ANALYSIS OF IRREGULAR FACTORIAL FRACTIONS

by

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1. Introduction and Summary.

Factorial fractions are finding increasing use in agricultural, biological and industrial experimentation. (For a general introduction to the various aspects of the subject see [2, 11, 13, 18]). Though in many situations we are interested only in the main effects of the different factors, in most others a study of two-factor interactions becomes necessary or desirable. However, good fractions are at present not available for most of the asymmetrical and even symmetrical factorials. By a good fraction we mean that it should possess each of the following features to a reasonable degree. Firstly it should be economic; i.e. should involve as few observations as desired. Secondly, the correlations between the estimates of various effects should be small, particularly those involving main effects. Thirdly the variances for the different main effects, or for the interactions should not widely differ from one pair of factors to another, i.e. there should be some kind of symmetry or balance in the fraction. This requirement is connected with a fourth one, viz. that the fraction should be analyzable with reasonable ease, and should be well interpretable.

A fraction is called irregular if it is not orthogonal, i.e. if some of the effects are correlated. The large amount of work (see for example [3, 5, 7, 8, 9, 10, 12, 14, 15, 16]) already done in this area shows that only a few orthogonal fractions are economic. Thus for even the vast majority of cases likely to arise in practice, good irregular fractions are still unknown.

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In this paper we develop the theory for irregular symmetrical and asymmetrical factorial fractions. To aid in the development of the theory, a $\lambda$-function has been introduced. Using this, it has been shown how to compute the matrix $(M, \text{say})$ to be inverted for solving the normal equations. Although in this paper we mainly discuss analysis of fractions, the theory developed here paves the way for the construction of good fractions by throwing light directly on the nature of $M$ and hence on the variance-covariance matrix.

In a separate paper [6] we discuss the analysis of a special class of fractions, called partially balanced, where we discuss special methods of inverting $M$. Based on these two papers will be several later communications, published elsewhere in which the construction of good irregular fractions will be considered.

Most of the theory contained herein will be found in the unpublished report [17], and a rather detailed summary in [4].

2. Definition of interactions: Normal equations.

Consider an $s^m$ factorial experiment. Treatment combinations can be written in the lexicographic order $a_1^{j_1} a_2^{j_2} ... a_m^{j_m}$, $0 \leq j_r \leq s-1$, $r = 1, 2, ..., m$. If in the symbol for a treatment, the exponent $j_r$ of $a_r$ is zero, we shall omit this $a_r$ from the symbol. If $j_r = 0$ for all $r$, the symbol will be written $\varnothing$.

Exactly similar notation will be used for interactions except that $a$'s will be replaced by A's and $\varnothing$ by $\mu$.

It is well known (see, for example [2, 11]) that each interaction degree of freedom can be expressed as a linear contrast of all treatment combinations:

$$
\sum_{j_1, j_2, \ldots, j_m} A_1^{j_1} A_2^{j_2} \ldots A_m^{j_m} = \sum_{k_1, k_2, \ldots, k_m} \beta_{k_1, k_2, \ldots, k_m} (a_1^{k_1} a_2^{k_2} \ldots a_m^{k_m}).
$$
Let \( \mathbf{a} \) denote the column vector of all assemblies in the above mentioned lexicographic order:

\[
(2) \quad \begin{bmatrix} a_1^2 & a_2^2 & \ldots & a_m^2; a_1^2 & a_2^2 & \ldots & a_m^2; & \ldots; & a_1^2 & a_2^2 & \ldots & a_m^2; \\
 a_1 & a_2 & \ldots & a_m; & \ldots; & a_1 & a_2 & \ldots & a_m; & \ldots; & a_1 & a_2 & \ldots & a_m; \\
 a_1^2 & a_2^2 & a_3^2 & \ldots & a_m^2; & \ldots; & a_1^2 & a_2^2 & a_3^2 & \ldots & a_m^2; & \ldots; & a_1^2 & a_2^2 & a_3^2 & \ldots & a_m^2 \end{bmatrix}
\]

Let \( \mathbf{A} \) denote the column vector of \( \mathbf{A}'s \) in the same order. Then equations (1) could be written in matrix notation as:

\[
(3) \quad \mathbf{A} = \mathbf{D} \mathbf{a},
\]

where \( \mathbf{D} \) is an \( s^m \times s^m \) matrix. It is known [2] that the sum of products of the corresponding elements in any two rows of \( \mathbf{D} \) is zero. Let \( \delta^2_1 \) denote the sum of squares of the elements in the \( i \)-th row of \( \mathbf{D} \). To make \( \mathbf{D} \) orthogonal we divide \( i \)-th row by \( \delta^2_1 \). Let \( \Delta \) be an \( s^m \times s^m \) diagonal matrix with \( \delta^{-1}_1 \) in the cell \( (i,i) \). Then \( \mathbf{C} = \Delta \mathbf{D} \) is an orthogonal matrix. Hence using (3) we get

\[
(4) \quad \mathbf{a} = \mathbf{C}' \Delta \mathbf{A} = \mathbf{D}' \Delta^2 \mathbf{A}.
\]

Let \( \mathbf{A} \) be partitioned as

\[
(5) \quad \mathbf{A}' = (\mathbf{L}' : \mathbf{I}'_o),
\]

where \( \mathbf{L} \) is the column vector of all interactions up to and including all the 2-factor interactions (say \( v \) in number), and \( \mathbf{I}_o \) is the vector of 3-factor and higher order interactions. We shall suppose hence forth that all elements of \( \mathbf{I}_o \) are assumed to be zero. The effect on the estimate of \( \mathbf{L} \) of replacing this last
assumption on \( I_0 \) by other assumptions, will be studied in a separate paper. It can be easily checked that

\[
(6) \quad \nu = 1 + m(s-1) + \binom{m}{2}(s-1)^2.
\]

Under the assumption that elements of \( I_0 \) are zero, we get from (4) and (5):

\[
(7) \quad \underline{a} = (D' \Delta^2)_o L,
\]

where \((D' \Delta^2)_o\) is the matrix obtained by cutting out from \((D' \Delta^2)\) the \((s^m - \nu)\) columns which correspond to \( I_0 \) in (4). Let \( \Delta^2_o \) be obtained from \( \Delta^2 \) by cutting out the last \((s^m - \nu)\) columns and rows from \( \Delta^2 \), and \( D'_o \) from \( D \) by cutting out the \((s^m - \nu)\) columns. Then it can be easily checked that

\[
(8) \quad \underline{a} = D'_o \Delta^2_o L.
\]

Let \( T \) be a fraction, i.e. a set of assemblies in which any given treatment combination may not occur or occur once or more times. Let the expected values of the assemblies in \( T \) written in the form of a vector be denoted by \( \underline{y}^* \) where, for simplicity, we are not assuming any block effects. Let \( E' \) be the matrix obtained from \( D'_o \) by cutting out the last \((s^m - \nu)\) columns corresponding to \( I_0 \), and also by omitting (or repeating) the rows corresponding to treatment combinations omitted (or repeated) from \( \underline{a} \) to get \( \underline{y}^* \). Note that the rows of \( E' \) are arranged in such a way as to correspond to the elements of \( \underline{y}^* \). Let \( \underline{y} \) be the vector of observations corresponding to \( \underline{y}^* \). Since we are assuming no block effects, it is clear that

\[
(9) \quad \text{Exp} (\underline{y}) = \underline{y}^* = E' \Delta^2_o L = E' \underline{p}, \quad \text{say}.
\]

The normal equations for estimating \( \underline{p} \) can then be written
(10) \((EE') \hat{E} = E Y\),

where \(\hat{E}\) stands for the estimate of \(E\). Hence if \((EE')\) is nonsingular, we get \(L\) as the best linear unbiased estimate of \(L\), where

\[(11) \quad L = \Delta^{-2}_0 (EE')^{-1} E Y = \Delta^{-2}_0 (EE')^{-1} x, \quad \text{say.}\]

We shall now demonstrate a useful and easy way of calculating \(x\). From (3) and (5) and the definition of \(D_0\) we get

\[(12) \quad L' = a' D_0 \quad \text{or} \quad L = D_0 a.\]

Corresponding to \(a\), construct a vector \(z\) of length \(s^m\) in the following way.

Take any element \(\theta\) of \(a\). Let the element in \(z\) corresponding to the treatment combination \(\theta\) be denoted by \(z(\theta)\). We define

\[(13) \quad z(\theta) = 0, \quad \text{if} \quad \theta \notin T \quad \text{and} \quad z(\theta) = \text{total yield of} \quad \theta \quad \text{(from all repetitions of} \quad \theta \quad \text{in} \ T) \quad \text{if} \quad \theta \in T.\]

Then from the definition of \(E'\), it is easy to check that

\[(14) \quad x = E Y = D_0 z.\]

Comparing (12) and (14) we find that \(x\) is obtained from \(z\) in the same way as \(L\) is obtained from \(a\), and that we do not need to write down \(E\) explicitly.

Finally, the calculation of \(L\) from \(a\) or of \(x\) from \(z\) can be easily done by making two-way tables for "the total yields" of various level-combinations of each pair of factors.

**Example.**

Consider the following 10 assemblies of a \(2^3\) factorial: \(a_1 a_2 a_3 (13, 15),\)

\(a_1 a_2 a_3 (10, 12), a_0 a_2 a_3 (17, 15)\) (twice each), and \(a_0 a_0 a_0 (5), a_1 a_2 a_3 (25),\)
The figures in brackets are (artificial) observed values. The set of all interactions is

\[ A' = (\mu; A_1, A_2, A_3; A_1A_2, A_1A_3, A_2A_3; A_1A_2A_3). \]

The vector \( \mathbf{a} \) is defined similarly. The matrix \( D \) in (3) is (usually) defined as

\[
D = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-1 & 1 & -1 & -1 & 1 & 1 & -1 \\
-1 & -1 & 1 & -1 & 1 & -1 & 1 \\
-1 & -1 & -1 & 1 & -1 & 1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & 1 & -1 & -1 & -1 & 1 & 1 \\
-1 & 1 & 1 & 1 & -1 & -1 & -1
\end{bmatrix}.
\]

Clearly \( \delta_i^2 = 9 \), for each \( i \). Also \( \mathbf{I}_o \) here is the vector with the single element \( (A_1A_2A_3) \), \( \nu = 1 + 3 + 3 = 7 \), and \( D'_o \) is the transpose of the matrix above the horizontal line in \( D \). From \( D'_o \) we get

\[
E = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
-1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 \\
-1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 \\
-1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1
\end{bmatrix},
\]
where the columns correspond to the assemblies in the order given above. To calculate $x$ we proceed as explained earlier by calculating $z$, which here equals

$$z = (5, 25, 12, 7, 13+15, 10+12, 17+15; 0).$$

Using the above in (14) and (11), one gets $L$.

Consider (11) again, which gives the solution of the normal equations. Since the computation of $x$ is easy, and the diagonal matrix $A_o$ can be easily evaluated the main problem in the analysis of a fraction $T$ reduces therefore to the evaluation of the matrix $EE'$ and its inversion. The matrix $EE'$ corresponding to a set of assemblies $T$ will be denoted by $(EE')_T$.

3. $\lambda$-operator and the calculation of $(EE')$.

Let $\Omega_{m,s}$ denote the total set of $s^m$ assemblies when we have $m$ factors each at $s$ levels. Let $(a_r^j)$ denote the sum of 'observed yields' of all treatments of $\Omega_{m,s}$ in which the symbol $(a_r^j)$ occurs. Then we define the main effect

$$A_r = d_0(a_r^0)a_r^0 + d_1(a_r^1)a_r^1 + \ldots + d_{s-1}(a_r^{s-1}),$$

where $d_i's$ are real numbers satisfying

$$\sum_{j=0}^{s-1} d_j(K) = 0, \quad 0 \leq K \leq s-1,$$

and

$$\sum_{j=0}^{s-1} d_j(K) d_j(K') = 0, \quad K \neq K', \quad 0 \leq K, K' \leq s-1,$$

and

$$d_j(0) = 1, \quad \text{for all } j.$$

Then in (1) the interaction $A_1 A_2 \ldots A_m$ is defined (see for example [3]) such that
\( d_{k_1,k_2,\ldots,k_m}^{j_1,j_2,\ldots,j_m} = d_{j_1}^{k_1} d_{j_2}^{k_2} \cdots d_{j_m}^{k_m} \).

Let \( \lambda_{i_1,i_2,\ldots,i_r}^{j_1,j_2,\ldots,j_r} \) be the number of times the symbol \( a_{i_1}^{j_1} a_{i_2}^{j_2} \cdots a_{i_r}^{j_r} \) (1 \( \leq \) \( r \) \( \leq \) \( m \)) occurs among the treatment combinations in the fraction \( T \) that we have. So far as the latter set of symbols are concerned we assume that they can be operated upon, over the field of real numbers, like ordinary algebraic products of the indeterminates \( a_{i_r}^{j_r} \) (1 \( \leq \) \( r \) \( \leq \) \( m \)). Thus we assume for example,

\[
(\beta_1 a_{i_1}^{j_1} + \beta_2 a_{i_2}^{j_2})(\beta_3 + \beta_4 a_{i_2}^{j_2})(a_{i_4}^{j_4} + 1)
= (\beta_3 \beta_1) a_{i_1}^{j_1} + (\beta_3 \beta_2) a_{i_2}^{j_2} + (\beta_1 \beta_4) a_{i_1}^{j_1} a_{i_2}^{j_2} + (\beta_2 \beta_4) a_{i_2}^{j_2} a_{i_3}^{j_3} + (\beta_3 \beta_1) a_{i_2}^{j_2} a_{i_4}^{j_4} +
+ (\beta_3 \beta_2) a_{i_2}^{j_2} a_{i_4}^{j_4} + (\beta_1 \beta_4) a_{i_1}^{j_1} a_{i_2}^{j_2} a_{i_4}^{j_4} + (\beta_2 \beta_4) a_{i_2}^{j_2} a_{i_3}^{j_3} a_{i_4}^{j_4}
\]

where \( \beta \)'s are real numbers.

Consider the set \( \mathcal{P} \) of all polynomials in the symbols \( a_{i_r}^{j_r} \) (1 \( \leq \) \( r \) \( \leq \) \( m \)) such that no term in any polynomial in \( \mathcal{P} \) contains any symbol \( a_{i_r}^{j_r} \) at a power higher than unity. Thus for example, terms like \( (a_{i_1}^{3}) a_{i_2}^{j_2}, a_{i_1}^{j_1} a_{i_2}^{j_2} a_{i_3}^{j_3}, a_{i_2}^{j_2} a_{i_3}^{j_3} a_{i_4}^{j_4} \) do not occur in these polynomials.

We define a \( \lambda \)-operator on the polynomials in \( \mathcal{P} \) by:

\[
\lambda^{(18)}\left(a_{i_1}^{j_1} a_{i_2}^{j_2} \cdots a_{i_r}^{j_r}\right) = \lambda_{i_1,i_2,\ldots,i_r}^{j_1,j_2,\ldots,j_r},
\]

\[
\lambda[\beta(a_{i_1}^{j_1} a_{i_2}^{j_2} \cdots a_{i_r}^{j_r})] = \beta \lambda_{i_1,i_2,\ldots,i_r}^{j_1,j_2,\ldots,j_r}.
\]
\[ \lambda \left[ \beta(a_1^{j_1} a_2^{j_2} \ldots a_r^{j_r}) + \beta'(a_1'^{i_1} a_2'^{i_2} \ldots a_u^{i_u}) \right] = \beta a_1^{j_1} a_2^{j_2} \ldots a_r^{j_r} + \beta' a_1'^{i_1} a_2'^{i_2} \ldots a_u^{i_u}, \]

where \( 1 \leq r, u \leq m \) and \( \beta \)'s are real numbers. Thus if \( P \) denotes any polynomial in \( \mathcal{C} \), \( \lambda(P) \) is a real number which naturally depends on \( T \). To emphasize this last fact we sometimes write \( \lambda(P) \) as \( \lambda(P, T) \). We shall say that \( P \) is in excess, is in defect, or is zero in \( T \) according as \( \lambda(P, T) > 0 \), \( < 0 \), or \( = 0 \).

Suppose \( T_1 \) and \( T_2 \) are two fractions or sets of assemblies. By \( T_1 + T_2 \) we denote the set of assemblies in which a particular assembly \( \theta \) is repeated \( q \) times provided that \( \theta \) occurs exactly \( q_1 \) times in \( T_1 \) and \( q_2 \) times in \( T_2 \) and \( q_1 + q_2 = q \). Then from the definition of the \( \lambda \)-function, given in (18), it is obvious that if \( P \in \mathcal{C} \), then

\[
(19) \quad \lambda(P, T_1 + T_2) = \lambda(P, T_1) + \lambda(P, T_2)
\]

It is useful to note that the \( \lambda \)-function defines a kind of homomorphism from the set \( \mathcal{C} \) of polynomials to the set of real numbers. Studies on this will however be made elsewhere.

Given any fraction \( T \), the matrix \( EE' \) can be directly expressed in terms of the \( \lambda \)-function. Every row of \( E \) and hence every column of \( E' \) corresponds to exactly one element of \( L \). Hence the element \( e(i, j) \) in the cell \( (i, j) \) of \( EE' \) corresponds to the elements in the \( i \)-th and \( j \)-th row of \( L \). Consider any two elements of \( L \), say \( A_1^{j_1} A_2^{j_2} \) and \( A_1^{j_3} A_4^{j_4} \). Note that there is no loss of generality in taking the elements in this form, since the main effects \( A_1^j \) and the general mean \( \mu \) are obtainable by choosing one or both of the \( j \)'s as zero. Let
\( \varepsilon(A_{1}^{1} A_{12}^{1}, A_{13}^{1} A_{14}^{1}) \) be the element in \( \mathbb{E}^{1} \) which stands at the intersection of
the row corresponding to \( A_{1}^{1} A_{12}^{1} \) and column corresponding to \( A_{13}^{1} A_{14}^{1} \). For the
evaluation of the said element one has to consider three different cases.

**Theorem 3.1.** (i) If \((i_1, i_2)\) has no factor common with \((i_3, i_4)\) then

\[
\varepsilon(a_{i_1}^{1}, a_{i_2}^{1}, a_{i_3}^{1}, a_{i_4}^{1}) = \lambda \left[ \sum_{r=1}^{s-1} \sum_{k=0}^{s-1} d_k (j_r) a_{i_r}^k \right]
\]

(ii) If \(i_1 = i_3\) and \(i_2 = i_4\), then

\[
\varepsilon(a_{i_1}^{1}, a_{i_2}^{1}, a_{i_3}^{1}, a_{i_4}^{1}) = \lambda \left[ \sum_{r=1}^{2} \sum_{k=0}^{s-1} d_k (j_r) a_{i_r}^k \right]
\]

(iii) If \((i_1, i_2)\) has exactly one factor common with \((i_3, i_4)\)

say \(i_1 = i_4\), then

\[
\varepsilon(a_{i_1}^{1}, a_{i_2}^{1}, a_{i_3}^{1}, a_{i_4}^{1}) = \lambda \left[ \sum_{k=0}^{s-1} d_k (j_1) d_k (j_1') a_{i_1}^k \right] \sum_{r=2}^{s-1} d_k (j_r) a_{i_r}^k
\]

**Proof:** The value of the left hand side of (19) is obtained by multiplying the
corresponding elements in the row \( R_1 \) against the element \( A_{1}^{1} A_{12}^{1} \) in \( L \), and
the row \( R_2 \) against the element \( A_{13}^{1} A_{14}^{1} \) in \( a_1 a_2 \cdots a_m \) in the vector \( a \),
are respectively

\[
\ell_1, \ell_2, \cdots, \ell_1, \cdots, \ell_2, \cdots, \ell_m
\]

\[
d_{0,0,0,\ldots,0,j_1,j_2,o,\ldots,0}
\]

where, as indicated, \(j_1, j_2, j_3\) and \(j_4\) stand respectively at \(i_1\)-th, \(i_2\)-th,
\(i_3\)-th and \(i_4\)-th places. Now using (3) and (4), these two elements reduce
respectively to

\[
d_{q_1} (j_1) d_{q_2} (j_2) \text{ and } d_{q_3} (j_3) d_{q_4} (j_4)
\]
The product of these two is \( \delta \), where

\[
\delta = d_{i_1}^{(j_1)} d_{i_2}^{(j_2)} d_{i_3}^{(j_3)} d_{i_4}^{(j_4)}.
\]

The number of times we get the same product \( \delta \) by multiplying two corresponding elements of \( R_1 \) and \( R_2 \) is equal to the number of times the symbol

\[
\begin{array}{cccc}
\kappa_{i_1} & \kappa_{i_2} & \kappa_{i_3} & \kappa_{i_4} \\
a_{i_1} & a_{i_2} & a_{i_3} & a_{i_4}
\end{array}
\]

occurs in the treatment contained in the vector \( \mathbf{v} \). By definition, this number is

\[
\lambda_{i_1,i_2,i_3,i_4}.
\]

Hence we get,

\[
\epsilon \left( a_{i_1}^{j_1}, a_{i_2}^{j_2}, a_{i_3}^{j_3}, a_{i_4}^{j_4} \right)
\]

\[
= \sum_{\kappa_{i_1}, \kappa_{i_2}, \kappa_{i_3}, \kappa_{i_4}} d_{i_1}^{(j_1)} d_{i_2}^{(j_2)} d_{i_3}^{(j_3)} d_{i_4}^{(j_4)} \times
\begin{array}{cccc}
\kappa_{i_1} & \kappa_{i_2} & \kappa_{i_3} & \kappa_{i_4} \\
a_{i_1} & a_{i_2} & a_{i_3} & a_{i_4}
\end{array}
\times \lambda_{i_1,i_2,i_3,i_4}
\]

\[
= \sum_{\kappa_{i_1}, \kappa_{i_2}, \kappa_{i_3}, \kappa_{i_4}} d_{i_1}^{(j_1)} d_{i_2}^{(j_2)} d_{i_3}^{(j_3)} d_{i_4}^{(j_4)} \times
\begin{array}{cccc}
\kappa_{i_1} & \kappa_{i_2} & \kappa_{i_3} & \kappa_{i_4} \\
a_{i_1} & a_{i_2} & a_{i_3} & a_{i_4}
\end{array}
\times \lambda_{i_1,i_2,i_3,i_4}
\]
\begin{align*}
&= \lambda \left[ \prod_{r=1}^{4} \left( \sum_{k=0}^{s-1} d_k(j_r) a^k_{i_r} \right) \right],
\end{align*}

as desired.

**Case II.** \( i_1 = i_2 \) and \( i_3 = i_4 \)

Then
\[
\varepsilon(\begin{array}{cc} j_1 & j_2 \\ a_1 & a_2 \end{array}, \begin{array}{cc} j_1' & j_2' \\ a_1 & a_2 \end{array})
\]
\[
= \lambda \left[ \prod_{r=1}^{2} \left( \sum_{k=0}^{s-1} d_k(j_r) a^k_{i_r} \right) \right]
\]

**Case III.** \((i_1, i_2)\) has exactly one factor common with \((i_3, i_4)\), say \(i_1 = i_4\).

Then
\[
\varepsilon(\begin{array}{cc} j_1 & j_2 \\ a_1 & a_2 \end{array}, \begin{array}{cc} j_1 & j_3 \\ a_1 & a_3 \end{array})
\]
\[
= \lambda \left[ (\sum_{k=0}^{s-1} d_k(j_1) a^k_{i_1} ) \prod_{r=2}^{3} \left( \sum_{k=0}^{s-1} d_k(j_r) a^k_{i_r} \right) \right]
\]

Proof of cases II and III follows on lines similar to case I.

4. **Further remarks on the \( \lambda \)-operator.**

From the last result we see that each element of \( EE' \) is expressible as a linear function of the \( \lambda \)'s. In this section we propose to demonstrate a number of techniques that could produce further simplification in the calculation of these elements.

**I.** Since for all \( i \), we have \( \lambda(\sum_{k=0}^{s-1} a^k_i) = n \), we shall create a new symbol \( a \) defined by

\[
(23) \quad a = \sum_{k=0}^{s-1} a^k_i, \quad \lambda(a) = n.
\]
Consider $\lambda \left\{ \left( \sum_{k=0}^{s-1} \lambda_{i_1}^{1/k} a_{i_2} \right) a_{i_2} \right\}$ where $i_1 \neq i_2$. From (18) this quantity equals

$$\sum_{k=0}^{s-1} \lambda_{i_2}^{1/k} a_{i_2} = \lambda_{i_2}^{k} \lambda_{i_2}^{k} = \lambda_{i_2}^{k}.$$

Hence we observe an important property of a viz:

$$a_{i_1}^{k} a_{i_2} = a_{i_1}^{k} a_{i_2} = a_{i_2}^{k}, \quad a a = a,$$

$$a_{i_1}^{k} a_{i_2}^{k} = a_{i_1}^{k} a_{i_2}^{k} \quad \text{etc}.$$

II. Let $T_1, T_2, \ldots, T_k$ be $k$ sets of assemblies and let $T = T_1 + \ldots + T_k$. Then using (19) - (22), one easily finds that

$$\left(EE'\right)_T = \left(EE'\right)_{T_1} + \left(EE'\right)_{T_2} + \ldots + \left(EE'\right)_{T_k}$$

III. Let $T$ be a set of assemblies expressible in the form $T = U_1 \odot U_2$, where $\odot$ denotes Kronecker product, $U_1$ is a set of assemblies involving say $k$ factors, and $U_2$ the remaining $m-k$ factors. Let the first set of $k$ factors be denoted by $G_1$ and the second set by $G_2$. Let the factors $i_1, i_2, \ldots, i_k$ belong to $G_1'$ and $i_1', i_2', \ldots, i_k'$ belong to $G_2$. Then it is easy to check that

$$\lambda \left\{ \left( a_{i_1}^{j_1} a_{i_2}^{j_2} \ldots a_{i_k}^{j_k} a_{i_1'}^{j_1'} a_{i_2'}^{j_2'} \ldots a_{i_k'}^{j_k'} \right), T \right\}$$

$$= \lambda \left\{ a_{i_1}^{j_1} a_{i_2}^{j_2} \ldots a_{i_k}^{j_k}, U_1 \right\} \cdot \lambda \left\{ a_{i_1'}^{j_1'} a_{i_2'}^{j_2'} \ldots a_{i_k'}^{j_k'}, U_2 \right\}$$

The above result has a useful corollary. Let $P \in \mathcal{P}$, and suppose $P = P_1 P_2'$, where $P_1$ is a polynomial involving factors from $G_1$ and $P_2'$ from $G_2$. Then
(27) $\lambda[p, T] = \lambda[p_1, p_2, u_1 \otimes u_2] = \lambda [p_1, u_1] \cdot \lambda [p_2, u_2]$

IV. Suppose $T$ is a set of assemblies which satisfy a certain set of equations over a finite field $GF(s)$ with $s$ elements (we are now assuming $s$ to be a prime power). We shall state a well known result and prove another one, both of which shall be useful in the computation of the $\lambda(p, T)$.

**Lemma 4.1.** Let there be $m$ factors $a_1, a_2, ..., a_m$, each at $s$ levels, where $s$ is a prime power. Suppose we obtain a $s^{-k}$ fractional replication $T$ by taking in $EG(m, s)$ $k$ independent linear equations

\[
\begin{align*}
\sum_{i=1}^{s} g_{1i} x_1^{i} + \sum_{i=2}^{s} g_{12i} x_2^{i} & + \cdots + \sum_{i=m}^{s} g_{1m} x_m^{i} = c_1 \\
\sum_{i=1}^{s} g_{2i} x_1^{i} + \sum_{i=2}^{s} g_{22i} x_2^{i} & + \cdots + \sum_{i=m}^{s} g_{2m} x_m^{i} = c_2 \\
\vdots & \\
\sum_{i=1}^{s} g_{ki} x_1^{i} + \sum_{i=2}^{s} g_{k2i} x_2^{i} & + \cdots + \sum_{i=m}^{s} g_{km} x_m^{i} = c_k
\end{align*}
\]

where all symbols represent elements in $GF(s)$. Suppose all linear combinations of the above equations contain $d$ or more $x$'s which have non-zero coefficients $g$'s. Then

(i) (for the case $d = 2t$) all interactions up to $t$-factors can be estimated assuming interactions of $(t+1)$-factors and higher orders to be negligible.

(ii) (for the case $d = 2t+1$) all interactions up to $t$-factors can be estimated assuming interactions of $(t+2)$-th and higher orders to be negligible.

(iii) the symbol $a_1^{j_1} a_2^{j_2} ... a_r^{j_r}$, where $1 \leq r \leq d$ and $j_1, j_2, ..., j_r = 0, 1, 2, ..., s-1$ occurs $\lambda^{j_1 j_2 ... j_r}$ times in the assemblies of the fraction $T$, where $\lambda^{j_1 j_2 ... j_r}$ is independent of $i_1, i_2, ..., i_r$ for $1 \leq r \leq d$. 
(iv) $\lambda_{j_1 j_2 \cdots j_r} = s^{m-r-k}$.

In the context of the above theorem, let us find the value of

$$\lambda_{i_1 i_2 \cdots i_r},$$

when $r > d$ (for the case of the fraction $T$). Let $\alpha_0, \ldots, \alpha_{s-1}$ be the $s$ elements of GF$(s)$, and let the level $j$ of any factor correspond to the element $\alpha_j$. Then for finding the value of $\lambda_{j_1 j_2 \cdots j_r}$, we have to find how many solutions of the equations (28) are such that

$$x_{i_1} = j_1, \quad x_{i_2} = j_2, \quad \ldots, \quad x_{i_r} = j_r.$$

(29)

For obtaining $\lambda_{j_1 \cdots j_r}$, we therefore substitute the values (29) in equations (28). Two cases may arise:

Case I. Substitute the values (29) in equations (28). If the resulting equations are inconsistent, then

$$\lambda_{i_1 \cdots i_r} = 0.$$

Case II. After substitution we may find $k'(\leq k)$ equations to be independent. By the previous theorem, we shall then have

$$\lambda_{i_1 \cdots i_r} = \text{number of points on an (m-r-k') flat in } \mathbb{E}G(m,s)$$

$$= s^{m-r-k'}$$

The above gives us
Lemma 4.2. Let there be $k$ equations in $GF(s)$, say as at (28), which generate an orthogonal array $T$ of strength $d$. Let $S$ be the set of all equations obtained by taking the linear combinations of equations (28). Let $S'$ be that subset of $S$ in which only $x_{i_1}, ..., x_{i_r}$ occur, and let the number of independent equations in $S'$ be $k'$. Then exactly $s^{r-k'}$ of the $\lambda^{j_1 j_2 \cdots j_r}_{i_1 i_2 \cdots i_r} (r > d)$ are nonzero, namely those, for which

$$x_{i_k} = \alpha_j^{j_k}$$

satisfies the equations $S'$. Further if the substitution (29) in equations (28) leaves $k'$ independent equations, then

$$\lambda^{j_1 j_2 \cdots j_r}_{i_1 i_2 \cdots i_r} = s^{m-r-k'}$$

We shall now describe a practical technique for the case $s = 2$, (for higher $s$ the generalisation is obvious) of computing $(EE'_T)_T$, when $T$ is an orthogonal array of strength $d$. First consider the case $d = 2$. Let (28) be written

$$Gx = c$$

in an obvious matrix notation, where $G$ is a $k \times m$ matrix. Let $\bar{G}$ be an $(m-k) \times m$ matrix of rank $(m-k)$ whose rows are orthogonal to those of $G$. Then it is well known (see for example [2]) that $\bar{G}$ possesses the property $P_d$ viz that no $d$ columns of $\bar{G}$ are linearly dependent. (In fact one way to construct equations (30) for obtaining an array of strength $d$ is to start from $\bar{G}$, and then get the matrix $G$ which is orthogonal to it). Assume now that $\bar{G}$ is given to us. It has $m$ columns which we may denote by $\bar{g}_1, \bar{g}_2, \cdots, \bar{g}_m$. Let the $i$-th column correspond to the $i$-th factor. Form an $m \times m$ table, the $i$-th row or $i$-th column of which corresponds to the $i$-th factor. In the $(i,j)$ cell of the table, put the vector $(\bar{g}_i + \bar{g}_j)$. Notice that only the cells above or in the diagonal of the table need to be computed. In the process of constructing economic fractions, we
shall encounter only those cases where \( m-k \) is small. For such cases an eye examination of this table is very revealing. If \( d < 3 \), then if a vector \( \vec{e}_{ij} \) occurs in any cell of the table, say in the \((i,j)\) cell, then it can be easily checked (using Lemma 3.2) that we must have

\[
\lambda(P_3, T) = \pm 2^{m-k}, \quad \text{where}
\]

\[
P_3 = (a_i^1 - a_i^0) (a_j^1 - a_j^0) (a_k^1 - a_k^0).
\]

The same holds for all values of the pair \((i,j)\) in which cell \( \vec{e}_{ij} \) occurs. If \( d < 4 \), then we shall similarly have

\[
\lambda(P_4, T) = \pm 2^{m-k}, \quad \text{where}
\]

\[
P_4 = (a_{11}^1 - a_{11}^0) (a_{12}^1 - a_{12}^0) (a_{12}^1 - a_{12}^0) (a_{13}^1 - a_{13}^0),
\]

provided that the cells \((i_1, j_1)\) and \((i_2, j_2)\) of our table have the same vector \( \vec{e}_k \) in them.

An examination of the formulae (20) - (22) shows that if \( d \) is at least 2, then only the expressions of the form (21) and (32) need to be evaluated, since in that case we assume \( d_1(1) = 1, \quad d_0(1) = -1. \)

If \( c \) is the zero vector, then it can be easily checked that one must use the signs \( - \) and \( + \) on the r.h.s. of (31) and (32) respectively. In other cases an examination of \( i\)-th, \( j\)-th, \( k\)-th (for (31)) and \( i_1\)-th, \( j_1\)-th, \( i_2\)-th, \( j_2\)-th (for (32)) columns of \( G \) is usually quick enough for this purpose.

V. We shall consider now the matrix \( EE' \) in detail for the cases \( s=2 \) and \( s=3. \)

Case I. \( s = 2. \)

Here we shall write
where the $M'$s are submatrices such that the 1st row block (and the first column block) refer to the general mean $\mu$, the second row and column blocks correspond to the main effects, and the third ones to the interactions, all expressed in their natural order, as in $L$. Thus for example the matrix $M_5$ is a $m \times \binom{m}{2}$ matrix whose rows refer to the main effects and columns to the interactions. When we wish to be more expressive, we shall refer to this fact by saying that $M_5$ corresponds to $[A_1, A_2, \ldots, A_m; A_{12}, A_{13}, \ldots, A_{m-1,m}]$, where for $2^n$ designs we shall abbreviate the symbol $A_iA_j$ to $A_{ij}$. Similarly $M_2$ corresponds to $[\mu; A_1, A_2, \ldots, A_m]$ and the matrix $[M_2 M_3]$ to $[\mu; A_1, A_2, \ldots, A_m; A_{12}, \ldots, A_{m-1,m}]$ etc.

Since for the $2^n$ case, we have $d_1(1) = 1$, $d_0(1) = -1$, the diagonal elements of $M_4$ are given by $\varepsilon(A_1, A_1)$ which equals $\lambda(a^1_1 + a^0_1) = n$. Similarly $M_1$ and the diagonal elements of $M_6$ are all equal to $n$. The elements of $M_2$ are of the type $\varepsilon(\mu, A_1)$, which equal $\lambda(a^1_1 - a^0_1)$, written for short $\lambda(1^1 - 0^0)_1$ or even $\lambda(1^1 - 0^0)$ it being understood that $i$-th factor is under consideration. The prime in $1^1$ or $0^0$ or $1'$ or $0'$ denotes that they are ordinary algebraic symbols representing levels of a factor, and that they can be manipulated in the same way as $a^1_1$ or $a^0_1$. Thus instead of writing $2a^1_1$, we shall write $2(1')$ or even $21'$. Hence the elements of $M_2$ are of the type

$$\lambda(1^1 - 0^0) = \lambda(a - 20') = n - 2\lambda^0,$$

where $1$ has been omitted from $\lambda^0_1$. Similarly the elements of $M_3$ are of the type $\varepsilon(\mu, A_iA_j)$ which equals $\lambda(1^1 - 0^0)(1^1 - 0^0) = (\lambda^{1^1} + \lambda^{00}) - (\lambda^{01} + \lambda^{10})$, 

\[
(33) \quad (EE') = \begin{bmatrix} M_1 & M_2 & M_3 \\ M'_2 & M_4 & M_5 \\ M'_3 & M'_5 & M_6 \end{bmatrix},
\]
in an obvious extension of our notation. Also

\( \varepsilon(A_i, A_j) = \lambda(l' - o') \varepsilon(l' - o') \) for \( k \neq i, j \).

For the full operational case of this symbol, we must note:

\[
(35) \quad (\beta_1 l' + \beta_2 0') \varepsilon(\beta_3 l' + \beta_4 0') = (\beta_1 \beta_3) l' + \beta_2 \beta_4 0'.
\]

Thus we have

\[
n = \lambda(a) = \lambda(l' + 0') = \lambda(l' - 0') \varepsilon(l' - 0'),
\]

which together with (34) gives

\[
\varepsilon(A_i, A_j) = \varepsilon(A_k, A_k), \text{ for } k \neq i, j.
\]

Elements of \( M_0 \) are of the type \( \varepsilon(A_i, A_i) \) or \( \varepsilon(A_k, A_k) \), which respectively equal \( \lambda(l' - o') \) and \( \lambda(l' - o') (l' - o') \). Similarly the elements of type \( \varepsilon(A_i A_j, A_k A_k) \) in \( M_0 \) equal \( \lambda(l' - o') \).

Let us call a fraction \( T \) to be \((l, 0)\) symmetric with respect to triplets

\[
\varepsilon(a_1 a_2 a_3, T) = \lambda(a_1 a_2 a_3, T), \text{ where each } j' = l+j (\text{mod } 2).
\]

This means for example that for any triplet of factors, the combination \(101\) occurs as many times in \( T \) as \((010)\), i.e. \( \lambda^{101} = \lambda^{010} \). Similarly \( \lambda^{000} = \lambda^{111} \), \( \lambda^{100} = \lambda^{011} \) etc. Also we have

\[
\lambda(l' - o') = \lambda(l' - o')
\]

\[
= \lambda \left[ l' + o', l' + o', k - o', l' + o', l' + o', k \right]
\]

\[
= \lambda \left[ (l' l' l' + l' o' o' + l' l' o' + l' l' l') - (o' o' o') - (o' o' l') \right]
\]

\[
= 0.
\]
Hence $\lambda^1 = \lambda^0$. Thus a sufficient condition that the matrices $M_2$ and $M_5$ are zero is that the fraction be $(1,0)$ symmetric with respect to triplets. This condition is necessary also, as can be verified by starting from the above equations and solving back.

**Case II. $s = 3$.**

We shall use an obvious extension of the notation for the case $s = 2$, without further explanation. The matrix $(EE')$ is broken up as

\[
EE' = \begin{bmatrix}
M_1 & M_2 & M_3 & M_4 & M_5 & M_6 \\
M_7 & M_8 & M_9 & M_{10} & M_{11} \\
M_{12} & M_{13} & M_{14} & M_{15} \\
M_{16} & M_{17} & M_{18} \\
\text{Sym.} & M_{19} & M_{20} \\
M_{21}
\end{bmatrix},
\]

where the row blocks (and the column blocks) respectively correspond to

$[\mu]$, $[B_1, B_2, \ldots, B_m]$, $[B_1^2, B_2^2, \ldots, B_m^2]$, $[B_1B_2, B_1B_3, \ldots, B_{m-1}B_m]$; $[B_1^2E_2, B_2^2E_2, \ldots, B_{m-1}E_m]$ and $[B_1^2B_2, B_1^2B_3, \ldots, B_{m-1}^2B_m]$ written for short $[\mu]$, $[B_1]$, $[B_1^2]$, $[B_1B_j]$, $[B_1^2B_j]$ and $[B_1^2B_j]$. Thus for example $M_{21}$ is an $m(m-1) \times m(m-1)$ matrix corresponding to $[[B_1^2B_j]]$ and $M_9$ is $m \times \binom{m}{2}$ and corresponds to $[[B_1] : [B_1B_j]]$. The calculation of various elements of $EE'$ will be illustrated for some of the matrices $M$. To apply (20) - (22), we shall define

\[
\begin{align*}
\delta_0(1) &= -1, & \delta_1(1) &= 0, & \delta_2(1) &= 1 \\
\delta_0(2) &= 1, & \delta_1(2) &= -2, & \delta_2(2) &= 1.
\end{align*}
\]

Then we have for example
(a) \[ M_1: \quad \varepsilon(\mu, \mu) = n \]

(b) \[ M_7: \quad \varepsilon(B_i, B_j) = \lambda(2^1 - 0^1) \text{I}_i (2^1 - 0^1) \text{I}_j = \lambda(2^1 + 0^1) \text{I}_i \]
\[ = \lambda(a - 1^1_i) = n - \lambda^1_i \]
\[ \varepsilon(B_i, B_j) = \lambda^{22}_{ij} + \lambda^{00}_{ij} - \lambda^{02}_{ij} - \lambda^{20}_{ij}. \]

(c) \[ M_{10}: \quad \varepsilon(B_i, B_j) = \lambda(2^1 - 0^1) \text{I}_i (2^1 - 2^1 + 0^1) \text{I}_j (2^1 - 2^1 + 0^1) \text{I}_j \]
\[ = \lambda(2^1 - 0^1) \text{I}_i (a - 3 1^1_i) = \lambda^{21}_{ij} - \lambda^{02}_{ij} - 3(\lambda_{1j}^{21} - \lambda_{1j}^{00}) \]
\[ \varepsilon(B_i, B_j) = \lambda(2^1 - 0^1) \text{I}_i (a - 3 1^1_i) (a - 3 1^1_k) \]
\[ = \lambda(2^1 - 0^1) \text{I}_i (a - 3 1^1_i - 3 1^1_k + 6(1^1_i 1^1_k) \text{I}_j \text{I}_k) \]
\[ = (\lambda^{21}_{1i} - \lambda^{01}_{1i}) - 3(\lambda^{21}_{1j} - \lambda^{01}_{1j}) + 3(\lambda^{21}_{1k} - \lambda^{01}_{1k}) + 3(\lambda^{21}_{1jk} - \lambda^{01}_{1jk}) \]

(d) \[ M_{21}: \quad \varepsilon(B_i^2, B_j^2) = \lambda [(2^1 - 2^1 + 0^1) \text{I}_j ^2 (2^1 - 0^1) \text{I}_j ^2 \]
\[ = \lambda(2^1 + 4 1^1 + 0^1) \text{I}_i (2^1 + 0^1) \text{I}_j = \lambda(a + 3 1^1_i) (a 1^1_j) \]
\[ = n + 3\lambda^{11}_{1i} - \lambda^{11}_{1j} + 3\lambda^{11}_{1ij} \]
\[ \varepsilon(B_i^2, B_j^2) = \lambda(2^1 - 2^1 + 0^1) \text{I}_i (2^1 - 0^1) \text{I}_j (2^1 - 2^1 + 0^1) \text{I}_j \]
\[ = \lambda(2^1 - 0^1) \text{I}_i (2^1 - 0^1) \text{I}_j = \varepsilon(B_i, B_j). \]
\[ \varepsilon(B_i^2, B_j^2) = \lambda(a + 3 1^1_i) (2^1 - 0^1) \text{I}_j (2^1 - 0^1) \text{I}_k \]
\[ = \varepsilon(B_j B_k) + 3\lambda [1^1_i (2^1 - 0^1) \text{I}_j (2^1 - 0^1) \text{I}_k] \]
\begin{align*}
e(B_{1j}^2, B_{j1}^2) &= \lambda(2^1 - 0^1)_i (2^1 - 0^1)_j (a + 3 l_i^1 - 6 l_k^1) \\
&= \epsilon(B_{1j}^2, B_{1j}^2) - 6 \lambda [l_i^1(2^1 - 0^1)_1 (2^1 - 0^1)_j] \\
e(B_{1j}^2, B_{j1}^2) &= \lambda(a - 3 l_i^1)(a - 1 j_1^1)(a - 3 l_k^1) \\
&= n - 3(\lambda_i^1 + \lambda_k^1) + 3(\lambda_{i1}^{11} + \lambda_{j1}^{11} + \lambda_{k1}^{11}) - 9 \lambda_{i1j1}^{11} \\
e(B_{1j}^2, B_{j1}^2) &= \lambda(a - 3 l_i^1)(a - 3 l_k^1)(2^1 - 0^1)_j (2^1 - 0^1)_j' \), \text{ etc.}
\end{align*}

A fraction \( T \) will be called \((2,0)\) symmetric with respect to triplets if
\[\lambda(a_{i1}^{j1} a_{i2}^{j2} a_{i3}^{j3}, T) = \lambda(a_{i1}^{j1} a_{i2}^{j2} a_{i3}^{j3}, T), \text{ where each } j^1 = 2 + 2 j \pmod{3}. \] This means that for example \( \lambda^{100}_{122} = \lambda^{202}_{122}, \lambda^{012}_{210} = \lambda^{210}_{210} \) etc. Using the above \( \lambda \)-calculus, we can easily check that this is a necessary and sufficient condition in order that the effects \( \{B_{1j}^1\} \) and \( \{B_{1j}^2\} \) are uncorrelated with the rest, i.e. in order that the matrices \( M_{j1}^2, M_{j2}^2, M_{j3}^2, M_{j4}^2, M_{j5}^2, M_{j6}^2, M_{j7}^2, M_{j8}^2, M_{j9}^2, M_{j10}^2, M_{j11}^2, M_{j12}^2, M_{j13}^2, M_{j14}^2, M_{j15}^2, M_{j16}^2, M_{j17}^2, M_{j18}^2, M_{j19}^2 \) are all zero matrices. In fact this breaks \( \text{EE}' \) into two matrices of orders \( m^2 \times m^2 \) and \( (m^2 + 1) \times (m^2 + 1) \).

The material in this section is useful not only in the analysis of fractions, but also in their construction. In many situations for constructing fractions, one has a number of sets of assemblies \( T_1, T_2, \ldots \). The formulae in this section help one to readily evaluate \((\text{EE}')\) for each \( T_1 \). By examining the nondiagonal elements of \( \text{EE}' \) for each \( T \), one can then piece together a few of the \( T \)'s in such a way that the resulting fraction \( T_1 + T_2 + \ldots + T_k \) say is economic and the nondiagonal elements are small enough. Achievement of \((1,0)\) or \((2,0)\) symmetry introduces further simplicity in inverting \((\text{EE}')\) and decreases the correlations of the estimates, some of them to zero.
6. Examples for symmetrical fractions.

To illustrate the preceding theory we shall present two fractions,

Example: A $2^7$ fraction in 44 assemblies. ($v = 29$)

The fraction is $T = T_1 + T_2 + T_3 + T_4$ where the $T_i$ are given by

$$
\begin{align*}
T_1 &= \begin{bmatrix} 1100 \\ 1010 \\ 1001 \\ 0101 \\ 0110 \\ 0011 \end{bmatrix} \\
T_2 &= \begin{bmatrix} 1111 \\ 1111 \\ 1111 \\ 1111 \\ 1111 \\ 1111 \\ 1111 \end{bmatrix} \\
T_3 &= \begin{bmatrix} 1111 \\ 1111 \\ 1111 \\ 1111 \\ 1111 \\ 1111 \\ 1111 \end{bmatrix} \\
T_4 &= \begin{bmatrix} 1111 \\ 1111 \\ 1111 \\ 1111 \\ 1111 \\ 1111 \\ 1111 \end{bmatrix} \\
\end{align*}
$$

It is evident that $T_2$ is completely $(1,0)$ symmetric. Also $T_4 = T_3$, i.e. $T_4$ is obtained by interchanging 1 and 0 in $T_3$. Hence the whole fraction $T$ is $(1,0)$ symmetric. Let the two sets of factors be denoted by $A_1, A_2, A_3, A_4$ and $B_1, B_2, B_3$. Consider now the matrix $M_4$ in $(EE')$ in (33). We have for $i \neq j$,

$$
\epsilon(A_i, A_j) = 2 + 2 \times (2-4) + 3 \times (3 - 2) + 3 \times (3 - 2) = 4 ,
$$

where the different terms correspond to the $T_i$'s. Similarly

$$
\epsilon(B_i, B_j) = 2 + 6 \times (2 - 0) + 5(1 - 2) + 5(1 - 2) = 4 .
$$

For $\epsilon(A_i, B_j)$ we use the property stated in sec. 5. II. We get

$$
\epsilon(A_i, B_j) = \sum_{T' s} (\lambda^{1}_{A_i} - \lambda^{0}_{A_i}) (\lambda^{1}_{B_j} - \lambda^{0}_{B_j}) = 2 + (0 \times 0) + (3-2)(1-2) + (-3+2)(2-1) = 0
$$

Hence $M_4$ could be written
\[ M_4 = \begin{bmatrix} 40 \ I_4 + 4 \ J_{44} & (0) \ J_{43} \\ (0) \ J_{34} & 40 \ I_3 + 4 \ J_{33} \end{bmatrix} \]

This can be easily inverted to get

\[ M_4^{-1} = \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} \]

where \( P_1 = \frac{1}{40} \ I_4 - \frac{1}{560} \ J_{44}, \ P_2 = (0) \ J_{43}, \) and \( P_3 = \frac{1}{40} \ I_3 - \frac{1}{520} \ J_{33}. \)

An examination of \( M_4^{-1} \) shows that the main effects, which are correlated only among themselves, have a small correlation of the magnitude 0.08 nearly.

It is interesting to compare this with a \( 2^7 \) fraction in 48 assemblies proposed by Addelman [1]. In the latter plan, 5 main effects are correlated with some of the interactions, the correlation being \( \frac{1}{3} \) for four of them and \( \frac{1}{2} \) for the remaining one, but the other correlations are lesser than in our plan.

Let us now consider \( \mu \) and the interactions. Using (34), we find that all terms of \( M_1, M_2 \) and \( M_6 \) are already known except those which involve 4 factors. These are

\[ \epsilon(A_1A_2, A_3A_4) = 2 + (6-0) \times 2 + (1-4) \times 3 + (1-4) \times 3 = -4 \]

\[ \epsilon(A_1B_1, B_2B_3) = \sum_{T_1} (\lambda_{a_1}^1 - \lambda_{a_1}^0) \ \epsilon(B_1, B_2B_3) \]

\[ = 2 + (3-3)(1-1) + (3-2)(3-0) + (2-3)(0-3) = 8 \]

\[ \epsilon(A_1A_j, B_kB_\ell) = 2 + (2-4)(2-0) + (3-2)(1-2) + (3-2)(1-2) = -4 \]

\[ \epsilon(A_1A_j, A_kB_\ell) = 2 + (3-3)(1-1) + (1-4)(1-2) + (4-1)(2-1) = 8. \]
Let the submatrix of $BB^t$ which corresponds to $[\mu; A_1A_2, A_1A_3, A_1A_4, A_2A_3, A_2A_4, A_3A_4; B_1B_2, B_1B_3, B_2B_3, A_1B_1, \ldots, A_4B_1, A_1B_2, \ldots, A_4B_2, A_1B_3, \ldots, A_4B_3]$ in this order be denoted by $\Omega$. Then from the above calculations

$$\Omega = \begin{bmatrix} 4 \mu & 4 J_{1,6} & 4 J_{1,3} & (0) J_{1,12} \\ Q_1 & (-4) J_{6,3} & Q_2 \\ Q_3 & Q_4 \\ \text{Sym.} & & Q_5 \end{bmatrix}$$

where $I_n^t = (n \times n)$ identity matrix

$J_{m,n}$ or $I_{mn} = (m \times n)$ matrix with unity everywhere

$$Q_1 = 40 I_6 + 4 J_{6,6} - 8 I_6^t,$$

$I_6^t$ being a $n \times n$ matrix

with unity in the cross diagonal and zero elsewhere,

$$Q_2 = [Q_6 \ 0 \ 0], \quad Q_6 = \begin{bmatrix} 0 & 0 & 6 & 6 \\ 0 & 6 & 0 & 6 \\ 0 & 6 & 6 & 0 \\ 6 & 0 & 0 & 6 \\ 6 & 0 & 6 & 0 \\ 6 & 6 & 0 & 0 \end{bmatrix}$$

$$Q_3 = 40 I_3 + 4 J_{3,3}$$
\[ Q_4 = \begin{bmatrix} 0 J_{1,4} & 0 J_{1,4} & 8 J_{1,4} \\ 0 J_{1,4} & 8 J_{1,4} & (0) J_{1,4} \\ 8 J_{1,4} & (0) J_{1,4} & (0) J_{1,4} \end{bmatrix} \]

\[ Q_5 = \begin{bmatrix} 40 I_4 + 4 J_{1,4} & 8 I_4 - 4 J_{1,4} & 8 I_4 - 4 J_{1,4} \\ 8 I_4 - 4 J_{1,4} & 40 I_4 + 4 J_{1,4} & 8 I_4 - 4 J_{1,4} \\ 8 I_4 - 4 J_{1,4} & 8 I_4 - 4 J_{1,4} & 40 I_4 + 4 J_{1,4} \end{bmatrix} \]

The inverse of the matrix (38) which could easily be obtained by using the algorithm for the analysis of multidimensional partially balanced designs is given in sec. 8 of [6].

**Example 2.** A \( \frac{1}{4} \) fraction in 45 assemblies. \( (v = 33) \). This fraction \( T \) which is symmetric with respect to the different factors is given below.

\[
\begin{array}{ccccccc}
0001 & 0021 & 1002 & 2120 & 1120 & 2110 \\
0010 & 0012 & 2001 & 2021 & 1102 & 0112 \\
0100 & 0120 & 2010 & 1022 & 1210 & 2011 \\
1000 & 0210 & 1020 & 0122 & 1012 & 0211 \\
2221 & 0102 & 2201 & 1220 & 1201 & 1111 \\
2122 & 1200 & 2102 & 0212 & 0121 & 2222 \\
1222 & 2100 & 2012 & 1202 & 2101 & 0000 \\
\end{array}
\]

As indicated, apart from the assembly \( (1111) \), the fraction consists of four assemblies of the types \( (0001) \) and \( (2221) \) each, and twelve assemblies of the types \( (0021), (2201) \) and \( (1120) \) each. Obviously, the fraction is \( (2,0) \) symmetric.
We shall first compute the diagonal elements of \((EE')_T\):

\[
e(\mu, \mu) = n = 48
\]

\[
e(B_1, B_1) = \lambda(2' + 0') = \lambda(2' + 1' + 0' - 1') = \lambda(b - 1') = 48 - 16 = 32,
\]

where for convenience, we write \((2' + 1' + 0') = b\), for \(3^n\) factorials, in place of using \(a\).

\[
e(B_1^2, B_1^2) = \lambda(2' - 2 1' + 0')_1 (2' - 2 1' + 0')_1 = \lambda(2' + 4 1' + 0')_1 = \lambda(b + 3 1') = 96.
\]

\[
e(B_1 B_2, B_1 B_2) = \lambda(2' + 0')_1 (2' + 0')_2 = \lambda(b - b_1^1)(b - b_2^1)
\]

\[
= \lambda(b - b_1^1 - b_2^1 + b_1^1 b_2^1) = 48 - 32 + 4 = 20.
\]

\[
e(B_1^2 B_2, B_1^2 B_2) = \lambda(b + 3 b_1^1)(b + 3 b_2^1)
\]

\[
= n + 6 \lambda' + 9 \lambda'' = 48 + 96 + 36 = 180.
\]

\[
e(B_1 B_2, B_1 B_2) = \lambda(b + 3 b_1^1)(b - b_2^1) = 48 + 48 - 16 - 12 = 68.
\]

Next we turn to the nondiagonal elements. The computations shall be presented separately for each submatrix \(M_i\) of \((EE')_T\) as at (40).

(1) \(M_7\)

\[
e(B_1, B_2) = \lambda(2'2' + 0'0' - 0'2' - 2'0') = 2 \lambda(2'2' - 0'2') = 2 \lambda(5 - 5) = 0
\]

(2) \(M_{11}\)

\[
e(B_1, B_1 B_2) = \lambda(b - b_1^1)(b - 3 b_2^1) = 48 - 16 - 48 + 12 = -4
\]

\[
e(B_1, B_2 B_2) = e(B_1, B_2) = 0.
\]

\[
e(B_1, B_2 B_3) = \lambda(2' - 0')(2' - 0')(b - 3 b_1^1) = 0 - 6(\lambda^{122} - \lambda^{120}) = 6
\]
\( (3) \ M_{21} \)

\[
\begin{align*}
\epsilon(B_1 B_2^2, B_1^2 B_2) &= \epsilon(B_1 B_2) = 0 \\
\epsilon(B_1^2 B_2, B_1^2 B_3) &= \epsilon(B_1 B_2^2 B_3) = 6 \\
\epsilon(B_1 B_2^2, B_1 B_3^2) &= \lambda(b - b^1)(b - 3 b^1)(b - 3 b^1) = \lambda(b - b^1)(b - 3 b^1)^2,
\end{align*}
\]

say,

\[
= 48 - 112 + 60 - 18 = -22
\]

\[
\begin{align*}
\epsilon(B_1^2 B_2, B_1 B_3) &= \lambda(b + 3 b^1)(2^o - 0')(2^o - 0') \\
&= 2 \lambda \left[(b(2^o - 0')(2^o - 0')) - \lambda(b - 3 b^1)(2^o - 0')(2^o - 0')
\end{align*}
\]

\[
= 0 - \epsilon(B_1, B_2 B_3^2) = -6
\]

\[
\begin{align*}
\epsilon(B_1^2 B_2, B_3 B_4) &= \lambda(b - 3 b^1)(b - 3 b^1)(2^o - 0')(2^o - 0') \\
&= \epsilon(B_1, B_2 B_3^2) - 3 \lambda \left[1^o(2^o - 0')(2^o - 0')
\end{align*}
\]

\[
= 6 - 6(2 - 3) + 9 \times 2 (\lambda_{122} - \lambda_{120}) = -6
\]

\( (4) \ M_6 \)

\[
\begin{align*}
\epsilon(\mu, B_1^2 B_2^2) &= \lambda(b - 3 b^1)(b - 3 b^1) \\
&= \lambda(b - 6 b^1 + 9 b^{11}) = 48 - 96 + 36 = -12
\end{align*}
\]

\( (5) \ M_{12} \)

\[
\begin{align*}
\epsilon(B_1^2, B_2^2) &= \epsilon(\mu, B_1^2 B_2^2) = -12
\end{align*}
\]

\( (6) \ M_{16} \)

\[
\begin{align*}
\epsilon(B_1 B_2, B_1 B_3) &= \lambda(b - b^1)(2^o - 0')(2^o - 0') \\
&= \epsilon(B_2, B_3) - 2(\lambda_{122} - \lambda_{120}) = 0 - 2(2 - 3) = 2
\end{align*}
\]

\[
\begin{align*}
\epsilon(B_1 B_2, B_3 B_4) &= \lambda(2^o - 0')(2^o - 0')(2^o - 0')(2^o - 0') = 2
\end{align*}
\]
(7) $M_{19}$

$\epsilon(B^2_1 B^2_2, B^2_1 B^2_3) = \lambda(b + 3 b^1)(b - 3 b^1)^2$

$= \lambda [-3(b-b^3) + 4b](b - 3b^1)^2 = 66 + 4(-12) = 18$

$\epsilon(B^2_1 B^2_2, B^2_3 B^2_4) = \lambda(b - 3 b^1)^4$

$= \lambda [b + 4(-3 b^3) + 6(-3 b^1)^2 + 4(-3 b^1)^3 + (-3 b^1)^4]$

$= n - 12 \lambda^1 + 54 \lambda^{11} - 108 \lambda^{111} = 81 \lambda^{1111}$

$= 48 - 192 + 216 - 216 + 162 = 18$.

(8) $M_{13}$

$\epsilon(B^2_1, B_1 B_2) = \epsilon(B_1, B_2) = 0$

$\epsilon(B^2_1, B_2 B_3) = \epsilon(B_2, B^2_1 B_3) = 6$

(9) $M_{14}$

$\epsilon(B^2_1, B^2_1 B_2) = \lambda(b + 3 b^1)(b - 3 b^1) = n - 9 \lambda^{11} = 12$

$\epsilon(B^2_1, B^2_2 B_3) = \lambda(b - 3 b^1)^3$

$+ \lambda[b + 3(-3 b^1) + 3(-3 b^1)^2 + (-3 b^1)^3]$

$= n - 9 \lambda^1 + 27 \lambda^{11} - 27 \lambda^{111}$

$= 48 - 144 + 108 - 54 = -42$

(10) $M_{17}$

$\epsilon(B_1 B_2, B^2_1 B_2) = \epsilon(B_1 B_2, \mu) = 0$

$\epsilon(B_1 B_2, B^2_1 B_3) = \epsilon(B_1 B_2, B^2_3) = 6$

$\epsilon(B_1 B_2, B^2_3 B_4) = \epsilon(B_1 B^2_3, B_2 B^2_4) = -6$
The two matrices $\Omega_1$ and $\Omega_2$ corresponding to $[[B_i]; [B^2_iB_j]]$ and $[[\mu]; [B_i]; [B^2_iB_j]; [B^2_iB^2_j]]$ in which $(EE_i)'_T$ is split because of $(2,0)$ symmetry can now be easily written down.

Ex 3. For a more theoretical example, consider a balanced fraction $T$ from a $j^m$ factorial, i.e. a fraction $T$ for which $(EE_i)'_T$ is symmetrical with respect to all the $m$ factors. We define $T$ to be commutative, if for all unequal $i, j$ and $k$, we have

$$\varepsilon(B^2_iB^2_j, B^2_iB^2_k) = \varepsilon(B^2_iB^2_j, B^2_iB^2_k).$$

Let us obtain an equivalent condition (used in sec. 5 of [6]) in terms of the $\lambda$-function. The above could be written

$$\lambda[(b - b^1_i)(b - 3 b^1_j)(b - 3 b^1_k)]$$

$$= \lambda[(b + 3 b^1_i)(b^2_j - b^0_j)(b^2_k - b^0_k)]$$

This simplifies to

$$\lambda(2'0'0' + 0'2'2' + 0'1'1' + 2'1'1') = 2\lambda(1'0'0' + 1'2'2'),$$

the suffixes being dropped because of symmetry.

7. The asymmetrical case.

The theory for this can be developed on lines parallel to the symmetrical case given in the preceding sections. For lack of space only the results will be stated and the details will be omitted, restricting ourselves to $m_1 \times m_2$ factorials.

A general treatment combination (and also its 'true effect') could be denoted by $i_1 i_2 \cdots i_m, b_1 b_2 \cdots b_m$, where $i$'s $\in (0,1)$ and $j$'s $\in (0,1,2)$.

Given a set of assemblies $T$, the number of times the symbol

$$\Theta = a_{i_1} a_{i_2} \cdots a_i b_{j_1} b_{j_2} \cdots b_{j_r},$$

occurs among the assemblies in $T$ will be
denoted by $\lambda_{i_1,i_2,\ldots,i_r}$; $j_1^i$, $j_2^i$, $\ldots$, $j_r^i$, or by $\lambda(\emptyset)$, extending the notation of section 3.

The set of all possible assemblies will be denoted by $\Omega_{2,3}$. The set of effects in which one is interested fall in three groups: (i) Pure effects of factors at 2 levels each, or pure A-effects (ii) Pure B-effects (iii) Mixed AB effects, i.e. those which involve at least one A factor and one B factor. We shall adopt an obvious generalization of the notation and terminology of the previous sections, and define a general interaction of our mixed factorial by

$$
\begin{align*}
&k_1 \quad k_2 \quad \ldots \quad k_m \\
&A_{11} \quad A_{12} \quad \ldots \quad A_{1m} \\
&\ldots \\
&B_{11} \quad B_{12} \quad \ldots \quad B_{1m} \\
&\ldots \\
&k_{m1} \quad k_{m2} \\
&\ldots \\
&\ldots \\
&k_{m1} \quad k_{m2} \\
&= \sum_{j_1,j_2,\ldots,j_m} \sum_{k_1,k_2,\ldots,k_m} \left( a_1 \cdots a_m \right) \left( b_1 \cdots b_m \right),
\end{align*}
$$

where

$$
\begin{align*}
&j_1,\ldots,j_m; j_1^i,\ldots,j_m^i \\
&d_{k_1,\ldots,k_m}; k_1^i,\ldots,k_m^i
\end{align*}
$$

where the $d_j(k)$ refer to the factors A (at 2 levels each) and are the same as the $d_j$'s defined at (46), and the $d_j^i(k)$ refer to the factors B (at 3 levels each) and are the same as the corresponding $d_j$'s defined at (37).

In writing the symbol for the interactions, we shall stick to the earlier convention that if any $k$ or $k^i$ is zero, we drop that factor from the symbol.

The normal equations for the mixed case can be set up by approaching exactly as for the symmetrical case. It can be seen that corresponding to the matrix $EE'$, which we had to invert for the symmetrical case, in the mixed case we shall have to invert another matrix which may be denoted by $FF'$. The nature of $FF'$ will be explained below without proofs.
As for EE', the rows and columns of FF' correspond to the effects:

(i) $\{\mu\}$, the general mean,  (ii) $\{A_i\}$, $\{A_iA_j\}$, the pure A-effects,  (iii) $\{B_i\}$, $\{B_i^2\}$, $\{B_iB_j\}$, $\{B_i^2B_j\}$, the pure B-effects, and (iv) $\{A_iB_i\}$, $\{A_iB_i^2\}$, the mixed AB effects. The form of FF' is exhibited below

$$FF' = \begin{bmatrix} \Omega_1 & \Omega_3 \\ \Omega_3^\prime & \Omega_2 \end{bmatrix}$$

where $\Omega_1$ is of the form:

$$\{\mu\} \{B_i\} \{B_i^2\} \{B_iB_j\} \{B_i^2B_j\} \{A_iB_i\} \{A_iA_j\}$$

$$\begin{bmatrix} M_1 & M_2 & M_3 & M_4 & M_5 & M_6 & M_7 \\ M_2 & M_{11} & M_{12} & M_{13} & M_{14} & M_{15} & M_{16} \\ M_3 & M_{12} & M_{20} & M_{21} & M_{22} & M_{23} & M_{24} \\ M_4 & M_{13} & M_{21} & M_{28} & M_{29} & M_{30} & M_{31} \\ M_5 & M_{14} & M_{22} & M_{29} & M_{35} & M_{36} & M_{37} \\ M_6 & M_{15} & M_{23} & M_{30} & M_{36} & M_{41} & M_{42} \\ M_7 & M_{16} & M_{24} & M_{31} & M_{37} & M_{42} & M_{46} \end{bmatrix}$$
\( \Omega_2 \) is of the form:

\[
\begin{bmatrix}
\{A_1\} & \{A_1B_j\} & \{A_1B_j^2\} \\
\{A_1\} & M_{50} & M_{51} & M_{52} \\
\{A_1B_j\} & M_{51} & M_{53} & M_{54} \\
\{A_1B_j^2\} & M_{52} & M_{54} & M_{55}
\end{bmatrix}
\]

(42)

and finally \( \Omega_3 \) can be represented as:

\[
\begin{bmatrix}
\{A_1\} & \{A_1B_j\} & \{A_1B_j^2\} \\
\{\mu\} & M_{8} & M_{9} & M_{10} \\
\{B_1\} & M_{17} & M_{18} & M_{19} \\
\{B_1^2\} & M_{25} & M_{26} & M_{27} \\
\{B_1B_j\} & M_{32} & M_{33} & M_{34} \\
\{B_1B_j^2\} & M_{38} & M_{39} & M_{40} \\
\{B_1^2B_j\} & M_{43} & M_{44} & M_{45} \\
\{A_1A_j\} & M_{47} & M_{48} & M_{49}
\end{bmatrix}
\]

(43)

The order of the whole matrix \( FF' \) is \( v_o \times v_o \) where obviously

\[ v_o = 1 + \frac{m(m+1)}{2} + 2n^2 + 2mn. \]

The whole matrix \( FF' \) has 100 submatrices. The set or sets of interactions to which a submatrix corresponds are shown on the left of the rows and above the columns. Thus for example \( M_{53} \) corresponds to \([\{A_1B_j\}; \{A_1B_j\}]\) and is therefore
of size mn x mn, $M_{ij}$ corresponds to $[\{B_i^2B_j^2\}; \{A_iB_j^2\}]$ and is of size $^{m\choose 2} \times mn$, etc. Each element of $\mathbb{F}^t$ thus corresponds to a unique pair of interaction. The element in the cell corresponding to the pair of interactions $(x,y)$ will be written $\epsilon(x,y)$. Thus for example $\epsilon(B_1^2B_2^2, B_3^2B_4^2)$ lies in $M_{36}$, $\epsilon(B_1B_2, B_1^2)$ in $M_{21}$ etc.

The following lemma can be easily established on the lines of Theorem 3.1.

Lemma 7.1

\[ \epsilon(A_1^{k_i} A_j^{k_j}, B_1^{r_i} B_j^{r_j}) = \lambda[A_1^{k_i} A_j^{k_j}, B_1^{r_i} B_j^{r_j}] \]

where, (i) out of the four symbols $k_i, k_j, r_i, r_j$, at most two are nonzero; and similarly for $k_i, k_j, r_i, r_j$;

(ii) $P_1(A)$ is a polynomial in $a_i^j$ defined by

\[ \epsilon(A_1^{k_i} A_j^{k_j}, A_1^{r_i} A_r^{r_j}) = \lambda[P_1(A)] \]

and similarly $P_2(B)$ is defined by

\[ \epsilon(B_1^{k_i} B_j^{k_j}, B_1^{r_i} B_r^{r_j}) = \lambda[P_2(B)] \]

(iii) after expanding the product $P_1(A)P_2(B)$ algebraically, the value on the r.h.s. of (44) is obtained by using in the usual way, the definition of $\lambda(\theta)$ for any assembly $\theta$.

We shall now illustrate the calculation of elements of $\mathbb{F}^t$. 
Ex. 1.  \( \epsilon(B_2^2, B_2A_1) \)

\[ = \lambda(b_1^2 - b_1^0)(b_2^2 - 2b_2^1 + b_2^0)(a_1^1 - a_1^0) \]

\[ = \lambda(b_1^2 - b_1^0)(b_2^2 - b_2^0)(a_1^1 - a_1^0) \]

\[ = \frac{22;1}{12;1} - \frac{22;0}{12;1} - \frac{20;1}{12;1} + \frac{20;0}{12;1} - \frac{02;1}{12;1} + \frac{02;0}{12;1} + \frac{00;1}{12;1} - \frac{00;0}{12;1} \]

Ex. 2.  \( \epsilon(A_1B_2, A_1B_2^0) \)

\[ = \lambda(a_1^1 - a_1^0)(b_2^2 - b_1^2)(a_1^1 - a_1^0)(b_2^2 - 2b_2^1 + b_2^0) \]

\[ = \lambda(a_1^1 + a_1^0)(b_2^2 - b_2^0) = \lambda(b_2^2 - b_2^0) = \lambda_{2,2}^{2,0} , \]

where the dot shows that no factors of the type are involved.

The \((1,0)\) and \((2,0)\) symmetry are defined as before. The following lemma is an extension of previous results.

**Lemma 7.2.** Let \( T \) be a set of assemblies from the \( m_1 \times 3^m_2 \) factorial. Then

(i) if \( T \) is \((1,0)\) symmetric with respect to triplet of the factors \( A \), the matrix \( FF' \) may be broken up into two orthogonal parts, one corresponding to the set of effects \( \{\mu\}, \{B_1\}, \{B_1^2\}, \{B_1B_j\}, \{B_1^2B_j\}, \{B_2B_j\} \) and \( \{A_1A_j\} \) and the other corresponding to the set \( \{A_1\}, \{A_1B_j\} \) and \( \{A_1B_2^0\} \),

(ii) if \( T \) is \((2,0)\) symmetric with respect to triplets of factors \( B \), the two sets in which \( FF' \) is broken are \( \{\mu\}, \{B_1^2\}, \{B_1B_j\}, \{B_1^2B_j\}, \{A_1A_j\}, \{A_1\}, \{A_1B_2^0\} \) and \( \{B_1\}, \{B_1^2B_j\}, \{A_1B_j\} \).

(iii) if it has both the above properties, then \( FF' \) is broken into 4 parts corresponding to (a) \( \{\mu\}, \{B_1^2\}, \{B_1B_j\}, \{B_1^2B_j\}, \{A_1A_j\} \), (b) \( \{B_1\}, \{B_1^2B_j\} \), (c) \( \{A_1B_j\} \), and (d) \( \{A_1\} \) and \( \{A_1B_2^0\} \).
All the above results will be amply illustrated in the next section.

8. Examples of the analysis of asymmetrical factorial fractions.

**Example 1.** Suppose we have an orthogonal array \( \Gamma^* \) of strength 4 in \( m = m_1 + m_2 \) factors, with \( s = 3 \) and \( N \) assemblies. From this array, one could obtain a fraction for a \( 2^{m_1} \times 3^{m_2} \) factorial by making the transformation

\[
T_1 = \begin{pmatrix}
0 & 1 & 2 \\
0 & 1 & 1
\end{pmatrix}
\]

of the levels of \( m_1 \) of the factors. This means that for these \( m_1 \) factors we replace the level symbol 2 by 1. In this example, we shall illustrate the above developed method of analysis of asymmetrical fractions by considering the analysis of certain fractions obtained by such transformations.

Let the set of \( N \) assemblies obtained after making the above transformation be denoted by \( \Gamma_1 \). Obtain \( N \) more assemblies \( \Gamma_2 \) by making the complementary transformation \( T_2 = (0 \ 1 \ 2) \). Consider the analysis of \( \Gamma = \Gamma_1 + \Gamma_2 \).

Obviously, \( \Gamma \) is \((1,0)\) symmetric with respect to the factors A. Hence by using Lemma 7.2, and equation (40), we see that \( \Omega \) in \((FF')_T \) is a zero matrix. Also \( \Omega_1 \) in (40) can be written as

\[
\Omega_1 = \begin{bmatrix}
(FF')_3 & L \\
L' & M_{46}
\end{bmatrix},
\]

where \((FF')\) represents the matrix \( EE' \), the \( 3^{m_2} \) factorial fraction obtained by ignoring the factors A in the mixed fraction \( \Gamma \). However since \( \Gamma \) is an array of strength 4 in factors B, the matrix \( EE' \) is a diagonal nonsingular matrix whose elements can be calculated as in Ex. 6.2. Also from (41), the matrix \( L' \) must equal

\[
L' = \begin{bmatrix}
M_1^1 & M_1^1 & M_1^1 & M_1^1 & M_1^1 & M_1^1
\end{bmatrix}.
\]
We now show that except $M_7$, all other matrices in $L$ are null. It is sufficient to observe that each element in all such matrices is of the form 
\( \lambda[P_1(A)P_2(B)] \), where $P_1(A)$ is of the form $(a_i^{1} - a_i^{0})(a_j^{1} - a_j^{0})$ and $P_2(B)$ is a product of one or two polynomials, each of which are of the form 
$(b_k^{2} - b_k^{0})$ or $(b_k^{2} - 2b_k^{1} + b_k^{0})$. Now

\[ (45) \quad \lambda[P_1(A)P_2(B), \Gamma_1] = \lambda[(a_i^{1}a_j^{1} + a_i^{0}a_j^{0} - a_i^{1}a_j^{0} - a_i^{0}a_j^{1})P_2(B), \Gamma_1] \]

\[ = \lambda[(4b_i^{1}b_j^{1} + b_i^{0}a_j^{0} - 2b_i^{1}b_j^{0} - 2b_i^{1}b_j^{0})P_2(B), \Gamma^*] , \]

by going back on the transformation $T_1$. Since $\Gamma^*$ is an orthogonal array of strength 3, the above expression is clearly zero. Similarly for $\Gamma_2$. This completes the proof of our assertion.

Hence $\Omega_1$ is split up into a diagonal matrix and the matrix

\[ (46) \quad \begin{bmatrix} M_1 & M_7 \\ M_7^t & M_{46} \end{bmatrix} \]

Any element of $M_7$ is $\lambda[(a_i^{1} - a_i^{0})(a_j^{1} - a_j^{0}), \Gamma]$ which from (45) equals

\[ \lambda[(5b_i^{1}b_j^{1} + 5b_i^{0}b_j^{0} - 4b_i^{1}b_j^{0} - 4b_i^{1}b_j^{0}), \Gamma^*] = (5 + 5 - 4 - 4). \frac{N}{9} = \frac{2N}{9} , \]

since $\lambda_{ii}^{ij}$ must all equal $\frac{N}{9}$ in $\Gamma^*$. Also from this it follows that the elements in cells of the type $(A_iA_j, A_iA_k)$ in $M_{46}$ also equal $2N/9$. For elements in $M_{46}$ involving 4 factors one gets in a similar manner.
\[
\lambda \left[ \prod_{r=1}^{4} (a_{1r}^1 - a_{1r}^0) , \Gamma_1 + \Gamma_2 \right] \\
= \lambda \left[ \prod_{r=1}^{4} (b_{1r}^2 + b_{1r}^1 - b_{1r}^0) \right] + \left[ \prod_{r=1}^{4} (-b_{1r}^2 - b_{1r}^1 + b_{1r}^0) \right], \Gamma^* \right] \\
= \lambda \left[ \prod_{r=1}^{4} (b - 2b_{1r}^0) \right] + \left[ \prod_{r=1}^{4} (2b_{1r}^0 - b) \right], \Gamma^* \right] \\
= 2\lambda \left[ \prod_{r=1}^{4} (b - 2b_{1r}^0) \right], \Gamma^* \right] \\
= 2\lambda (b - 4b_{1r}^0) + 6(4b_{i1j}^0 - 4b_{i1j}^{oo}) + 16b_{i1jk}^{ooo} ,
\]

because of symmetry with respect to quadruplets of factors. The last expression evidently equals

\[
2(N - 8 \cdot \frac{N}{3} + 2\cdot \frac{N}{9} + 32 \cdot \frac{N}{27} + 16 \cdot \frac{N}{81}) = 2N/81
\]

Formulæ by which the inverse of \( M_{46} \) and hence of (46) can be quickly obtained are available in sec. 5, [6], and will not be reproduced here. However it would be of interest to know the correlations between the various interactions revealed by the inverse of (46). For pairs of interactions of the type \( (A_i A_j, A_i A_k) \) they vary from \((-0.085)\) for the case \( m_1 = 4 \) to about \((-0.065)\) for \( m_1 = 8 \) while for other pairs it varies from \((0.052)\) to \((0.015)\) between the same range of values of \( m_1 \).

The diagonal matrix in \( \Omega_1 \) can be easily calculated as in Example 2, and its inverse presents no problem. We are therefore left with \( \Omega_2 \). Here also, an argument of the above form will demonstrate that \( M_{51}, M_{52} \) and \( M_{54} \) are null, and that \( M_{53} \) and \( M_{55} \) are diagonal matrices.

We are finally left with \( M_{50} \) which corresponds to the main effects of the factors \( A \). Its nondiagonal and diagonal elements are respectively \( 2N \) and \( 2N/9 \). Thus the main effects of factors \( A \) are correlated and the correlation between any pair is found to equal \((-1/m_1 + 8)\).
Example 2. A $2^4 \times 3^2$ fraction in 48 assemblies ($v = 35$). The fraction is constructed as follows (Method II of [5, 17]). Consider the equation

$$2x_1 + 2x_2 + 2x_3 + 2x_4 + x_5 + x_6 = 0 \pmod{2}$$

where $x$'s take the three values 0, 1, 2. Out of the 243 assemblies of the fraction of a $3^6$ factorial that we get this way, we eliminate all the assemblies that contain the level 2 of any of the factors represented here by $x_1$, $x_2$, $x_3$ and $x_4$. We then get the 48 assemblies $T = T_1 + T_2 + T_3$, exhibited below.

$$
\begin{array}{ccc}
T_1 & & T_2 \\
\begin{bmatrix}
0000 \\
1110 \\
1101 \\
1011 \\
0111 \\
\end{bmatrix} & x & \begin{bmatrix}
00 \\
12 \\
21 \\
\end{bmatrix} & \begin{bmatrix}
1111 \\
0001 \\
0010 \\
0100 \\
1000 \\
\end{bmatrix} & x & \begin{bmatrix}
01 \\
10 \\
22 \\
\end{bmatrix} & \begin{bmatrix}
1100 \\
1010 \\
1001 \\
0110 \\
0101 \\
0011 \\
\end{bmatrix} & x & \begin{bmatrix}
11 \\
20 \\
02 \\
\end{bmatrix}
\end{array}
$$

Since each of the 16 combinations of the levels of $A$-factors occurs exactly 3 times, it is clear that the effects $\{A_i\}$, $\{A_iA_j\}$ are all completely orthogonal.

Hence leaving these two sets of effects aside, and noting that the fraction is also $(2, 0)$ symmetric, we find from lemma 7.2 that $FF'$ is broken into four parts corresponding to: (i) $\{\mu\}$, $\{B_i^2\}$, $\{B_iB_j\}$, $\{B_i^2B_j^2\}$, (ii) $\{B_i\}$, $\{B_i^2B_j\}$, (iii) $\{A_iB_j\}$ and (iv) $\{A_iB_j^2\}$. We shall take up each part separately.

Part (i) This corresponds to

$$
\begin{bmatrix}
M_1 & M_3 & M_4 & M_6 \\
M_3 & M_20 & M_21 & M_23 \\
M_4 & M_21 & M_28 & M_30 \\
M_6 & M_23 & M_30 & M_41 \\
\end{bmatrix} = \begin{bmatrix}
48 & 0 & 0 & -2 & 6 \\
96 & 6 & -2 & -6 \\
96 & -2 & -6 \\
22 & -2 \\
\end{bmatrix}
$$

$$
\text{Sym.} \quad 198
$$
To see this, we notice that leaving $M_{21}$, $M_{23}$, and $M_{20}$, the rest of the matrices are all $(1 \times 1)$, and

\[ M_1 = \epsilon(\mu, \mu) = 48, \]

\[ M_4 = \epsilon(\mu, B_1^1 B_2^1) = \lambda(2^1 - 0^1)(2^1 - 0^1) = \lambda(22 + 00 - 02 - 20) = 5 \times 2 - 6 \times 2 = -2, \]

\[ M_6 = \epsilon(\mu, B_1^2 B_2^2) = \lambda(b - 3b_1^1)(b - 3b_2^1) = 48 - 3(\lambda_1 + \lambda_2) + 9\lambda_1^{11} \]

\[ = 48 - 3 \times 2 \times 16 + 9 \times 6 = 6, \]

\[ M_{28} = \epsilon(B_1^1 B_2, B_1^1 B_2) = \lambda(b - b_1^1)(b - b_2^1) = 48 - 2 \times 16 + 6 = 22, \]

\[ M_{41} = \epsilon(B_1^1 B_2^2, B_1^2 B_2^2) = \lambda(b + 3b_1^1)(b + 3b_2^1) = 48 + 2 \times 16 \times 3 + 6 \times 9 = 198, \]

\[ M_{30} = \epsilon(B_1 B_2, B_1^2 B_2^2) = \lambda(2^1 - 0^1)(2^1 - 0^1) = -2. \]

Also clearly

\[ M_{20} = \begin{bmatrix} 96 & 6 \\ 6 & 96 \end{bmatrix}, \quad M_3 = [0 \ 0] \]

since

\[ \epsilon(B_1^1 B_1^2) = \lambda(b + 3b_1^1) = 48 + 16 \times 3 = 96 \]

\[ \epsilon(B_1^2 B_2^2) = \epsilon(\mu, B_1^2 B_1^2) = 6 \]

\[ \epsilon(\mu, B_1^2) = \lambda(b_1^2 - 2b_2^1 + b_4^0) = 0 \]

The inverse of $\Omega_1^*$ is given by

\[ \Omega_1^{*-1} = \frac{1}{2} \begin{bmatrix} \frac{1}{24} & 0 & 0 & \Gamma_1 \\ 0 & 0 & \Gamma_2 & \Gamma_3 \\ 0 & \Gamma_1 & \Gamma_3 & \Gamma_4 \end{bmatrix}, \]
where \( \Gamma_2 = \frac{1}{17 \times 45} \begin{bmatrix} 16 & -1 \\ -1 & 16 \end{bmatrix} \)

\[
\begin{bmatrix}
\Gamma_1^* & \Gamma_3^* & \Gamma_4^*
\end{bmatrix} = \frac{136}{135 \times 9800} \begin{bmatrix}
4 & 2 & 99 & 1 \\
-\frac{4}{3} & \frac{2}{3} & \frac{2}{3} & 1 \\
\end{bmatrix}
\]

**Part (ii)** Here we get

\[
\Omega_{20}^* = \begin{bmatrix}
M_{11} & M_{14} \\
\text{Sym.} & M_{35}
\end{bmatrix} = \begin{bmatrix}
32 & -2 & -2 & 2 \\
-2 & 32 & 2 & -2 \\
\text{Sym.} & 62 & -2 & 62
\end{bmatrix}
\]

by calculations similar to above, and the inverse is

\[
\begin{bmatrix}
68 & 4 & 2 & -2 \\
4 & 68 & -2 & 2 \\
2 & -2 & 35 & 1 \\
-2 & 2 & 1 & 35 \\
\end{bmatrix}
\]

**Part (iii)** We have

\[
M_{33} = \begin{bmatrix}
\Omega_{31}^* & \Omega_{32}^* \\
\text{Sym.} & \Omega_{33}^*
\end{bmatrix}
\]

where the partitioning corresponds to \( \{ A_1 B_1 \} \) and \( \{ A_1 B_2 \} \). Now

\[
\epsilon(A_1 B_1, A_1 B_1) = \epsilon(B_1, B_1) = 32
\]

\[
\epsilon(A_1 B_1, A_2 B_1) = \lambda(1-0)_1 (1-0)_2 (2 + 0)_1
\]

\[
= \lambda(1-0)_1 (1-0)_2 (b - b^1_1) = -\lambda(11 + 00 - 01 - 10)_{a_1 a_2} b^1_1
\]

\[
= -(\lambda^{11,1} + \lambda^{00,1}) + (\lambda^{01,1} + \lambda^{10,1})
\]

\[
= -(3 + 3 + 2) + (2 + 2 + 4) = 0
\]
\[
\epsilon(A_{1B_1}, A_{1B_2}) = \epsilon(B_{1}, B_{2}) = -2
\]
\[
\epsilon(A_{1B_1}, A_{2B_2}) = \lambda[(1-0)(1-0)]_{a_1a_2} [(2-0)(2-0)]_{b_1b_2}
\]
\[
= [(3+3) + 4x2] - [(2+2) + 2x2] = 6
\]

Hence
\[
\Omega_{31}^* = \Omega_{33}^* = 32 I_4, \quad \Omega_{32}^* = 6 J_{44} - 8 I_4,
\]
and
\[
M_{33}^{-1} = \frac{1}{1480}
\begin{bmatrix}
16 I_4 + J_{44} & 4 I_4 - \frac{7}{2} J_{44} \\
\downarrow & \downarrow \\
16 I_4 + J_{44} & 16 I_4 + J_{44}
\end{bmatrix}
\]

**Part (iv) Here**

\[
M_{55} = \begin{bmatrix}
\Omega_{41}^* & \Omega_{42}^* \\
\text{Sym.} & \Omega_{43}^*
\end{bmatrix},
\]

where
\[
\Omega_{41}^* = \Omega_{43}^* = 96 I_4, \quad \Omega_{42}^* = 24 I_4 - 18 J_{44}.
\]

Also
\[
M_{55}^{-1} = \frac{1}{1440}
\begin{bmatrix}
16 I_4 + J_{44} & -4 I + \frac{7}{2} J \\
\downarrow & \downarrow \\
\text{Sym.} & 16 I_4 + J_{44}
\end{bmatrix}
\]
REFERENCES


