 \nu_4 + 60 \nu_2 \mu_2 + 30 \nu_1 \nu_4 - 30 \nu_1 \mu_4 + 1260 \nu_1 \nu_2 \nu_2^2\]

\[-840 \nu_1 \mu_2^3 - 240 \nu_1 \nu_2 \mu_2 - 300 \nu_1 \nu_3 \mu_2^2 + 420 \nu_1 \mu_2 \mu_3 + 120 \nu_1 \nu_2 \nu_3\]

\[-450 \nu_1 \nu_2 \mu_2 + 30 \nu_2^2 - 270 \nu_1 \nu_2 + 630 \nu_1 \mu_2 + 900 \nu_1 \nu_2 \nu_2\]

\[+ 120 \nu_1 \nu_3 - 120 \nu_1 \nu_3 \nu_2 + 360 \nu_1 \nu_2 - 360 \nu_1 \mu_2 - 120 \nu_1 \nu_2 + 120\]

\[\gamma_{1,7} = \nu_1 \gamma_7 - 7 \nu_1 \mu_6 - 168 \nu_2 \mu_6 - 280 \nu_2 \mu_3 + 28 \nu_1 \mu_6\]

\[+ 378 \nu_1 \mu_5 + 1260 \nu_1 \mu_2 \mu_4 + 280 \nu_1 \mu_3 - 56 \nu_1 \mu_3 \mu_3\]

\[-6300 \nu_1 \mu_2 \mu_3^2 - 280 \nu_1 \mu_3^2 + 21 \nu_2 \mu_5 - 420 \nu_2 \mu_2 \mu_4 - 3780 \nu_2 \mu_2 \mu_3\]

\[+ 2520 \nu_2 \mu_2 \mu_3^2 + 1890 \nu_2 \mu_2 \mu_4 - 1260 \nu_2 \mu_3^2 - 3150 \nu_1 \mu_3 \mu_4\]

\[+ 17325 \nu_1 \mu_5^2 + 35 \nu_1 \mu_4 - 35 \nu_1 \mu_4^2 + 1260 \nu_1 \mu_4^3 + 10395 \nu_2 \mu_2^5\]
\[-10395v_{1\mu_2}^6 - 4725v_{2\mu_2}^4 - 560v_{4\mu_2} - 210v_{5\mu_2}^2 + 35v_{2\mu_3}^5 \]
\[+ 21v_{6\mu_2} + 7v_{1\nu_6} - 7v_{1\nu_6}^2 + 840v_{1\nu_2\mu_2}^2 - 63v_{1\nu_2\mu_5} \]
\[+ 1260v_{1\nu_2\mu_4} - 2520v_{1\nu_2\mu_2}^2 + 168v_{1\nu_2\mu_5}^2 - 1890v_{1\nu_2\mu_4} \]
\[+ 280v_{1\nu_3\mu_4} + 140v_{1\nu_3\mu_4}^2 + 2240v_{1\nu_3\mu_2\mu_3} - 11340v_{1\nu_2\mu_3}^2 \]
\[+ 12600v_{1\nu_2\mu_3}^3 - 5040v_{1\nu_3\mu_3}^3 + 14175v_{1\nu_2\mu_2}^4 - 10395v_{1\nu_2\mu_2}^5 \]
\[+ 1050v_{1\nu_4\mu_2}^2 - 175v_{1\nu_4\mu_3} - 126v_{1\nu_5\mu_2} + 21v_{1\nu_5\mu_5} - 105v_{1\nu_3\mu_4} \]
\[+ 1680v_{2\nu_2\mu_3}^2 + 2100v_{2\nu_2\mu_3} - 3780v_{2\mu_2\mu_3}^3 - 350v_{2\nu_3\mu_3} \]
\[-315v_{2\nu_4\mu_2} + 35v_{2\nu_4\mu_2}^2 - 210v_{2\nu_3\mu_2}^2 - 560v_{2\mu_2\mu_3}^2 - 140v_{1\nu_2\mu_3}^2 \]
\[-6300v_{1\nu_2\mu_2}^2 - 9450v_{1\nu_3\mu_2}^3 + 700v_{1\nu_3\mu_3}^2 - 6720v_{1\nu_2\mu_2}^2 \]
\[+ 7560v_{1\nu_2\mu_3}^2 + 2520v_{1\nu_2\mu_2}^2 - 4200v_{1\nu_3\mu_2}^2 \]
\[+ 15120v_{1\nu_2\mu_2}^2 + 42v_{1\nu_4\mu_5}^2 - 42v_{1\nu_5\mu_2} + 420v_{1\nu_4\mu_4} - 840v_{1\nu_2\mu_4} \]
\[+ 630v_{1\nu_2\mu_2}^2 - 210v_{1\nu_2\nu_4} + 1050v_{1\nu_2\mu_3}^2 - 210v_{1\nu_2\nu_3} \]
\[+ 630v_{1\nu_2\mu_2}^2 + 210v_{1\nu_4\mu_4} - 210v_{1\nu_4\mu_4} + 12600v_{1\nu_2\mu_2}^2 \]
\[+ 3360v_{1\nu_2\mu_3}^2 - 7560v_{1\nu_2\mu_2}^2 - 2100v_{1\nu_2\mu_3}^2 - 2520v_{1\nu_3\mu_2}^2 \]
\[ \gamma_{8,8} = \nu_8 - \nu_{18} + 315 \nu_2 v_4^2 - 8 \nu_2 v_7 + 252 \nu_2 v_6^2 + 504 \nu_2 v_5 \nu_3 \nu_5 \]

- \[ 1575 \nu_1 v_2 v_4^2 + 360 \nu_1 v_2 v_7 - 630 \nu_1 v_2 v_6^2 - 2520 \nu_1 v_2 \nu_3 \nu_5 \]

- \[ 84 \nu_1 v_3 v_6 - 2100 \nu_1 v_3 v_4 + 15400 \nu_1 v_3 v_2^2 + 34650 \nu_1 v_3 v_2 v_3 \nu_4 \]

- \[ 28 v_3 v_6^2 + 756 v_3 v_2 v_7 + 1260 v_3 v_2 v_6^2 - 12600 v_3 v_2 v_5 \nu_3 \]

- \[ 9450 v_3 v_2 v_4^2 - 3780 v_2 v_4 v_2 v_5 + 12600 v_2 v_3 \nu_4 + 69300 v_2 v_2 v_2 \nu_3 \]

- \[ 34650 v_2 v_2 v_4 + 69300 v_1 v_2 v_3 v_4 - 138600 v_1 v_2 v_3 v_2 - 51975 \nu_1 v_2 v_4 \]

- \[ 17325 \nu_4 v_2^2 + 840 \nu_4 v_3^2 - 12600 v_4 v_2 \nu_3 + 1260 v_4 \nu_2 v_4 \]

- \[ 56 \nu_4 \nu_3 + 126 \nu_4 v_4 + 135135 v_2 v_6 - 207900 \nu_2 v_6 \nu_3 \]

- \[ 135135 v_1 v_2^2 + 270270 v_1 v_2 v_3 - 2800 v_3 v_2^3 - 62370 v_3 v_2^5 \]

+ \[ 69300 v_3 v_2 v_5 - 70 v_5 v_4 + 1260 v_5 v_2 v_3^2 - 3150 v_5 v_2^3 + 378 v_6 v_2^2 \]
\[-56 \nu_6 \mu_3 - 28 \nu_7 \mu_2 - 315 \nu_1 \mu_4^2 - 69300 \nu_1 \mu_2 \mu_3 - 135135 \nu_1 \mu_2\]

\[+ 12600 \nu_1 \mu_2 \mu_3 \mu_4 - 34650 \nu_1 \mu_2 \mu_4 + 207900 \nu_1 \mu_2 \mu_3 + 8 \nu_1 \nu_1\]

\[-252 \nu_6 \nu_1 \nu_1 \nu_1 - 504 \nu_1 \mu_2 \mu_3 + 3780 \nu_1 \mu_2 \mu_3 + 2800 \nu_1 \mu_3\]

\[+ 84 \nu_1 \nu_1 \nu_1 - 2268 \nu_1 \nu_1 \mu_2 \mu_3 - 3780 \nu_1 \nu_1 \mu_2 \mu_3 + 28350 \nu_1 \nu_1 \mu_2 \mu_3\]

\[+ 37800 \nu_1 \nu_1 \mu_2 \mu_3 - 207900 \nu_1 \nu_1 \mu_2 \mu_3 + 187110 \nu_1 \nu_1 \mu_2\]

\[-56 \nu_6 \nu_1 \mu_2 + 1512 \nu_1 \mu_2 \mu_3 + 2520 \nu_1 \mu_3 \mu_4 - 18900 \nu_1 \mu_2 \mu_4\]

\[-2520 \nu_1 \mu_2 \mu_3 + 138600 \nu_1 \mu_2 \mu_3 - 124740 \nu_1 \mu_2 \mu_3 - 69300 \nu_1 \nu_3 \mu_2\]

\[-3360 \nu_1 \nu_3 \mu_2 + 50400 \nu_1 \nu_3 \mu_2 - 5040 \nu_1 \nu_3 \mu_2\]

\[+ 224 \nu_1 \nu_3 \mu_5 - 51975 \nu_1 \nu_3 \mu_2 - 2520 \nu_1 \nu_3 \mu_2 + 37800 \nu_1 \nu_3 \mu_2\]

\[-3780 \nu_1 \nu_3 \mu_2 + 168 \nu_1 \nu_2 \mu_5 + 207900 \nu_1 \nu_2 \mu_3 + 10080 \nu_1 \nu_2 \mu_3\]

\[-151200 \nu_1 \nu_2 \mu_3 - 103950 \nu_1 \nu_2 \mu_3 - 672 \nu_1 \nu_2 \mu_5 - 5040 \nu_1 \nu_2 \mu_3\]

\[+ 75600 \nu_1 \nu_2 \mu_3 + 15120 \nu_1 \nu_2 \mu_3 - 1680 \nu_1 \mu_4\]

\[-6300 \nu_1 \nu_4 \mu_2 \mu_3 + 4200 \nu_1 \nu_4 \mu_2 \mu_3 - 12600 \nu_1 \nu_3 \mu_2\]

\[-]$
\[-2100v_1v_2^2\mu_4 + 25200v_1^2v_3\mu_2\mu_3 - 7560v_1^4\mu_2\mu_4 + 1344v_1^4\mu_3\]

\[+ 350v_1v_4\mu_4 + 700v_2v_3\mu_4 - 1400v_1^2v_3\mu_4 + 15750v_1v_4\mu_2^3\]

\[+ 37800v_1v_2^2\mu_2\mu_3 - 75600v_1^2v_2\mu_2\mu_3 + 31500v_2v_3\mu_2^3\]

\[+ 30240v_1^5\mu_2\mu_3 - 63000v_1^2v_3\mu_2^3 - 3780v_2^2\mu_2^1 + 11340v_1v_4\mu_2^2\]

\[-94500v_1^2v_2\mu_3 + 189000v_1v_3\mu_2^3 - 75600v_1^5\mu_2^3 - 2268v_1v_5\mu_2^2\]

\[-5670v_2v_4\mu_2^2 + 45360v_1v_2v_3\mu_2^2 - 45360v_1^3v_3\mu_2\]

\[+ 11340v_2^3\mu_2 - 102260v_1v_2^2\mu_2 + 136080v_1^4v_2^2 + 45360v_1^6\mu_2^2\]

\[+ 336v_1v_5\mu_3 + 840v_2v_4\mu_3 - 1680v_1^2v_4\mu_3 + 196v_1v_6\mu_2\]

\[+ 6720v_1^6\mu_3 + 84v_2v_5\mu_2 + 560v_3^2\mu_2 - 6720v_1v_2v_3\mu_3\]

\[-168v_1^2v_3\mu_2 + 6720v_1^3v_3\mu_3 - 1680v_2^3\mu_3 + 980v_3v_4\mu_2\]

\[-5880v_1v_2v_4\mu_2 + 5880v_1^3v_4\mu_2 + 15120v_1v_2\mu_3 - 20160v_1v_4\mu_3\]

\[-3920v_1v_3\mu_2 - 630v_2^4 - 6720v_1^5v_3 - 5880v_2^3v_3\mu_2 + 13440v_1^2v_3\]

\[+ 35280v_1^2v_3\mu_2 - 5040v_1^2v_3\mu_2 - 1680v_1v_3\mu_2 + 560v_2v_3^2\]

\[-23520v_1v_3\mu_2 + 17640v_1v_2\mu_2 + 1680v_1^4v_4 - 2520v_1v_2v_4\]

\[+ 420v_2v_4 + 560v_1v_3v_4 - 35v_4^2 - 70560v_1^2\mu_2 + 70560v_5v_2\mu_2\]
1.5 Check on the Formal Calculations.

In the work so far, apart from the assumption of finiteness of the appropriate moments, it has been implicitly assumed that the distribution function $F(x)$ is such that for some $k$ its convolution $F_k(x)$ has an absolutely continuous component.

Let $G_n$ be the class of functions $F(x)$, which can be written as

$$1 - \frac{1}{n} \left[ -\frac{x}{\nu_1} + -\frac{x}{\nu_2} + \ldots + -\frac{x}{\nu_n} \right], x \geq 0$$

and the $\nu_i$'s are real and positive and let

$$G = \bigcup_{n=1}^{\infty} G_n.$$  

Smith[20] has shown that if $F(x) \in G$, then

(i) $F^*(s)$ is analytic in some open neighborhood of $s=0$, and

(ii) there is a unique function $z_1(s)$ defined on this neighborhood, vanishing at $s=0$ and satisfying the relation

(1.5.1) \[ s - 1 + F^*(z_1(s)), \text{ and further} \]

(1.5.2) \[ z_1(s) = \sum_{n=1}^{\infty} \frac{\alpha_n}{n!} s^n, \]

where \[ \alpha_n = \gamma_n \text{ of (1.3.22)}. \]

We will now prove that if $F(x) \in G$, $\gamma_{n,n}$ can be expressed in terms of $\gamma_n(=\alpha_n)$ and the cumulants of $K(x)$, and this relationship
provides a check on the calculations of the previous section. We prove

**Theorem 1.5.1**

If $F(x) \in G$, then

$$
\sum_{n=1}^{\infty} \frac{k_n}{n!} (-1)^n \left[ \sum_{m=1}^{\infty} \frac{a_m}{m!} s^m \right]^n = \sum_{n=1}^{\infty} \frac{a_n}{n!} s^n .
$$

**Proof:**

We have

$$K^*(s) = \sum_{n=0}^{\infty} \lambda_n \left[ Q(s) \right]^n = \Lambda(Q(s)),$$

where

$$\Lambda(z) = 1 + \lambda_1 z + \lambda_2 z^2 + \ldots .$$

Also

$$\log \Lambda(z) = \sum_{n=1}^{\infty} \frac{a_n}{n!} z^n .$$

therefore

$$\log \Lambda(Q) = \sum_{n=1}^{\infty} a_n \left[ Q(s) \right]^n .$$

but

$$\log K^*(s) = \sum_{n=1}^{\infty} (-1)^n \frac{k_n}{n!} s^n ,$$

where $\lambda_n$ is the $n$th cumulant of $K(x)$

$$\therefore \sum_{n=1}^{\infty} (-1)^n \frac{k_n}{n!} s^n = \sum_{n=1}^{\infty} \frac{a_n}{n!} \left[ Q(s) \right]^n$$

$$= \sum_{n=1}^{\infty} \frac{a_n}{n!} \left[ 1 - F^*(s) \right]^n$$

**Putting $s = z_1(s)$ in the above and making use of (1.5.1) and**

**1.5.2 we get**
\[ (1.5.3) \quad \sum_{n=1}^{\infty} \frac{\kappa_n}{n!} (-1)^n \left( \sum_{m=1}^{\infty} \frac{\gamma_m}{m!} s^m \right)^n = \sum_{n=1}^{\infty} \frac{a_n}{n!} s^n. \]

Comparing the coefficients of the first eight powers of \( s \) on either side of (1.5.3) we get

\[
\begin{align*}
a_1 &= -k_1 \\
a_2 &= -k_1 \alpha_2 + k_2 \\
a_3 &= -k_1 \alpha_3 + 3 \alpha_2 k_2 - k_3 \\
a_4 &= -k_1 \alpha_4 + k_2 (3 \alpha_2^2 + 4 \alpha_3) - 6 \alpha_2 k_3 + k_4 \\
a_5 &= -k_1 \alpha_5 + k_2 (5 \alpha_4 + 10 \alpha_2 \alpha_3) - k_3 (15 \alpha_2^2 + 10 \alpha_3) \\
&\quad + 10 k_4 \alpha_2 - k_5 \\
a_6 &= -k_1 \alpha_6 + k_2 (10 \alpha_2^2 + 6 \alpha_5 + 15 \alpha_2 \alpha_4) \\
&\quad - k_3 (15 \alpha_4 + 60 \alpha_2 \alpha_3 + 15 \alpha_2^3) + k_4 (45 \alpha_2^2 + 20 \alpha_3) \\
&\quad - 15 k_5 \alpha_2 + k_6 \\
a_7 &= -k_1 \alpha_7 + k_2 (7 \alpha_6 + 21 \alpha_2 \alpha_5 + 35 \alpha_3 \alpha_4) \\
&\quad - k_3 (70 \alpha_2^2 + 21 \alpha_5 + 105 \alpha_2 \alpha_4 + 105 \alpha_3 \alpha_2^2) \\
&\quad + k_4 (35 \alpha_4 + 105 \alpha_2^3 + 210 \alpha_2 \alpha_3) \\
&\quad - k_5 (105 \alpha_2^2 + 35 \alpha_3) + 21 k_6 \alpha_2 - k_7
\end{align*}
\]
\[ a_8 = - \kappa_1 \alpha_8 + \kappa_2 (35 \alpha_4^2 + 8 \alpha_7 + 28 \alpha_2 \alpha_6 + 56 \alpha_3 \alpha_5) \]

\[- \kappa_3 (28 \alpha_6 + 168 \alpha_2 \alpha_5 + 280 \alpha_3 \alpha_4 + 280 \alpha_3^2 \alpha_2 \alpha_4 + 210 \alpha_2^2 \alpha_4) \]

\[+ \kappa_4 (105 \alpha_4^4 + 280 \alpha_2 \alpha_2 \alpha_5^2 + 540 \alpha_2 \alpha_5^2 \alpha_4 + 420 \alpha_2 \alpha_4 \alpha_5) \]

\[+ 56 \alpha_5 \] \[+ \kappa_5 (70 \alpha_4^5 + 560 \alpha_2 \alpha_5^3 + 420 \alpha_5^3 \alpha_4) + \kappa_6 (210 \alpha_2^2 \alpha_5^2 + 56 \alpha_5^3) \]

\[- \kappa_7 (28 \alpha_2) + \kappa_8. \]

Now \[ \gamma_{n,n} = a_n + (n-1)! \cdot . \]

Smith \[ 26 \] has tabulated the first eight \( \alpha^i \)'s and Kendall \[ 15 \] has tabulated the cumulants. Using these two tables all the individual terms in \( \gamma_{n,n} \) tabulated in the previous section for \( n=1,2,\ldots,8 \) are checked and they all passed the check. Another obvious check is that \( \gamma_{n,n} \equiv o \), when \( F(x) = K(x) \), which is verified in the case of the \( \gamma_{n,n} \).


Consider a Renewal process which has been in progress for a long time. Fix a point \( t \) on the time axis (it is not a matter of necessity that the event \( t \) of the Renewal process happen at the arbitrarily chosen point \( t \) on the time axis). Let us denote
by Y(t) the time that elapses from t on to the next succeeding event of the Renewal process. Let K_t(x) denote the distribution function of Y(t). What we now observe from the instant t onwards is a general Renewal process whose "residual life time" distribution is K_t(x); the intervals between events being independently identically distributed random variables with the same distribution function F(x). Cox and Smith [6] have shown that

\[ \lim_{t \to \infty} K_t(x) = \frac{1}{\mu_1} \int_0^x \left[ 1 - F(y) \right] dy = K(x), \text{ say} \]

In other words, if a Renewal process with distribution function F(x) of the intervals between successive events, has been in progress a long time since, and if observations are made starting from an arbitrary origin t, then what we are actually observing is a General Renewal Process whose "Residual life time" distribution is given by K(x) of (1.6.1). This particular G.R.P. is called the "Equilibrium process". We can now write down the cumulants of the Equilibrium process as a particular case of the General Renewal Process. We put small "e" on \( \gamma_{n,n} \) to stand for the equilibrium process.

We have from Section 1.5 that

\[ (1.6.2) \quad \gamma^e_{n,n} = s^e_n + (n-1)! , \]

where \( s^e_n \) are given by (1.5.4) with the modification that \( \kappa \)'s should be replaced by \( \kappa^e \). We will tabulate the first eight \( \kappa^e \)'s of \( K(x) \) given by (1.6.1) assuming \( \mu_1 = 1 \).
\[
\kappa^e_1 = \frac{\mu_2}{2} - \frac{\mu_2}{4}
\]
\[
\kappa^e_2 = \frac{\mu_4}{3} - \frac{\mu_2}{4}
\]
\[
\kappa^e_3 = \frac{\mu_4}{4} - \frac{\mu_2}{2} + \frac{\mu_2}{3}
\]
\[
\kappa^e_4 = \frac{\mu_5}{6} - \frac{\mu_2}{2} + \frac{\mu_2}{2} + \frac{\mu_2}{2} + \frac{3\mu_2}{4}
\]
\[
\kappa^e_5 = \frac{\mu_6}{6} - \frac{\mu_2}{2} + \frac{\mu_2}{2} + \frac{\mu_2}{2} + \frac{3\mu_2}{4} + \frac{3\mu_2}{8}
\]
\[
(1.6.3) \quad \kappa^e_6 = \frac{1\mu_7}{7} - \frac{1\mu_2}{2} + \frac{\mu_2}{2} + \frac{3\mu_2}{2} + \frac{3\mu_2}{2} + \frac{5\mu_2}{2} + \frac{5\mu_2}{8} + \frac{5\mu_2}{2}
\]
\[
- \frac{15\mu_2}{4} + \frac{10\mu_3}{9} - \frac{15\mu_2}{2} + \frac{15\mu_2}{2} - \frac{15\mu_2}{2} + \frac{15\mu_2}{2} + \frac{15\mu_2}{2} - \frac{15\mu_2}{2}
\]
\[
\kappa^e_7 = \frac{1\mu_8}{8} - \frac{1\mu_2}{2} + \frac{7\mu_2}{2} - \frac{7\mu_2}{2} - \frac{7\mu_2}{2} + \frac{7\mu_2}{2} + \frac{7\mu_2}{2}
\]
\[
- \frac{21\mu_2}{4} + \frac{3\mu_2}{2} + \frac{3\mu_2}{2} + \frac{3\mu_2}{2} + \frac{10\mu_2}{2} + \frac{10\mu_2}{2} + \frac{10\mu_2}{2}
\]
\[
- \frac{3\mu_2}{2} + \frac{3\mu_2}{2} - \frac{10\mu_2}{4} + \frac{10\mu_2}{4} + \frac{10\mu_2}{4}
\]
\[
+ \frac{3\mu_2}{2} - \frac{3\mu_2}{2} + \frac{3\mu_2}{2} + \frac{3\mu_2}{2} + \frac{3\mu_2}{2} + \frac{3\mu_2}{2}
\]
\[
\kappa^e_8 = \frac{1\mu_9}{9} - \frac{1\mu_2}{2} + \frac{\mu_2}{2} + \frac{\mu_2}{2} + \frac{2\mu_2}{2} + \frac{2\mu_2}{2} - \frac{7\mu_2}{3} + \frac{7\mu_2}{3} + \frac{2\mu_2}{3}
\]
\[
- \frac{7\mu_2}{3} + \frac{7\mu_2}{3} + \frac{7\mu_2}{3} + \frac{7\mu_2}{3} + \frac{7\mu_2}{3} + \frac{7\mu_2}{3}
\]
\[
+ \frac{21\mu_2}{4} + \frac{3\mu_2}{3} + \frac{2\mu_2}{3} + \frac{2\mu_2}{3} + \frac{2\mu_2}{3} + \frac{2\mu_2}{3}
\]
\[
+ \frac{21\mu_2}{3} + \frac{3\mu_2}{4} + \frac{3\mu_2}{4} + \frac{3\mu_2}{4} + \frac{3\mu_2}{4} + \frac{3\mu_2}{4}
\]
\[- \frac{105\mu_2^5}{2} \mu_3 - \frac{70\mu_3^4}{9} + \frac{280\mu_2^2\mu_3^3}{3} - 175\mu_2^4\mu_3 + 105\mu_2^6\mu_3 \]

\[- \frac{312}{16} \mu_2^8 \]

Substituting (1.6.3) in (1.5.4) and (1.5.4) in (1.6.2) we obtain for the first four \(\gamma^{e}_{n,n}\) the following

\[\gamma^{e}_{11} = 1 - \frac{\mu_2}{2} \]

\[(1.6.4) \quad \gamma^{e}_{22} = \frac{1}{3} \mu_3 - \frac{3\mu_2}{4} + 1 \]

\[\gamma^{e}_{33} = 2\mu_2\mu_3 - \frac{\mu_4}{4} - \frac{5}{2}\mu_2^3 + 2 \]

\[\gamma^{e}_{44} = \frac{1}{5} \mu_5 - \frac{5}{2}\mu_2\mu_4 + 15\mu_2^2\mu_3 - \frac{105}{8}\mu_2^4 - \frac{5}{3}\mu_3^2 + 6. \]
PART II
ESTIMATION

2.0. Summary

Empirical investigations on the estimation of the variance-time curve of a Renewal process were carried out earlier by Cox and Smith [67]. In this part, a statistic is proposed as an estimate of the variance-time curve at a given point on the time-axis, based on observations in a given interval of a Renewal process. "Equilibrium" is assumed. Some sampling properties are derived when the basic renewal process is arbitrary. Under the Null Hypothesis that the underlying renewal process is purely random (Poisson Process), the consistency and asymptotic normality of the statistic are established; the latter result incidentally provides a test of significance for randomness of the underlying renewal process. Some difficulties involved in certain non-null distributions of the statistic are discussed.

2.1. The Statistic and Properties

The variance-time curve is the graph of the variance of $N_t$ plotted against $t$, the time parameter. Neglecting the remainder which tends to zero as $t \to \infty$, the explicit expression for the variance-time curve in terms of the moments of the underlying renewal process is given, assuming equilibrium, by
(2.1.1) \[ V(t) = \text{Var}(N_t) = \alpha t + \beta', \] where
\[
\alpha = \frac{\mu_2 - \mu_1^2}{\mu_1^3},
\]
(2.1.2) and \[ \beta' = \frac{3\mu_2^2 - 2\mu_1\mu_3}{6\mu_1}. \]

Suppose observations on the process are made in the interval \((o,T)\). We propose as estimate of \( V(\tau), 0 < \tau < T \), the statistic \( \hat{\tau} \), defined thus:

(2.1.3) \[ \hat{\tau} = \frac{1}{T-\tau} \int_0^{T-\tau} \left( N(t+\tau,t) \right)^2 dt - \frac{\tau^2}{T^2} \left( N(o,T) \right)^2, \]

where \( N(t+\tau,t) \) denotes the number of events in the interval \((t,t+\tau)\) and \( N(o,T) \), the number of events in \((o,T)\).

Let \( N_t^e \) denote the number of events of an Equilibrium process in \((o,t)\). Taking expectations on both sides of (2.1.3) we obtain

\[
E(\hat{\tau}) = \frac{1}{T-\tau} \int_0^{T-\tau} E\left( N(t+\tau,t) \right)^2 dt - \frac{\tau^2}{T^2} E\left( N(o,T) \right)^2
\]

\[
= \frac{1}{T-\tau} \int_0^{T-\tau} E\left( N_t^e \right)^2 dt - \frac{\tau^2}{T^2} E\left( N_T^e \right)^2
\]

\[
= E\left( N_t^e \right)^2 - \frac{\tau^2}{T^2} E\left( N_T^e \right)^2
\]

\[
= E\left( N_t^e \right)^2 - \frac{\tau^2}{T^2} \left[ \alpha T + \beta' + \alpha' T^2 + o(1) \right],
\]
as \( T \to \infty \), where \( \alpha' = \frac{1}{\mu_1} \). Thus

\[
E(\hat{V}(\tau)) = E(N^e_T) - \tau^2 \alpha'^2 + o(1), \text{ as } T \to \infty
\]

\[(2.1.4) \quad = E N^e_T - E^2(N^e_T) + E^2(N^e_T) - \tau^2/\mu_1^2 + o(1).
\]

But we have from page 23 of Part I, that \( E\left(N_t\right) = \phi'(t) \), and

\[
\phi'_1(s) = k^* \phi'_1 = \frac{k^*}{1-F^*}.
\]

However, assuming equilibrium,

\[
k^* = \frac{1-F^*}{\mu_1}\]

and so,

\[
\phi'_1(s) = \frac{1}{\mu_1 s}.
\]

Hence

\[(2.1.5) \quad E\left(N^e_t\right) = \frac{t}{\mu_1}, \text{ exactly,}
\]

and thus

\[
E^2\left(N^e_T\right) - \frac{\tau^2}{\mu_1^2} \equiv o.
\]

We have, therefore, that

\[(2.1.6) \quad E\left(\hat{V}(\tau)\right) = V(\tau) + o(1), \text{ as } T \to \infty.
\]

In other words, \( \hat{V}(\tau) \) is an asymptotically unbiased estimate of \( V(\tau) \), for any arbitrary underlying renewal process.

In the particular case when the underlying renewal process is Poisson with parameter \( \lambda \), we obtain
\[ E \left( \hat{V}(\tau) \right) = E \left( N_T^e \right)^2 - \frac{\tau^2}{T^2} E(N_T^e)^2 \]

\[ = \lambda T + \lambda^2 T^2 - \frac{\tau^2}{T^2} \left( \lambda T + \lambda^2 T^2 \right) \]

\[ = \lambda T + \lambda^2 T^2 - \frac{\tau^2 \lambda}{T} - \lambda^2 T^2 \]

(2.1.7) \[ = \lambda T \left( 1 - \frac{\tau}{T} \right) \]

It is evident from (2.1.7) that \[ \frac{T}{T-\tau} \hat{V}(\tau) \] is an exactly unbiased estimate of \[ V(\tau) \] in this case and this estimator is also asymptotically unbiased in the general case.

Evidently,

(2.1.8) \[ \text{Cov} \left( \hat{V}(\tau), \hat{V}(\tau+h) \right) = E \left( \hat{V}(\tau) - E \hat{V}(\tau) \right) \left( \hat{V}(\tau+h) - E \hat{V}(\tau+h) \right) \]

for arbitrary \( h \).

However,

\[ \hat{V}(\tau) - E \hat{V}(\tau) = \frac{1}{T-\tau} \int_0^{T-\tau} \left( N^2(t+\tau,t) - E N^2(t+\tau,t) \right) dt \]

\[ - \frac{\tau^2}{T^2} \left( N^2(0,T) - E N^2(0,T) \right) \]

and if we write

\[ \Delta N^2(t+\tau,t) = N^2(t+\tau,t) - E N^2(t+\tau,t), \]

\[ \Delta N^2(T) = N^2(0,T) - E N^2(0,T), \]

then we have that

(2.1.9) \[ \left( \hat{V}(\tau) - E \hat{V}(\tau) \right) \left( \hat{V}(\tau+h) - E \hat{V}(\tau+h) \right) \]
\[ \frac{1}{T-\tau} \left[ \int_0^{T-\tau} \Delta N^2(t+\tau,t) \, dt - \frac{\tau^2}{T^2} \Delta N^2(T) \right] \]

\[ \times \left[ \int_0^{T-\tau-h} \Delta N^2(t+\tau+h,t) \, dt - \frac{(\tau+h)^2 \Delta N^2(T)}{T^2} \right] \]

\[ = I_1 - I_2 - I_3 + I_4, \text{ say} \]

where

\[ I_1 = \frac{1}{(T-\tau)(T-\tau-h)} \int_{t=0}^{t=T-\tau} \int_{t'=0}^{t'=T-\tau-h} \Delta N^2(t+\tau,t) \Delta N^2(t+\tau+h,t') \, dt \, dt' \]

\[ I_2 = \frac{t^2}{T^2(T-\tau-h)} \int_0^{T-\tau-h} \Delta N^2(T) \Delta N^2(t+\tau+h,t) \, dt, \]

\[ (2.1.10) \]

\[ I_3 = \frac{(\tau+h)^2}{T^2(T-\tau)} \int_0^{T-\tau} \Delta N^2(T) \Delta N^2(t+\tau,t) \, dt, \]

\[ I_4 = \frac{\tau^2(\tau+h)^2}{T^4} \left( \Delta N^2(T) \right)^2 \]

Upon taking expectations on both sides of \((2.1.9)\), we discover that

\[ (2.1.11) \quad \text{Cov} \left( \hat{V}(\tau), \hat{V}(\tau+h) \right) = E(I_1) - E(I_2) - E(I_3) + E(I_4). \]
We will evaluate $E(I_2)$, $E(I_3)$ and $E(I_4)$ first, and then obtain
the more complicated $E(I_1)$.

Clearly,

$$E(I_2) = \frac{\tau^2}{T^2(T-\tau-h)} \int_0^{T-\tau-h} \Delta N^2(T) \Delta N^2(t+\tau+h,t) \, dt$$

$$(2.1.12) \quad = \frac{\tau^2}{T^2(T-\tau-h)} \int_0^{T-\tau-h} \text{Cov} \left( N^2(T), N^2(t+\tau+h,t) \right) \, dt.$$

Consider the diagram below:

```
\begin{tikzpicture}
  \node (I1) at (0,0) {$I_1$};
  \node (I2) at (2,0) {$I_2$};
  \node (I3) at (4,0) {$I_3$};
  \node (O) at (0,-0.5) {$O$};
  \node (T) at (4,-0.5) {$T$};
  \draw (O) -- (I1) node [midway, above] {$o, t$};
  \draw (I1) -- (I2) node [midway, above] {$t, t+\tau+h$};
  \draw (I2) -- (I3) node [midway, above] {$t+\tau+h, T$};
\end{tikzpicture}
```

In this figure:

$$I_1 = (o,t]$$

$$I_2 = (t,t+\tau+h]$$

and

$$I_3 = (t+\tau+h,T]$$

Let $N_I$ stand for the observed number of events in the interval $I$

Then

$$(2.1.13) \quad \text{Cov} \left( N^2(T), N^2(t+\tau+h,t) \right)$$

$$= \text{Cov} \left( \frac{N_{I_2}^2}{N_{I_2}}, \frac{(N_{I_1} + N_{I_2} + N_{I_3})^2}{N_{I_2}} \right)$$
\[
= \text{cov} \left( N_{I_2}^2, N_{I_1}^2 \right) + \text{Var} \left( N_{I_2}^2 \right) + \text{cov} \left( N_{I_2}^2, N_{I_3}^2 \right) \\
+ 2 \text{cov} \left( N_{I_2}^2, N_{I_1} N_{I_2} \right) + 2 \text{cov} \left( N_{I_2}^2, N_{I_2} N_{I_3} \right) \\
+ 2 \text{cov} \left( N_{I_2}^2, N_{I_2} N_{I_3} \right).
\]

So far we do not know how to evaluate expressions like \( \text{E} \left( N_{I_1}^2 N_{I_2}^2 \right) \) for an arbitrary underlying R.P. However, when the underlying renewal process is purely random (Poisson), since events in disjoint intervals are independent, we can evaluate the terms of the R.H.S. of (2.1.13).

Thus, when the underlying process is Poisson, (2.1.13) reduces to

\[
(2.1.14) \quad \text{cov} \left( N^2(T), N^2(t+\tau+h,t) \right) = \text{Var} \left( N_{I_2}^2 \right) = \text{Var} \left( N_{\tau+h}^2 \right),
\]

and so

\[
E(I_2) = \frac{\tau^2}{T^2(T-\tau-h)} \text{Var} \left( N_{\tau+h}^2 \right),
\]

\[
(2.1.15) \quad = \frac{\tau^2}{T^2} \text{Var} \left( N_{\tau+h}^2 \right).
\]

Similarly, when the underlying process is Poisson, we find

\[
(2.1.16) \quad E(I_3) = \frac{(\tau+h)^2}{T^2} \text{Var} \left( N_{\tau}^2 \right),
\]

\[
(2.1.17) \quad E(I_4) = \frac{\tau^2(\tau+h)^2}{T^4} \text{Var} \left( N_{T}^2 \right).
\]
On putting $\lambda t = \mu$, we have, for a Poisson process,

$$ E(N_t) = \mu $$

(2.1.18) $$ E(N_t^2) = \mu + \mu^2 $$

$$ E(N_t^3) = \mu + 3\mu^2 + \mu^3 $$

$$ E(N_t^4) = \mu + 7\mu^2 + 6\mu^3 + \mu^4 $$

Thus

$$ \text{Var}(N_T^2) = E(N_T^4) - E^2(N_T^2) $$

$$ = \mu + 6\mu^2 + 4\mu^3, $$

$$ = \lambda T + 6\lambda^2 T^2 + 4\lambda^3 T^3. $$

From this it follows that

(2.1.19) $$ E(I_h) = \frac{4\lambda^3 \tau^2 (\tau+h)^2}{T} + \frac{6\lambda^2 \tau^2 (\tau+h)^2}{T^2} + \frac{\lambda \tau^2 (\tau+h)^2}{T^3}. $$

Before we calculate the more complicated expression

$$ \text{cov}(\hat{V}(\tau), \hat{V}(\tau+h)) $$

we will obtain $\text{Var}(\hat{V}(\tau))$. The only quantity still to be calculated for obtaining $\text{Var}(\hat{V}(\tau))$, is

$$ E(I_1), \text{ when } h = 0. $$

We have

(2.1.20) $$ E(I_1) = \frac{1}{(T-\tau)^2} \int_0^{T-\tau} \int_0^{T-\tau} \text{cov}(N^2(t+\tau,t), N^2(t'+\tau,t')) \, dt \, dt'. $$

Case I: $t > t'$

a) $t > t' + \tau$
In this case we evidently have
\[
\text{cov} \left( N^2(t, t), N^2(t', t') \right) = 0.
\]

b) \( t' < t < t'+\tau \)

Let
\[
I_1 = (t', t] \quad \quad I_2 = (t, t'+\tau], \quad I_3 = (t'+\tau, t+\tau].
\]

Then
\[
N(t+\tau t) = N_{I_2}^2 + N_{I_3}^2,
\]
\[
N(t'+\tau t) = N_{I_1}^2 + N_{I_2}^2.
\]

Therefore,
\[
\text{cov} \left( N^2(t, t), N^2(t', t') \right)
= \text{cov} \left( N_{I_2}^2 + N_{I_3}^2 + 2N_{I_2}N_{I_3}, N_{I_1}^2 + N_{I_2}^2 + 2N_{I_1}N_{I_2} \right)
= \text{Var} \left( N_{I_2}^2 \right),
\]
when the underlying process is Poisson, and therefore
\[
= \text{Var} \left( N_{t'-t+\tau}^2 \right).
\]
Case II: \( t' > t \).

a) \( t' > t + \tau \)

In this case
\[
\text{cov}\left( N^2(t + \tau, t), N^2(t' + \tau, t') \right) = 0.
\]

b) \( t < t' < t + \tau \).

In this case one obtains, as before,
\[
\text{cov}\left( N^2(t + \tau, t), N^2(t' + \tau, t') \right) = \text{Var}\left( N^2_{t-t'+\tau} \right).
\]

The Domain of Integration.
Now

\[
(2.1.21) \quad \int_0^T \int_0^T \text{cov} \left( N^2(t+\tau, t), N^2(t'+\tau, t') \right) \, dt \, dt'
\]

\[
= \int_{\mathcal{D}_1} \text{Var} \left( N^2_{t-t+\tau} \right) \, dt \, dt' + \int_{\mathcal{D}_2} \text{Var} \left( N^2_{t-t'+\tau} \right) \, dt \, dt'.
\]

By symmetry, we discover that

\[
(2.1.22) \quad \int_0^T \int_0^T \text{cov} \left( N^2(t+\tau, t), N^2(t'+\tau, t') \right) \, dt \, dt'
\]

\[
= 2 \int_{t'=0}^{t'=T-2\tau} \int_{t=t''}^{t=t'+\tau} \text{Var} \left( N^2_{t'-t+\tau} \right) \, dt \, dt' + 2 \int_{t'=T-\tau}^{t'=T-2\tau} \int_{t=t}^{t=t'+\tau} \text{Var} \left( N^2_{t'-t+\tau} \right) \, dt \, dt'.
\]

If we write

\[
\phi(t, t') = \text{Var} \left( N^2_{t'-t+\tau} \right),
\]

then

\[
\phi(t, t') = \lambda(t'-t+\tau) + 6\lambda^2(t'-t+\tau)^2 + 4\lambda^3(t'-t+\tau)^3,
\]

\[
= A + Bt' + Ct' + Dt'^3 - E + F + Ft^2 - Dt^3
\]

\[
+ Et't - Ftt't^2 + Ft^2t',
\]

where
\[ A = \lambda t + 6\lambda^2 t^2 + 4\lambda^3 t^3, \]
\[ B = \lambda + 12\lambda^2 t + 12\lambda^3 t^2, \]
\[ C = 6\lambda^2 + 12\lambda^3 t, \]
\[ D = 4\lambda^3, \]
\[ E = -(12\lambda^2 + 24\lambda^3 t), \]
\[ F = 12\lambda^3. \]

On integrating \( \phi(t, t') \) with respect to \( t \) and denoting the result by \( \Phi(t, t') \), we obtain
\[
\Phi(t, t') = \left[ A + B t' + C t' t^2 + D t' t^3 \right] t - \frac{B t^2}{2} + \frac{C t^3}{3} - \frac{D t^4}{4} + \frac{E}{2} t' t^2 - \frac{F}{2} t' t^2 t^2 + \frac{F}{3} t' t^3.
\]

Therefore,
\[
\Phi(t, t') = \left. \left( A t - \frac{B t^2}{2} + \frac{C t^3}{3} - \frac{D t^4}{4} \right) + \left( C t^2 - D t^3 + \frac{E t^2}{2} + \frac{F t^3}{3} \right) t' + \left( 2C t - \frac{3D t^2}{2} + E t + \frac{F t^2}{2} \right) t'^2 \right|_{t=t'}
= \frac{\lambda t^2}{2} + 2\lambda^2 t^3 + \lambda^3 t^4 = \frac{\lambda t^2}{2} \text{ say,}
\]

after substituting for \( A, B, C, D, E, F \) from \((2.1.24)\). Hence
\[(2.1.25)\]

\[
\begin{align*}
2 & \int_{t'=0}^{t'=T-2\tau} \int_{t=t'}^{t'=t'+\tau} \text{Var} \left( \xi_{t', t+\tau} \right) \, dt \, dt' \\
& = 2 \int_{t'=0}^{t'=T-2\tau} \mathcal{C} \, dt' \\
& = 2 \mathcal{C} (T - 2\tau) \\
& = (T - 2\tau) \left( \lambda^2 \tau^2 + 4 \lambda^2 \tau^3 + 2 \lambda^2 \tau^4 \right).
\end{align*}
\]

Now

\[
\phi(t, t') \begin{cases} 
\phi(t, t') & t = T - \tau \\
\phi(t, t') & t = t' 
\end{cases}
= \left[ A + Bt' + Ct^2 + Dt^3 \right] (T - \tau) - \frac{B}{2} (T - \tau)^2 \\
+ \frac{C}{3} (T - \tau)^3 - \frac{D}{4} (T - \tau)^4 + \frac{E}{2} t'(T - \tau)^2 - \frac{F}{2} t'^2 (T - \tau)^2 \\
+ \frac{F}{3} t'(T - \tau)^3 - At' - Bt'^2 - Ct'^3 - Dt'^4 + \frac{Bt'^2}{2} - Ct'^3 \\
+ \frac{D}{4} t'^4 - \frac{E}{2} t'^3 + \frac{F}{2} t'^4 - \frac{F}{3} t'^4 \\
= \left[ A(T - \tau) - \frac{B(T - \tau)^2}{2} + \frac{C(T - \tau)^3}{3} - \frac{D(T - \tau)^4}{4} \right] \\
+ t' \left[ B(T - \tau) + \frac{E}{2} (T - \tau)^2 + \frac{F}{3} (T - \tau)^3 - A \right] \\
+ t'^2 \left[ C(T - \tau) - \frac{F}{2} (T - \tau)^2 - \frac{B}{2} \right] + t'^3 \left[ D(T - \tau) - \frac{4C}{3} - \frac{E}{2} \right].
\]
\[ + t^{4} \left[ \frac{F}{6} - \frac{3D}{4} \right] . \]

Hence

\[
\begin{aligned}
(2.1.26) & \quad \int_{t'=T-2\tau}^{T-\tau} \int_{t'=T-2\tau}^{T-\tau} \text{Var} \left( N_{t'-t+\tau} \right) \, dt \, dt' \\
& = \left[ A (T-\tau) - \frac{B}{2} (T-\tau)^2 + \frac{C}{3} (T-\tau)^3 - \frac{D}{4} (T-\tau)^4 \right] \tau \\
& + \frac{1}{2} \left[ B (T-\tau) + \frac{E}{2} (T-\tau)^2 + F (T-\tau)^3 - A \right] \left[ (T-\tau)^2 - (T-2\tau)^2 \right] \\
& + \frac{1}{3} \left[ C (T-\tau) - \frac{F}{2} (T-\tau)^2 - B \right] \left[ (T-\tau)^3 - (T-2\tau)^3 \right] \\
& + \frac{1}{4} \left[ D (T-\tau) - \frac{4C-E}{3} \right] \left[ (T-\tau)^4 - (T-2\tau)^4 \right] \\
& + \frac{1}{5} \left[ \frac{F}{6} - \frac{3D}{4} \right] \left[ (T-\tau)^5 - (T-2\tau)^5 \right] .
\end{aligned}
\]

After a straightforward and laborious simplification the right hand side of (2.1.26) reduces to

\[ \frac{\lambda \tau^3}{3} + \frac{3 \lambda^2 \tau^4}{2} + \frac{4 \lambda^3 \tau^5}{5} . \]

Hence

\[
(2.1.27) \quad 2 \int_{T-2\tau}^{T-\tau} \left. \int_{t'=T-2\tau}^{T-\tau} \text{Var} \left( N_{t'-t+\tau} \right) \, dt \, dt' \right. \\
& = \frac{2 \lambda \tau^3}{3} + \frac{3 \lambda^2 \tau^4}{2} + \frac{8 \lambda^3 \tau^5}{5} ,
\]
and on substituting (2.1.27) and (2.1.25) in (2.1.22), we obtain

\[
E(I_1) = \frac{1}{(T-\tau)^2} \int_0^{T-\tau} \int_0^{T-\tau} \text{cov} \left( N^2(t+\tau, t), N^2(t'+\tau, t') \right) dt \, dt'
\]

\[
= \frac{(T-2\tau)}{(T-\tau)^2} \left( \lambda T^2 + 4 \lambda^2 T^3 + 2 \lambda^3 T^4 \right)
\]

\[
+ \frac{1}{(T-\tau)^2} \left( \frac{2}{3} \lambda T^3 + 3 \lambda^2 T^4 + 8 \lambda^3 T^5 \right)
\]

\[
= \frac{\lambda T^2 + 4 \lambda^2 T^3 + 2 \lambda^3 T^4}{T-\tau} - \frac{\lambda T^3 + \lambda^2 T^4 + 2 \lambda^3 T^5}{(T-\tau)^2}
\]

(2.1.28) \quad = \frac{\lambda T^2 + 4 \lambda^2 T^3 + 2 \lambda^3 T^4}{T} + o\left(\frac{1}{T}\right).

Upon combining (2.1.15), (2.1.16), (2.1.19) and (2.1.28), we obtain

(2.1.29) \quad \text{Var}\left( \hat{V}(\tau) \right) = \frac{\lambda T^2 + 4 \lambda^2 T^3 + 2 \lambda^3 T^4}{T} + \frac{4}{T} \frac{\lambda^3 T^4}{T} + o\left(\frac{1}{T}\right)

\[
= \frac{\lambda T^2 + 4 \lambda^2 T^3 + 6 \lambda^3 T^4}{T} + o\left(\frac{1}{T}\right), \quad \text{as}
\]

\[ T \to + \infty. \]

From (2.1.29) it is evident that \( \text{Var}\left( \hat{V}(\tau) \right) \) tends to zero as \( T \) tends to \( \infty \), and combining this result with the asymptotic unbiasedness of \( \hat{V}(\tau) \), we have the following.
Theorem 2.1

When the underlying renewal process is Poisson $\hat{V}(\tau)$ is a consistent estimator of $V(\tau)$.

We shall now obtain the correlogram of the process $\hat{V}(\tau)$, namely, $\text{cov}(\hat{V}(\tau), \hat{V}(\tau+h))$. Our main problem hinges upon evaluating

\[(2.1.30) \quad E(I_1) = \frac{1}{(T-\tau)(T-\tau-h)} \int_{t=0}^{T-\tau} \int_{t'=0}^{T-\tau-h} \text{cov} \left( N^2(t+\tau,t), N^2(t'+\tau+h,t') \right) dt \, dt'. \]

To calculate $\text{cov} \left( N^2(t+\tau,t), N^2(t'+\tau+h,t') \right)$, we consider the following cases.

**Case I: $t' > t$**

a) $t' > t + \tau$

In this case the covariance is zero for an underlying Poisson Process.

b) $t < t' < t + \tau$
In this case, for an underlying Poisson process,

$$\text{cov}\left( N^2(t+\tau,t), N^2(t'+\tau+h,t') \right) = \text{Var}\left( N^2_{t-t'+\tau} \right).$$

**Case II: \( t > t' \)**

a) \( t > t' + \tau + h \).

In this case the covariance vanishes.

b) \( t' + h < t < t' + \tau + h \).

In this case

$$\text{cov}\left( N^2(t+\tau,t), N^2(t'+\tau+h,t') \right) = \text{Var}\left( N^2_{t'-t+\tau+h} \right)$$

c) \( t' < t < t' + h \)

In this case

$$\text{cov}\left( N^2(t+\tau,t), N^2(t'+\tau+h,t') \right) = \text{Var}\left( N^2_{\tau} \right).$$

Hence

$$(2.1.31) \quad \frac{1}{(T-\tau)(T-\tau-h)} \int_0^{T-\tau} \int_0^{T-\tau-h} \text{cov}\left( N^2(t+\tau,t), N^2(t'+\tau+h,t') \right) \, dt \, dt'$$
\[ \frac{1}{(T-\tau)(T-\tau-h)} \int_{D_1} \text{Var}(n^2_{t-t'+\tau}) \, dt \, dt' \]

\[ + \frac{1}{(T-\tau)(T-\tau-h)} \int_{D_2} \text{Var}(n^2_{\tau}) \, dt_1 \, dt_2 \]

\[ + \frac{1}{(T-\tau)(T-\tau-h)} \int_{D_3} \text{Var}(n^2_{t|t+\tau+h}) \, dt \, dt' \]

where the domains of integration \( D_1, D_2 \) and \( D_3 \) are as given in the diagram below.
Now

\[(2.1.32)\]

\[
\frac{1}{(T-\tau)(T-\tau-h)} \int_{D_1} \int_{t=T-2\tau}^{t=T-\tau} \int_{t'=t}^{t'=t+\tau} \text{Var} \left( N_{t-t+\tau}^2 \right) \, dt \, dt'
\]

\[
= \frac{1}{(T-\tau)(T-\tau-h)} \int_{t=T-2\tau}^{t=T-\tau} \int_{t'-t}^{t'-t+\tau} \text{Var} \left( N_{t-t+\tau}^2 \right) \, dt \, dt'
\]

\[
= \frac{\lambda_\tau^2 + 2 \lambda_\tau^2 + \lambda_\tau^4}{2} + o(1) \frac{T}{T}
\]

in view of \((2.1.28)\).

Also,

\[
\int_{D_2} \text{Var} \left( N_{\tau}^2 \right) \, dt_1 \, dt_2 = \text{Area of } D_2 \times \text{Var} \left( N_{\tau}^2 \right)
\]

\[
= \left[ \frac{h^2}{2} + (T-\tau-h)h \right] \times \text{Var} \left( N_{\tau}^2 \right)
\]

Hence

\[
\frac{1}{(T-\tau)(T-\tau-h)} \int_{D_2} \text{Var} \left( N_{\tau}^2 \right) \, dt_1 \, dt_2
\]

\[(2.1.33)\]

\[
= h \text{Var} \left( N_{\tau}^2 \right) + o(1) \frac{1}{T}
\]

Now

\[(2.1.34)\]

\[
\int_{D_3} \text{Var} \left( N_{t'-t+\tau+h}^2 \right) \, dt \, dt'
\]
\[ t' = T - 2\tau - h \quad t = t' + \tau + h \]
\[ t' = T - 2\tau - h \quad t = t' + h \]
\[ t = T - \tau \]
\[ \int_{t' = 0}^{t'} \int_{t = t' + h}^{t'} \text{Var} \left( N_{t' - t + \tau + h}^2 \right) \, dt \, dt' \]
\[ + \int_{t' = T - 2\tau - h}^{t'} \int_{t = t' + h}^{t'} \text{Var} \left( N_{t' - t + \tau + h}^2 \right) \, dt \, dt' \]

If we put \( t'' = t' + h \) then (2.1.34) becomes

\[ t'' = T - 2\tau \quad t = t'' + \tau \]
\[ (2.1.35) \int_{t'' = h}^{t''} \int_{t = t''}^{t''} \text{Var} \left( N_{t'' - t + \tau}^2 \right) \, dt \, dt'' \]
\[ t'' = T - \tau \quad t = T - \tau \]
\[ + \int_{t'' = T - 2\tau}^{t''} \int_{t = t''}^{t''} \text{Var} \left( N_{t'' - t + \tau}^2 \right) \, dt \, dt'' . \]

But, from the calculations immediately preceding (2.1.25), we obtain

\[ t = t'' + \tau \]
\[ (2.1.36) \int_{t = t''}^{t'' + \tau} \text{Var} \left( N_{t'' - t + \tau}^2 \right) \, dt \, dt'' = \frac{\lambda \tau^2}{2} + 2 \frac{\lambda^2 \tau^3}{3} + \lambda^2 \tau^4 , \]

and the second part of (2.1.35) is, in view of (2.1.27), given by

\[ t'' = T - \tau \quad t = T - \tau \]
\[ (2.1.37) \int_{t'' = T - 2\tau}^{t''} \int_{t = t''}^{t''} \text{Var} \left( N_{t'' - t + \tau}^2 \right) \, dt \, dt'' \]
\[ \begin{align*}
\frac{\lambda r^3}{3} + 3 \frac{\lambda^2 r^4}{2} + 4 \frac{\lambda^3 r^5}{5}.
\end{align*} \]

Thus

\[ (2.1.38) \frac{1}{(T-\tau)(T-\tau-h)} \int \int_{D_3} \text{Var} \left( \frac{N^2_{t', t+\tau+h}}{N^2_t} \right) dt \, dt'. \]

\[ = \frac{\lambda r^2}{2} + 2 \frac{\lambda^2 r^3}{T} + \frac{\lambda^3 r^4}{T} + o(1). \]

Upon combining (2.1.38), (2.1.33), (2.1.32) and (2.1.31) we obtain

\[ (2.1.39) \frac{1}{(T-\tau)(T-\tau-h)} \int \int_{0}^{T-\tau} \int \int_{0}^{T-\tau-h} \text{cov} \left( \frac{N^2(t+\tau,t'+t')}{N^2(t', t+\tau+h, t')} \right) dt \, dt'. \]

\[ = \frac{\lambda r^2}{T} + 4 \frac{\lambda^2 r^3}{T} + 2 \frac{\lambda^3 r^4}{T} + \lambda \text{Var} \left( \frac{N^2_t}{T} \right) + o(1). \]

Upon combining (2.1.39), (2.1.15), (2.1.16) and (2.1.19) we obtain

\[ (2.1.40) \text{cov} \left( \frac{\hat{V}(\tau)}{V(\tau+h)} \right) \]

\[ = \frac{\lambda r^2}{T} + 4 \frac{\lambda^2 r^3}{T} + 2 \frac{\lambda^3 r^4}{T} + 4 \lambda^3 r^4 (\tau+h)^2 + \lambda \text{Var} \left( \frac{N^2_t}{T} \right) \]

\[ + o(1). \]

Note that, when \( h=0 \), (2.1.40) reduces to the formula for \( \text{Var} \left( \frac{\hat{V}(\tau)}{V(\tau)} \right) \)

obtained in (2.1.29).
2.2. Asymptotic Normality of the Statistic $\hat{V}(\tau)$ when the underlying process is Poisson.

We have seen that

\begin{equation}
\hat{V}(\tau) = \hat{V}_1(\tau) - \hat{V}_2(\tau),
\end{equation}

where

\begin{equation}
\hat{V}_1(\tau) = \frac{1}{T-\tau} \int_0^{T-\tau} (N(t+\tau,t))^2 \, dt,
\end{equation}

and

\begin{equation}
\hat{V}_2(\tau) = \frac{\tau^2}{T^2} \left( N(o,T) \right)^2.
\end{equation}

When the underlying R.P. is Poisson we have proved that

\begin{equation}
\begin{cases}
E \left( \hat{V}_1(\tau) \right) = \lambda \tau + \lambda^2 \frac{\tau^2}{T} \\
Var \left( \hat{V}_1(\tau) \right) = \frac{\lambda \tau^2 + 4 \lambda^2 \tau^3 + 2 \lambda^3 T^4}{T} + o(1)
\end{cases}
\end{equation}

and

\begin{equation}
\begin{cases}
E \left( \hat{V}_2(\tau) \right) = \lambda^2 \tau^2 + \frac{\lambda^2 T^2}{T} \\
Var \left( \hat{V}_2(\tau) \right) = \frac{4 \lambda^3 T^4}{T} + o(1)
\end{cases}
\end{equation}

and finally

\begin{equation}
\text{cov} \left( \hat{V}_1(\tau), \hat{V}_2(\tau) \right) = o\left(\frac{1}{T}\right).
\end{equation}

We will establish the asymptotic normality of $\hat{V}(\tau)$ in several steps.

**Step 1:** $\hat{V}_1(\tau)$ is asymptotically normal.

Let

\begin{equation}
W_t = \int_0^{t-\tau} \left( N(t+\tau,t) \right)^2 \, dt.
\end{equation}
Consider the development of the underlying Poisson process on the non-negative time axis. Let us say the event "E" happens at $t=t_0$ if there are no Poisson events in the interval $(t_0, t_0-\tau)$. Assume "E" happened at the origin 0. It is clear that the intervals separating the occurrences of the events "E" are independent and identically distributed, and the points on the time axis corresponding to the occurrences of "E" mark the development of a renewal process say $\{t_i\}$. Let

$$T_n = t_1^+t_2^+ \ldots t_n$$

when the underlying renewal process is Poisson we will show that $W_t$ defined by (2.2.6) is a cumulative process in the sense of Smith [21,22] with respect to the regeneration points $\{T_n\}$. 

Let us write

$$Y_n = \Delta_n W_t = W_{Tn} - W_{T_{n-1}}$$

$$= \int_{T_{n-1}}^{T_n-\tau} \left( N(t+\tau,t) \right)^2 dt.$$

Since the number of Poisson events occurring in disjoint intervals are mutually independently distributed, and since the $Y_n$'s are made up of Poisson events in disjoint intervals, the lengths of which are independently identically distributed, the $Y_n$'s are therefore independently identically distributed. Also, since the number of Poisson events in any finite $t$ interval is finite with probability one, $W_t$ of (2.2.6) is with probability
one of bounded variation in every finite \( t \)-interval. Since \( W_t \) is non-negative condition \((C_3)\) of Smith has also satisfied. We have thus proved that \( W_t \) is a cumulative process w.r.t. the regeneration points \( T_n \).

When \( \mu_2 < \infty \), Smith has proved that

\[
(2.2.8) \quad \sqrt{t} \left( \frac{W_t}{t} - \frac{\kappa_1}{\mu_1} \right) \xrightarrow{D} \mathcal{N}(0, \theta^2)
\]

where \( \kappa_1 \) is the mean of a \( Y_n \).

On combining (2.2.8) with (2.2.4) we find that

\[
(2.2.9) \quad \sqrt{T} \left( \hat{\lambda}_1(\tau) - \lambda \tau - \lambda^2 \tau^2 \right) \xrightarrow{D} \mathcal{N}(0, \theta^2),
\]

where

\[
(2.2.10) \quad \theta^2 = \lambda \tau^2 + 4 \lambda^2 \tau^3 + 2 \lambda^3 \tau^4.
\]

Step II: \( W'_t = N(0,t) \) is a cumulative process w.r.t. the same sequence of regeneration points \( T_n \).

For \( Y'_n = \Delta_n W'_t = N(T_{n-1}, T_n) \), are evidently independently identically distributed r.v's, for an underlying Poisson process.

Therefore, according to the theorem of Smith on the joint asymptotic normality of cumulative processes referred to the same sequence of regeneration points we have that

\[
(2.2.11) \quad \sqrt{T} \left( \hat{\lambda}_1(\tau) - \lambda \tau - \lambda^2 \tau^2 \right) \quad \text{and} \quad \sqrt{T} \left[ (\hat{\lambda}_2(\tau)^{1/2} - \lambda \tau \right]
\]

are asymptotically distributed according to a bivariate normal law.

Step III: \( \sqrt{T} \left[ \left( \frac{\hat{\lambda}_2(\tau)^{1/2}}{\lambda \tau} \right) \right] \) is asymptotically normally distributed.
For \( \left( \hat{V}_2(\tau) \right)^{1/2} = \frac{\tau}{T} N(0,T) \), where under the present hypotheses, \( N(0,T) \) is distributed in Poisson form with parameter \( \mu_T = \lambda T \). But

\[
(2.2.12) \quad \frac{N(o,T) - \mu_T}{\sqrt{\mu_T}} \xrightarrow{T \to \infty} N(0,1),
\]

from which the desired result is evident.

**Step IV**

As a consequence of the above three steps we have

\[
(2.2.13) \quad \sqrt{T} \left[ \hat{V}_1(\tau) - E(\hat{V}_1(\tau)) \right] - \sqrt{T} \left[ \left( \hat{V}_2(\tau) \right)^{1/2} - E(\hat{V}_2(\tau))^{1/2} \right]
\]

is asymptotically normally distributed.

**Step V:**

\[
(2.2.14) \quad \sqrt{T} \left[ \hat{V}_2(\tau) - E(\hat{V}_2(\tau)) \right] = \sqrt{T} \left[ \left( \hat{V}_2(\tau) \right)^{1/2} - E(\hat{V}_2(\tau))^{1/2} \right]
\]

\[+ \quad o_p(1),\]

where \( o_p(1) \) is a quantity which converges to zero in probability.

To see this, we refer to corollary 3, page 225 of Mann and Wald \( \text{[16]} \), putting \( r=1, j=1, \gamma_n=1, \) and \( \chi_n = \chi_n^{1/\lambda_n} \), where \( \chi_n \) is Poisson with parameter \( \lambda_n \).

Let \( f(n) = \frac{1}{\sqrt{\lambda_n}} \), \( g(x) = x^p \), \( p \) real and \( \neq 0 \). Then their

\[
T_1(x,a) = g'(a)(x-a) \quad \text{and this implies that}
\]

\( X_n - 1 = o_p(1) \),

\[\chi_n \]

\[\lambda_n\]
\[ \frac{X'_n - \lambda_n}{\sqrt{\lambda_n}} = o_p(1), \]

which follows from the fact that

\[ \frac{X'_n - \lambda_n}{\sqrt{\lambda_n}} \overset{D}{\longrightarrow} N(0,1), \quad \lambda_n \rightarrow \infty \]

The Corollary then states that

\[ (2.2.15) \quad \frac{X'^{P}_n - \lambda'^{P}_n}{p\lambda_n^{1/2}} = \frac{X'_n - \lambda_n}{\lambda_n^{1/2}} + o_p(1). \]

Upon putting \( p = 2 \) in the above, (2.2.14) is evident.

**Step VI:**

\[ (2.2.16) \quad \sqrt{T} \left( \hat{V}_1(\tau) - E(\hat{V}_1(\tau)) \right) - \sqrt{T} \left( \hat{V}_2(\tau) - E(\hat{V}_2(\tau)) \right) \]

is asymptotically normal.

For, the difference between (2.2.13), which is asymptotically normal, and (2.2.16) above, converges to zero in probability as is proved in Step V.

We have therefore proved

**Theorem 2.2**

When the underlying renewal process is Poisson, then

\[ (2.2.17) \quad \sqrt{T} \frac{\left( \hat{V}(\tau) - \lambda \tau \right)}{\left( \lambda \tau^2 + 4\lambda^2 \tau^3 + 6\lambda^3 \tau^4 \right)^{1/2}} \]

is asymptotically \( N(0,1) \). Theorem 2.2, incidentally provides a test of significance for the randomness of an underlying renewal process.
2.3 Some difficulties involved in certain non-null distributions of the statistic $\hat{V}(\tau)$.

Cox and Smith considered the following problem. Suppose we observe a Stochastic process which is assumed to be the superposition of $N$ independent renewal processes each having the same underlying distribution function $F(x)$. How to estimate the number of underlying renewal processes? Assuming equilibrium, and writing $C^2$ for the coefficient variation of the underlying R.P., then as $t \to \infty$,

\begin{equation}
(2.3.1) \quad V_p(t) \sim C^2 t \lambda,
\end{equation}

where $\lambda^{-1} = \frac{\mu}{N}$, the mean interval between events in the pooled output, and $V_p(t)$ is the variance of $N_t$ for the pooled output. Cox and Smith further assumed $F(x)$ to be of the $\chi^2$-type, in which case

\begin{equation}
(2.3.2) \quad V_p(t) \sim C^2 t \lambda + N(1 - C^4),
\end{equation}

and then equated the empirical (observed) slope of the asymptote, and the intercept of asymptote, to the corresponding theoretical ones given in (2.3.1) and (2.3.2) to obtain estimates of $C^2$ and of $N$. In order to obtain confidence limits for $C$ and $N$ one needs to know the distribution of $\hat{V}(\tau)$, say, when the underlying process is of the $\chi^2$-type. Even in this simple case, we do not know how to evaluate terms like

$$E \left( N^2(t+\tau,t), N^2(t'+\tau, t) \right),$$
for arbitrary $t$ and $t'$. Furthermore little is known about the joint distribution of events in different intervals, not necessarily consecutive. With this in mind let us define "estimability" of functions of events in different intervals on the time axis as follows:

**Definition:**

Let $I_1, I_2, \ldots, I_k$ be a set of consecutive intervals on the time axis and let $N_{I_j}^{(i)}$, $j=1,2,\ldots,k$ denote the number of events in the interval $I_j$ of a certain renewal process. Equilibrium may be assumed. Let $\mathcal{C}$ be the class of intervals which consists of $I_1, I_2, \ldots, I_k$ and the intervals which are unions of any $r$ consecutive of them. A function $N_{I_{I_1}}^{(m_1)} \ldots N_{I_{I_r}}^{(m_r)}$ of $N_{I_1}^{(i)}$, $N_{I_2}^{(i)}$, $\ldots$, $N_{I_r}^{(i)}$ where $m_1, m_2, \ldots, m_r$ are non-negative integers and $I_{I_1}, I_{I_2}, \ldots, I_{I_r}$ are any $r$ of $I_j$, $j=1,2,\ldots,k$, is said to be "estimable" if it can be identically represented as a linear combination with constant coefficients (being functionally independent of $N_i$'s) of the $(m_1 + m_2 + \ldots + m_r)^{th}$ powers of $N_I$ where $I \in \mathcal{C}$.

Thus if a function $N_{I_{I_1}}^{(m_1)} N_{I_{I_2}}^{(m_2)} \ldots N_{I_{I_r}}^{(m_r)}$ of the $N_i$'s is estimable its expected value can be represented as a finite linear
combination with constant coefficients of the expected values of
the \((m_1+m_2^+ \ldots + m_r)^{th}\) powers of \(N_I\) where \(I \in \mathcal{C}\). These
expected values of the powers of \(N_I\) where \(I \in \mathcal{C}\) are explicitly
known in terms of the moments of the underlying renewal process.

These functions are therefore aptly termed "estimable" in
that their expected values for any arbitrary underlying renewal
process can be explicitly written down without the complication
of running into intractable distribution problems:

**Examples of Estimable functions:**

\[ 0 \quad \tau_1 \quad \tau_1 + \tau_2 \quad t \]

Consider the two intervals
\[ I_1 = (0, \tau_1] \]
and
\[ I_2 = (\tau_1, \tau_1 + \tau_2]. \]

In this case the class \(\mathcal{C}\) consists of the three intervals \(I_1, I_2\)
and \(I = I_1 + I_2 = (0, \tau_1 + \tau_2]. \)

Squaring both sides of \(N_I = N_{I_1} + N_{I_2}\) and rearranging we get

\[ N_{I_1} N_{I_2} = \frac{1}{2} \left( N_{I_1}^2 - N_{I_1}^2 - N_{I_2}^2 \right). \]

Hence \(N_{I_1} N_{I_2}\) is estimable and

\[ E(N_{I_1} N_{I_2}) = \frac{1}{2} \left( E(N_{I_1}^2) - E(N_{I_1}^2) - E(N_{I_2}^2) \right). \]
Similarly
\[ N_{I_1 I_2}^2 + N_{I_1 I_2}^2 = \frac{1}{3} \left( N_{I_1}^3 - N_{I_1}^3 - N_{I_2}^3 \right) \]

Hence the symmetric function of \( N_{I_1} \) and \( N_{I_2} \) on the L.H.S. of

the above is estimable and

\[ E(N_{I_1 I_2}^2 + N_{I_1 I_2}^2) = \frac{1}{3} \left( E(N_{I_1}^3) - E(N_{I_1}^3) - E(N_{I_2}^3) \right) \]

Examples of Nonestimable functions:

The function \( N_{I_1 I_2}^2 \) is not estimable. For, if it is estimable
then \( l_1, l_2, l_3 \), not functionally dependent on \( N_{I_1} \), where \( I \in C \),
should exist such that

\[ N_{I_1 I_2}^2 \equiv l_1 N_{I_1}^3 + l_2 N_{I_2}^3 + l_3 N_{I_3}^3 \]

This is possible only when \( l_1 = l_2 = c, l_3 = -c \)
where

\[ c = -\frac{1}{3} \frac{N_{I_1 I_2}^2}{N_{I_1 I_2}^2 + N_{I_1 I_2}^2} \]

which contradicts the requirement that the \( l \)'s should be functionally independent of the \( N_{I_1} \)'s. Similarly \( N_{I_1 I_2}^2 \) is not estimable.

However in the particular case when the underlying process is
Poisson, since events in disjoint intervals are independently dis-
tributed, all the above functions are estimable. For example in
formula (2.1.13) on page 56, we come across several nonestimable
functions on the R.H.S. for the case of arbitrary underlying renewal
processes.
PART III

SOME GENERALIZATIONS OF THE RENEWAL PROCESS

3.0 Summary

This part is more or less expository in nature, the main object being to show that a certain generalization of renewal processes proposed by Hammersley [21], though unnecessarily complicated in the form considered by him, is an easy extension of Smith's theory of cumulative processes. By slightly changing the definition of Hammersley's proposed generalization we will show that Smith's formulae for the asymptotic mean and variance of a cumulative process, extended by us to a further degree of accuracy, can be directly applied to yield the asymptotic mean and variance of the random variable considered in the generalization. Also, to examine Hammersley's generalization in the light of Smith's theory of cumulative processes, we need to modify slightly the random variable considered by Smith [22]. Furthermore, we will demonstrate the necessity of the "global" assumption, namely, that the basic distribution function $F(x)$, is such that for some $k$ its $k^{th}$ convolution has an absolutely continuous component. Both Smith and we have found this assumption necessary in order to obtain the asymptotic representation theorems for the cumulants of the renewal process and general renewal process respectively.
3.1. The asymptotic mean and variance of the random variable $Y^S_t$

Let $\{t_i\}, i=1,2, \ldots$ be a renewal process. Let $T_n = t_1 + t_2 + \ldots + t_n$, $n=1,2, \ldots$; $T_n$ is the time instant corresponding to the occurrence of an event 'E'. Let $W_t$ be a cumulative process defined w.r.t. the sequence of regeneration points $\{T_n\}$. Then

$$y_n = \Delta_n W_t = W_{T_n} - W_{T_{n-1}}, \ n=1,2, \ldots$$

are independently identically distributed random variables.

Let

$$\kappa_r = E y_n^r \quad \text{and} \quad \mu_r = E t_n^r, \ \mu_{i,j} = E(t_n^i y_n^j)$$

Smith has considered the random variable $Y^S_t$, where

$$Y^S_t = i_{i=1}^{N_t+1} y_i,$$

$N_t$ being the number of events up to time $t$ of the R.P. $\{t_i\}$.

For the random variable $Y^S_t$, Smith has shown that

(A) if $\kappa_1 < \infty, \ \mu_1 < \infty$

$$E(Y^S_t) = \frac{\kappa_1}{\mu_1} t + o(t) \quad \text{as} \quad t \to \infty$$

(B) if $\mu_2 < \infty, \ \kappa_2 < \infty$, and $\mu_{11} = E(t_i y_i) < \infty$,

$$\text{Var}(Y^S_t) = \frac{t}{\mu_1} \left( \frac{\kappa_2 + \mu_2}{\mu_1^2} \frac{\kappa_1^2 - 2\kappa_1 \mu_{11}}{\mu_1} \right) + o(t) \quad \text{as} \quad t \to \infty$$

Before we consider Hammersley's random variable $Y^H_t$, we need to extend formulae (A) and (B) to a further degree of accuracy.
Let $G(t, y)$ denote the joint distribution of $t_1$ and $y_1$.

Let
\begin{equation}
\Psi^S_t(y) = \Pr \left( Y^S_t \leq y \right),
\end{equation}
in other words $\Psi^S_t(y)$ denotes the distribution function of $Y^S_t$.

Then it may be seen that
\begin{equation}
\Psi^S_t(y) = \int_{-\infty}^{\infty} \int_{-\infty}^{t} \Psi^S_{t-t'}(y-y') \, d_{t', y'} G(t', y').
\end{equation}

Let
\begin{equation}
G^*_s(\omega) = \int_{t=0}^{\infty} \int_{y=-\infty}^{\infty} e^{st+i\omega y} \, d_{t, y} G(t, y).
\end{equation}

and write
\begin{equation}
\Psi^S_s(y) = s \int_{0}^{\infty} e^{-st} \Psi^S_t(y) \, dt,
\end{equation}
for the Laplace-Stieltjes transform of $\Psi^S_t(y)$. It is easily seen that, for a fixed $s$, $\Psi^S_s(y)$ is a distribution function in $y$. Also
\begin{equation}
\Psi^S_s(\omega) = \int_{-\infty}^{\infty} e^{i\omega y} \, d_y \Psi^S_s(y),
\end{equation}
is a characteristic function.

From (3.1.6) we see that $G^*_s(\omega)$ is the characteristic function of the random variable $Y_n$, and if we write $F(t) = G(t, \omega)$ for the
distribution function of \( t_n \), then \( G^*(\omega) = F^*(s) \), the Laplace-
Stieltjes transform of \( F(t) \).

Applying the transform (3.1.6) on both sides of (3.1.5), we
discover that

\[
(3.1.9) \quad \mathcal{L}_{s}^{S^*}(\omega) = G^*(\omega) - G^*(\omega) + \mathcal{L}_{s}^{S^*}(\omega) G^*(\omega),
\]

and hence

\[
(3.1.10) \quad \mathcal{L}_{s}^{S^*}(\omega) = \frac{G^*(\omega) - G^*(\omega)}{1 - G^*(\omega)}.
\]

Noticing that the L.H.S. of (3.1.10) is the characteristic
function of the distribution function \( \mathcal{L}_{s}^{S^*}(y) \) and also that for
\( \Re(s) > 0 \), if \( \kappa \) \( \omega < \omega \), the R.H.S. is differentiable w.r.t. \( \omega \),
we obtain

\[
(3.1.11) \quad i \int_{-\infty}^{\infty} y \, dy \mathcal{L}_{s}^{S^*}(y) = \left. \frac{G^*(\omega)}{(1 - G^*(\omega))} \right|_{\omega=\omega} = \frac{i \kappa}{1 - F^*(s)}
\]

Hence

\[
(3.1.12) \quad \int_{-\infty}^{\infty} y \, dy \mathcal{L}_{s}^{S^*}(y) = \frac{\kappa}{1 - F^*(s)}
\]

If \( \mu_2 < \omega \), we know from a basic lemma of Smith that there exists
another distribution function \( F(2)(t) \) such that

\[
F^*(s) = 1 - \mu_1 s + \mu_2 s^2 \frac{F^*(s)}{(2)}
\]

Hence

\[
\frac{\kappa}{1 - F^*(s)} \mu_1 s = \frac{\kappa_1}{\mu_1 s} \left( 1 + \frac{\mu_2 s}{2 \mu_1} F^*(s) + o(s) \right)
\]

\[
= \frac{\kappa_1}{\mu_1 s} \frac{1}{2 \mu_1} F^*(s) + o(1)
\]
\[ = \frac{\kappa_1}{\mu_1 s} + \frac{\kappa_1 \mu_2}{2\mu_1^2} + \frac{\kappa_1 \mu_2}{2\mu_1^2} (F^*(s) - 1) + o(1) \]

\[ (3.1.13) = \frac{\kappa_1}{\mu_1 s} + \frac{\kappa_1 \mu_2}{2\mu_1^2} + o(1), \text{ as } s \to +\infty. \]

But the L.H.S. of (3.1.12) is the L.S.T of

\[ (3.1.14) \quad E \left( Y_t^g \right) = \int_{-\infty}^{\infty} y \, d_y \psi_t^g (y). \]

Hence

If \( \kappa_1 < \infty \), \( \mu_2 < \infty \), then

\[ (A') \quad E(Y_t^S) = \frac{\kappa_1}{\mu_1} t + \frac{\kappa_1 \mu_2}{2\mu_1^2} + o(1), \text{ as } t \to \infty. \]

Now for \( \text{Re}(s) > 0 \), if \( \kappa_2 < \infty \), the R.H.S. of (3.1.10) is twice differentiable w.r.t. \( \omega \) and we find, as earlier shown by Smith \( \sqrt{27} \),

\[ (3.1.15) \quad \int_{-\infty}^{\infty} y^2 \, d_y \psi_S^S(y) = \frac{\kappa_2}{1 - F^*(s)} + 2\kappa_1 \frac{R^*(s)}{(1 - F^*)^2} \]

where \( R^*(s) \) is the L.S.T. of a function of B.V. and

\[ (3.1.16) \quad R^*(s) = \frac{1}{l} \frac{d}{d\omega} \left( G^*(\omega) \right) \bigg|_{\omega = 0} \]

Now if \( \mu_{21} < \infty \), then the L.S.T. \( R^*(s) \) of \( R(t) \) can be written as
\[(3.1.17) \quad R^*(s) = \kappa_1 - s\mu_{11} + \frac{\mu_{21}}{2!} s^2 + o(s^2), \text{ as } s \to +\infty.\]

Also, if \( \mu_3 < \infty \),

\[(3.1.18) \quad \frac{2}{(1-F^*)^2} = \frac{2}{\mu_1^2} s + \frac{2\mu_2}{\mu_1^3} + \frac{3\mu_2^2}{2\mu_1^4} - \frac{2\mu_3}{3\mu_1^3} + o(1) \text{ as } s \to +\infty.\]

Also

\[(3.1.19) \quad \frac{\kappa_2}{1 - F^*(s)} = \frac{1}{\mu_1^2} \frac{\mu_2}{2\mu_1} + o(1) \text{ as } s \to +\infty.\]

Combining (3.1.15), (3.1.17), (3.1.18) and (3.1.19) we find that if \( \mu_3 < \infty, \mu_{21} < \infty \), then

\[(3.1.20) \quad \int_{-\infty}^{\infty} y^2 d_y \mathcal{P}_s^*(y) = \frac{2}{s^2} \left( \frac{\kappa_1^2}{\mu_1^2} \right) + \frac{1}{s} \left( \frac{2\kappa_1^2 \mu_2}{\mu_1^3} - 2 \kappa_1 \mu_{11} + \frac{\kappa_2}{\mu_1} \right) + \frac{3 \kappa_1^2 \mu_2^2}{2\mu_1^4} - \frac{2\kappa_1^2 \mu_3}{3\mu_1^3} - \frac{2 \kappa_1 \mu_2 \mu_{11}}{\mu_1^3} + \frac{\kappa_1 \mu_{21}}{\mu_1^2} + \frac{\kappa_2 \mu_2}{2\mu_1^2} + o(1) \text{ as } s \to +\infty.\]
Noticing that the L.H.S. of (3.1.20) is the L.S.T. of $E(Y_t^S)^2$ we discover that

$$(3.1.21) \quad E(Y_t^S)^2 = \frac{t^2}{\mu_1^2} \kappa_1^2 + \frac{t}{\mu_1} \left( \kappa_2 + \frac{2\kappa_1^2\mu_2}{\mu_1^2} - \frac{2\kappa_1\mu_{11}}{\mu_1} \right)$$

$$+ \frac{3\kappa_1^2\mu_2}{2\mu_1^4} - \frac{2\kappa_1^2\mu_3}{3\mu_1^3} - \frac{2\kappa_1\mu_2\mu_{11}}{\mu_1^3} + \frac{\kappa_1\mu_{21}}{\mu_1^2} + \frac{\kappa_2\mu_2}{2\mu_1} + o(1)$$

as $t \to +\infty$.

Now when $\kappa_1 < \infty$, $\mu_2 < \infty$, we have proved that

$$(3.1.22) \quad E(Y_t^S) = \frac{\kappa_1}{\mu_1} t + \frac{\kappa_1\mu_2}{2\mu_1} + o(1) \text{ as } t \to \infty.$$.

If we square both sides of (3.1.22) we obtain a term like $t \times o(1)$ where behaviour as $t \to +\infty$, we do not know so far.

We now appeal to the cumulant representation theorem of Smith [207] for the R P., proved under the global assumption that if $\mu_{n+p+1} < \infty$, $p > 0$, then

$$\Psi_n(t) = \gamma_n t + \gamma_{n+1} + \frac{\lambda(t)}{(1+t)^p}$$

where $\lambda(t) \in \mathbb{R}$.

Since $\frac{1}{1-F^*} = \Phi^* = \Psi_1^*$, and since $\mu_2 < \infty$, the remainder in

$$(3.1.22) \text{ is actually } o(1)$$

and hence under the global assumption,
if \( \mu_3 < \infty \), the term \( tx o(1) \) above tends to zero as \( t \to +\infty \). Thus if \( F(x) \) satisfies the global assumption and if \( \mu_3 < \infty ; \kappa_1 < \infty \), then

\[
(3.1.23) \quad E(Y_t^S)^2 = \frac{\kappa_1^2}{\mu_1^2} t^2 + \frac{\kappa_1^2 \mu_2^2}{4\mu_1^4} + t \frac{\kappa_1^2 \mu_2}{\mu_1^3} + o(1)
\]

as \( t \to \infty \).

Subtracting (3.1.23) from (3.1.21) we find that, if \( F(x) \) satisfies the global assumption \( \int \) we omit this statement hereafter to avoid tedious repetition \( \int \), and if

\[ \kappa_2 < \infty, \mu_3 < \infty, \text{ and } \mu_{21} < \infty, \text{ then} \]

\[
(B') \quad \text{Var} \quad Y_t^S = \frac{t}{\mu_1} \frac{\kappa_2}{\mu_1} + \frac{\kappa_1 \mu_2}{\mu_1^2} - \frac{2\kappa_1 \mu_{11}}{\mu_1} + \frac{5\kappa_1^2 \mu_2^2}{4\mu_1^4} - \frac{2\kappa_1 \mu_3}{3\mu_1^3} - \frac{2\kappa_1 \mu_2 \mu_{11}}{\mu_1^3} + \frac{\kappa_1 \mu_{21}}{\mu_1^2} + \frac{\kappa_2 \mu_2}{2\mu_1^2} + o(1) \text{ as } t \to \infty.
\]

Thus (A') and (B') are extensions of Smith's (A) and (B) to an additional degree of accuracy. If we put \( y_n = 1 \), (A') and (B') reduce to Smith's original formulae for the expectation and
variance of \( N_t \), which has also incidentally provided us with a check on our formulae \((A')\) and \((B')\).

3.2 The asymptotic mean and variance of the one dimensional version of Hammersley's random variable \( Y_t^H \).

Let \( \{t_n, y_n(pxl)\}, n=1,2, \ldots \) be a sequence of independently, identically distributed random variables with the same distribution function \( G(t, y_n(pxl)) \), where \( t_n \) is a one dimensional non-negative random variable while \( y_n(pxl) \) is a \( p \)-dimensional random vector.

Then Hammersley [17] considers the random variable

\[
(3.2.1) \quad \eta_1(t) = \sum_{n=1}^{N_t} y_n(pxl)
\]

where \( N_t \), as before, is the number of events up to a specified time instant \( t \) w.r.t. to the renewal process \( \{t_n\} \).

In this section we will only consider the one-dimensional version of Hammersley's generalization (3.2.1) and in the next section we will show that by a certain "linearization" technique we can at once obtain the results corresponding to the asymptotic covariances of the components of \( y_n(pxl) \) from the one dimensional results of this section. We will denote the one dimensional version of \( \eta_1(t) \) of (3.2.1) by \( Y_t^H \) where

\[
(3.2.2) \quad Y_t^H = \sum_{n=1}^{N_t} y_n
\]

\( \{y_n\} \) being independently and identically distributed random variables.
Notice that the only difference between $Y_t^S$ of Smith and $Y_t^H$ of Hammersley is that while Smith takes the summation from 1 to $N_t + 1$, Hammersley takes it from 1 to $N_t$. This simple difference introduces a lot of complication in the calculations as will be shown below. This complication has been foreseen by the former author, but evidently not by the latter. We will adhere to the notation of the previous section. As before, $G(t, y)$ denotes the joint distribution of $t_1$ and $y_1$.

Let

$$\psi_t^H(y) = \Pr \left( Y_t^H \leq y \right)$$

Then

$$(3.2.3) \quad \psi_t^H(y) = 1 - G(t, \infty) + \int_0^\infty \int_{-\infty}^t \psi_{t-t'}^H(y-y') \, dt' \, dG(t', y')$$

Applying the transform (3.1.6) on both sides of (3.2.3), we find

$$\psi_S^H(\omega) = 1 - G^*(\omega) + \psi_S^H(\omega) \, G^*(\omega).$$

Hence

$$(3.2.4) \quad \psi_S^H(\omega) = \frac{1 - G^*(\omega)}{1 - G_S^*(\omega)}$$

Bearing in mind that the L.H.S. of the above is the characteristic function of the distribution function $\psi_S^H(y)$ and that for $\Re(s) > 0$, the R.H.S. is differentiable w.r.t. $\omega$,
we find that the first absolute moment of $\psi_{s}^{H*}(y)$ is finite and that

$$\int_{-\infty}^{\infty} \psi_{s}^{H*}(y) = \left( \frac{1 - C^*(\omega)}{G^*(\omega)} \right) \left( \frac{G^*(\omega)}{1 - G^*(\omega)} \right)^2$$

$$= \frac{i R^*(s)}{1 - F^*(s)}$$

Thus

$$\int_{-\infty}^{\infty} \psi_{s}^{H*}(y) = \frac{R^*(s)}{1 - F^*(s)}$$

$$= \frac{k_1}{\mu_1} + \frac{k_1 \mu_2}{2 \mu_1} - \frac{\mu_{11}}{\mu_1} + o(1) \text{ as } s \to +\infty,$$

(Note we have assumed $k_1 < \infty$, $\mu_2 < \infty$ and $\mu_{11} < \infty$, and then the calculations are as before).

Since the L.H.S. of (3.2.5) is the L.S.T. of $E(Y_t^H)$, we find that if $k_1 < \infty$, $\mu_2 < \infty$ and $\mu_{11} < \infty$ then

$$E(Y_t^H) = t \frac{k_1}{\mu_1} + \frac{k_1 \mu_2}{2 \mu_1} - \frac{\mu_{11}}{\mu_1} + o(1) \text{ as } t \to \infty.$$

Similarly,
\[- \int_{-\infty}^{\infty} y^2 \frac{d_y \Psi_s^H(y)}{1 - F^*(s)} = \left. \frac{\dot{G}_s^*(\omega)}{1 - F^*(s)} \right|_{\omega = 0} + 2 \left[ \frac{\dot{G}_s^*(\omega)}{(1 - F^*(s))^2} \right] \]

\[= - \frac{Q^*(s)}{1 - F^*(s)} + 2 \frac{(i R^*(s))^2}{(1 - F^*(s))^2} \]

where \(Q^*(s) = \left. \frac{\dot{G}_s^*(\omega)}{1 - F^*(s)} \right|_{\omega = 0}\) is the L.S.T. of \(Q(t) = \int_{-\infty}^{\infty} y^2 \frac{d_y}{1 - F^*(s)} G(t, y),\)

a function of \(B.V.\) Hence

\[(3.2.7) \quad \int_{-\infty}^{\infty} y^2  \frac{d_y \Psi_s^H(y)}{1 - F^*(s)} = \frac{Q^*(s)}{1 - F^*(s)} + 2 \frac{(i R^*(s))^2}{(1 - F^*(s))^2} . \]

After a straightforward calculation we find

\[(3.2.8) \quad E(Y_t^H)^2 = \frac{k_1^2 k_2^2}{\mu_1^2} + \left( \frac{2 \mu_2^2 - 2 k_1^2 \mu_{11} + k_2}{3 \mu_1^2} \right) t \]

\[+ \frac{3 k_1^2 \mu_2}{2 \mu_1^2} - \frac{2 \mu_1^2}{3 \mu_1^2} + \frac{2 \mu_{11}^2 + 2 k_1 \mu_{21}}{2 \mu_1^2} \]

\[- \frac{4 k_1 \mu_{11} \mu_2}{\mu_1^3} + \frac{\mu_2 k_2}{2 \mu_1^2} - \frac{\mu_{12}}{\mu_1} + o(1), \]

as \(t \to \infty.\)
if $\kappa_2 < \infty$, $\mu_3 < \infty$, $\mu_{21} < \infty$ and $\mu_{12} < \infty$.

We finally discover, combining (3.2.8) and (3.2.6), that if $\kappa_2 < \infty$, $\mu_3 < \infty$, $\mu_{21} < \infty$ and $\mu_{12} < \infty$ then

$$
(3.2.9) \quad \text{Var} \left( Y_t^H \right) = t \left( \frac{\kappa_2}{\mu_1} + \frac{\kappa_2^2 \mu_2}{\mu_1^3} \right.
- \left. 2 \frac{\kappa_1 \mu_{11}}{\mu_1^2} \right)
+ \frac{5 \kappa_2^2 \mu_2^2}{4 \mu_1^4} - \frac{2 \kappa_1 \mu_3}{3 \mu_1^3} - \frac{3 \kappa_1 \mu_{11} \mu_2}{\mu_1^3} + \frac{\mu_{11}^2}{\mu_1^2}
+ \frac{2 \kappa_1 \mu_{21}}{\mu_1^2} + \frac{\mu_2^2}{2 \mu_1^2} - \frac{\mu_{12}}{\mu_1} + o(1) \text{ as } t \to \infty.
$$

In the special case $\gamma_n = 1$, both $E(Y_t^H)$ and $\text{Var}(Y_t^H)$ reduce to Smith's formulae for the asymptotic mean and variance of $N_t$. By taking the summation from 1 to $N_t$, we become involved in the complication of dealing with $Q^*(s)$, which could be easily avoided by taking the summation from 1 to $N_t + 1$, as in $Y_t^S$.

3.3 The multidimensional cumulative process treated as a special case of the one dimensional cumulative process.

Consider the multidimensional version of $Y_t^H$, namely $\eta_1(t)$ given by (3.2.1). Suppose we want to obtain the asymptotic covariance of $\eta_{1j}(t)$ and $\eta_{1k}(t)$, $j \neq k$ for any two components.
\[ \eta_{lj} \text{ and } \eta_{lk} \text{ of the } p\text{-dimensional vector } \eta_n. \]

Let

\[ (3.3.1) \quad \eta_{lj}(t) = \sum_{n=1}^{N_t} y_n^{(j)} \]

and

\[ (3.3.2) \quad \eta_{lk}(t) = \sum_{n=1}^{N_t} y_n^{(k)} \]

where \( y_n^{(j)} \) and \( y_n^{(k)} \) are the \( j^{th} \) and \( k^{th} \) components of the vector \( y_n(p \times 1) \). Consider

\[ (3.3.3) \quad \eta(t) = \eta_{lj}(t) + \eta_{lk}(t) = \sum_{n=1}^{N_t} y_n \]

where

\[ (3.3.4) \quad y_n = y_n^{(j)} + y_n^{(k)} \]

clearly \( \{y_n\} \) are independently identically distributed random variables, since \( \{y_n(p \times 1)\} \) are independently identically distributed; and furthermore the summation in (3.3.1), (3.3.2) and (3.3.3) is over the number of events w.r.t. the same renewal process \( \{t_i\} \). Hence taking the variance of both sides of (3.3.3) we obtain

\[ \text{Var} \left( \eta(t) \right) = \text{Var} \left( \eta_{lj}(t) \right) + \text{Var} \left( \eta_{lk}(t) \right) \]

\[ + 2 \text{ cov} \left( \eta_{lj}(t), \eta_{lk}(t) \right), \]

and so

\[ (3.3.5) \quad \text{cov} \left( \eta_{lj}(t), \eta_{lk}(t) \right) \]

\[ = \frac{1}{2} \left[ \text{Var} \left( \eta(t) \right) - \text{Var} \left( \eta_{lj}(t) \right) - \text{Var} \left( \eta_{lk}(t) \right) \right]. \]
In the notation of Hammersley let us write

\[ \mu_{\alpha\beta\gamma} = \mathbb{E} \left[ y_n^{\alpha(j)} \right] \alpha \mathbb{E} \left[ y_n^{\beta(k)} \right] \beta \mathbb{E} \left[ y_n^{\gamma}\right] \gamma \]

and

\[ \rho = \frac{1}{\mu_1} = \frac{1}{\mu_{001}} \]

Then our formula (3.2.9) for \( \text{Var} (\eta(t)) \) in the above notation becomes

(3.3.6) \( \text{Var} (\eta(t)) = t \left[ \rho (\mu_{200} + \mu_{020} + 2\mu_{110}) + 2\rho^2(\mu_{100} + \mu_{010}) \right. \]

\[ \left. \times (\mu_{101} + \mu_{011}) + \rho^3(\mu_{100} + \mu_{010})^2 \mu_{002} \right] \]

\[ + \frac{5\rho^4}{4}(\mu_{100} + \mu_{010})^2 \mu_{002}^2 - \frac{2\rho^3(\mu_{100} + \mu_{010})^2 \mu_{003}}{3} \]

\[ - 3\rho^3(\mu_{100} + \mu_{010})(\mu_{101} + \mu_{011}) \mu_{002} \]

\[ + \rho^2(\mu_{101} + \mu_{011})^2 + 2\rho^2(\mu_{100} + \mu_{010})(\mu_{102} + \mu_{012}) \]

\[ + 2\rho^2 \mu_{002}(\mu_{200} + \mu_{020} + 2\mu_{110}) - \rho(\mu_{201} + \mu_{021} + 2\mu_{111}) + o(1) \text{ as } t \to \infty. \]

\( \text{Var} (\eta_{1j}(t)) \) becomes

(3.3.7) \( \text{Var} \eta_{1j}(t) = t (\rho \mu_{200} + \rho \mu_{020} + 2\rho^2(\mu_{100} + \mu_{101}) \]

\[ + \frac{5\rho^2}{4} \mu_{002}^2 - \frac{2\rho^3}{3} \mu_{003} \]

\[ - 3\rho^3 \mu_{100} \mu_{010} \mu_{002} + \rho^2 \mu_{101}^2 + 2\rho^2 \mu_{102} \mu_{012} \]

\[ + 2\rho^2 \mu_{002} \mu_{200} - \rho \mu_{201} + o(1) \text{ as } t \to \infty. \]
and

\[(3.3.8) \quad \text{Var} \left( \eta_{2j}(t) \right) = t \left( \rho \mu_{020} + \rho^2 \mu_{010} \mu_{020} - 2 \rho^2 \mu_{010} \mu_{011} \right) + \frac{5 \rho^2}{4} \mu_{010} \mu_{002} - \frac{2 \rho^3}{3} \mu_{010} \mu_{003} \]

\[\quad - 3 \rho^3 \mu_{010} \mu_{011} \mu_{002} + \rho^2 \mu_{011} + 2 \rho^2 \mu_{010} \mu_{012} \]

\[\quad + \frac{2 \rho^2}{2} \mu_{002} \mu_{020} - \rho \mu_{021} + o(1) \text{ as } t \to \infty.\]

Combining (3.3.5), (3.3.6), (3.3.7) and (3.3.8) we find

\[(3.3.9) \quad \text{cov} \left( \eta_{1j}, \eta_{1k} \right) = \rho t \left( \mu_{110} - \rho \mu_{101} \mu_{010} - \rho \mu_{011} \mu_{100} + \rho^2 \mu_{100} \mu_{010} \mu_{002} \right) + \frac{5 \rho^4}{4} \mu_{100} \mu_{100} \mu_{002} - \frac{2 \rho^3}{3} \mu_{100} \mu_{003} \]

\[\quad - \frac{3 \rho^3}{2} \mu_{100} \mu_{011} \mu_{002} - \frac{3 \rho^3}{2} \mu_{010} \mu_{101} \mu_{002} + \rho^2 \mu_{101} \mu_{011} \]

\[\quad + \rho^2 \mu_{100} \mu_{012} + \rho^2 \mu_{010} \mu_{102} + \frac{\rho^2}{2} \mu_{002} \mu_{110} - \rho \mu_{111} + o(1) \text{ as } t \to \infty.\]

Using our formula (B') for \( \text{Var} \left( Y_t^S \right) \), we find, after simplification, that

\[(3.3.10) \quad \text{cov} \left( \eta_{1j}^S, \eta_{1k}^S \right) = \rho t \left( \mu_{100} - \rho \mu_{101} \mu_{010} - \rho \mu_{011} \mu_{100} + \rho^2 \mu_{100} \mu_{010} \mu_{002} \right) + \frac{5 \rho^4}{4} \mu_{100} \mu_{100} \mu_{002} - \frac{2 \rho^3}{3} \mu_{100} \mu_{003} \]

\[\quad - \rho^3 \left( \mu_{100} \mu_{011} \mu_{002} + \mu_{010} \mu_{101} \mu_{002} \right) \]
\[ + \frac{\sigma^2}{2} \left( \mu_{100}^2 + \mu_{010}^2 + \mu_{002}^2 + \mu_{102}^2 + \mu_{110}^2 \right) + o(1) \text{ as } t \to \infty. \]

Both (3.3.9) and (3.3.10) reduce to Smith's original limit for \( \text{Var } N_t \) when \( \nu_n(p \times 1) = 1 \), and this has also incidentally provided a check on the formulae.

Actually there is nothing special about formula (3.3.9) which is not in (3.3.10) and what we have essentially demonstrated is that our extended formulae \( A' \) and \( B' \) (which are extensions of Smith's formulae \( A \) and \( B \)) can be exploited to meet completely the requirements of Hammersley's proposed generalization, which is rather a complication, as Smith very aptly terms in his symposium.
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