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CENTRAL LIMIT THEOREM FOR SUMS  
OVER SETS OF RANDOM VARIABLES

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The convergence of weighted sums of random variables taken in arbitrary order out of a set  $F$  is defined. Necessary and sufficient conditions on  $F$  and the weight coefficients for convergence to the normal law are given.

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CENTRAL LIMIT THEOREM FOR SUMS  
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Introduction and summary. The classical problem of the probabilistic limit laws is to determine conditions for random variables in order that their sums for increasing number of terms tend to a limit law. In statistical estimation the situation occurs frequently where the distributions of these random variables are not exactly known [1]. They may be assumed to be members of a certain family of distribution functions. This case has been considered for instance by A. Wald and by E. Parzen [3], and conditions for the uniform convergence of these families have been stated. In a more general approach presented in this paper the family is replaced by a set  $F$  of random variables. Then conditions are given for this set and the linear coefficients occurring in the sums in order that these sums converge for any selection of random variables out of  $F$  to a normal law. It seems to be much more difficult if not impossible to find similar simple conditions for the convergence to other limit laws.

1. Convergence to the normal law.

Definition: Let  $\{b_n\}$  be a sequence  $b_1, b_2, \dots$  of random variables (r.v.), and let each  $b_n$  be a function  $b_n(\epsilon_1, \epsilon_2, \dots)$  of at most countably

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many other r.v.-s  $\epsilon_i$ . For all  $n$  the  $\epsilon$ 's are taken out of the same sequence  $\{\epsilon_i\}$ . Let furthermore all  $\epsilon$ 's be elements in a set  $F$  of r.v.-s. Then  $\{b_n\}$  is said to converge on  $F$  (in a sense to be specified) if this convergence holds for each possible choice of the sequence  $\{\epsilon_i\}$ .

Let  $A = \{a_{nk}\}$  be an infinite real matrix of generalized triangular shape (type (T) say), i.e., each of its rows contains only a finite but positive number of non-zero elements. Further, there is a sequence of integers  $k_1, k_2, \dots \rightarrow \infty$  such that  $a_{ni} = 0$  for all  $i > k_n$ . Finally, for any  $k$  let there be infinitely many  $a_{nr} \neq 0$  (this will be seen to be no essential restriction).

Henceforth, each element  $\epsilon$  in any considered set  $F$  shall have zero mean and positive, finite variance. Let any sequence  $\epsilon_1, \epsilon_2, \dots, \epsilon_i \in F$ , be written as an (infinite) column vector  $\underline{\epsilon}$ . Consider now the sequence of r.v.-s  $\zeta_1, \zeta_2, \dots$  given by the vector

$$(1) \quad \underline{z} = B^{-1} A \underline{\epsilon}$$

where  $B$  is a diagonal matrix the square of whose  $n$ -th diagonal element is

$$(2) \quad B_n^2 = \sum_{k=1}^{k_n} a_{nk}^2 \sigma_k^2, \quad \sigma_k^2 = \text{var } \epsilon_k.$$

Then  $\text{var } \zeta_n = 1$  for all  $n$ . We now make use of the Lindeberg-Feller version [2, p. 103] of the Central limit theorem:

In order that the distribution functions of the sums

$$(3) \quad \zeta_N = \xi_{N1} + \xi_{N2} + \dots + \xi_{NN}$$

of independent random variables  $\xi_{N1}, \xi_{N2}, \dots, \xi_{NN}$  with

$$(4) \quad E \xi_{Nk} = 0 \quad \text{for } k = 1, 2, \dots, N, \text{ and}$$

$$(5) \quad \sum_{k=1}^N \text{var } \xi_{Nk} = 1$$

converge to the normal distribution  $N(0,1)$ :

$$\Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^x e^{-z^2/2} dz$$

and that the r.v.-s  $\xi_{Nk}$  be infinitesimal it is necessary and sufficient that Lindeberg's condition

$$(6) \quad \sum_{k=1}^N \int_{|z| > \delta} z^2 dF_{Nk}(z) \rightarrow 0 \quad \text{for } N \rightarrow \infty$$

be satisfied for every  $\delta > 0$ .  $F_{Nk}(z)$  is the distribution function of  $\xi_{Nk}$ .

If one puts  $\xi_{nk} = B_n^{-1} a_{nk} \epsilon_k$ ,  $\zeta_n$  as in (1), conditions (3), (4), and (5) are satisfied, and the following theorem can be concluded:

Theorem: Let  $\tilde{F}$  be the space of all random variables with zero mean and positive finite variance. Let  $F$  be a non-empty subset of  $\tilde{F}$ . Then it is necessary and sufficient for the convergence of  $(\zeta_n^*)$  to the normal law  $N(0,1)$  on the set  $F$  for  $n \rightarrow \infty$  that the following conditions regarding  $F$  and a matrix  $A$  of type (T) are simultaneously satisfied:

$$(I) \quad \max_{k=1, \dots, k_n} \frac{a_{nk}^2}{\sum_{k=1}^{k_n} a_{nk}^2} \rightarrow 0 \quad \text{for } n \rightarrow \infty$$

(II) There exists a bounded function  $g(c)$  for  $c \geq 0$  with  $\lim_{c \rightarrow \infty} g(c) = 0$

\* Footnote: If the fourth moments of all  $\epsilon \in F$  are uniformly bounded then their  $\xi$ , a statistic similar to  $\zeta_n$  yet not containing the  $\sigma_v$ 's and tending to  $N(0,1)$ ; see [1]

such that for each r.v.  $\epsilon \in F$  with distribution function  $G(\epsilon)$ , say,

$$\int_{c \leq |\epsilon|} \epsilon^2 dG(\epsilon) < g(c)$$

holds.

(III) There exists a positive constant  $m$  such that for each  $\epsilon \in F$

$$\text{var } \epsilon > m$$

holds.

As the theorem shows, uniform boundedness of all variances from above is not sufficient.

Proof: (for a special matrix  $A$  a proof was already given in [1]).

1) Sufficiency: Let  $\underline{a}'_n$  be the  $n$ -th row vector of  $A$ . Then for any  $k \leq k_n$  by (I) and (III)

$$q_{nk}^2 = \frac{a_{nk}^2}{B_n^2} \leq \frac{1}{m} \max_k \frac{a_{nk}^2}{a_n^2} \rightarrow 0 \quad \text{for } n \rightarrow \infty.$$

Let  $\underline{\epsilon}$  be any sequence of "errors"  $\epsilon_i \in F$ . We see that the sufficient condition (6) in the central limit theorem is satisfied because for any

$\delta > 0$

$$\sum_{k=1}^{k_n} \int_{|z| > \delta} z^2 dF_{nk}(z) \leq \frac{a_n^2}{B_n^2} \max_{k=1, \dots, k_n} \int_{|\epsilon| > \delta/q_{nk}} \epsilon^2 dG(\epsilon)$$

which tends to zero. We have namely  $B_n^{-2} a_n^2 < m^{-1}$ , and by (II), with  $G_k$  the distribution function of  $\epsilon_k$ , for any  $k \leq k_n$

$$\int_{|\epsilon| > \delta/q_{nk}} \epsilon^2 dG_k(\epsilon) < g(\delta/q_{nk}) \rightarrow 0$$

because  $\max_{k=1, \dots, k_n} q_{nk} \rightarrow 0$ .

2) The necessity of both of the conditions (II) and (III) is proved in several steps. Condition (I) is seen to be identical with the infinitesimality of the terms  $\xi_{nk}$  in  $\zeta_n$ . The latter could have been required instead of (I) from the beginning. By choosing all r.v.-s  $\epsilon_k$  to be identically distributed like  $\epsilon_0$ , say, we have from

$$\max_k P(|q_{nk}\epsilon_0| \geq \delta) = \max_k P(|\epsilon_0| \geq \frac{\delta}{q_{nk}}) \rightarrow 0$$

for any  $\delta > 0$  that  $\max_k q_{nk}^2 = (\text{var } \epsilon_0)^{-1} \max_k \frac{a_{nk}^2}{a_n} \rightarrow 0$ .

3) We assume that the variances of the r.v.-s in  $F$  are not bounded where  $F$  is any subset of  $\tilde{F}$ . We construct a sequence  $\{\epsilon_v\}$  with  $\sigma_v^2 \rightarrow \infty$  such that (8) is bounded away from zero for all  $n$  for which  $a_{nk_n} \neq 0$ .

To this end, we write (8) in the form

$$\sum_{k=1}^{k_n-1} q_{nk}^2 \int_{|z| > \delta/q_{nk}} z^2 dG_k(z) + q_{nk_n}^2 \int_{|z| > \delta/q_{nk_n}} z^2 dG_{k_n}(z)$$

and, assuming  $\delta^2 < 1/2$ , we determine the last integration interval so that

$$\frac{\delta^2}{a_{nk_n}^2} B_n^2 < 1/2 \left( \sum_{k=1}^{k_n-1} \frac{a_{nk}^2}{a_{nk_n}^2} \sigma_k^2 + \sigma_{k_n}^2 \right) < \gamma^2 \sigma_{k_n}^2.$$

This is always possible for  $a_{nk_n} \neq 0$  and a constant  $\gamma < 1$  and already fixed  $\sigma_k$ 's,  $k=1, \dots, k_n-1$ , by making  $\sigma_{k_n}$  sufficiently large. Then, taking only the  $k_n$ -th term, (6) is larger than

$$\frac{a_{nk_n}^2}{B_n^2} \int_{|z| > \gamma \sigma_{k_n}} z^2 dG_{k_n}(z) > (1 - \gamma^2) \frac{a_{nk_n}^2 \sigma_{k_n}^2}{B_n^2} > \frac{1 - \gamma^2}{2\gamma^2} > 0.$$

Hence all the r.v.-s in  $F$  have uniformly bounded variances.

4) Let  $\epsilon$  be any element in  $F \subseteq \tilde{F}$ . Its distribution function may be denoted by  $G_\epsilon(z)$ , and a uniform bound for the variances of all the  $\epsilon$ 's let be  $M > \text{var } \epsilon$ . Let  $f_\epsilon(c) = \int_{|z| > c} z^2 dG_\epsilon(z)$  and consider for  $c \geq 0$  the

non-increasing function

$$M(c) = \limsup_{\epsilon \in F} f_\epsilon(c).$$

$F$  has property (II) if and only if  $\lim_{c \rightarrow \infty} M(c) = 0$ . We assume now that  $F$  does not have property (II). We will then construct a sequence of errors such that (6) does not tend to zero, thus coming to a contradiction. Obviously,  $F$  must have infinitely many elements, and for any sequence  $c_v \rightarrow \infty$  we can find a sequence of r.v.-s  $\epsilon_v \in F$  and functions  $f_v(c)$  such that for  $v=1,2,\dots$

$$f_v(c_v) = \int_{|z| > c_v} z^2 dG_v(z) > C > 0$$

where  $\lim_{c \rightarrow \infty} M(c) > C > 0$ ,  $C$  a suitable constant;

$$(9) \quad C < \sigma_v^2 = \text{var } \epsilon_v < M.$$

By selection of a suitable sub-sequence  $\{\epsilon_{v_k}\}$  from  $\epsilon_v$  and identifying them with the components of  $\underline{\epsilon}$  we show that at least for a sub-sequence

$n_\rho$ ,  $\rho=1,2,\dots$  in the set of positive integers, (6) is not true. Because of (9) we have

$$(10) \quad \frac{a_{nk}^2}{M a_{-n}^2} < q_{nk}^2 < \frac{a_{nk}^2}{C a_{-n}^2}$$

which tends necessarily to zero because of 2). Let  $\eta$  be a positive constant  $< C^{-1}$ . Let for each  $n$   $\kappa_n$  be the largest integer such that

$$\frac{1}{C a_{-n}^2} \sum_{k=1}^{\kappa_n} a_{nk}^2 < \eta$$

yet such that the sequence  $\{\kappa_n\}$  is non-decreasing. Evidently  $\kappa_n \rightarrow \infty$  because of (10).

Now determine an infinite sequence  $n_1, n_2, \dots \rightarrow \infty$  such that for  $\rho=1,2,\dots$

$$K_\rho \leq \kappa_{n_{\rho+1}}$$

Here to simplify the notation we have written  $K_\rho$  for  $\kappa_{n_\rho}$ .

Let now  $\epsilon_{v_k}$  be the same r.v. for  $k = \kappa_{n_{\rho+1}}, \dots, K_\rho$ , say  $\epsilon_{K_\rho}$ ,

with

$$c_{v_k}^2 = c_{K_\rho}^2 > \frac{M a_{-n_\rho}^2}{\min_j a_{n_\rho j}^2}.$$

Here  $j$  runs from  $\kappa_{n_{\rho+1}}$  to  $K_\rho$  except those  $j$  for which  $a_{n_\rho j} = 0$ . Because

of  $c_v \rightarrow \infty$ , a  $c_{v_k} < \infty$  can always be found.

We now get for (6) a contradiction because for each  $n_\rho$  and any  $\delta$ ,

$$0 < \delta < 1, \\ \sum_{k=1}^{K_\rho} q_{n_\rho k}^2 \int_{|z| > \delta/q_{n_\rho k}} z^2 dG_k(z) \geq \left( \sum_{k=\kappa_{n_{\rho+1}}+1}^{K_\rho} q_{n_\rho k}^2 \right) \int_{|z| > c_{K_\rho}} z^2 dG_{K_\rho}(z) >$$

$$\frac{C}{M} \left( 1 - \sum_{k=1}^{\kappa} \frac{a_{nk}^2}{a_n^2} \right) > \frac{C}{M} (1 - C\eta) > \text{const} > 0.$$

Thus the necessity of (II) has been proved.

5) It remains to prove the necessity of (III). We assume that an arbitrary subset  $F \subseteq \tilde{F}$  does not have property (III) and construct a contradiction by showing that (6) remains above a positive constant for a sequence of positive numbers  $n_\rho \rightarrow \infty$ . Any such set  $F$  contains necessarily infinitely many r.v.-s and especially sequences  $\{\epsilon_\nu\}$  with

$$\liminf_{\nu \rightarrow \infty} \sigma_\nu = 0$$

We first give the proof for a special set  $F$  of this kind, say  $S$ , with r.v.-s whose distributions are 2-step functions  $S_\sigma(z)$  each having steps at  $+$  and  $-$   $\sigma$  with weight  $1/2$  each. The  $k$ -th term in (6) for a sequence  $\{S_k\} \subset S$  and variance  $\sigma_k$

$$\sum_{k=1}^{\kappa} \frac{\sigma_k^2}{a_{nk}^2} \int_{|z| > \delta/a_{nk}} z^2 dS_k(z)$$

gives the contribution

$$(11) \quad \frac{\sigma_k^2 a_{nk}^2}{B_n^2} > \delta^2$$

if only we have for the integration interval  $\sigma_k > \delta/a_{nk}$  (which is the same inequality as (11)). If we can construct an error sequence  $\{\epsilon_\nu\}$

with  $\epsilon_\nu \in S$  such that (11) holds for some  $\delta_0 > 0$  and for infinitely many  $n_\rho$  and for at least one  $k$  at each  $n_\rho$  then we have a contradiction. Clearly,

if (11) holds for some  $\delta_0 > 0$  it holds also for any positive  $\delta < \delta_0$ .

We write (11) in the form, putting  $\delta_0^2 = \eta$ :

$$(12) \quad (1 - \eta) \sigma_k^2 a_{nk}^2 - \eta \sum_{\substack{j=1 \\ j \neq k}}^n \sigma_j^2 a_{nj}^2 > 0.$$

Let

$$A_n^2 = \max_{k=1, \dots, k_n} a_{nk}^2$$

and consider for fixed  $k$  and  $n \rightarrow \infty$  the sequences  $s_k \equiv \left\{ (a_{nk}/A_n)^2 \right\}$ .

Subcase 5a): Let there exist a number  $k = \kappa$  for which  $s_k$  has a positive limit value greater than  $c < 1$ , say. Select from  $s_k$  a subsequence all whose terms are greater than  $c' < c$  for  $n = n_1, n_2, \dots \rightarrow \infty$ . Then for these  $n_\rho$  and  $\kappa$ , (12) is greater than

$$((1-\eta)\sigma_\kappa^2 c' - \eta \sum_{k=1}^{\infty} \sigma_k^2) A_{n_\rho}^2$$

which is positive if we choose  $\sum_{j=1}^{\infty} \sigma_j^2 < \infty$  and  $\eta > 0$  to be a sufficiently small constant independent of  $n_\rho$ . Thus (11) is a lower bound for (6) and under our assumption about  $s_k$ , (III) is proved to be necessary.

Subcase 5b): We now assume that there is no  $\kappa$  as defined above, i.e.

each sequence  $s_k$  tends to zero. It follows that there can be found nondecreasing sequences of integers  $m_n \rightarrow \infty$  (for  $n_0 \leq n \rightarrow \infty$ ) such that

$$\sum_{k=1}^{m_n} \frac{a_{nk}^2}{A_n^2} < \mu < 1.$$

Determine now for a suitably selected sequence  $m_n \rightarrow \infty$  a sequence of integers  $n_\rho \rightarrow \infty$  from  $m_{n_\rho+1} = k_{n_\rho} \equiv K_\rho$  ( $K_\rho$  as defined in 4.). Let  $A_n^2$

be taken on by  $a_{nk}^2$ . Clearly  $m_n < k_n^*$ . If now  $\sigma_1, \sigma_2, \dots, \sigma_{m_n}$  for  $n=n_\rho$  are any constants we then determine inductively  $\sigma_{m+1}, \dots, \sigma_{k_n}$  such that for (12) the lower bound holds, putting  $k=k_n^*$ ,  $n=n_\rho$ ,

$$A_n^2 \left\{ (1-\eta) \sigma_{k_n^*}^{-\eta} \mu \max_{k=1, \dots, m_n} \sigma_k^{2-\eta} \sum_{\substack{k=m+1 \\ k \neq k_n^*}}^{k_n} \sigma_k^2 \right\} > 0.$$

This can be done simultaneously for any  $n$  equal to an  $n_\rho$  by choosing

(i)  $\sigma_{k_n^*} = \text{const} > 0$ , (ii)  $\sum_{k=m+1}^{k_n} \sigma_k^2 < \text{const} < \infty$ , (iii)  $\eta \leq \eta_0$  a sufficiently

small constant (all constants independent of  $n_\rho$ ). Hence also in this

subcase condition (III) is shown to be necessary for the set  $S$ .

6) We now turn back to an arbitrary subset  $F$  not having property (III). There are sequences of r.v.-s with distributions  $G_\nu(z)$  in  $F$  such that for their variances  $\liminf_{\nu \rightarrow \infty} \sigma_\nu = 0$  holds. Consider for each  $G_\nu(z)$  a

step function  $S_\nu(z)$  as defined in 5) which has the same variance  $\sigma_\nu$ . We then have for any  $\nu$ , any  $\delta > 0$  and any number  $q_{nk}$

$$\int_{|z| \geq \delta/q_{nk}} z^2 dG_\nu(z) \geq 1/2 \int_{|z| \geq \delta'/q_{nk}} z^2 dS_\nu(z)$$

with  $\delta' = \sqrt{2\delta}$ . It is sufficient to prove this for  $\frac{\delta'}{q_{nk}} = \sigma_\nu - \epsilon$ . The

left side is not smaller than  $\sigma_\nu^2(1 - 1/2)$  which is also the value of the right side. It now follows from 5) that for any matrix  $A$  of type (T) we can find a sequence of errors with d.f.-s  $G_k$  such that for  $n=n_\rho$ ,  $\rho \rightarrow \infty$

$$\sum_{k=1}^K \frac{1}{q_{n\rho k}^2} \int_{|z| > \delta/q_{n\rho k}} z^2 dG_k(z) \geq 1/2 \sum_{k=1}^K \frac{1}{q_{n\rho k}^2} \int_{|z| > \sqrt{2} \delta/q_{n\rho k}} z^2 dS_k(z) > 2\delta^2 > 0.$$

This completes the proof of theorem 2.

## 2. The law of large numbers for sums over spaces F.

While it turned out to be possible in the above case 1. of convergence to the standard normal distribution to find simple separate conditions for the matrix A and the space F, this does not seem to be the case for convergence to other limit distributions. So for instance for the convergence to the law of large numbers over a space F there are here given only sufficient conditions under rather restrictive a priori conditions placed upon F. If one wants necessary and sufficient conditions it seems unlikely that they can be separated into conditions for A and F only. In fact, the limit theorems to be used suggest this for the case of general limit laws. The central limit theorem thus represents an exceptional situation. The following well known theorem [2, p.134] is used to obtain conditions for sums of random variables in order that they obey the law of large numbers. Let

$$\xi_{n1}, \xi_{n2}, \dots, \xi_{nk_n}, \quad n=1,2,\dots$$

be a doublesequence of in each row independent r.v.-s, and let  $F_{nk}(x)$ ,  $m_{nk}$ ,  $\mu_{nk}$ ,  $\sigma_{nk}^2$  be resp. the cumulative d.f., the median, the mean and the variance of  $\xi_{nk}$ .

Theorem: In order that the sums  $\zeta_n = \xi_{n1} + \xi_{n2} + \dots + \xi_{nk_n}$  obey the law of large numbers (LLN), it is necessary and sufficient that as  $n \rightarrow \infty$

$$\begin{aligned}
 & 1) \quad \sum_{k=1}^{k_n} \int_{|x| \geq 1} dF_{nk}(x + m_{nk}) \rightarrow 0 \\
 & 2) \quad \sum_{k=1}^{k_n} \int_{|x| < 1} x^2 dF_{nk}(x + m_{nk}) \rightarrow 0 \\
 (13) \quad & 3) \quad \sum_{k=1}^{k_n} \left\{ \int_{|x| < \tau} x dF_{nk}(x + m_{nk}) + m_{nk} \right\} \rightarrow 0
 \end{aligned}$$

for every  $\tau > 0$ .

We now put as before  $\xi_{nk} = a_{nk} \xi_k$ ,  $\xi_k \in F$ , and with  $F_k(x)$  being the d.f. of  $\xi_k$  it comes

$$F_{nk}(x) = P(\xi_{nk} < x) = F_k(x/a_{nk}).$$

One observes the following

Lemma: If the LLN holds over  $F$  and  $F$  contains at least one non-degenerate r.v. (i.e. its d.f. is not a unit step function) then  $\sum_{k=1}^{k_n} a_{nk}^2 \rightarrow 0$  as  $n \rightarrow \infty$  is necessary.

Proof: Because of

$$\int_{|x| < 1} x^2 dF_{nk}(x + m_{nk}) = a_{nk}^2 \int_{m_k - a_{nk}^{-1}}^{m_k + a_{nk}^{-1}} (z - m_k)^2 dF_k(z)$$

where  $m_k = \frac{m_{nk}}{a_{nk}}$  = median of  $\xi_k$ , condition 2) in the above theorem has for

identical, non-degenerate r.v.-s  $\xi_k$  the lower bound

$$\int_{m_1 - \alpha_n^{-1}}^{m_1 + \alpha_n^{-1}} (z - m_1)^2 dF_1(z) \sum_{k=1}^k a_{nk}, \quad \alpha_n = \max_k a_{nk}.$$

As the first factor remains positive the second one must tend to zero .

Lemma: If the LLN holds over  $F$  then the medians of all elements in  $F$  are necessarily uniformly bounded.

Proof: If the assertion is not true then  $F$  contains infinitely many elements, and there are sequences of r.v.-s such that the moduli of their medians  $|m_k| \rightarrow \infty$  . We now select any sequence of non-zero matrix elements  $a_{n^*k^*} \rightarrow 0$  for  $k^* \rightarrow \infty$  from among the elements

$a_{1k_1}, a_{2k_2}, \dots, a_{nk_n}, \dots$  and determine the  $\xi_{k^*}$  inductively such that

$$(14) \quad |m_{k^*}| > \frac{C + \tau + \left| \sum_{l=1}^{k^*-1} \right|}{|a_{n^*k^*}|}$$

which is always possible. Here  $C$  is a positive constant and  $\left| \sum_{l=1}^{k^*-1} \right|$  means

the modulus of the sum (13) up to the  $(k^*-1)$ -st term. We then have for

(13) putting  $n=n^*$

$$\left| \sum_{k=1}^{k^*} a_{n^*k} \left\{ m_k + \int_{m_k - \tau a_{n^*k}^{-1}}^{m_k + \tau a_{n^*k}^{-1}} (z - m_k) dF_k(z) \right\} \right| \geq$$

$$|a_{n^*k^*}| \left\{ |m_{k^*}| - \frac{\tau}{|a_{n^*k^*}|} \right\} - \left| \sum_{k=1}^{k^*-1} \right| > C$$

according to (14) for any choice of  $\xi_1, \dots, \xi_{k^*-1}$ . Thus there is a contradiction because (13) should tend to zero.

It seems to be difficult if not impossible to find necessary conditions to be imposed on the means and variances of the elements of  $F$  independent of the given matrix  $A$ . This problem is considerably simplified if not only the sequences of r.v.-s  $\xi_1, \xi_2, \xi_3, \dots$  are left undetermined

but also the matrix  $A$ . The problem then would be to find properties of a class of matrices  $A$  and of sets  $F$  which are necessary or sufficient (or both) in order that the LLN holds uniformly over these classes. Yet this case is of less practical importance. Without proofs (which by the way are similar to that of lemma 2) we observe that in this situation the following conditions are necessary:

$$(15) \quad a \int_{-1/a}^{1/a} \xi \, dF(\xi) \rightarrow 0$$

$$a^2 \int_{-1/a}^{1/a} \xi^2 \, dF(\xi) \rightarrow 0$$

uniformly for all r.v.-s  $\xi \in F$  for  $a \rightarrow 0$ . If the matrix  $A$  is fixed then these conditions are required only for  $a$ -values out of certain sets of numbers  $\{a_{nk}\}$  resp.  $\{\tau_{nk}\}$ . If certain weak smoothness conditions of the  $F(\xi)$ 's are satisfied then the distinction between these discrete sets and the continuum of  $a$ -values does not make any difference. Thus it is practically reasonable to consider (15) as necessary (but these conditions are still far from being sufficient).

In order to find sufficient conditions for  $A$  and  $F$  one may start with the strong a priori assumption that for all  $\xi \in F$  holds

$$\text{var } \xi < \text{const} < \infty .$$

Besides the above shown necessary uniform boundedness of the medians  $m$  we then also have uniform boundedness for the means  $\mu$ , namely

$$|\mu - m| \leq \sqrt{2} \sigma .$$