ON SOME NONPARAMETRIC GENERALIZATIONS OF WILKS' TESTS
FOR $H_m$, $H_V$ AND $H_{MV C}'$

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ON SOME NONPARAMETRIC GENERALIZATIONS OF WILKS' TESTS
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Summary. This paper is concerned with some nonparametric generalizations of the well known likelihood ratio tests, proposed and studied by S. S. Wilks [29](and also by Vogel [28]), for testing the joint hypothesis of equality of means, equality of variances and equality of covariances of multivariate normal distributions. In this part of the paper, we shall be specifically concerned with the proposal and study of a class of nonparametric rank order tests for the equality of location parameters of multivariate distributions of unspecified forms, and consider the application of this result to some problems in the two factor analysis of variance problem (in the orthogonal case). In the second part, the general problem of nonparametric generalizations of $H_{VC}$ and $H_{MVC}$ will be considered.

1. Introduction. Let $X_a = (X_{1a}, ..., X_{na})$, $a = 1, ..., n$, be independent and identically distributed (vector valued) random variables (i.i.d.r.v.'s), distributed according to a continuous p-variate cumulative distribution function (cdf) $F(x)$, where $x = (x_1, ..., x_p)$. In the sequel it will be assumed that $p \geq 2$.

In the particular case when $F$ is a multinormal cdf ($\mathcal{G}$, say), it is completely specified by its mean vector $\mu = (\mu_1, ..., \mu_p)$ and dispersion matrix $\Sigma = (\sigma_{ij})_{1 \leq i, j \leq p}$. It is also well-known that $\sigma_{ii} (i=1, ..., p)$ are measures of dispersion of the variates.

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and $\xi_{ij} = \sigma_{ij}/\{\sigma_{ii}\sigma_{jj}\}^{\frac{1}{2}}(i \neq j = 1, \ldots, p)$ are measures of association of them.

The hypothesis of compound symmetry ($H_{MVC}$) as sketched by Wilks [29], relates to

\begin{equation}
H_{MVC} = H_M \cap H_{VC},
\end{equation}

where

\begin{equation}
H_M: \xi = \xi(1, \ldots, 1) \text{ assuming } \tilde{\Sigma} = ((\delta_{ij}\sigma^2 + (1 - \delta_{ij})\rho \sigma^2)_{i,j=1,\ldots,p},
\end{equation}

$\delta_{ij}$ being the usual Kronecker delta and $-1/(p-1) < \rho < 1$;

\begin{equation}
H_{VC}: \tilde{\Sigma} = ((\sigma^2\delta_{ij} + (1 - \delta_{ij})\rho \sigma^2)_{i,j=1,\ldots,p}.
\end{equation}

Thus, test for $H_M$ is really based on the assumption that $H_{VC}$ holds. Wilks [29] has proposed and studied the likelihood ratio tests $L_{MVC}$, $L_M$ and $L_{VC}$ for testing the hypotheses $H_{MVC}$, $H_M$ and $H_{VC}$ respectively. Further generalizations on this line are due to Votaw [28], who has also considered tests for the above hypotheses in the presence of an external criterion variable. The object of the present investigation is to propose and study some nonparametric generalizations of these tests. By analogy with the parametric case, we define, in any convenient way, the location and scale parameters of $x_1$ in $F(x)$ by $\mu_1$ and $\delta_1$, $(i = 1, \ldots, p)$ respectively. We then rewrite $F(x)$ as

\begin{equation}
F(x) = F_0(x_1 - \mu_1)/\delta_1, \ldots, [x_p - \mu_p]/\delta_p.
\end{equation}

We also denote by $F_0$ the class of all $p$-variate continuous cdf's $\{F(u)\}$, where $F(u)$ is a symmetric function of its arguments $u = (u_1, \ldots, u_p)$. Now, in the nonparametric generalizations of $H_{MVC}$, $H_M$ and $H_{VC}$, we proceed as follows:

\begin{equation}
H_M: \mu_1 = \ldots = \mu_p, \text{ assuming } \delta_1 = \ldots = \delta_p \text{ and } F_0 \in F_0;
\end{equation}
(1.6) \( H_{VC} : \delta_1 = \ldots = \delta_p \) and \( F_0 \in \mathcal{F}_0 ; \)

(1.7) \( H_{MVC} = H_{M} \cap H_{VC} \) i.e., \( F \in \mathcal{F}_0 . \)

In this paper, we shall specifically consider nonparametric tests for the hypothesis \( H_M \) in (1.5), and subsequently, we shall utilize these results in the analysis of variance problems related to two orthogonal factors. In this context, nonparametric analogs of the usual analysis of variance test as well as other multiple comparison procedures are also studied. All these procedures are based on a class of rank order statistics and are shown to be genuinely distribution free under some permutation model. This study also generalizes the existing theory of rank order tests and estimates to the case of \( p \) matched samples. The asymptotic properties of the proposed methods are studied and compared with those of the parametrically optimum ones. In the second part of this paper, nonparametric generalizations of \( L_{MVC} \) and \( L_{VC} \) tests will be considered.

2. Nonparametric generalizations of \( L_M \) test for \( H_{M} \). Let us pool the \( n \) vector valued observations \( \bar{X}_a, \ a = 1, \ldots, n \) into a combined set of \( N = np \) variables. We denote these \( N \) variables by

\[
(2.1) \quad Z_N = (Z_1, \ldots, Z_N),
\]

where we adopt the convention that

\[
(2.2) \quad Z_{(a-1)p+j} = X_{ja} \quad \text{for} \ a = 1, \ldots, n, \ j = 1, \ldots, p.
\]

We then arrange the \( N \) observations in (2.1) in order of magnitude, and denote them by
(2.3) \( Z_{N,1} < \ldots < Z_{N,N} \);

by virtue of the assumed continuity of \( F(x) \), the possibility of ties in (2.3),
may be ignored in probability. Now, for any positive integer \( n \), we define a
sequence of rank functions (which depends on \( N = np \) in an explicit manner) by

(2.4) \( \mathbf{E}_N = (E_{N,1}, \ldots, E_{N,N}), \)

where we adopt the Chernoff-Savage [7] convention, and define

(2.5) \( E_{N,a} = J_N(a/(N+1)); \quad 1 \leq a \leq N. \)

The function \( J_N \) need be defined only at \( a/(N+1) \) for \( a = 1, \ldots, N \), but may have
its domain of definition extended to \((0,1]\) by the convention in ([7], [11]). For
the \( i \)th variate, we define an indicator function

(2.6) \( \mathbf{C}^{(i)}_{N,a} = \begin{cases} 1, & \text{if } Z_{N,a} \text{ is an } X_{ip}(p = 1, \ldots, n); \\ 0, & \text{otherwise}, \end{cases} \)

for \( a = 1, \ldots, N \) and \( i = 1, \ldots, p \). Thus, we have

(2.7) \( \sum_{a=1}^{N} \mathbf{C}^{(i)}_{N,a} = n, \quad \sum_{a=1}^{N} \mathbf{C}^{(i)}_{N,a} \mathbf{C}^{(j)}_{N,a} = n \delta_{ij} \) for \( i, j = 1, \ldots, p, \)

where \( \delta_{ij} \) is the usual Kronecker delta, and finally

(2.8) \( \sum_{i=1}^{p} \sum_{a=1}^{N} \mathbf{C}^{(i)}_{N,a} = N. \)

Now, we consider a \( p \)-vector

(2.9) \( \mathbf{T}_N = (T_{N,1}, \ldots, T_{N,p}); \)
(2.10) \( T_{N,i} = \frac{1}{n} \sum_{\alpha=1}^{N} C(i) E_{N,\alpha} \), \( i = 1, \ldots, p \).

It may be noted that by virtue of (2.8), we have

(2.11) \( \frac{1}{p} \sum_{i=1}^{p} T_{N,i} = \frac{1}{N} \sum_{\alpha=1}^{N} E_{N,\alpha} = E_{N} \) (say),

where \( E_{N} \) is a non-stochastic constant depending only on \( E_{N} \). Thus, \( T_{N,\alpha} \) can contain at most (\( p-1 \)) linearly independent quantities. Our proposed test is based on the stochastic vector \( T_{N} \). It may be noted that the null hypothesis \( H_{M} \) in (1.5) implies that \( F(\mathbf{x}) \) is a symmetric function of its arguments. Thus, the problem reduces to testing the interchangability of the \( p \) variates \( x_{1}, \ldots, x_{p} \) in \( F(\mathbf{x}) \).

In \( H_{M} \), we shall be particularly interested in the set of alternatives that \( \mu_{1}, \ldots, \mu_{p} \) in (1.4) are not all equal. To develop a strictly distribution-free test, we shall extend the idea of bivariate interchangability, derived by the author in an earlier paper ([26]), to the \( p(\geq 2) \) variate case, and consider an analogous permutation procedure.

3. Permutationally distribution-free test for \( H_{M} \). With reference to the order statistic (2.3), let us denote the rank of \( X_{ia} \) by \( R_{ia} \) for \( i = 1, \ldots, p \), \( a = 1, \ldots, n \). Then, the rank \( p \)-tuple corresponding to the vector \( \mathbf{X}_{a} \) is denoted by

(3.1) \( R_{a} = (R_{1a}, \ldots, R_{pa}), \alpha = 1, \ldots, n \).

We now consider the collection (rank) matrix, which we define as

(3.2) \( R_{p\times n} = (R_{1}, \ldots, R_{n}) = \begin{pmatrix} R_{11} & R_{12} & \cdots & R_{1n} \\ R_{21} & R_{22} & \cdots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ R_{p1} & R_{p2} & \cdots & R_{pn} \end{pmatrix} \).
The \( N \) elements of \( \sim_N \) are the \( N \) natural integers \((1, \ldots, N)\), permuted in some way. The matrix \( \sim_N \) consists of \( n \) random rank \( p \)-tuplets which constitute the \( n \) columns of it; naturally, \( \sim_N \) is a stochastic matrix. Two such collection matrices, say, \( \sim_N \) and \( \sim_N^{*} \), are said to be equivalent when it is possible to arrive at \( \sim_N \) from \( \sim_N^{*} \) by a number of inversions of the columns of the later. This implies that if instead of taking the observations \( x_{\alpha} \) in natural order \((\alpha = 1, \ldots, n)\), we take in any other order, say, \( x_{i_1}, \ldots, x_{i_n} \), where \((i_1, \ldots, i_n)\) is a permutation of \((1, \ldots, n)\), the two collection matrices will be equivalent. Thus, the total number of non-equivalent realizations that \( \sim_N \) may have is equal to \((np)!/n! \).

The set of all these realizations of \( \sim_N \) is denoted by \( \mathcal{R}_N \), so that \( \sim_N \in \mathcal{R}_N \). Now, there are \( p \) elements in each column of \( \sim_N \). These \( p \) elements can be permuted among them in \( p! \) ways. Thus, any given \( \sim_N \) may be used to derive a set of \((p!)^n \) realizations of such collection matrices, simply by permuting the elements within each column of it. This set of \((p!)^n \) realizations corresponding to the given \( \sim_N \) is denoted by \( S(\sim_N) \), and is termed the permutation set of \( \sim_N \). Thus, \( S(\sim_N) \) is a subset of \( \mathcal{R}_N \), and the total number of non-equivalent subsets \( S(\sim_N) \) in \( \mathcal{R}_N \) is evidently
\[(np)!/hn!(p!)^{n^2} \]. Consequently, for any \( \sim_N \) in \((2.1)\),
\[(3.3) \quad \sim_N \in S(\sim_N) \subset \mathcal{R}_N \].

The probability distribution of \( \sim_N \) over \( \mathcal{R}_N \) (defined on an additive class of subsets \( \mathcal{A}_N \) of \( \mathcal{R}_N' \)) will evidently depend on the cdf \( F \), even when \( H_M \) holds. However, if \( H_M \) in \((1.5)\) holds, then given \( x_{\alpha} \), all possible permutations of \((x_{1\alpha}, \ldots, x_{p\alpha})\) in the \( p \) places of the vector, will be conditionally equally likely, each having the permutational probability \( 1/p! \). Thus, conditionally on \( \sim_N \) in \((3.1)\), under \( H_M \) in \((1.5)\), the \( p! \) possible permutations of the \( p \) rank elements
among themselves, will be equally likely, each having the same conditional probability $1/p!$. Since $\{x_\alpha, \alpha = 1, \ldots, n\}$ are mutually stochastically independent, this implies that given $z_N$ in (3.2), we may have $(p!)^n$ possible realizations derived from it, and under $H_M$, these $(p!)^n$ realizations are equally (conditionally) likely. Now, this set of $(p!)^n$ realizations of $z_N$ is nothing but $S(z_N)$. Hence, we may put the same statement in an alternative way. Corresponding to the permutation set $S(z_N)$ being held fixed, there will be a set of $(p!)^n$ possible realizations $\{z_N\}$, which are conditionally equally likely, viz.,

$$P\left\{z_N \mid S(z_N), H_M^2\right\} = (p!)^{-n},$$

for any $S(z_N)$. Thus, if we now correspond the rank function $E_{N,\alpha}$ to the rank $a$ for $\alpha = 1, \ldots, N$, it follows that for each $z_N$ there will be a matrix whose elements will be $E_{N,\alpha}$ instead of $a$, in (3.2). Thus, for each $z_N$, we will have a value of $T_N$, defined in (2.9) and (2.10). Hence, corresponding to the set $S(z_N)$, we will have a set of $(p!)^n$ values of $T_N$, which we denote by $T_N[S(z_N)]$. Consequently, from (3.4) we get that conditionally on the set $T_N[S(z_N)]$, the permutation distribution of $T_N$ (over the $(p!)^n$ equally likely realizations) would be uniform under $H_M$ in (1.5). Let us denote this permutational probability measure by $P_n$, and consider a test function $\phi(z_N)$, which with each observed $z_N$ (in (2.1),) associates a probability of rejecting $H_M$ in (1.5) with the aid of the completely specified probability measure $P_n$. Thus, we can always select $\phi(z_N)$, in such a manner that

$$E\left\{\phi(z_N) \mid P_n\right\} = \varepsilon: 0 < \varepsilon < 1,$$
where \( \varepsilon \) is the preassigned level of significance of the test. (3.5) implies that \( \mathbb{E} \left[ \phi(Z_N) \right]_{H_M} = \varepsilon \), and hence, \( \phi(Z_N) \) is a distribution free similar test of size \( \varepsilon \).

Now, for the convenience in actual practice, we would prefer in using a single valued test statistic (say \( W_N \)), which may be used to specify the test function \( \phi(Z_N) \) in a precise way. We shall see in the next section that the permutation distribution of \( T_N \) (under the probability measure \( P_n \)) has asymptotically a multinormal form. This suggests that an appropriate (though may not be optimum) way of arriving at a suitable test statistic may be to consider the quadratic form associated with this multinormal (permutation) distribution. It is easily shown that

\[
(3.6) \quad \mathbb{E}(T_{N,i} \mid P_n) = \bar{E}_N, \text{ for } i = 1, \ldots, p,
\]

where \( \bar{E}_N \) is defined in (2.11). Also, it can be easily shown that

\[
(3.7) \quad \text{Cov}(T_{N,i}, T_{N,j} \mid P_n) = \frac{1}{n} \left( \delta_{ij} \frac{p-1}{p-1} \right) \cdot \sigma_N^2(R_N), \text{ for } i, j = 1, \ldots, p,
\]

where \( \delta_{ij} \) is the usual Kronecker delta, and

\[
(3.8) \quad \sigma_N^2(R_N) = \frac{1}{N} \sum_{\alpha=1}^{n} \sum_{i=1}^{p} (E_{N,R_{i\alpha}} - E_{N,R_{\alpha}})^2,
\]

with \( E_{N,R_{\alpha}} \) being defined as

\[
(3.9) \quad E_{N,R_{\alpha}} = \frac{1}{p} \sum_{i=1}^{p} E_{N,R_{i\alpha}} \quad \text{for } \alpha = 1, \ldots, n.
\]
Thus, \( \sigma^2_N(\mathbf{R}_N) \) depends upon the collection matrix, but remains invariant under \( S(\mathbf{R}_N) \). Thus, if we work with the inverse of the (permutational) covariance matrix of \( T_{N,i} \), \( i = 1, \ldots, p - 1 \), and consider the associated quadratic form then by using (2.11), the same is shown to reduce to the following simple form

\[
W_N = n[(p-1)/p] \sum_{i=1}^{p} (T_{N,i} - \mathbb{E}_N)^2 / \sigma^2_N(\mathbf{R}_N).
\]

Now, under \( H_M \), \( T_N \) will have the location vector \( \mathbb{E}_N(1, \ldots, 1) \) (permutationally), and hence, it can be shown that if \( \sigma^2_N(\mathbf{R}_N) \) is finite and non-zero, then under the permutational probability measure \( \mathbb{P}_n \), \( W_N \) will have \( (p!)^N \) possible realizations, which are equally likely. On the other hand, if \( H_M \) does not hold and the \( p \) variates have locations, not all equal, then at least one of \( T_{N,i} \) will be stochastically different from \( \mathbb{E}_N \) (this will be made clear in a later section), and hence \( W_N \), being a positive semi-definite quadratic form in \( \mathbb{N}_N \), will be stochastically larger. Thus, it appears to be reasonable to base our permutation test on the following rejection rule:

\[
\phi(\mathbb{Z}_N) = \begin{cases} 
1, & \text{if } W_N > W_N,_{\mathbf{R}_N} \\
\gamma_N(\mathbf{R}_N), & \text{if } W_N = W_N,_{\mathbf{R}_N} \\
0, & \text{if } W_N < W_N,_{\mathbf{R}_N},
\end{cases}
\]

where \( W_N,_{\mathbf{R}_N} \) and \( \gamma_N(\mathbf{R}_N) \) are so chosen that

\[
\mathbb{E} \left\{ \phi(\mathbb{Z}_N) \mid \mathbb{P}_n \right\} = \epsilon.
\]

Thus, if in actual practice \( n \) is not large, we can consider the set \( T_N[S(\mathbf{R}_N)] \) of \((p!)^N\) values of \( \mathbb{N}_N \) (and hence, of \( W_N \)), which will provide us with the permutational
distribution function of \( W_N \), and the same may be used to find out \( W_N, \mathcal{E}(R_N) \) and \( \gamma_N(R_N) \). This test will naturally be a strictly distribution free similar size \( \varepsilon \) test. However, if \( n \) is not very small, the labor involved in this procedure increases tremendously. To obviate this drawback, we shall now consider the asymptotic permutation test and also show how the same is also asymptotically equivalent to some unconditional test for \( H_M \) which may be based on the same rank order statistic (vector) \( T_N \).

4. Asymptotic permutation distribution of \( W_N \). As in the case of the study of the asymptotic theory of rank order tests for various other problems of statistical inference ([7], [11], [20], [21], [26], [27]), we shall impose certain regularity conditions on \( E_N \) in (2.4) as well as on \( F(x) \). Let us define

\[
F_{[i]}(x) = \frac{1}{n} \text{[Number of } X_{ia} \leq x], i = 1, \ldots, p,
\]

\[
H_N(x) = \frac{1}{p} \sum_{i=1}^{p} F_{[i]}(x);
\]

\[
F_{[i,j]}(x, y) = \frac{1}{n} \text{[Number of } (X_{ia}, X_{ja}) \leq (x, y)], i \neq j = 1, \ldots, p,
\]

\[
H_N(x, y) = \binom{p}{2}^{-1} \sum_{1 \leq i < j \leq p} F_{[i,j]}(x, y).
\]

Again, let \( F_{[i]}(x) \) and \( F_{[i,j]}(x, y) \) be respectively the marginal cdf of \( X_{ia} \) and \( (X_{ia}, X_{ja}) \), for \( i \neq j = 1, \ldots, p \), and we define

\[
H(x) = \frac{1}{p} \sum_{i=1}^{p} F_{[i]}(x),
\]
\begin{align}
H^*(x, y) &= (P_1)^{-1} \sum_{1 \leq i < j \leq p} F_{i, j}(x, y).
\end{align}

Then, we define $J_N$ as in (2.5) and assume that

\begin{align}
(4.7) \quad \text{lim}_{N \to \infty} J_N(H) &= J(H) \text{ exists for all } 0 < H < 1 \text{ and is not a constant}
\end{align}

\begin{align}
(4.8) \quad \frac{1}{N} \sum_{a=1}^{N} \left[ J_N\left( \frac{a}{N+1} \right) - J\left( \frac{a}{N+1} \right) \right] &= o(N^{-1}),
\end{align}

\begin{align}
(4.9) \quad \int_{-\infty}^{\infty} [J_N\left( \frac{N}{N+1} H_N(x) \right) - J\left( \frac{N}{N+1} H_N(x) \right)] \, dF_N[1](x) &= o_p(N^{-1}), \quad i = 1, \ldots, p.
\end{align}

(c.3) $J(H)$ is absolutely continuous in $H: 0 < H < 1$, and

\begin{align}
(4.10) \quad |J^{(r)}(H)| &= \left| \frac{d^r}{dH^r} J(H) \right| \leq K[H(1-H)]^{-r+\delta},
\end{align}

for $r = 0, 1$, and some $\delta > 0$, where $K$ is a constant.

For the permutation distribution theory, we require two more mild regularity conditions for the existence and convergence of $\sigma_N^2(\hat{R}_N)$ in (3.8). These, we state below.

\begin{align}
(4.11) \quad \frac{1}{N} \sum_{a=1}^{N} [J_N^2\left( \frac{a}{N+1} \right) - J(\frac{a}{N+1})]^2 &= o(1),
\end{align}

\begin{align}
(4.12) \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [J_N\left( \frac{N}{N+1} H_N(x) \right) J_N\left( \frac{N}{N+1} H_N(y) \right) - J(\frac{N}{N+1} H_N(x)) J(\frac{N}{N+1} H_N(y))] \, dF_N[1, 1](x, y) = o_p(1)
\end{align}

for all $i \neq j = 1, \ldots, p$. 

Finally, we define

\[ (4.13) \quad \psi_{ij}(F) = \int \int_{-\infty}^{\infty} J(H(x))J(H(y))dF[i,j](x,y) \quad \text{for } i, j = 1, \ldots, p; \]

\[ (4.14) \quad \psi(F) = (\psi_{ij}(F))_{i,j=1,\ldots,p} \]

\[ (4.15) \quad \text{(c.5) \quad Rank of } \psi(F) \geq 2. \]

It may be noted that for testing the hypothesis \( H_0 \) we shall consider the class of rank order tests for which \( J(h) \) is monotonic in \( h \): \( 0 < h < 1 \) (this point will be made clear at a later stage.), and hence, it can be shown that if the scatter of \( \tilde{X} \) in \( F(x) \) is not confined to any one dimensional space on the \( p \)-dimensional Euclidean space, then (c.5) holds.

**Lemma 4.1.** Let us define

\[ (4.16) \quad A^2 = \int_0^1 J^2(u)du, \quad \text{and} \]

\[ (4.17) \quad \tilde{\psi} = (\psi)_{i,j=1,\ldots,p} = (\psi_{ij}(F)). \]

Then, if (c.5) holds,

\[ (4.18) \quad A^2 - \tilde{\psi} > 0. \]

**Proof.** It follows from (4.13) that

\[ (4.19) \quad \sum_{i=1}^{p} \psi_{ii}(F) = \sum_{i=1}^{p} \int_{-\infty}^{\infty} J^2(H(x))dF[i](x) = \int_{-\infty}^{\infty} J^2(H(x))dH(x) = pA^2. \]

Thus, we get from (4.16), (4.17) and (4.19) that
\[ A^2 - \delta = \frac{1}{p} \sum_{i=1}^{p} \gamma_{ii}(F) - \frac{1}{p(p-1)} \sum_{i \neq j=1}^{p} \gamma_{ij}(F). \]

\[ = \frac{1}{p} \sum_{i=1}^{p} \gamma_{ii}(F) - \frac{1}{p(p-1)} \sum_{i=1}^{p} \sum_{j=1}^{p} \gamma_{ij}(F). \]

(4.20)

\[ = \frac{1}{p-1} \left[ \frac{1}{p} \sum_{i=1}^{p} \gamma_{ii}(F) - \frac{1}{p^2} \sum_{i=1}^{p} \sum_{j=1}^{p} \gamma_{ij}(F) \right]. \]

Now, \[ \left| \frac{1}{p^2} \sum_{i=1}^{p} \sum_{j=1}^{p} \gamma_{ij}(F) \right| \leq \left( \left( \frac{1}{p} \sum_{i=1}^{p} \gamma_{ii}(F) \right)^{\frac{1}{2}} \right)^2, \]

the equality sign holds only when \( \gamma_{ij}(F) = [ \gamma_{ii}(F), \gamma_{jj}(F) ]^{\frac{1}{2}} \),

for all \( i, j = 1, \ldots, p \). Now, the condition (c.5) implies that there is at least one \((i,j)|i \neq j = 1, \ldots, p\) for which

\[ \left| \gamma_{ij}(F) \right| < [ \gamma_{ii}(F), \gamma_{jj}(F) ]^{\frac{1}{2}}. \]

Thus, under (c.5), we have

(4.21)

\[ \left| \frac{1}{p^2} \sum_{i=1}^{p} \sum_{j=1}^{p} \gamma_{ij}(F) \right| < \left( \frac{1}{p} \sum_{i=1}^{p} \gamma_{ii}(F) \right)^{\frac{1}{2}}. \]

Again, by elementary inequality relating to moments, we have

(4.22)

\[ \left( \frac{1}{p} \sum_{i=1}^{p} [ \gamma_{ii}(F) ]^{\frac{1}{2}} \right)^2 \leq \frac{1}{p} \sum_{i=1}^{p} \gamma_{ii}(F), \]

where the equality sign holds only when \( \gamma_{11}(F) = \ldots = \gamma_{pp}(F) \).

Thus, from (4.20), (4.21) and (4.22), we get that under (c.5), the right hand side of (4.20) will be strictly positive.

Hence, the lemma.
THEOREM 4.2. Under conditions (c.1) through (c.5), \( \sigma^2_N(r_N) \) defined in (3.8), converges in probability to \([(p-1)/p][\bar{A}^2 - \bar{\delta}] > 0 \), where \( \bar{A}^2 \) and \( \bar{\delta} \) are defined in (4.16) and (4.17), respectively.

PROOF. We may rewrite \( \sigma^2_N(r_N) \) in (3.8) as

\[
(4.23) \quad \frac{p-1}{p} \cdot \frac{1}{N} \sum_{a=1}^{N} E_{N,a}^2 - \frac{1}{p^2} \sum_{i,j=1}^{p} \left\{ \frac{1}{n} \sum_{a=1}^{n} E_{N,R_{ia}} E_{N,R_{ja}} \right\}.
\]

Now, by virtue of condition (c.4), it can be easily shown that

\[
(4.24) \quad \frac{1}{N} \sum_{a=1}^{N} E_{N,a}^2 \to \int_{0}^{1} \int_{0}^{1} \frac{1}{J^2}(u)du = \bar{A}^2.
\]

Again, \( \frac{1}{n} \sum_{a=1}^{n} E_{N,R_{ia}} E_{N,R_{ja}} \) may also be written (after using (4.12),) as

\[
(4.25) \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} J\left(\frac{N}{N+1} H_N(x)\right) dF_{N[i,j]}(x,y) + o_p(1),
\]

for \( i \neq j = 1, \ldots, p \). It may be noted now that \( F_{N[i]}(i=1, \ldots, p) \) are the sample empirical cdf's based on \( n \cdot i \cdot d \cdot r \cdot v \). Hence, on using the well-known result on Kolmogorov-Semirnov statistic, we have

\[
(4.26) \quad n^{1/2} \left[ \sup_{x} \left| F_{N[i]}(x) - F_{[i]}(x) \right| \right] \text{ bounded in probability},
\]

for each \( i = 1, \ldots, p \). Consequently on using (4.2) and (4.5), we get that

\[
(4.27) \quad \sup_{x} n^{1/2} \left| \frac{N}{N+1} H_N(x) - H(x) \right| \leq p^{-1/2} \sum_{i=1}^{p} \left\{ \sup_{x} n^{1/2} \left| \frac{F_{N[i]}(x) - F_{[i]}(x)}{p^{1/2}} \right| \right\}
\]

is also bounded in probability (by Poincare's theorem on total Probability.).
Again, by elementary algebra, we have

\[(4.28) \quad 0 \leq dF_N[i](x) \leq p dH_N(x), \quad 0 \leq d[1-F_N[i](x)] \leq p d[1-H_N(x)],\]

for all \(i = 1, \ldots, p\). Hence, proceeding precisely on the same line as the proof of theorem 5.2 of Puri and Sen [21], and omitting the details of the derivation, we will arrive at the stochastic convergence of \((4.25)\) to \(\mathcal{V}_{ij}\), defined in \((4.13)\), for all \(i, j = 1, \ldots, p\). Consequently, from \((4.23)\) and \((4.24)\) and the convergence of \((4.25)\) to \(\mathcal{V}_{ij}\), we arrive at the following

\[(4.29) \quad \sigma^2_N \to \frac{\Sigma}{p} \frac{A^2 - 1}{p^2} \frac{\Sigma}{i \neq j} = \frac{P}{p} \frac{A^2 - 1}{p} \frac{\Sigma}{j} > 0,\]

where \(\Sigma\) is defined in \((4.17)\) and where by lemma 4.1, \(A^2 - 1 > 0\).

Hence, the theorem.

**Theorem 4.3.** If conditions (c.1) through (c.5) hold then under the permutational
probability measure \(\mathcal{\Theta}_n\), the statistic \(W_N\), in \((2.10)\), has asymptotically, in
probability, a chi-square distribution with \((p-1)\) degrees of freedom.

(It may be noted that the permutation distribution of \(W_N\) is essentially a conditional distribution, depending on \(Z_N\) in \((2.1)\). Hence, the implication of the above theorem is that it holds, in probability, i.e., for almost all \(Z_N\).)

**Proof.** Let us first prove that under \(\mathcal{\Theta}_n\), \(\{n_T \Sigma (T_N, i - E_N), i = 1, \ldots, p\}\) has asymptotically, in probability, a multinormal distribution of rank \((p-1)\). As in
\((2.11)\), we have shown that \(\Sigma (T_N, i - E_N) = 0\), it follows that the rank of \(T_N\)
may be at most equal to \(p-1\). So, if we can show that for any non-null real
\(\delta = (\delta_1, \ldots, \delta_{p-1})\), \(\Sigma_{i=1}^{p-1} \delta_i n_T (T_N, i - E_N)\) has a non-degenerate and asymptotically normal (permutation) distribution, our desired result will follow. Now, using
(2.11), we can also write

\[ (4.30) \quad n^{\frac{1}{q}} \sum_{i=1}^{p} \delta_i (T_N, i - E_N) = n^{\frac{1}{q}} \sum_{i=1}^{p} \delta'_i T_N, i, \]

where

\[ (4.31) \quad \sum_{i=1}^{p} \delta'_i = 0 \text{ and at least one of } (\delta'_1, \ldots, \delta'_p) \neq 0. \]

Thus, it is sufficient to show that \( n^{\frac{1}{q}} \) times any arbitrary contrast in \( T_N \), has asymptotically a normal (permutation) distribution. Now, using (2.9), (2.10) and condition (c.2), we may write (4.30) as

\[ (4.32) \quad n^{\frac{1}{q}} \sum_{\alpha=1}^{n} \sum_{i=1}^{p} \delta'_i J(\frac{R_{i\alpha}}{N+1}) + o_p(1). \]

Let us then define

\[ (4.33) \quad Y_{N, \alpha} (R_N) = \sum_{i=1}^{p} \delta'_i J(\frac{R_{i\alpha}}{N+1}), \alpha = 1, \ldots, n. \]

It thus follows from (4.32) and (4.33) that we are only to show that \( n^{\frac{1}{q}} \sum_{\alpha=1}^{n} Y_{N, \alpha} (R_N) \) has asymptotically (under \( \mathcal{O}_n \)) a non-degenerate normal distribution, in probability. Now, under \( \mathcal{O}_n \), there are \( p! \) equally likely permutations of \( (R_{1\alpha}, \ldots, R_{p\alpha}) \) among themselves, and hence, \( Y_{N, \alpha} (R_N) \) can have only \( p! \) equally likely permuted values, each with probability \( 1/p! \). Thus,

\[ (4.34) \quad \mathbb{E} \{ Y_{N, \alpha} (R_N) \mid \mathcal{O}_n \} = \sum_{i=1}^{p} \delta'_i \left( \sum_{\alpha=1}^{p} J(\frac{R_{i\alpha}}{N+1}) \right) = 0, \]
(4.35) \[ E \left\{ Y_{N,a}^2(R_N) \bigg| \mathcal{P}_n \right\} = \frac{p}{\Sigma} (\xi'_i)^2 \left\{ \frac{1}{p-1} \Sigma \left[ J_{(N+1)}^a \right] - \frac{1}{p} \Sigma \left[ \frac{R(a)}{N+1} \right]^2 \right\}, \]

for \( a = 1, \ldots, n \). Since, for each \( a \), the pi permutations have nothing to do with the permutations for other \( a' \) \( (a \neq a' = 1, \ldots, n) \), \( \{ Y_{N,a}(R_N) \mid a = 1, \ldots, n \} \) are (under \( \mathcal{P}_n \)) stochastically independent. To prove the central limit theorem for \( \{ Y_{N,a}(R_N) \mid a = 1, \ldots, n \} \) (under \( \mathcal{P}_n \)), we shall now use the Berry-Essen theorem (cf. Loeve [19, p. 288]), which may be stated as follows:

Let \( \{ \hat{w}_i \} \) be any sequence of independent random variables with means \( \{ \mu_i \} \), variances \( \{ \sigma_i^2 \} \) and absolute third order moments \( \{ \rho_i \} \); let then

\[ S_n^2 = \Sigma_{i=1}^n \sigma_i^2, \quad \rho_n = \Sigma_{i=1}^n \rho_i. \]

Also, let \( G_n(x) \) be the cdf of \( \Sigma_{i=1}^n (\hat{w}_i - \mu_i)/S_n \), and \( \Phi(x) \) be the standardized normal cdf. Then there exists a finite constant \( c(<\infty) \), such that for all \( x \)

\[ |G_n(x) - \Phi(x)| < c \rho_n S_n^3. \]

(4.36) \[ |G_n(x) - \Phi(x)| < c \rho_n S_n^3. \]

(It may be noted that instead of a single sequence of stochastic variables, we may have a double sequence \( \{ \hat{w}_{n,i} \} \) with means \( \{ \mu_{n,i} \} \), variances \( \{ \sigma_{n,i}^2 \} \) etc., and the theorem also applies to this situation.). Now, from (4.35) we get the following precisely on the same line as in theorem 4.2 that

\[ \frac{1}{n} S_n^2 = \frac{1}{n} \Sigma_{a=1}^n V(Y_{N,a}(R_N) \bigg| \mathcal{P}_n) \]

\[ = \frac{p}{\Sigma} (\xi'_i)^2 \cdot \frac{p}{p-1} \left\{ \frac{1}{N} \Sigma \left[ J_{(N+1)}^a \right] - \frac{1}{p} \Sigma \left[ \frac{R(a)}{N+1} \right]^2 \right\}, \]

(4.37) \[ \frac{p}{\Sigma} (\xi'_i)^2. (A^2 - \tilde{S}) > 0, \]
by lemma 4.1. Again,

\[
\frac{1}{n} \frac{1}{\nu} = \frac{1}{n} \sum_{\alpha=1}^{n} \mathbb{E} \left[ \frac{Y_{N,\alpha} (R_N)}{\nu} \right] \leq \left\{ \frac{P}{\nu} \sum_{i=1}^{\max_{1 \leq \alpha \leq N} J(\frac{\alpha}{N+1})} \right\} \frac{1}{n} \sum_{\alpha=1}^{N} \mathbb{E} \left[ Y_{N,\alpha} (R_N) \right] \left\{ \frac{2}{\nu} \right\} \leq (i=1) \delta_1 \|N \delta \cdot \frac{S^2}{n}, \text{ by condition (c.3)}.
\]

Consequently, from (4.37) and (4.38), we get that

\[
(4.39) \quad \frac{1}{n} \frac{1}{\nu} S_n^{-2} \leq KN^{-\delta} \sum_{i=1}^{P} \frac{\delta}{\nu_i} \left| \frac{1}{s_n^2} \right| = O_p(N^{-\delta}).
\]

Hence, from (4.6), we may conclude that \( n^\delta \sum_{i=1}^{P} \delta \left( T_{N,i} - E_N \right) \) has asymptotically, in probability, a normal (permutation) distribution, for any \((\delta_1, \ldots, \delta_P)\) satisfying (4.31). This proves that \( n^\delta \left( \left( T_{N,i} - E_N \right), i = 1, \ldots, p - 1 \right) \) has asymptotically, in probability, a \( p-1 \) variate normal distribution (under \( \mathbb{P} \)). Now by considering the exponent of this asymptotic multinormal distribution and using some well-known results on the limiting distribution of continuous functions of random variables (cf. Cramer [8]), it can be readily shown that under the permutational probability measure \( \mathbb{P} \), the statistic \( W_N \) in (3.10), has asymptotically, in probability, a \( \chi^2 \) distribution with \( (p-1) \) d.f.

Hence, the theorem.

Let us now denote by \( \chi^2_{p-1, \varepsilon} \) the upper \( 100\varepsilon \% \) point of the chi-square distribution with \( p-1 \) d.f. Then from (3.11), (3.12) and theorem 4.3, we readily arrive at the following.

**Theorem 4.4.** Under the permutational probability measure \( \mathbb{P} \),

\[
W_{N, \varepsilon} \xrightarrow{P} \chi^2_{p-1, \varepsilon} \text{ and } \gamma_{N, \varepsilon} (R_N) \xrightarrow{P} 0.
\]
By virtue of theorem 4.4, the exact permutation test in (3.11) reduces asymptotically to

\[
\phi(Z_N) = \begin{cases} 
1, & \text{if } W_N \geq \chi^2_{p-1, \varepsilon} \\
0, & \text{otherwise}
\end{cases}
\]

(4.40)

In the sequel, (4.40) will be termed the large sample permutation test while (3.11) as the exact permutation test. For the study of the asymptotic properties of the proposed permutation tests, it appears that we may use (4.40) instead of (3.11). Now, the study of the asymptotic properties will require the knowledge of the asymptotic form of the unconditional distribution of \( W_N \), which we shall proceed to consider in the next section.

5. Asymptotic multinormality of the standardized form of \( T_N \). We adopt here precisely the same notations as in the beginning of section 4, and with the help of (2.5), (4.1) and (4.2), we rewrite \( T_N, i \) as

\[
T_{N, i} = \int_{-\infty}^{\infty} J_N(x) \prod_{j=1}^{N} H_N(x_j) dF_{N[i]}(x), \quad i = 1, \ldots, p.
\]

(5.1)

Apparently, (5.1) has the same form as that of Chernoff-Savage [7] type of rank order statistics related to the multisample case, considered by Puri [20](also [11]). However, in their case, all the samples are stochastically independent, while in our case, it is a \( p \)-variate sample. Let us then introduce the following definitions.

\[
\mu_{N, i} = \int_{-\infty}^{\infty} J(H(x)) dF_{i}[x], \quad i = 1, \ldots, p;
\]

(5.2)
(5.3) \[ \beta_{i,k}(x) = \int_{\infty < y < \infty} F_i(x)[1-F_i(y)]J'(H(x))J'(H(y))dF_k(x)dF_k(y) \]
\[ + \int_{\infty < y < \infty} F_i(x)[1-F_i(y)]J'(H(x))J'(H(y))dF_k(x)dF_k(y); \]

(5.4) \[ \beta_{i,j,k}(x) = \int_{\infty < y < \infty} [F_{i,j}(x,y)-F_i(x,F_{j,y}(y))]J'(H(x))J'(H(y))dF_k(x)dF_k(y) \]
for \( i \neq j = 1, \ldots, p, k, \lambda = 1, \ldots, p \). Finally, let

(5.5) \[ \beta^*_{i,j} = \frac{1}{p} \left\{ \sum_{k=1}^{p} \sum_{\lambda=1}^{p} [\beta_{k,i,j} + \beta_{i,j,k} - \beta_{k,j,i} - \beta_{i,k,j}] \right\}, \]
for \( i, j = 1, \ldots, p \), and

(5.6) \[ \beta^* = (\beta^*_{i,j})_{i,j = 1, \ldots, p}. \]

**Theorem 5.1.** If the conditions (c.1), (c.2) and (c.3) of section 4 hold, then the random vector \( N^p[(T_{N,i}-\mu_{N,i}), i = 1, \ldots, p] \) has asymptotically a multinormal distribution with a null mean vector and a dispersion matrix \( \beta^* \).

(It may be noted that by virtue of (2.11), the above multinormal distribution will be essentially singular having a rank less than or equal to \( p-1 \).)

**Proof.** We proceed precisely on the same line as in the proof of theorem 5.1 of [11] and write

(5.4) \[ T_{N,i} = \mu_{N,i} + E_{1,N}^{(i)} + B_{2,N}^{(i)} + \sum_{\lambda=1}^{p} C_{\lambda,N}^{(i)}, i = 1, \ldots, p, \]
where
(5.8) $B_{1,N}^{(i)} = \int J(H)d[F_N[i](x) - F[i](x)],$

(5.9) $B_{2,N}^{(i)} = \int [H_N(x) - H(x)] J'(H(x))dF[i](x),$

(5.10) $C_{1,N}^{(i)} = \frac{1}{N+1} \int H_N(x)J'(H(x))dF_N[i](x),$

(5.11) $C_{2,N}^{(i)} = \int [H_N(x) - H(x)]J'(H(x))d[F_N[i](x) - F[i](x)],$

(5.12) $C_{3,N}^{(i)} = \int [J(\frac{N}{N+1}H_N(x)) - J(H(x)) - (\frac{N}{N+1}H_N(x) - H(x))J'(H(x))]dF_N[i](x),$

and

(5.13) $C_{4,N}^{(i)} = \int [J(\frac{N}{N+1}H_N(x)) - J(\frac{N}{N+1}H_N(x))]dF_N[i](x);$

in all expressions the range of integration is over $-\infty$ to $\infty$. Now, by condition (c.2), (5.13) will be $o_p(n^{-\frac{1}{2}})$, and precisely on the same line as in the treatment of $C_{13N}$ of Chernoff and Savage [7, p. 988] it is easily seen that $C_{1,N}^{(i)}$ is also $o_p(n^{-\frac{1}{2}})$, uniformly in $i = 1, \ldots, p$. Further, it has been shown by the present author ([26]) that in the case of $p = 2$, $C_{2,N}^{(i)}$ and $C_{3,N}^{(i)}$ (for $i = 1$) are both $o_p(n^{-\frac{1}{2}})$. Essentially, the same argument holds for the general case of $p \geq 2,$ and hence, avoiding the details of these, we may write

(5.14) $N^{\frac{1}{2}} \mid (T_{N,i} - \mu_{N,i}) - (B_{1,N}^{(i)} + B_{2,N}^{(i)}) \mid = o_p(1),$

for all $i = 1, \ldots, p$. Consequently, it is sufficient to prove that $\{N^{\frac{1}{2}}(B_{1,N}^{(i)} + B_{2,N}^{(i)}), i = 1, \ldots, p\}$ has asymptotically a multinormal distribution. Now, by partial integration of (5.8), we readily arrive at the following expression, through a few simple steps.
(5.15) \[ B_{1,N}^{(1)} + B_{2,N}^{(1)} = \frac{1}{P} \sum_{k=1}^{P} \left\{ \frac{1}{n} \sum_{a=1}^{n} \left[ B_{1:k}^{(1)}(X_{1a}) - B_{k:1}^{(1)}(X_{ka}) \right] \right\} , \]

where

(5.16) \[ B_{k:q}^{(1)}(X_{ka}) = \int_{0}^{\infty} \left[ F_{1[k]}^{(a)}(x) - F_{[k]}^{(a)}(x) \right] J'(H(x)) dF_{[q]}(x), \]

(5.17) \[ F_{1[k]}^{(a)}(x) = \begin{cases} 0 & \text{if } x < X_{ka} \\ 1 & \text{if } x \geq X_{ka} \end{cases} , \]

for \( k, q = 1, \ldots, p \) and \( a = 1, \ldots, n \). We shall now show that for any non-null real \( p \)-vector \( \delta = (\delta_{1}, \ldots, \delta_{p}) \), the random variable

(5.18) \[ N^{\frac{1}{2}} \sum_{i=1}^{P} \delta_{i} \left[ B_{1,N}^{(i)} + B_{2,N}^{(i)} \right] \]

has asymptotically a normal distribution. Now, we may also rewrite

(5.18) as \( N^{\frac{1}{2}} \sum_{k=1}^{P} \sum_{q=1}^{P} \delta_{kq}^{*} \left[ \frac{1}{n} \sum_{a=1}^{n} B_{k:q}^{(1)}(X_{ka}) \right] \), where \( \delta^{*} = (\delta_{11}^{*}, \ldots, \delta_{pp}^{*}) \) is also non-null and real. Now, if we write

(5.19) \[ B(X_{\sim a}; \delta^{*}) = \frac{1}{P} \sum_{k=1}^{P} \sum_{q=1}^{P} \delta_{kq}^{*} B_{k:q}^{(1)}(X_{ka}) , \quad a = 1, \ldots, n , \]

it follows from the discussion made above that (5.18) may also be written as

(5.20) \[ N^{\frac{1}{2}} \left\{ \frac{1}{n} \sum_{a=1}^{n} B(X_{\sim a}; \delta^{*}) \right\} , \]

which apart from the factor \( N^{\frac{1}{2}} \), is the average of \( n \) independent and identically distributed random variables \( \left\{ B(X_{\sim a}; \delta^{*}), a = 1, \ldots, n \right\} \). Hence, to apply the
classical central limit theorem under Lindeberg condition, it is sufficient to show that $B(X_a, S^*)$ has finite first and second order moments. We shall prove a slightly stronger result that for any $\eta: 0 < \eta < S$ (defined in (4.10)),

$E \left| B(X_a, S^*) \right|^{2+\eta} < \infty$, uniformly in $F[1], \ldots, F[p]$. Now, using (5.19) and some well-known inequalities, we have

$$E \left| B(X_a, S^*) \right|^{2+\eta} \leq P \sum_{k=1}^{p} \sum_{q=1}^{p} \left| S_{kq}^* \right|^{2+\eta} E_{B_k,q}(X_{ka})^{2+\eta},$$

and proceeding precisely on the same line as in the case of univariate several sample observations (for instance, see [11, (4.18) and (4.19)]), we can easily prove that for all $0 < \eta < S$,

$$E \left| B_k,q(X_{ka}) \right|^{2+\eta} \leq K \left| u(1-v) \right| \left| J'(u) \right| \left| J'(v) \right| du dv < \infty,$$

by condition (c.3) in (4.10). Thus, from (5.21) and (5.22), we conclude that $B(X_a, S^*)$ has a finite moment of the order $2 + \eta$, $\eta > 0$, and this in turn, implies that the first two moments of the same are finite. Hence, we arrive at the asymptotic normality of the variable in (5.18), and this implies the asymptotic normality of the joint distribution of $N_0(T_{N,i} - \mu_{N,i})$, $i = 1, \ldots, p$. Again, from (5.16), we get by an application of Fubini's theorem that

$$\text{Cov}(B_{i,j}(X_{ia}), B_{k,q}(X_{ka})) = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ \beta_{ik,jq} & \text{if } \alpha = \beta, \end{cases}$$

(where $\beta_{ij,kq}$ is defined in (5.3) and (5.4),) for $i, j, k, q = 1, \ldots, p; \alpha, \beta = 1, \ldots, n$. From (5.15), (5.16) and (5.23), we readily arrive at
(5.24) \[ \lim_{N \to \infty} \left\{ N \text{ Cov}(T_N, i, T_N, j) \right\} = \beta^*_{i,j}, \ i, \ j = 1, \ldots, p, \]

where \( \beta^* = (\beta^*_{i,j}) \) is defined in (5.6).

Hence, the theorem.

It has already been pointed out earlier that the asymptotic multinormal distribution, derived in theorem 5.1, is singular and is of rank at most equal to \( p - 1 \). If the null hypothesis (1.5) holds, and we define \( H(x), H^*(x, y), A^2 \) and \( \delta \) as in (4.5), (4.6), (4.16) and (4.17), respectively, then it will readily follow from (5.24), (5.3) and (5.4) (though a few simple steps) that

(5.25) \[ \lim_{N \to \infty} \left\{ N \text{ Cov}(T_N, i, T_N, j \ \big| H_M \text{ in (1.5)}) \right\} = (\delta_{i,j})^{p-1}(A^2 - \delta), \]

for \( i, j = 1, \ldots, p \), where \( \delta_{i,j} \) is the usual Kronecker delta. Consequently, with the help of lemma 4.1, we readily arrive at the following.

**COROLLARY 5.1.1.** If \( H_M \text{ in (1.5)} \) holds and the conditions of theorem 5.1 holds, then under (c.5) in (4.5), \( [N^2(T_N, i - \mu), i = 1, \ldots, p] \) has a singular multinormal distribution of rank \( p - 1 \), (where \( \mu = \int_{-\infty}^{\infty} J(u)du \)).

We shall now consider the usual type of Pitman's translation alternatives, and for this, we replace the parent cdf \( F(x) \) by a sequence of cdf's \( F_{N_i}^{(x)} \), such that the marginal cdf's of \( \{F_{N_i}^{(x)}(x)\} \) satisfy the sequence of alternatives \( \{H_N\} \), where

(5.25) \[ H_N: F_{[i]} \{N_i\} (x) = H(x + N^{-\frac{1}{2}} \Theta_i), i = 1, \ldots, p, \]

where \( H(x) \) is assumed to be an absolutely continuous (univariate) cdf having a continuous density function \( h(x) \), and where the assumption of equality of scales and symmetry in (1.5) are also assumed to hold for the sequence of cdf's \( \{F_{N_i}^{(x)}\} \). Let us then define
\[ (5.26) \quad \mathcal{L}(H) = \lim_{x \to \infty} \frac{d}{dx} J(H(x))dH(x). \]

Then, it is easy to verify that

\[ (5.27) \quad \lim_{N \to \infty} \left[ N^p \mathbb{E}[\{T_{N,i} - \mu\} | H_N] \right] = \Theta_i \mathcal{L}(H), \quad i = 1, \ldots, p; \]

\[ (5.28) \quad \lim_{N \to \infty} \left[ \text{Cov}(T_{N,i}, T_{N,j} | H_N) \right] = (\delta_{ij} p - 1)(\Lambda^2 - \bar{v}), \]

for \( i, j = 1, \ldots, p \). Consequently, it follows from theorem 5.1, that under \( \{H_N\}, \{N^p (T_{N,i} - \mu), \quad i = 1, \ldots, p - 1 \} \) has asymptotically a \((p - 1)\) variate normal distribution with a mean vector \( \mathcal{L}(H)(\Theta_1, \ldots, \Theta_p) \) and a dispersion matrix \( (\delta_{ij} p - 1)(\Lambda^2 - \bar{v}) \).

\[ (5.29) \quad (\delta_{ij} p - 1)(\Lambda^2 - \bar{v}). \]

Hence, it readily follows that under \( \{H_N\} \)

\[ (5.30) \quad W_N^g = \frac{n}{A^2 - \bar{v}} \sum_{i=1}^{p} (T_{N,i} - \bar{E}_N)^2 \]

has asymptotically a non-central \( \chi^2 \) distribution with \((p-1)\) d.f. and the noncentrality parameter

\[ (5.31) \quad \Delta_N = \left\{ \frac{\mathcal{L}(H)^2}{(\Lambda^2 - \bar{v})} \right\} \left\{ \frac{1}{p} \sum_{i=1}^{p} (\Theta_i - \bar{\Theta})^2 \right\}, \]

where \( \bar{\Theta} = \frac{1}{p} \sum_{i=1}^{p} \Theta_i. \)

Now, from (3.10), theorem 4.2 and (5.30), we readily get that under \( \{H_N\} \)

\[ W_N \overset{\sim}{\sim} W_N^g, \]

where \( \overset{\sim}{\sim} \) means asymptotically equivalent, in probability. Hence, we arrive at the following.
THEOREM 5.2. Under the sequence of alternative hypotheses \( \{ H_N^2 \} \) in (5.25), the statistic \( W_N \) in (3.10), has asymptotically a non-central \( \chi^2 \) distribution with \( (p - 1) \) d.f., and the noncentrality parameter \( \Delta_w \), defined in (5.31).

At this stage, we may consider also some asymptotically distribution free tests for \( H_M^2 \) in (1.5). This may be formulated as follows. Let \( S^2 \) be some consistent estimator of \( \bar{\alpha}^2 - \bar{\gamma} \), in the sense that

\[
S^2 \xrightarrow{P} \alpha^2 - \gamma \text{ for all } \Gamma_0 \in \mathbf{F}_c, \text{ in (1.4)}.
\]

Then, it follows from (5.30) and a well-known limit theorem by Cramer [8, p. 253] that under \( \{ H_N^2 \} \) in (5.25)

\[
(5.33) \quad \hat{W}_N = \frac{n}{S^2} \sum_{i=1}^{P} (T_{N,i} - E_N)^2 \xrightarrow{P} \hat{W}_N^*.
\]

Hence, the test based on \( \hat{W}_N \) will be asymptotically a distribution-free test for \( H_M^2 \) in (1.5). It further follows from theorem 5.2, that the test based on \( \hat{W}_N \) in (3.10) will be asymptotically power equivalent to the one based on \( \hat{W}_N \), for any sequence of alternatives of the type \( \{ H_N^2 \} \) in (5.25).

Thus, the permutation test based on \( \hat{W}_N \) in (3.10) also appears to be power equivalent (asymptotically) to unconditional tests based on stochastically equivalent statistics. Since, we have shown that the permutation test has the advantage of being exactly distribution-free even for small sample sizes, we may advocate the unrestricted use of the same for all sample sizes.

6. Asymptotic efficiency of rank order tests. We shall now consider the asymptotic relative efficiency (A.R.E.) of our proposed rank order tests with respect to the likelihood ratio (\( L_M \)) test, considered by Wilks (1946). It is easily seen that
under the sequence of alternatives \( \{H_N\} \) in (5.25), Wilks' \( L_M \) statistic has asymptotically (actually, \(-2 \log_e L_M\)) a noncentral chi-square distribution with \((p - 1)\) d.f. and the noncentrality parameter

\[
(6.1) \quad \Delta_L = \frac{1}{\sigma^2(1-\bar{\rho})} \left\{ \frac{1}{p} \sum_{i=1}^{p} (\theta_i - \bar{\theta})^2 \right\},
\]

where \( \bar{\rho} \) is the average (over the \( \binom{p}{2} \) possible pairs) correlation coefficient between \( X_i \) and \( X_j \), \( i \neq j = 1, \ldots, p \), and \( \sigma^2 \), the common variance.

Let us now write \( A^2 - \bar{\rho} \) (in (5.31)) in the form \( A^2(1 - \bar{\rho}_o) \), where \( \bar{\rho}_o = \bar{\rho}/A^2 \) is the average score-correlation of the \( p \)-variates. Then from theorem 5.2 and (6.1), we get that the A.R.E. of the \( W \)-test with respect to the \( L_M \)-test is given by

\[
e(\mathcal{W}, L_M) = a \cdot \sigma^2 (1 - \bar{\rho})/A^2(1 - \bar{\rho}_o)
\]

(6.2). 

The first factor on the right-hand side of (6.2) is solely dependent on the marginal distribution \( H(x) \) in (4.5), while the second factor depends on the joint distribution \( F(x) \), through the bivariate marginal cdf \( H^*(x, y) \), in (4.6). Various bounds for the first factor are available in the literature for various common types of \( J(u) \): 0 \(< u \< 1 \), and for an excellent account of these the reader is referred to Hodges and Lehmann ([12], [13]), and Chernoff and Savage [7]. The bounds for the second factor will depend in a quite involved manner on the parent cdf \( F(x) \). However, implicitly here we are dealing with the case of total symmetry of multivariate cdf's, and for this case, Bickel [6] has considered
bounds for \((1 - \rho)/(1 - \rho_j)\) for bivariate normal cdf and for the specific cases of median and rank sum tests which are characterized by

\[
J(u) = \begin{cases} 
1 & u \leq \frac{1}{2} \\
0, & u > \frac{1}{2} 
\end{cases}, \quad \text{and} \quad J(u) = u: 0 < u < 1,
\]

respectively. For normal scores, extension of this efficiency-result to multivariate normal cdf's, is due to Bhattacharyya [3]. However, no general conclusions can be made about (6.2), for arbitrary multivariate cdf's. For the specific case of multinormal cdf's, however, the reader is referred to the works of Bickel [6] and Bhattacharyya [5].

7. Nonparametric analysis of orthogonal two factor layouts. We shall now use the results of earlier sections to derive a simple method of obtaining nonparametric rank order analysis of variance tests for two factor (orthogonal) designs, and proceed to study its efficiency aspects. Let us denote the two factors by block and treatment, and consider a randomized block layout, where each block contains \(p\) plots and the \(p\) treatments are allocated at random in these \(p\) plots. Let then \(X_{ij}\) stand for the yield of the plot in the \(i\)th block receiving the \(j\)th treatment, for \(i = 1, \ldots, n, j = 1, \ldots, p\). We write

\[
X_{ij} = \mu + a_i + \tau_j + e_{ij},
\]

where \(\mu\) is the mean effect, \(a_i\), the effect particular to the \(i\)th block, \(\tau_j\) the effect specific to the \(j\)th treatment, and \(e_{ij}\) is the residual error component which we assume to be mutually independent for different \((i,j), i = 1, \ldots, n, j = 1, \ldots, p\), and to have a common continuous cdf \(G(e)\). We may relax this assumption to some extent as that the joint cdf of \((e_{i1}, \ldots, e_{ip})\) is a symmetric
function of its arguments, for all \( i = 1, \ldots, n \). The block effects \( a_i, i = 1, \ldots, n \) may be non-stochastic or may even be stochastic in nature, and we may without any loss of generality, put

\[
(7.2) \quad \sum_{j=1}^{p} \tau_j = 0.
\]

The null hypothesis, to be tested, relates to

\[
(7.3) \quad H_0: \tau_1 = \ldots = \tau_p = 0.
\]

For this problem, Friedman [10] has suggested the use of rankings within each of the \( n \) blocks and consider a test based on these \( np \) ranks. His procedure has been generalized to balanced incomplete block designs by Durbin [9] and to general blocks by Benard and Elteren [1]. Brown and Mood [6] have considered the median procedure for the same problem, and further extensions on this line are due to Bhapkar [2]. In all these cases, the rankings are done separately within each block, and it has been argued by Hodges and Lehmann [14] that as these do not utilize any information contained in the interblock comparisons, some rank procedure which preserves this information will be presumably more efficient, at least for more general designs that are not orthogonal. For this purpose, they have suggested the use of ranking after alignment, and have also considered the use of Wilcoxon's and Kruskal-Wallis' rank sum tests, in this context. The object of the present investigation is to generalize this procedure to a more general class of rank order statistics and to throw light on the efficiency aspects of this method. In the present paper, the case of orthogonal designs (with equal number of observations per cell) will be considered, while the general case of non-orthogonal design will be taken up in a separate issue.
Now, by definition of $X_{ij}$ in (7.1), any intra-block contrast (i.e., $\sum_{j=1}^{p} C_{ij} X_{ij}$ where $\sum_{j=1}^{p} C_{ij} = 0$) is a contrast in treatment effects $\tau_j$, $(j = 1, \ldots, p)$ and in the error components $e_{ij}$'s, but is independent of $\mu$ and $\alpha_i$, $i = 1, \ldots, n$. Thus, if we define

$$(7.4) \quad C_{ij} = \delta_{ij} - 1/p, \quad j = 1, \ldots, p,$$

where $\delta_{ij}$ is the usual Kronecker delta, then, it is easily seen that on defining

$$(7.5) \quad \theta_{ij} = \sum_{j=1}^{p} C_{ij} \tau_j, \quad j = 1, \ldots, p,$$

the random variable

$$(7.6) \quad Y_{i\lambda} = \sum_{j=1}^{p} C_{ij} X_{ij} = \theta_{ij} + \sum_{j=1}^{p} C_{ij} e_{ij}, \quad \lambda = 1, \ldots, p$$

are linearly dependent, while any $p - 1$ of them will be linearly independent. Now, if the null hypothesis (7.3) holds, then from (7.5), we have $\theta_{ij} = 0$ for all $\lambda = 1, \ldots, p$, and hence, from (7.6), we may conclude that the joint distribution of $Y_{i1}, \ldots, Y_{ip}$ will be a symmetric function of them. On the other hand, if $H_0$ in (7.3) does not hold, then at least one of $\theta_{ij} \neq 0$ for $\lambda = 1, \ldots, p$, and hence, the joint cdf of $(Y_{i1}, \ldots, Y_{ip})$ will only be symmetric in them, if they are adjusted by appropriate locations which are not all identical. Thus, the problem of testing the hypothesis of equality of means ($H_0$) in (1.5), as applied to $(Y_{i1}, \ldots, Y_{ip})$, $i = 1, \ldots, n$, is equivalent to the problem of testing the null hypothesis (7.3).

So according to our procedure in section 2, we compute the values of $Y_{i\lambda}$ for $\lambda = 1, \ldots, p$, $i = 1, \ldots, n$, and arrange these observations in order of magnitude. Let then $R_{i\lambda}$ stand for the rank of $Y_{i\lambda}$ in this set of $N = np$ values, and we define
as in (2.4), $T_N$ as in (2.9) and (2.10), $v^2_{\alpha N}$ as in (3.8), and the test statistic $W_N$ as in (3.10). Thus, for small values of $n$, we may use the exact permutation test in (3.11), while for large $n$, this will be the large sample permutation test in (4.40). It may be noted here, that for the computation of $W_N$ in (3.10), we follow the usual procedure in the two way analysis of variance test with the only change that the values of $Y_{ij}$ ($i = 1, \ldots, n; j = 1, \ldots, p$) are only replaced by the corresponding values of rank-order functions $E_{N, R_{ij}}$'s.

Let us now proceed to study the asymptotic efficiency aspects of the proposed method. For this, we consider the class of alternatives (in the usual Pitman's translation sense)

$$H_0: \tau_j = \tau_{j, N} = N^{-\frac{1}{2}} \eta_j, j = 1, \ldots, p,$$

where $\eta_j = (\eta_1, \ldots, \eta_p)$ a $p$ vector with real and finite elements which satisfy the constraint $\sum_j \eta_j = 0$ (by (7.2)). Consequently, it readily follows from (7.5) and (7.7) that equivalently

$$H_0: \Theta = \Theta_{\eta, N} = N^{-\frac{1}{2}} \eta_j \text{ for } j = 1, \ldots, p.$$

Now, let us denote the joint distribution of $(Y_{11}, \ldots, Y_{1p})$, under $H_N$ by $F_N(y_1, \ldots, y_p)$, and the marginal cdf of $Y_{ij}$ corresponding to $F_N$ by $F_{[j], N}(y)$, $j = 1, \ldots, p$. From what has been discussed after (7.6), it follows that under $H_N$

$$F_{[j], N}(y) = H(y - N^{-\frac{1}{2}} \eta_j) \text{ for } j = 1, \ldots, p,$$

where $H(y)$ is a continuous cdf. To express the power efficiency in an elegant form, we further assume that $H(x)$ is absolutely continuous with a continuous
density \( h(x) \). Similarly, if we define by \( F^{[J,M]} N(x, y) \) the joint (marginal) cdf of \( Y_{1j}, Y_{1x} \), and \( H_N(x, y) \) as in (4.6), (with an additional suffix \( N \), in order to tackle the case of sequence of distributions), then, it is easy to check that

\[
\lim_{N \to \infty} H_N(x, y) = H(x, y) = H(y, x) \text{ for all } (x, y) \in \mathbb{R}^2.
\]

Thus, if we define \( C_N(H) \) as in (5.26), and \( A^2, \beta \) as in (4.16) and (4.17), respectively, we get from theorem 5.2, the following.

**Theorem 7.1.** Under the sequence of alternatives \( \{H_N^2 \} \) in (7.7), if the conditions (c.1) through (c.5) in section 4 hold, then \( W_N \) in (3.10), as adapted in this section, has asymptotically a non-central chi-square distribution with \( (p - 1) \) degrees of freedom and the non-centrality parameter

\[
\Delta_N = \left( \frac{1}{p} \sum_{j=1}^{p} \eta_j^2 \right) \left[ \{C(H)\}^2 / (A^2 - \beta) \right].
\]

Now, if in (7.1), the cdf \( G(e) \) of \( e_{1j} \) has a finite variance \( \sigma^2 \), then it is well-known that the classical analysis of variance \( (F_{p-1, (n-1)(p-1)}) \) test has asymptotically, under \( \{H_N^2 \} \) in (7.4), a non central chi-square distribution with \( (p - 1) \) d.f. and the noncentrality parameter

\[
\Delta_F = \left( \frac{1}{p} \sum_{j=1}^{p} \eta_j^2 \right) / \sigma^2.
\]

Hence, using the usual definition of Pitman- efficiency, we arrive at the following.

**Theorem 7.2.** For the sequence of alternatives \( \{H_N^2 \} \) in (7.7), the asymptotic relative efficiency (A.R.E.) of \( W_N \)-test with respect to the classical analysis of variance test is given by
\[(7.13) \quad \sigma^2 \left[ \mathcal{L}(h) \right]^2 / (A^2 - J). \]

Now, it follows from (4.16) and (4.17) that we can rewrite \((A^2 - J)\) as

\[(7.14) \quad \left\{ \int_0^1 J^2(u)du - \left[ \int_0^1 J(u)du \right]^2 \right\} \left\{ 1 - p_J \right\}^2
= \sigma_J^2 (1 - p_J),\]

where \(\sigma_J^2\) is the first factor of the expression in (7.14), and \(p_J\) is termed the **rank order correlation** which is defined as

\[(7.15) \quad \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} J(H(y))J(H(y))dH(x, y) - \left[ \int_{-\infty}^{1} J(u)du \right]^2 \right\} / \sigma_J^2.\]

Further, if we assume the conditions of lemma 7.2 of Puri [20] to be fulfilled, then, we may write

\[(7.16) \quad \mathcal{L}(H) = \int_{-\infty}^{\infty} \frac{d}{dx} J(H(x))dH(x).\]

Thus, from (7.13) through (7.16), we arrive at the following expression for the A.R.E. in (7.13):

\[(7.17) \quad \frac{\sigma^2}{\sigma_J^2 (1 - p_J)} \left\{ \int_{-\infty}^{\infty} \frac{d}{dx} J(H(x))dH(x) \right\}^2.\]

Let us then prove the following simple lemma.

**Lemma 7.3.** If the cdf \(F(x_1, \ldots, x_p)\) is symmetric in \((x_1, \ldots, x_p)\) and the univariate marginal distribution corresponding to \(F\) is not degenerate, then \(p_J \geq 1/(p - 1)\), where \(p_J\) is defined in (7.15).

**Proof:** Let us consider the random variable
(7.18) \[ Z = \sum_{i=1}^{p} J(H(x_i)), \]

where \( H(x) \) is the marginal cdf of \( F(x_1, \ldots, x_p) \) and \( X = (X_1, \ldots, X_p) \) is a random variable distributed according to the cdf \( F(x_1, \ldots, x_p) \). Then, it follows readily that

(7.19) \[ V(Z) = \sigma_j^2 [1 + (p - 1) \rho_j] \geq 0, \]

where \( \sigma_j^2 > 0 \), by the conditions that \( H(x) \) is non-degenerate and \( J(H): 0 < H < 1 \) is not a constant [cf. (4.7)]. From (7.19), it readily follows that \( 1 + (p - 1) \rho_j \geq 0 \), i.e., \( \rho_j \geq -1/(p-1) \).

Hence, the lemma.

**Lemma 7.4.** If \( F(x_1, \ldots, x_p) \) is a totally symmetric \( p \)-variate singular distribution on the \((p-1)\) - flat \( \sum_{i=1}^{p} x_i = 0 \), while there is no lower dimensional space containing the scatter of the points of \( F(x_1, \ldots, x_p) \), then \( \rho_j = -\frac{1}{p-1} \), if and only if, \( J(H(x)) \) is a linear function of \( x: 0 < H(x) < 1 \).

**Proof.** Let us examine the conditions under which the equality sign in (7.19) holds. As \( \sigma_j^2 > 0 \), this implies (from (7.18),) that \( V(Z) \) will be exactly equal to zero, if \( Z \) is a constant with probability one. Now, \( \sum_{i=1}^{p} x_i = 0 \) is the only constraint in \((x_1, \ldots, x_p)\) which holds with probability one. Hence, in order that \( \sum_{i=1}^{p} J(H(x_i)) \) is a constant for all \((x_1, \ldots, x_p)\) satisfying \( \sum_{i=1}^{p} x_i = 0 \), we must have \( J(H(x)) \) a linear function of \( x \) almost everywhere.

Hence, the lemma.

**Lemma 7.5.** \( \frac{p}{(p-1)} (1 - \rho_j) \geq 1 \).

The proof follows immediately from the preceding two lemmas.
Let us now denote by $\sigma^2_H$, the variance of the marginal cdf $H(x)$. Then, the way in which the random variables $Y_{1i}, ..., Y_{ip}$ are defined in (7.6), it readily follows that $\sigma^2_H = [(p - 1)/p] \sigma^2$. Hence, from (7.17) and lemma 7.5, we get that the A.R.E. in (1.73) is at least as large as

$$\frac{\sigma^2_H}{\sigma_j^2} \left[ \int_{-\infty}^{\infty} \frac{dJ(\hat{H}(x))}{dx}J(\hat{H}(x)) d\hat{H}(x) \right]^2. \tag{7.20}$$

We shall now consider two specific type of rank order statistics, namely the rank-sum and the normal score statistics, and study their A.R.E. with respect to the F-test. Let us first consider the normal score statistic. Let $\Phi$ denote the cdf of a standardized normal variate, and let $a_{N,i}$ stand for the expected value of the ith (smallest) order statistic in a random sample of size $N$, drawn from a population having the cdf $\Phi$, $i = 1, ..., N$. For normal score test, we use

$$E_{N,a} = a_{N,a} \text{ for } a = 1, ..., N. \tag{7.21}$$

In this case, $\sigma_j^2$ in (7.14) will reduce to unity and (7.17) reduces to

$$\frac{\sigma^2_H}{1-\sigma_j^2} \left\{ \int_{-\infty}^{\infty} \frac{h^2(x)}{\Phi(\Phi^{-1}[H(x)])} dx \right\}^2, \tag{7.22}$$

where $\phi$ is the density function corresponding to the cdf $\Phi$. Now, by lemma 7.4, we get that $1 - \sigma_j \leq p/(p - 1)$, where the equality sign holds only when

$$\Phi^{-1}[H(x)] \text{ is a linear function of } x \text{ a.e.} \tag{7.23}$$

i.e., $H \equiv \Phi$. Consequently, with the help of lemma 7.5, we get that (7.22) is greater than or equal to

$$\sigma^2_H \left\{ \int_{-\infty}^{\infty} \frac{[h^2(x)/\Phi(\Phi^{-1}[H(x)])]}{dx} dx \right\}^2. \tag{7.24}$$
where the equality sign holds only for normal cdf's, and where $\sigma_H^2$ is the variance of the cdf $H(x)$. Thus, using a well-known result by Hodges and Lehmann [13], we get from (7.24) that the same is greater than or equal to 1 for all continuous cdf's. Hence, the normal score analysis of variance test based on rankings after alignment is also asymptotically as powerful as the F-test for normal alternatives and more powerful for non-normal cdf's. Let us now consider the rank-sum statistic. In this case, $E_{N,a} = a/(N+1)$ for $1 \leq a \leq N_2$ and consequently (7.17) reduces to

\[
(7.25) \quad 12\sigma^2 \left(1 - \frac{1}{p} \right) \frac{1}{\alpha} \left[ \int_0^\infty h^2(x)dx \right]^2,
\]

where $\rho$ is the grade correlation of any two variates in $F(x_1, \ldots, x_p)$. Using lemma 7.5 and as in (7.20), we get that (7.25) is at least as large as

\[
(7.26) \quad 12\sigma_H^2 \left[ \int_\infty^- h^2(x)dx \right]^2.
\]

Proceeding then precisely on the same line as in Hodges and Lehmann [12], it is easily seen that

\[
(7.27) \quad \inf_{H \in \mathcal{F}} 12 \sigma^2 \left[ \int_\infty^- h^2(x)dx \right]^2 = .864,
\]

where $\mathcal{F}$ is the family of all continuous univariate cdf's. Hence, the lower bound of A.R.E. for Wilcoxon's test obtained by Hodges and Lehmann [12] also remains valid for the rank-sum analysis of variance test based on rankings after alignment. When the error component $e_{ij}$ in (7.1) has a normal cdf, it readily follows from (7.6) that the marginal cdf $H(x)$ of $Y_{ij}(\ell = 1, \ldots, p)$ is also normal, with a variance equal to $(p - 1)\sigma^2/p$. Also, using the well-known result that for normal cdf $\rho_g = \frac{6}{\pi} \sin^{-1} \rho/2$, where $\rho$ is the correlation coefficient, and
noting that \( Y_{11}, \ldots, Y_{1p} \) are equally correlated with a value of the correlation coefficient equal to \(-1/(p-1)\) (by (7.6)), we get that for normally distributed errors, (7.25) reduces to

\[
(7.28) \quad \frac{2}{\pi} \cdot \frac{p}{(p-1)[1+(6/\pi)\sin^{-1}(1/2(p-1))]^{3/2}} = e_p \quad \text{(say)}.
\]

It may be noted that for \( p = 2 \), \( e_p \) is equal to \( 3/\pi \), the efficiency of the usual rank sum tests by Wilcoxon or Kruskal and Wallis for one criterion variance analysis. For two factor analysis of variance problem, the test by Friedman [10] has the A.R.E. (with respect to the classical F-test) equal to (cf. [14, p. 485])

\[
(7.29) \quad \frac{2}{\pi} \cdot \frac{p}{p+1} = e_p^* \quad \text{(say)}.
\]

Thus, the A.R.E. of our proposed rank-sum test based on rankings after alignment with respect to Friedman's [10] test is given by

\[
(7.30) \quad \frac{p+1}{(p-1)} \left[1 + \frac{6}{\pi} \sin^{-1}\left(\frac{1}{2(p-1)}\right)\right]^{-\frac{3}{2}} = e_p^{**}.
\]

It is evident that \( e_p^{**} \) is strictly greater than one for all \( p \geq 2 \), and, it tends to the limit 1 as \( p \to \infty \). The following table gives some idea of the values of \( e_p^*, e_p^{**} \) and \( e_p \).

<table>
<thead>
<tr>
<th>p</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>( \infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e_p )</td>
<td>.955</td>
<td>.966</td>
<td>.965</td>
<td>.963</td>
<td>.962</td>
<td>.961</td>
<td>.960</td>
<td>.960</td>
<td>.959</td>
<td>.958</td>
<td>.957</td>
<td>.955</td>
</tr>
<tr>
<td>( e_p^* )</td>
<td>.637</td>
<td>.716</td>
<td>.764</td>
<td>.796</td>
<td>.819</td>
<td>.835</td>
<td>.848</td>
<td>.860</td>
<td>.868</td>
<td>.895</td>
<td>.910</td>
<td>.955</td>
</tr>
<tr>
<td>( e_p^{**} )</td>
<td>1.500</td>
<td>1.349</td>
<td>1.263</td>
<td>1.210</td>
<td>1.175</td>
<td>1.150</td>
<td>1.132</td>
<td>1.116</td>
<td>1.105</td>
<td>1.070</td>
<td>1.052</td>
<td>1.000</td>
</tr>
</tbody>
</table>
This clearly indicates how the method of ranking after alignment results in
greater efficiency of the test. Incidentally, it confirms the remark of
cell.

Let us now consider the case of more than one but equal number of observations
per cell. Here, we will have \( r \) observations \( X_{ijk} \), \( k = 1, \ldots, r \) for each \( j = 1, \ldots, n \). If we define the block means by \( X_{i..} = \frac{1}{pr} \sum_{j=1}^{p} \sum_{k=1}^{r} X_{ijk} \), then we may
work with the variables

\[
(7.31) \quad Y_{ijk} = X_{ijk} - X_{i..} \quad \text{for} \quad k = 1, \ldots, r, \ j = 1, \ldots, p, \ i = 1, \ldots, n.
\]

Thus, we may rank these \( N' = np \) values in (7.31) in order of magnitude, define
the rank of \( Y_{ijk} \) in this set by \( R_{ijk} \) and allocate the rank function \( E_{N', R_{ijk}} \)
to \( Y_{ijk} \) for all \( i = 1, \ldots, p, \ k = 1, \ldots, r \). We then define

\[
(7.32) \quad T_{Nfj} = \frac{1}{nr} \sum_{i=1}^{n} \sum_{k=1}^{r} E_{N', R_{ijk}} \quad \text{for} \quad j = 1, \ldots, p.
\]

\[
(7.33) \quad \sigma_{N'}^2(R_{N'}) = \frac{1}{n(pr-1)} \left\{ \sum_{i=1}^{n} \left[ \sum_{j=1}^{p} \sum_{k=1}^{r} E_{N', R_{ijk}} \right] - \frac{1}{pr} \sum_{j=1}^{p} \sum_{k=1}^{r} E_{N', R_{ijk}} \right\}^2.
\]

Then, under the null hypothesis of no treatment effect, all possible \((pr)!\)
permutations of \((Y_{ijk}, j = 1, \ldots, p, k = 1, \ldots, r)\) among themselves are equally
likely, for each \( i = 1, \ldots, n \). Thus, we may readily extend here the permutation
agreement of section 3 with the only change that \( p \) has to be replaced by \( pr \).
Consequently, on denoting by \( \phi_n \), the permutational probability measure, we arrive
at

\[
(7.34) \quad \text{Cov}(T_{N'}, j, T_{N'}, \lambda) = \frac{1}{N'} \sigma_{N'}^2(R_{N'})(\delta_{j\lambda}p-1), \ j, \lambda = 1, \ldots, p
\]
where \( S_{jk} \) is the usual Kronecker delta. Hence, as in section 3, we may arrive at the test statistic

\[
W_{N'},r = nr \sum_{j=1}^{p} (T_{N',j} - \bar{E}_{N'})^2 / \sigma_{N'}^2 (R_{N'}),
\]

where

\[
\bar{E}_{N'} = \frac{1}{N'} \sum_{i=1}^{n} \sum_{j=1}^{p} \sum_{k=1}^{r} \bar{E}_{N',ij,k}.
\]

It can be shown similarly that under the null hypothesis (7.3), \( W_{N'},r \) has asymptotically a chi-square distribution with \( (p - 1) \) degrees of freedom, and by a straightforward generalization of theorems 5.1 and 5.2, it can be shown that for the sequence of alternatives in (7.8) (with \( N \) replaced by \( N' \)), \( W_{N'},r \) has asymptotically a non-central chi-square distribution with \( (p - 1) \) d.f. and the non-centrality parameter (7.11), where \( H(x) \) refers to the marginal cdf of \( Y_{ijk} \). Thus, the conclusions regarding the A.R.E. of the proposed test, derived in the particular case of \( r = 1 \), also remains valid for any \( r \geq 1 \), the only change in (7.28), (7.29) and (7.30) would be to replace \( p \) by \( pr \). This has a damping effect on \( e^{**} \), as would be evident from table 7.1.

8. **Nonparametric multiple comparisons for two factor designs.** Often, when the null hypothesis (7.3) is rejected, we may want to test for or estimate various contrasts among the treatment effects \( \tau_1, \ldots, \tau_p \). In the parametric case, the well-known T- and S-methods of multiple comparisons are usually employed for such a purpose. In the nonparametric case, there are only a few such multiple comparisons. For the one criterion analysis of variance problem, a review of the existing nonparametric multiple comparison methods along with the generalization of S- and
T-methods has been considered in [25], while for the two factor analysis of variance problem, the author is not aware of any development. In this paper, we shall consider an analog of T-method of multiple comparisons, as adapted in the nonparametric set up. The S-method will not be considered here explicitly, as the same will follow as in the one criterion problem. Moreover, for the situation of orthogonal designs, it will be (at least asymptotically) less efficient than the proposed analog of the T-method. For simplicity of discussion, we shall consider the case of a single observation per cell, while the more general case of \( r(\geq 1) \) observations per cell will follow exactly in the same manner.

Let us first consider the problem of simultaneous testing or estimation of all possible paired differences \( \tau_i - \tau_j \), for \( i \neq j = 1, \ldots, p \). We define the block-adjusted random variables \( Y_{i,j} \), \( j = 1, \ldots, p \), \( i = 1, \ldots, n \), as in (7.6), and then proceeding as in section 2, we define \( T_N = (T_{N,1}, \ldots, T_{N,p}) \) as in (2.9), where \( X_i \)'s are replaced by \( Y_i \)'s. We also define \( \sigma_N^2(\bar{X}_N) \) as in (3.8), and define a statistic

\[
S_N = \max_{1 \leq i, j \leq p} \left[ \frac{1}{n^\frac{1}{2}} \left| T_{N,i} - T_{N,j} \right| / \sigma_N(\bar{X}_N) \right].
\]

For small values of \( n \), we may proceed precisely on the same line as in section 3, and find out the permutation distribution function (say \( \Theta_n(s) \)) of \( S_N \). Let then \( \Theta_n(S_N, \bar{X}_N, \bar{X}_N) \) be a solution of

\[
G \Theta_n(S_N, \bar{X}_N, \bar{X}_N) < 1 - \varepsilon \leq G \Theta_n(S_N, \bar{X}_N, \bar{X}_N).
\]

Then, we may consider the following non-randomized multiple comparison test of size (less than or equal to) \( \varepsilon \): regard \( T_{N,i} \) and \( T_{N,j} \) to be significantly different
from each other (i.e., $\tau_i \neq \tau_j$), if for that pair $(i, j)$ $(i \neq j = 1, \ldots, p)$

$$n^{4} | T_{N,i} - T_{N,j} | \geq \sigma_{N}^{2} \alpha_{N} S_{N,e} \alpha_{N}.$$  

Of course, we may also consider a randomized procedure of size exactly $\varepsilon$, which differs from the above one, only at the point $n^{4} | T_{N,i} - T_{N,j} | = \sigma_{N}^{2} \alpha_{N} S_{N,e} \alpha_{N}$, in which case, we reject $\tau_i = \tau_j$ with a probability $\gamma_{N}^{2} \alpha_{N}^{2} \alpha_{N}$ chosen in such a way that

$$P \left\{ S_{N} > S_{N,e} \alpha_{N} \right\} + \gamma_{N}^{2} \alpha_{N}^{2} \alpha_{N} P \left\{ S_{N} = S_{N,e} \alpha_{N} \right\} = \varepsilon.$$  

It may be noted that the procedure of evaluating $S_{N,e} \alpha_{N}$ and $\gamma_{N}^{2} \alpha_{N}^{2} \alpha_{N}$ becomes prohibitively laborious as $n$ increases. For this, we consider the following theorem which simplifies the solution appreciably.

**THEOREM 8.1.** Under the null hypothesis (7.3),

$$S_{N,e} \alpha_{N} \xrightarrow{P} R_{p,e} \sqrt{ \frac{P}{P-1} } \text{ and } \gamma_{N}^{2} \alpha_{N}^{2} \alpha_{N} \xrightarrow{P} 0,$$

where $R_{p,e}$ is the upper $100\varepsilon\%$ point of the distribution of the sample range in a random sample of size $p$ drawn from a standardized normal distribution.

**PROOF.** It follows from (3.7) that

$$\text{Cov} \left\{ T_{N,i}, T_{N,j} \right\} \alpha_{N}^{2} \frac{p-1}{p} = \frac{1}{n} \frac{(\delta_{i,j} p-1)}{p-1} \sigma_{N}^{2} \alpha_{N}^{2}, \quad i, j = 1, \ldots, p,$$

where $\delta_{i,j}$ is the usual Kronecker delta. Also, it follows from theorem 4.3 that for large $n$, $\left\{ n^{4} (T_{N,i} - E_{N}) \right\}$, $i = 1, \ldots, p$ has sensibly a multinormal distribution, which is singular and of rank $p - 1$. Consequently, on applying theorem 3 of Scheffe' [22, p. 75], we get following a few simple steps that asymptotically, $S_{N}$ in (8.1)
has the distribution $\chi_p(s)$, where $\chi_p(s)$ is the distribution of the range of a sample of size $p$ drawn from a population having a normal distribution with variance $p/(p-1)$. Consequently, the upper $100\%$ point of $\chi_p(s)$ will be equal to $R_{p,\varepsilon} \sqrt{\frac{p}{p-1}}$, where $R_{p,\varepsilon}$ is defined in the statement of the theorem.

Hence, under $\Theta_n$,

$$S_{N,\varepsilon}(R_N) \xrightarrow{P} \sqrt{\frac{P}{P-1}} R_{p,\varepsilon},$$

and the continuity of $\chi_p(s)$ implies that $\chi_N(R_N) \xrightarrow{P} 0$.

Hence, the theorem.

It may be noted that using theorem 5.1, it can be shown that the unconditional distribution of $S_N$ in (6.1), under $H_0$ in (7.3), also converges to $\chi_p(s)$. Thus, the permutation multiple comparison procedure, considered earlier, also agrees with the asymptotic unconditional procedure. Thus, asymptotically, the paired comparison procedure reduces to the following one:

regard those pairs $(\tau_i, \tau_j)$ in (7.1) to be significantly different from each other, for which

$$(8.7) \quad \left| n^{\frac{1}{2}}(T_{N,i} - T_{N,j})/\sigma_N(R_N) \right| \geq R_{p,\varepsilon} \sqrt{\frac{P}{P-1}},$$

for $i \neq j = 1, \ldots, p$.

It may be noted that by theorem 4.2, $\sigma^2_N(R_N)$ converges, in probability, to a non-zero finite quantity, while by theorem 5.1, $T_{N,i}$ converges, in probability, to $\mu_{N,i}$; defined in (5.2), for $i = 1, \ldots, p$, no matter whether $H_0$ in (7.3) holds or not. Thus, it is easy to see that if the function $J(H)$, defined in (4.7) is $\uparrow$ in $H$: $0 < H < 1$, then for any given $\tau_i \neq \tau_j$, $T_{N,i} - T_{N,j}$ stochastically
converges to a non-null quantity. Consequently, from (8.7) we will get that the multiple comparison test will be consistent against the set of alternatives that \( \tau_i \neq \tau_j \) for at least one \( i \neq j = 1, \ldots, p \). To study the asymptotic efficiency of this method relative to the T-method, we again consider Pitman's type of translation alternatives, as in (7.7), (7.8) and (7.9). In this case, if we write \( T_{N,i} \) as in (5.7) and (5.14), then after a few simple steps, it can be shown that

\[
(8.8) \quad h_n \{ (T_{N,i} - \mu_{N,i}) - (T_{N,j} - \mu_{N,j}) \} = \frac{1}{\sqrt{n}} \sum_{a=1}^{n} \left[ B^*(X_{1a}) - B^*(X_{1a}) \right] + o_p(1),
\]

where

\[
(8.9) \quad B^*(X_{ka}) = \int_{-\infty}^{\infty} \left[ F_{1[k]}(x) - F_{[k]}(x) \right] H(x) dH(x), \quad k = 1, \ldots, p, \quad a = 1, \ldots, n,
\]

\( F_{1[k]} \) being defined in (5.17), and \( H(x) \) in (7.9). Thus, under the sequence of alternatives in (7.8) and (7.9), it is easily seen that \( E[B^*(X_{ka})] = 0 \) and

\[
(8.10) \quad \text{Cov}[B^*(X_{ka}), B^*(X_{qa})] = \sum_{a,b}^{n} \left[ \delta_{aq} + \rho_{j}(1 - \delta_{kq}) \right] \sigma_{j}^2,
\]

for \( k, q = 1, \ldots, p, a, b = 1, \ldots, n \), where \( \delta \)'s are the usual Kronecker delta, and \( \sigma_{j}^2 \) and \( \xi_{j} \) are defined in (7.14) and (7.15) respectively. Proceeding then precisely on the same line as in the proof of theorem 5.1, we arrive at the joint (asymptotic) normality of \( \left[ n^{-\frac{1}{2}} \sum_{a=1}^{n} B^*(X_{ka}) \right] \), \( k = 1, \ldots, p \), and hence, as in the proof of theorem 8.1 that

\[
(8.11) \quad \lim_{n=\infty} P \left\{ \max_{1 \leq i, j \leq p} n^{-\frac{1}{2}} \left| (T_{N,i} - T_{N,j}) - (\mu_{N,i} - \mu_{N,j}) \right| \leq R_{p, \epsilon} \sigma_{j} (1 - \rho_{j}) \right\} = 1 - \epsilon
\]

Finally, defining \( \zeta(H) \) as in (5.26), we get from (8.11) after a few simple steps
that the probability will be \( 1 - \varepsilon \) that

\[
\nu^\beta(T_{N,i} - T_{N,j}) - R_p, \xi \sigma_j (1 - \rho_j) \leq \zeta(H)[\eta_i - \eta_j]
\]

 simultaneously for all \( i \neq j = 1, \ldots, p \). It may be noted that under the sequence of alternatives in (7.9), \( \sigma^2_n(T_n) \) converges to \( \frac{1}{(p - 1)/p} \sigma^2_j (1 - \xi_j) \). Thus, the test in (8.7) may be regarded as asymptotically equivalent to the corresponding test derived from (8.12). If we now compare (8.12) with the usual T-method (cf. Scheffe' [22, pp. 73-75]), we readily arrive at the A.R.E. of the nonparametric method relative to the parametric method equal to

\[
\frac{\sigma^2}{\sigma^2_j (1 - \rho_j)} [\zeta(H)]^2,
\]

which is the same as in (7.17). Consequently, all that has been discussed about (7.17) in the preceding section also applies to (8.13), and hence, these are not reproduced again.

The simultaneous confidence region in (8.12) involves the unknown quantities \( \rho_j \) and \( \zeta(H) \). We shall now consider a method of constructing a simultaneous confidence region for \( \eta_i - \eta_j \) (for all \( i \neq j = 1, \ldots, p \)) which does not depend on the knowledge of \( \zeta(H) \). For this, we shall slightly modify the technique of estimating a shift parameter considered by Hodges and Lehmann [15], Lehmann [18], and Sen ([23], [24]), among others. Since, in our case, the rank order statistic \( \tau_n \) in (2.9) is based on the block-adjusted observations \( Y_{1k}, k = 1, \ldots, p, i = 1, \ldots, n, \) in (7.6), the ordinary sliding principle underlying the use of the above methods has to
modified slightly. It may be noted that if to each \( X_{ij} (i = 1, \ldots, n) \), we add a constant \( a \), then the values of \( Y_{ij} \) will be increased by the amount \((p - 1)a/p\), while those of \( Y_{ij'} (j' \neq j = 1, \ldots, p) \), will be decreased by the amount \( a/p \).

Thus is the case, we may conclude that by a real \( a \) to each \( X_{ij} (i = 1, \ldots, n) \), the relative magnitudes of \( Y_{ij'} (j' \neq j = 1, \ldots, p, i = 1, \ldots, n) \) are not affected, only the resulting values of \( Y_{ij} 's \) are shifted by the same amount from the others.

Let us now denote

\[
(8.14) \quad \Delta_{ij} = \tau_i - \tau_j \text{ for } i, j = 1, \ldots, p.
\]

Now, if instead of the observations \((X_{11}, \ldots, X_{ip})\), we work with the observations \((X_{11}, \ldots, X_{ij-1}, X_{ij} - a, X_{ij+1}, \ldots, X_{ip})\), \(i = 1, \ldots, n\), the resulting value of \( T_{N,N'}^{(a_j)} \), in (2.9), will be denoted by

\[
(8.15) \quad T_{N,N'}^{(a_j)} = (T_{N,1}^{(a_j)}, \ldots, T_{N,p}^{(a_j)}).
\]

It may be noted then that \( T_{N,j}^{(a_j)} \) is \( \downarrow \) in \( a_j \), while \( T_{N,p}^{(a_j)} \) \( \uparrow \) in \( a_j \) for all \( i \neq j = 1, \ldots, p \). Let us then define

\[
(8.16) \quad V_{N,ij}^{(a_j)} = T_{N,1}^{(a_j)} - T_{N,j}^{(a_j)}, i \neq j = 1, \ldots, p.
\]

Then, by definition, \( V_{N,ij}^{(a_j)} \) is \( \uparrow \) in \( a_j \). Let then

\[
(8.17) \begin{cases}
\hat{\Delta}_{ji}^{(1)} = \inf \{ a_j : V_{N,ij}^{(a_j)} > 0 \} , \\
\hat{\Delta}_{ji}^{(2)} = \sup \{ a_j : V_{N,ij}^{(a_j)} < 0 \} .
\end{cases}
\]

\[
(8.18) \quad \hat{\Delta}_{ji} = \frac{1}{2} (\hat{\Delta}_{ji}^{(1)} + \hat{\Delta}_{ji}^{(2)}) \text{ for } i \neq j = 1, \ldots, p.
\]
\( \hat{\Lambda}_{ji} \) is our proposed estimator of \( \Lambda_{ji} \) for all \( j \neq i = 1, \ldots, p \). Before we proceed to construct the simultaneous confidence interval for \( \Lambda_{ij} \), we would like to study some properties of these estimates. For this, we conventionally let

\[
(8.19) \quad \Delta_{1i} = \hat{\Delta}_{1i} = 0 \text{ for } i = 1, \ldots, p;
\]

\[
(8.20) \quad \hat{\Delta}_i \cdot = \frac{1}{p} \sum_{j=1}^{p} \hat{\Delta}_{ij}, \text{ for } i = 1, \ldots, p.
\]

Then by an adaptation of the same argument as in [16], we may propose the adjusted estimators of \( \Delta_{ij} \) as

\[
(8.21) \quad \Delta_{ij}^* = \hat{\Delta}_i \cdot - \hat{\Delta}_j \cdot \text{ for } i, j = 1, \ldots, p.
\]

The properties of translation invariance and consistency of the estimates in (8.21) follow precisely on the same line as in the univariate case, considered by Hodges and Lehmann [15] and Lehmann [16]. For the sake of completeness of discussion, we shall consider here the general case of contrasts and study the asymptotic normality of their estimates.

Any contrast \( \phi = \sum_{i=1}^{p} c_i \tau_i \) with \( \sum_{i=1}^{p} c_i = 0 \), may also be written as \( \sum_{i=1}^{p} \sum_{j=1}^{p} d_{ij} \Delta_{ij} \)

(e.g., \( d_{ij} = \frac{1}{p} c_i, j = 1, \ldots, p, i = 1, \ldots, p \)), and hence, a natural estimate of it may be \( \sum_{i=1}^{p} \sum_{j=1}^{p} d_{ij} \Delta_{ij}^* \), which we may also write as \( \sum_{i=1}^{p} c_i \hat{\Delta}_i \cdot = \hat{\phi} \) (say).

Let us now consider a set of \( (p - 1) \) linearly independent contrasts

\[
(8.22) \quad \phi_{\lambda} = \sum_{i=1}^{p} c_{i(\lambda)} \tau_i, \lambda = 1, \ldots, p - 1,
\]

and denote the corresponding estimators as \( \hat{\phi}_{\lambda}, \lambda = 1, \ldots, p - 1 \). We then define a square matrix \( \Phi = (Y_{\lambda',\lambda})_{\lambda,\lambda'=1,\ldots,p-1} \) with elements
where \( \sigma_j^2, \phi_j, \text{ and } \zeta(H) \) are defined in (7.14), (7.15) and (5.26) respectively.

Then, we have the following theorem.

**THEOREM 8.2.** If the cdf \( H \) in (7.9) is absolutely continuous, and the rank-order statistic \( T_N \) satisfies the conditions of theorem 5.1, then \( n^\hat{}([\phi_\lambda^\hat{} - \phi_\lambda^\prime], \lambda = 1, \ldots, p-1) \)

\[ \text{has asymptotically a multinormal distribution with a null mean vector and a dispersion matrix } \int, \text{ defined in (8.23).} \]

**PROOF.** Using theorem 5.1, we get by a simple adaptation of the lemma 2 of [16] with more or less obvious generalizations that

\[ n^\hat{}|\Delta_{1p}^\hat{} - \Delta_{1p}^\prime| \xrightarrow{p} 0, \text{ for all } i = 1, \ldots, p. \]

Consequently, we get that

\[ n^\hat{}(\phi_\lambda^\hat{} - \phi_\lambda^\prime) \sim \frac{p}{\sum c_{\lambda}^1} [n^\hat{}(\Delta_{1p}^\hat{} - \Delta_{1p}^\prime)], \lambda = 1, \ldots, p - 1. \]

Thus, it appears to be sufficient to show that the vector \( \{n^\hat{}(\Delta_{1p}^\hat{} - \Delta_{1p}^\prime), i = 1, \ldots, p - 1\} \)

has asymptotically a \(p - 1\) variate multinormal distribution.

To prove this, we consider the following theorem, whose proof is omitted because, the same follows precisely on the same line as in theorem 5.1.

**THEOREM 8.3.** Under the sequence of alternatives in (7.9), the distribution of \( \{n^\hat{}(T_{N,i} - T_{N,p}), i = 1, \ldots, p - 1\} \)

is asymptotically a \((p - 1)\)- variate normal one with a mean vector \( \{\zeta(H)(\eta_i^1 - \eta_i^p), i = 1, \ldots, p - 1\} \) and a dispersion matrix with elements

\[ \sigma_j^2(1 - \phi_j)(1 + \zeta_{ij}), i, j = 1, \ldots, p - 1, \]
where $\delta_{ij}$ is the usual Kronecker delta.

Now, with the help of theorem 8.3, we may readily generalize theorem 4 of Hodges and Lehmann [13, pp. 608-609] in a more or less straightforward manner. This generalized theorem will then relate to the asymptotic normality of the set of $(p - 1)$ variates $\{n^\frac{1}{2}(\hat{\Delta}_{ip} - \Delta_{ip}), i = 1, \ldots, p - 1\}$, under the regularity conditions of theorem 8.2. Once, this is obtained, the proof of the theorem 8.2 is also completed.

Now, if we compare the estimate $\hat{\phi}_\lambda$ of $\phi_\lambda$ with the standard least square estimate of the same contrast, we note that the ratio of the asymptotic variances of the two estimates, as taken to be a suitable measure of the asymptotic relative efficiency, again comes out to be the same as in (8.13). Hence, the conclusions of section 7 applies also to any contrasts among $\tau_1, \ldots, \tau_c$.

It follows from theorem 8.2 that for any fixed $\tau_1, \ldots, \tau_c$, the estimators $\hat{\phi}_\lambda$, $\lambda = 1, \ldots, p - 1$, converge, in probability to $\phi_\lambda$, $\lambda = 1, \ldots, p - 1$, respectively. Thus, any simultaneous confidence region for $\phi_\lambda$, $\lambda = 1, \ldots, p - 1$ will asymptotically degenerate to a point. So, when we are interested in such a simultaneous confidence region, we have really in mind the contrasts among $\eta_1, \ldots, \eta_p$, defined in (7.8). For this, let us define,

$$\Delta_N^{(1)} = \text{Sup} \left\{ a_j: V_{N,ij} < \frac{1}{V_N} \sigma_N(R_N)R_p, \sqrt{\frac{D}{P-1}} \right\},$$

$$\Delta_N^{(2)} = \text{Inf} \left\{ a_j: V_{N,ij} > -\frac{1}{V_N} \sigma_N(R_N)R_p, \sqrt{\frac{D}{P-1}} \right\}.$$

Then,

$$N^\frac{1}{2} \Delta_N^{(2)} \leq \eta_j - \eta_i \leq N^\frac{1}{2} \Delta_N^{(1)}, \ i \neq j = 1, \ldots, p$$

(8.27)
holds simultaneously with a probability which is asymptotically equal to 1 - \( \varepsilon \).

This is our desired simultaneous confidence region for \( \eta_1, \ldots, \eta_p \). Since,

\[ \sum_{i=1}^{p} \eta_i = 0 \] (by (7.2) and (7.7)), (8.27) will be a subspace (of rank \( p - 1 \)) in the \( p \)-dimensional Euclidean space of \( (\eta_1, \ldots, \eta_p) \). It may also be noted that by an adaptation of the same proof as in theorem 2 of [24], it can be shown (with the help of (8.12),) that

\[
(8.28) \quad N \left( \frac{\hat{V}^{(1)}_{N,j_1}}{n_j}, \frac{\hat{V}^{(2)}_{N,j_1}}{n_j} \right) \xrightarrow{P} 2R_p, \quad \mathcal{W}(H).
\]

Thus, a consistent estimate of \( \mathcal{W}(H) \) may be obtained as

\[
(8.29) \quad \hat{\mathcal{W}}(H) = 2R_p, \mathcal{W}[\frac{1}{2} \sum_{1 \leq i < j \leq p} \frac{\hat{V}^{(1)}_{N,j_1} - \hat{V}^{(2)}_{N,j_1}}{n_j}].
\]

Once, (8.29) is derived, with the help of theorems 4.2 and 8.3, and the usual technique underlying the derivation of the T-method of multiple comparisons (cf. Scheffe' [22, pp. 73-75]), we readily arrive at the following asymptotic simultaneous confidence interval to the contrast \( \psi = \sum_{i=1}^{p} C_i \eta_i \), where \( C = (C_1, \ldots, C_p) \) is orthogonal to \( I_p = (1, \ldots, 1) \):

the probability is asymptotically equal to 1 - \( \varepsilon \) that the statement

\[
(8.30) \quad \frac{1}{\hat{\mathcal{W}}(H)} \left[ \sum_{i=1}^{p} C_i T_{N,i} - \left( \frac{1}{2} \sum_{i=1}^{p} C_i \right) \sigma_N (R_N)^{-1} R_p, \mathcal{W} \frac{p}{\sqrt{p-1}} \right. \leq \sum_{i=1}^{p} C_i \eta_i
\]

\[
\left. \leq \frac{1}{\hat{\mathcal{W}}(H)} \left[ \sum_{i=1}^{p} C_i T_{N,i} + \left( \frac{1}{2} \sum_{i=1}^{p} C_i \right) \sigma_N (R_N)^{-1} R_p, \mathcal{W} \frac{p}{\sqrt{p-1}} \right. \right]
\]

holds simultaneously for all \( 0 < \alpha < 1 \).

This may be regarded as the nonparametric generalization of the usual T-method of multiple comparisons for the two-factor experiments.
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