THE ANALYSIS OF INCOMPLETE BLOCKS EXPERIMENTS WITH PARTICULAR EMPHASIS ON THE RELATIONSHIP BETWEEN THE RECOVERY OF INTER-BLOCK INFORMATION AND GENERALIZED LEAST SQUARES WHEN BLOCK EFFECTS ARE ASSUMED RANDOM

by

Gary G. Koch
University of North Carolina

Institute of Statistics Mimeo Series No. 464

March 1966

This research was supported by the National Institutes of Health Institute of General Medical Sciences Grant No. GM-12868-02 and by the Mathematics Division of the Air Force Office of Scientific Research Grant No. AF-AFOSR-760-65.

The paper is based on the notes of R. C. Bose in Statistics 150 and Statistics 254 at the University of North Carolina at Chapel Hill.

DEPARTMENT OF STATISTICS
UNIVERSITY OF NORTH CAROLINA
Chapel Hill, N. C.
An incomplete blocks experiment is characterized by a design which is simply the arrangement of $v$ objects called treatments into $b$ subsets of objects called blocks such that the $i$-th treatment occurs $n_{ij}$ times in the $j$-th block.\footnote{It is always assumed that within any given block, the treatments are assigned at random to the experimental units.}

We define the incidence matrix $N = N(v \times b)$ associated with the design as follows:

$$
N = \begin{bmatrix}
  n_{11} & n_{12} & \cdots & n_{1b} \\
  n_{21} & n_{22} & \cdots & n_{2b} \\
  \vdots & \vdots & \ddots & \vdots \\
  n_{v1} & n_{v2} & \cdots & n_{vb}
\end{bmatrix}
$$

(1)

The sum of the elements in the $i$-th row of $N$ is called the number of replications of the $i$-th treatment and is denoted by $r_i$; the sum of the elements in the $j$-th column of $N$ is called the size of the $j$-th block and is denoted by $k_j$; the sum of all the elements in $N$ is the total number of observations (or experimental units) in the experiment and is denoted by $n$. Hence, the following relations hold

\begin{align*}
(2-a) & \quad \sum_{j=1}^{b} n_{ij} = r_i \quad i = 1, 2, \ldots, v, \\
(2-b) & \quad \sum_{i=1}^{v} n_{ij} = k_j \quad j = 1, 2, \ldots, b, \\
(2-c) & \quad \sum_{i=1}^{v} \sum_{j=1}^{b} n_{ij} = \sum_{j=1}^{b} k_j = \sum_{i=1}^{v} r_i = n
\end{align*}

If we let $r = r(v \times 1)$ and $k = k(b \times 1)$ be defined as
and if we let \( \mathbf{1}_u = \mathbf{1}_u (u \times 1) \) denote a column vector of "ones," then (2-a), (2-b), and (2-c) may be re-written as

\[
(4-a) \quad N \mathbf{1}_b = \mathbf{r} ,
(4-b) \quad N' \mathbf{1}_v = \mathbf{k} ,
(4-c) \quad \mathbf{1}_v' N \mathbf{1}_b = \mathbf{k}' \mathbf{1}_b = \mathbf{1}_v' \mathbf{r} = n .
\]

We now assume that the observations arising from the experiment are described by the following linear statistical model

\[
y_u = y_{ij}^{(a)} = g + t_i + b_j + e_u \quad \text{for } u = 1, 2, \ldots, n
\]

\[
i = 1, 2, \ldots, v
\]

\[
\text{with } j = 1, 2, \ldots, b
\]

\[
\alpha = 1, 2, \ldots, n_{ij}
\]

where it is realized that \( y_u \) is defined only for \( \alpha > 0 \). In words, (5) postulates that the \( u \)-th observation is a linear combination of a general mean effect \( g \), an effect \( t_i \) due to the \( i \)-th treatment, an effect \( b_j \) due to the \( j \)-th block, and a residual error \( e_u \). The \( \{ e_u \} \) are assumed to be uncorrelated with expected values equal to zero and variances equal to an unknown constant \( \sigma^2 \); the \( \{ b_j \}, \{ t_i \} \), and \( \alpha \) for the present, are assumed to be fixed constants. In order to write the regression model corresponding to (5), we define matrices \( H' = H' (n \times v) \) and \( L' = L' (n \times b) \) as follows
\[
H' = \begin{bmatrix}
    h_{11} & h_{21} & \cdots & h_{v1} \\
    h_{12} & h_{22} & \cdots & h_{v2} \\
    \cdots & \cdots & \cdots & \cdots \\
    h_{1n} & h_{2n} & \cdots & h_{vn}
\end{bmatrix}, \quad L' = \begin{bmatrix}
    l_{11} & l_{21} & \cdots & l_{b1} \\
    l_{12} & l_{22} & \cdots & l_{b2} \\
    \cdots & \cdots & \cdots & \cdots \\
    l_{1n} & l_{2n} & \cdots & l_{bn}
\end{bmatrix}
\]

where \( h_{iu} = \begin{cases} 
1 & \text{if the } i\text{-th treatment is applied to the } u\text{-th observation} \\
0 & \text{otherwise}
\end{cases} \)

\( l_{ju} = \begin{cases} 
1 & \text{if the } u\text{-th observation is in the } j\text{-th block} \\
0 & \text{otherwise}
\end{cases} \)

From the definitions of \( H' \) and \( L' \), it follows that

\[
(7-a) \quad \sum_{i=1}^{v} h_{iu} = 1 \quad u = 1, 2, \ldots, n
\]

\[
(7-b) \quad \sum_{j=1}^{b} l_{ju} = 1 \quad u = 1, 2, \ldots, n
\]

since any given experimental unit belongs to exactly one block and has exactly one treatment applied to it; also

\[
(8-a) \quad \sum_{u=1}^{n} h_{iu} = r_i \quad i = 1, 2, \ldots, v
\]

\[
(8-b) \quad \sum_{u=1}^{n} l_{ju} = k_j \quad j = 1, 2, \ldots, b
\]

since the \( i\)-th treatment is applied to \( r_i \) experimental units and the \( j\)-th block contains \( k_j \) experimental units. The relations (7-a), (7-b), (8-a), and (8-b) may be re-written as

\[
(9-a) \quad H'_{\cdot v} = h_n
\]

\[
(9-b) \quad L'_{\cdot b} = l_n
\]

\[
(9-c) \quad H_{\cdot n} = r
\]

\[
(9-d) \quad L_{\cdot n} = k
\]
If we let \( \mathbf{Y} = \mathbf{Y}(n \times 1) \) and \( \mathbf{e} = \mathbf{e}(n \times 1) \) be given by

\[
(10) \quad \mathbf{Y} = \begin{bmatrix}
    y_1' \\
    y_2' \\
    \vdots \\
    y_n'
\end{bmatrix} \quad \text{and} \quad \mathbf{e} = \begin{bmatrix}
    e_1' \\
    e_2' \\
    \vdots \\
    e_n'
\end{bmatrix},
\]

then the model (5) may be re-written as

\[
(11) \quad \mathbf{Y} = \mathbf{j}_n' \mathbf{g} + \mathbf{H}' \mathbf{t} + \mathbf{L}' \mathbf{b} + \mathbf{e}
\]

where

\[
(12) \quad \mathbf{e}(\mathbf{e}) = \mathbf{0}_n, \quad \mathbf{e}(\mathbf{ee}') = \mathbf{I}_n \sigma^2
\]

with \( \mathbf{0}_n \) being the \( n \)-dimensional null vector and \( \mathbf{I}_n \) the \( n \times n \) identity matrix.

At this point, we define the following sample quantities

\[
(13) \quad \mathbf{G} = \sum_{u=1}^{n} y_u = \sum_{i=1}^{v} \sum_{j=1}^{b} \sum_{\alpha=1}^{n_{ij}} y_{i,j}(\alpha),
\]

\[
(14) \quad \mathbf{T}_i = \sum_{u=1}^{n} h_{iu} y_u = \sum_{j=1}^{b} \sum_{\alpha=1}^{n_{ij}} y_{i,j}(\alpha) \quad i = 1, 2, \ldots, v
\]

\[
(15) \quad \mathbf{B}_j = \sum_{u=1}^{n} h_{ju} y_u = \sum_{i=1}^{v} \sum_{d=1}^{n_{id}} y_{i,d}(\alpha) \quad j = 1, 2, \ldots, b;
\]

\( \mathbf{G} \) is called the grand total, \( \mathbf{T}_i \) is called the \( i \)-th treatment total, and \( \mathbf{B}_j \) is called the \( j \)-th block total. Letting \( \mathbf{T} = \mathbf{T}(v \times 1) \) and \( \mathbf{B} = \mathbf{B}(b \times 1) \) be given by

\[
(16) \quad \mathbf{T} = \begin{bmatrix}
    T_1 \\
    T_2 \\
    \vdots \\
    T_v
\end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix}
    B_1 \\
    B_2 \\
    \vdots \\
    B_b
\end{bmatrix},
\]
then (13), (14), and (15) may be more compactly written as

\[
G = J_n Y
\]

\[
T = H Y
\]

\[
B = L Y
\]

(17)

The least squares (normal) equations of estimation are

(18) \[ A A' p = A Y \]

where for incomplete blocks experiments \( A \) and \( p \) are given by

(19) \[
A = \begin{bmatrix} H \\ L \end{bmatrix} \quad \text{and} \quad p = \begin{bmatrix} g \\ t \\ b \end{bmatrix}
\]

Hence, we find

(20) \[ A A' = \begin{bmatrix} J_n & J_n' H' & J_n' L' \\ H & J_n & H L' \\ L & H' & L L' \end{bmatrix} = \begin{bmatrix} n & r' & k' \\ r & H H' & H L' \\ k & L H' & L L' \end{bmatrix} \]

(21) \[ A Y = \begin{bmatrix} J_n' Y \\ H Y \\ L Y \end{bmatrix} = \begin{bmatrix} G \\ T \\ B \end{bmatrix} \]

by using (9-c), (9-d), and (17). Now

(22-a) \[ h_{i'u} = \begin{cases} 1 & \text{if } i = i' \\ 0 & \text{if } i \neq i' \end{cases} \]

(22-b) \[ u_{j'u} = \begin{cases} 1 & \text{if } j = j' \\ 0 & \text{if } j \neq j' \end{cases} \]

again because any given experimental unit belongs to exactly one block and has exactly one treatment applied to it. From (22-a) and (22-b), it follows that

(23-a) \[ \sum_{u=1}^{n} h_{i'u} h_{i' u} = \begin{cases} r_1 & \text{if } i = i' \\ 0 & \text{if } i \neq i' \end{cases} \]
\[(23-b) \sum_{u=1}^{n} k_{ju} \ell_{ju} = \begin{cases} k_j & \text{if } j = j' \\ 0 & \text{if } j \neq j' \end{cases}\]

and hence that
\[(24) \quad H H' = D_r, \quad L L' = D_k\]

where $D_r$ and $D_k$ are diagonal matrices whose diagonal elements are the elements of $r$ and $k$ respectively. Next let us observe that
\[(25) \quad h_{iu} \ell_{ju} = 1 \text{ if and only if the } u\text{-th experimental unit belongs to the } j\text{-th block and has the } i\text{-th treatment applied to it.}\]

Thus, we have
\[(26) \quad \sum_{u=1}^{n} h_{iu} \ell_{ju} = n_{ij}\]

and hence
\[(27) \quad H L' = N .\]

Substituting (24) and (27) into (26), the normal equations (18) may be written
\[(28) \quad \begin{bmatrix} n & r' & k' \\ r' & D_r & N \\ k & N' & D_k \end{bmatrix} \begin{bmatrix} g \\ t \\ b \end{bmatrix} = \begin{bmatrix} G \\ T \\ B \end{bmatrix} .\]

Note that the equations (28) contain two independent redundancies since if both sides are pre-multiplied by $R = R[2 \times (v + b + 1)]$
\[(29) \quad R = \begin{bmatrix} -1 & 0' & 0' \\ -1 & 0' & 0' \end{bmatrix},\]

we obtain
\[(30) \quad \begin{bmatrix} 0 & 0' & 0' \\ 0 & 0' & 0' \end{bmatrix} \begin{bmatrix} g \\ t \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} .\]
To estimate the treatment effects, we solve (28) for \( \overline{t} \) by working with the adjusted normal equations which are obtained by pre-multiplying both sides of (28) by \( F = F(v \times (v + b + l)) \)

\[
(31) \quad F = \begin{bmatrix} O_v & I_v - ND_k^{-1} \end{bmatrix}.
\]

If we define \( C = C(v \times v) \) and \( Q = Q(v \times l) \) by

\[
(32) \quad FAA' = \begin{bmatrix} O_v & D_x - ND_k^{-1}N' \end{bmatrix} \begin{bmatrix} Q_v & C & O_vb \end{bmatrix} = \begin{bmatrix} O_v & C & O_vb \end{bmatrix}
\]

\[
(33) \quad FAY = T - ND_k^{-1}B = Q,
\]

then the adjusted normal equations may be written

\[
(34) \quad C\overline{t} = Q.
\]

From (32) and (33), it follows that the elements of \( C \) and \( Q \) have the forms

\[
(35) \quad c_{ii'} = r_i \delta_{ii'} - \sum_{j=1}^{b} \frac{n_{ij} \delta_{ij}}{k_j} \quad \text{where} \quad \delta_{ii'} = \begin{cases} 1 & \text{if } i = i' \\ 0 & \text{if } i \neq i' \end{cases}
\]

\[
(36) \quad Q_i = T_i - \sum_{j=1}^{b} \frac{n_{ij}}{k_j} \overline{B}_j = T_i - \sum_{j=1}^{b} n_{ij} \overline{B}_j
\]

where \( \overline{B}_j = \frac{1}{k_j} B_j \) is the \( j \)-th block average. The quantity

\[
(37) \quad \sum_{j=1}^{b} n_{ij} \overline{B}_j = T_i - Q_i
\]

represents a weighted sum of block averages for the blocks in which the \( i \)-th treatment appears where the weight corresponding to the \( j \)-th block is the number \( n_{ij} \) of occurrences of the \( i \)-th treatment in the \( j \)-th block. Similarly the quantity

\[
(38) \quad \sum_{j=1}^{b} \frac{n_{ij} \delta_{ij}}{k_j} = r_i \delta_{ii'} - c_{ii'}
\]

represents a weighted sum of the number of distinct ordered pairs of units.
within a given block in which the first unit receives treatment i and the second receives treatment $i'$ where the weights are the respective inverses of block sizes. Finally, we observe

$$\epsilon(q) = \epsilon(FAy) = FAA'q = Ct$$

because of (32) and (33).

A solution of (34) is given by

$$\hat{q} = c^g q$$

where $c^g$ is any conditional inverse of $C$. Given (40), we claim that a solution to (28) is

$$\hat{q} = \begin{bmatrix} \hat{g} \\ \hat{t} \\ \hat{b} \end{bmatrix} = \begin{bmatrix} 0 \\ \hat{t} \\ \hat{b} \end{bmatrix} \begin{bmatrix} D_k^{-1}B - D_k^{-1}N'\hat{C} \end{bmatrix}$$

since

$$\begin{bmatrix} n & x' & k' \\ x & D_r & N \\ k & N' & D_k \end{bmatrix} \begin{bmatrix} \hat{g} \\ \hat{t} \\ \hat{b} \end{bmatrix} = \begin{bmatrix} x'r' + k'D_k^{-1}B - k'D_k^{-1}N'\hat{C} \\ D_r\hat{t} + ND_k^{-1}B - ND_k^{-1}N'\hat{C} \\ N'\hat{b} + B - N'\hat{C} \end{bmatrix} = \begin{bmatrix} k'D_k^{-1}B \\ C\hat{b} + T - Q \\ B \end{bmatrix} = \begin{bmatrix} G \\ T \\ B \end{bmatrix}$$

We now re-write (41) as

$$\hat{q} = \begin{bmatrix} 0 \\ \hat{t} \\ \hat{b} \end{bmatrix} \begin{bmatrix} D_k^{-1}B - D_k^{-1}N'\hat{C} \end{bmatrix} = \begin{bmatrix} 0 \\ c^g q \\ D_k^{-1}B - D_k^{-1}N'c^g q \end{bmatrix} = \begin{bmatrix} 0 \\ c^g T - c^g ND_k^{-1}B \\ -D_k^{-1}N'c^g T + (D_k^{-1}N'c^g ND_k^{-1})B \end{bmatrix}$$
\[
\begin{bmatrix}
\hat{\mathbf{g}} \\
\hat{\mathbf{t}} \\
\hat{\mathbf{b}}
\end{bmatrix} = 
\begin{bmatrix}
\mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{c}_v & \mathbf{c}_v' & \mathbf{c}_b' \\
\mathbf{D}_k^{-1} \mathbf{N}' \mathbf{c}_b & \mathbf{D}_k^{-1} \mathbf{D}_k^{-1} + \mathbf{D}_k^{-1} \mathbf{N}' \mathbf{c}_b' & \mathbf{D}_k^{-1} \mathbf{N}' \mathbf{c}_b'
\end{bmatrix}
\begin{bmatrix}
\mathbf{G} \\
\mathbf{T} \\
\mathbf{B}
\end{bmatrix}.
\]

As a result, we have that a conditional inverse of \( \mathbf{AA}' \), as given in (20) say, is

\[
(\mathbf{AA}')^S = 
\begin{bmatrix}
\mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{c}_v & \mathbf{c}_v' & \mathbf{c}_b' \\
\mathbf{D}_k^{-1} \mathbf{N}' \mathbf{c}_b & \mathbf{D}_k^{-1} \mathbf{D}_k^{-1} + \mathbf{D}_k^{-1} \mathbf{N}' \mathbf{c}_b' & \mathbf{D}_k^{-1} \mathbf{N}' \mathbf{c}_b'
\end{bmatrix}.
\]

Thus, from (44), it follows that the variance of the best linear estimate of an estimable linear function \( \mathbf{h}' \mathbf{t} \) of the treatment effects is given by

\[
\text{Var} (\mathbf{h}' \mathbf{t}) = \mathbf{h}' \mathbf{c}^S \mathbf{h} \sigma^2
\]

while the variance of the best linear estimate of an estimable linear function \( \mathbf{\ell}' \mathbf{b} \) of block effects is given by

\[
\text{Var} (\mathbf{\ell}' \mathbf{b}) = \mathbf{\ell}' \left( \mathbf{D}_k^{-1} + \mathbf{D}_k^{-1} \mathbf{N}' \mathbf{c}_b' \mathbf{D}_k^{-1} \right) \mathbf{\ell} \sigma^2.
\]

**Definition:** In an incomplete blocks design, a linear function \( \mathbf{h}' \mathbf{t} \) of the treatment effects is called a treatment contrast provided \( \mathbf{h}' \mathbf{d}_v = 0 \); similarly, a linear function \( \mathbf{\ell}' \mathbf{b} \) of the block effects is called a block contrast provided \( \mathbf{\ell}' \mathbf{d}_b = 0 \).

**Theorem 1:** If \( \mathbf{h}' \mathbf{t} \) is estimable, then it must be a treatment contrast.

**Proof:** If \( \mathbf{h}' \mathbf{t} \) is estimable, then there exists a row vector \( \mathbf{c}' = \mathbf{c}'(1 \times n) \) such that

\[
\mathbf{c}(\mathbf{a}' \mathbf{y}) = \mathbf{h}' \mathbf{t} \quad \text{or} \quad \mathbf{c}(\mathbf{a}' \mathbf{y}) = \mathbf{h}' \mathbf{t}.
\]
(48) \[ c' t_n + c' H' t + c' L' b = h' t \]

But (48) implies that the following relations hold

(49) \[ c' t_n = 0, \ c' H' = h', \ c' L' = 0' \]

Hence, using (9-a) and (49), we find

(50) \[ h' t' v = c' H' t' v = c' t' n = 0 \]

thus, \( h' t \) must be a treatment contrast.

**Corollary:** If \( A' b \) is estimable, then it must be a block contrast.

The proof may be obtained from that of Theorem 1 by interchanging \( b, \ t, \ H, \) and \( v \) with \( A, \ b, \ L \) and \( b \) respectively.

**Definition:** An incomplete blocks design is said to be connected if for every pair of treatments \((i_1, i_0)\), there exists a chain \((i_1, j_1, i_2, j_2, \ldots i_s, j_s, i_{s+1} = i_0)\) such that \( n_{i_{\gamma}j_{\gamma}} > 0 \) and \( n_{i_{\gamma+1}j_{\gamma}} > 0 \) for \( \gamma = 1, 2, \ldots, s \).

For example, suppose \( v = 4, b = 4, i_1 = 1, i_0 = 3 \), and

\[
N = \begin{bmatrix}
2 & 2 & 0 & 0 \\
0 & 0 & 1 & 2 \\
0 & 0 & 2 & 0 \\
0 & 1 & 0 & 2
\end{bmatrix}
\]

The figure shows the path connecting treatments 1 and 3. One can readily verify that the design is connected. However, if

---

\(^2\) For practical purposes, all incomplete blocks designs are connected because any design which is not connected can be decomposed into disjoint components, each of which is a connected incomplete blocks design.
the design is not connected for there is no chain which links treatments 1 and 3 together; as a result, there is no way of estimating the difference between the effects of the first and third treatments in an unbiased manner. With these examples in mind, we see that in a connected design the treatment and block differences are separable; otherwise, they are confounded. This observation leads to the following theorem.

**Theorem 2:** If an incomplete blocks design is connected, then all treatment contrasts are estimable.

**Proof:** If \( h' t \) is a treatment contrast, then

\[
\begin{align*}
    h'_i v &= \sum_{i=1}^{v} h_i = 0 \text{ or } h_v = -\sum_{i=1}^{v-1} h_i; \text{ hence} \\
    h'_i t &= \sum_{i=1}^{v} h_i t_i = \sum_{i=1}^{v-1} h_i (t_i - t_v).
\end{align*}
\]

Since the design is connected, there exists a chain linking treatments \( i \) and \( v \) for \( i=1, 2, \ldots, v-1 \). As a result, if \( i_1 = i \) and \( i_0 = v \) say, then

\[
\begin{align*}
    c \left( \sum_{\gamma=1}^{g} y_{i_1,\gamma}^{(1)} - \sum_{\gamma=1}^{g} y_{i_2,\gamma+1}^{(1)} \right) = t_i - t_v
\end{align*}
\]

may be written because of the existence of a chain such that \( n_{i,\gamma,\gamma} \geq 1 \) and \( n_{i,\gamma+1,\gamma} \geq 1 \) and the model (5). Thus, \( t_i - t_v \) is estimable for \( i = 1, 2, \ldots, v-1 \); and hence, it follows from (52) that \( h'_i t \) is estimable.

**Theorem 3:** If all treatment contrasts are estimable, then all block contrasts are estimable.
**Proof:** If $\mathbf{A} \mathbf{b}$ is a block contrast, then

$$
(54) \quad \mathbf{A} \mathbf{b} = \sum_{j=1}^{b} \mathbf{A}_j \mathbf{b}_j = \sum_{j=1}^{b-1} \mathbf{A}_j (\mathbf{b}_j - \mathbf{b}) \quad \text{since} \quad \sum_{j=1}^{b} \mathbf{A}_j = 0.
$$

Now there exist $i$ and $i'$ such that $n_{ij} \geq 1$ and $n_{i'b} \geq 1$ for otherwise one of these blocks would be empty; thus from the model (5), we may write

$$
(55) \quad \varepsilon (y_{ij}^{(1)} - y_{i'^{1}b}^{(1)}) = (t_{i} - t_{i'}) + (b_j - b_b). \quad (55)
$$

Since $(t_i - t_{i'})$ is estimable by hypothesis, (55) implies that $(b_j - b_b)$ is estimable for $j=1, 2, \ldots, b-1$. Hence, from (54), it follows that $\mathbf{A} \mathbf{b}$ is estimable. Note that the converse of Theorem 3 also holds, the proof being essentially identical to the one given above.

Combining Theorems 2 and 3 with our preceding results, we may state

**Theorem 4:** For a connected incomplete blocks design

i. if $\mathbf{A}_v = 0$, the best estimate of $\mathbf{b}$ is $\mathbf{A}_b$

ii. if $\mathbf{A}_b = 0$, the best estimate of $\mathbf{b}$ is $\mathbf{A}_v$

where $\mathbf{A}_v$ and $\mathbf{A}_b$ are given by (47).

With the above result in mind, we may now say something about the rank of the matrix $A$ specified by (19).

**Theorem 5:** If an incomplete blocks design is connected, then $\text{Rank} (A) = n_v = b + v - 1$.

$n_e = n - n_v = n - b - v + 1$.

**Proof:** Without loss of generality, we assume $n_{11} > 0$. Then the functions $g + t_1 + b_1, t_1 - t_2, t_1 - t_3, \ldots, t_1 - t_v, b_1 - b_2, b_1 - b_3, \ldots, b_1 - b_b$ are all estimable. As a result, there exists a matrix $U = U[(b+v-1) \times n]$ such that
\[ (56) \quad e(U_{\mathbf{Y}}) = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & -1 \end{bmatrix} \begin{bmatrix} t \\ b \end{bmatrix} = V_{\mathbf{P}} \quad \text{or} \]

\[ (57) \quad e(U_{\mathbf{Y}}) = UA'\mathbf{P} = V_{\mathbf{P}} \quad \text{or} \]

\[ (58) \quad UA' = V. \]

Hence, using the fact that \( \text{Rank} \ (UA') \leq \text{Min} \ [\text{Rank} \ (U), \ \text{Rank} \ (A')] \), we have

\[ (59) \quad n_o = \text{Rank} \ (A) = \text{Rank} \ (A') \geq \text{Rank} \ (UA') = \text{Rank} \ (V) = b + v - 1 \]

since \( V \) has \((b + v - 1)\) independent rows. However

\[ (60) \quad n_o = \text{Rank} \ (A) \leq b + v - 1 \]

because there exists a matrix \( R \) of rank 2 specified by (29) such that

\[ (61) \quad RA = \begin{bmatrix} O_n' \\ O_n' \end{bmatrix}. \]

Combining (59) and (60), we have \( n_o = b + v - 1 \).

We now focus attention on finding the sums of squares corresponding to various sets of linear functions of the observations. First, using (53) and (17), we define a matrix \( K = K(v \times n) \) by the relations

\[ (62) \quad Q = T - ND_k^{-1}B = H_{\mathbf{Y}} - ND_k^{-1}H_{\mathbf{Y}} = K_{\mathbf{Y}} \quad \text{i.e.,} \]
(63) $K = H - ND_k^{-1} L = FA.$

Theorem 6: $KK' = C$ where $C$ is defined by (32).

Proof: From (39), we have

(64) $\varepsilon'(q) = \varepsilon'(Kh) = K H_n' = K H' + KL' D = C t.$

But (64) implies that the following relations hold

(65) $K H_n' = C', K L' = 0_{vb}.$

Hence, we have

(66) $KK' = K (H' - L'D_k^{-1} N') = K H' - K L'D_k^{-1} N' = C.$

Corollary: The variance covariance matrix of $q$ is given by $C \sigma^2.$

Proof: $\text{Var}(q) = \text{Var}(Kh) = K I_n' K' \sigma^2 = C \sigma^2.$

The sum of squares due to $q$ is

(67) $SS(q_1, q_2, \ldots, q_v) = \varepsilon'(K'K) = \varepsilon' (C' G C = q_v \lambda; )$

thus, the sum of squares corresponding to the hypothesis

(68) $H_0: C t = 0_v$

is given by

(69) $SS(H_0: C t = 0) = SS(q_1, q_2, \ldots, q_v) = q_v \lambda.$

Theorem 7: For a connected incomplete blocks design, $C t = 0_v$ if and only if $h'k = 0$ for all $h$ such that $h'd_v = 0$ (i.e., all treatment contrasts vanish).

Proof: Assume every treatment contrast vanishes. Now

(70) $C d_v = (D_r - ND_k^{-1} N') d_v = r - ND_k^{-1} k = r - N d_b = r - r = 0_v.$
i.e., every row of $C$ is the coefficient vector of a treatment contrast. Hence, $C_{1} = 0_{v}$ since every treatment contrast vanishes. On the other hand, suppose $C_{i} = 0_{v}$ in a connected design. Since all treatment contrasts are estimable, $t_{1} - t_{2}, t_{1} - t_{3}, \ldots, t_{1} - t_{v}$ are estimable. Let $\Gamma = \Gamma((v-1) \times v)$ be given by

$$
\begin{bmatrix}
1 & -1 & 0 & \ldots & 0 \\
1 & 0 & -1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \ldots & -1
\end{bmatrix}
$$

(71) \[ \Gamma = \begin{bmatrix}
1 & -1 & 0 & \ldots & 0 \\
1 & 0 & -1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \ldots & -1
\end{bmatrix} \]

then $\Gamma_{t}$ is a set of estimable functions for which the corresponding set of best estimates is $\hat{\theta}^{\Gamma}_{\hat{C}}$ where $\hat{\theta}^{\star} = \hat{C}^{\Gamma}Q$. Now

(72) \[ \Gamma_{t} = \hat{\varepsilon}(\hat{\theta}^{\Gamma}_{\hat{C}}) = \hat{\varepsilon}(\hat{C}^{\Gamma}Q) = \hat{C}^{\Gamma}C_{t} \]
or

(73) \[ \Gamma = \hat{r}C^{\Gamma} \]

Hence, $C_{t} = 0_{v}$ implies $\hat{C}^{\Gamma}C_{t} = 0_{v-1}$ which implies $\Gamma_{t} = 0_{v-1}$ which implies that all treatment contrasts vanish.

**Corollary 1:** For a connected design, Rank $(C) = v-1$.

**Proof:** Since $C_{d_{v}} = 0_{v}$ from (70), Rank $(C) \leq v-1$; also

(74) \[ \text{Rank} (C) \geq \text{Min} \{ \text{Rank} (C), \text{Rank} (C^{\Gamma}), \text{Rank} (\Gamma) \} \geq \text{Rank} (\Gamma) = v-1. \]

Hence, Rank $(C) = v-1$.

At this point, note that if Rank $(C) = v-1$, then all treatment contrasts must be estimable because of (70). Also, it is useful to observe here that

(75) \[ Q^{\prime}d_{v} = T^{\prime}d_{v} - B^{\prime}D_{k}^{-1}N^{\prime}d_{v} = G - B^{\prime}D_{k}^{-1}k = G - B^{\prime}d_{b} = G - G = 0 \]

and that if $\hat{\theta}$ is a solution to (34), then for any $\theta$, $\hat{\theta} + \theta_{d_{v}}$ is also a solution since

(76) \[ C(\hat{\theta} + \theta_{d_{v}}) = \hat{\theta}^{\Gamma} + \theta C_{d_{v}} = \hat{\theta}^{\Gamma} = \hat{\theta}. \]
Corollary 2: If $S^2_t$ denotes the sum of squares for testing the hypothesis $t_1 = t_2 = \ldots = t_v$, then in a connected incomplete blocks design $S^2_t = Q^\Lambda_t$ with $(v-1)$ degrees of freedom; we call $S^2_t$ the adjusted treatment sum of squares.

Proof: $S^2_t = SS(H_0: t_1 = t_2 = \ldots = t_v) = SS(H_0: C_t = 0_v) = Q^\Lambda_t$ because of (69) and Theorem 7.

From here on, assume that the incomplete blocks design is connected unless the contrary is stated. As a result, we also have that

\begin{equation}
S^2_t = S^2_{ct} - S^2_e \tag{77}
\end{equation}

where $S^2_e$ is the sum of squares due to error associated with model (11) and $S^2_{ct}$ is the sum of squares due to conditional error associated with the hypothesis that all treatment contrasts vanish. But when this hypothesis is true, $t_1 = t_2 = \ldots = t_v$ and the model (11) may be re-written as

\begin{equation}
\mathbf{y} = L' \mathbf{b} + \mathbf{e} \tag{78}
\end{equation}

where, as before, $\mathbf{e}$ satisfies (12). The sum of squares due to error corresponding to the model (78) is

\begin{equation}
S^2_{ct} = \sum_{u=1}^{n} y^2_u - \sum_{j=1}^{b} \frac{1}{k_j} B^2_j = \mathbf{y}' \mathbf{Y} - B' D_k^{-1} B \tag{79}
\end{equation}

Thus, from (77), (79), and Corollary 2, we obtain

\begin{equation}
S^2_e = S^2_{ct} - S^2_t = \mathbf{y}' \mathbf{Y} - B' D_k^{-1} B - Q^\Lambda_t \tag{80}
\end{equation}

from Theorem 5, we recall that the number of degrees of freedom belonging to error is $n_e = n - v - b + 1$.

Proceeding in the opposite direction, we find the sum of squares $S^2_b$ for testing the hypothesis that all block contrasts vanish from the relation
(81) \[ s_{b}^2 = s_{cb}^2 - s_{e}^2 \]

where \( s_{cb}^2 \) is the sum of squares due to conditional error. But when this hypothesis is true, \( b_1 = b_2 = \ldots = b_b \) and the model (11) may be re-written as

(82) \[ y = H't + e \]

where, as before, \( e \) satisfies (12). The sum of squares due to error corresponding to the model (82) is

(83) \[ s_{cb}^2 = \sum_{u=1}^{n} y_u^2 - \sum_{i=1}^{v} \frac{1}{r_i} \tau_i = y'y - T'D_r^{-1}T \cdot \]

Thus, from (80), (81), and (83), we obtain

(84) \[ s_{b}^2 = s_{cb}^2 - s_{e}^2 = Z'D_k^{-1}Z - T'D_r^{-1}T + \hat{\theta}'\hat{\theta} \cdot \]

Since the design is connected, all block contrasts are estimable and hence the number of degrees of freedom corresponding to \( s_{b}^2 \) is \( (b-1) \).

If the \( e \) in the model specified by (11) have the multivariate normal distribution \( N(0, I_n\sigma^2) \), then from the normal theory of linear regression, it follows that the statistics

(85) \[ F_{v-1, n_e} = \frac{s_{e}^2/(v-1)}{s_{e}^2/n_e} \]

\[ F_{b-1, n_e} = \frac{s_{b}^2/(b-1)}{s_{e}^2/n_e} \]

have the \( F \)-distributions \( F(v-1, n_e) \) and \( F(b-1, n_e) \) respectively. Also, either from the general theory of the linear model or from substitution, we find

(86-a) \[ C(s_{t}^2) = C(\hat{\theta}'\hat{\theta}) = C(\hat{\theta}'C\hat{\theta}) = C(\hat{\theta}'KK\hat{\theta}) = (v-1)\sigma^2 + \hat{t}'C\hat{t} \]

(86-b) \[ C(s_{e}^2) = n_e\sigma^2 \]

(86-c) \[ C(s_{b}^2) = (b-1)\sigma^2 + b'(D_k - N'D_r^{-1}N)b = (b-1)\sigma^2 + b'\sigma b \]

Thus, we may form the following analysis of variance.
<table>
<thead>
<tr>
<th>Source of Variation</th>
<th>Degrees of Freedom</th>
<th>Sum of Squares</th>
<th>Mean Square</th>
<th>Expected Mean Square</th>
<th>Variance Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>1</td>
<td>$\frac{g^2}{n}$</td>
<td>$\frac{G^2}{n}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Blocks unadjusted</td>
<td>b-1</td>
<td>$S_b^2 = B'B_k^{-1}B_n - \frac{G^2}{n}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Treatments adjusted</td>
<td>v-1</td>
<td>$S_t^2$</td>
<td>$\frac{S_t^2}{v-1}$</td>
<td>$\sigma^2 + \frac{(t'Ct)}{v-1}$</td>
<td>$\frac{s_t^2}{s_e^2}$</td>
</tr>
<tr>
<td>Treatments unadjusted</td>
<td>v-1</td>
<td>$S_t^2 = T'T_n^{-1}T_k - \frac{G^2}{n}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Blocks adjusted</td>
<td>b-1</td>
<td>$S_b^2$</td>
<td>$\frac{S_b^2}{b-1}$</td>
<td>$\sigma^2 + \frac{(b'C_b,b)}{b-1}$</td>
<td>$\frac{s_b^2}{s_e^2}$</td>
</tr>
<tr>
<td>Error</td>
<td>n_e</td>
<td>$S_e^2$</td>
<td>$\frac{S_e^2}{n_e}$</td>
<td>$\sigma^2$</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>n</td>
<td>$\sum_{u=1}^{n} y_u^2$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Let us now review the preceding results by considering the special case of a balanced incomplete blocks (BIB) design.

**Definition:** An incomplete blocks design is said to be balanced if the following conditions are satisfied

1. each block contains \( k \) distinct treatments,
2. each treatment occurs in \( r \) distinct blocks,
3. each pair of distinct treatments occurs together in \( \lambda \) different blocks.\(^3\)

In a BIB design, the matrix \( N \) reduces to an array of 0's and 1's with

\[
N_{ij} = \begin{cases} 
1 & \text{if the } i\text{-th treatment occurs in the } j\text{-th block} \\
0 & \text{otherwise}
\end{cases}
\]

Also, since all the treatments are replicated equally often and all the block sizes are equal, we may write

\[
\pi = rI_v, \quad k = k_1, \quad D_r = rI_v, \quad D_k = kI_b
\]

Substituting (86) into the expressions for \( C \) and \( Q \) given in (32) and (33), we obtain

\[
(89) \quad C = rI_v - \frac{1}{k} NN', \quad Q = T - \frac{1}{k} NB ;
\]

hence the elements of \( C \) and \( Q \) have the forms

\[
(90) \quad c_{ii'} = r s_{ii'} - \frac{1}{k} \sum_{j=1}^{b} n_{ij} n_{i'j} \quad \text{where } s_{ii'} = \begin{cases} 
1 & \text{if } i=i' \\
0 & \text{if } i \neq i'
\end{cases}
\]

\[
(91) \quad Q_i = T_1 - \frac{1}{k} \sum_{j=1}^{b} n_{ij} \bar{B}_j = T_1 - \sum_{j=1}^{b} n_{ij} \bar{B}_j
\]

where \( \bar{B}_j \) is the \( j \)-th block average. The quantity

\[
(92) \quad \sum_{j=1}^{b} n_{ij} \bar{B}_j = T_1 - Q_i
\]

\(^3\)Note that this condition implies that all BIB designs are connected.
is now simply the sum of the block averages for the blocks in which the i-th treatment appears because of (87). Similarly, since

\[(93) \quad n_{ij} n_{i'j} = \begin{cases} 1 & \text{if both the i-th and i'}-\text{th treatments occur in the j-th block}, \\ 0 & \text{otherwise} \end{cases} \]

the quantity

\[(94) \quad \frac{1}{k} \sum_{j=1}^{b} n_{ij} n_{i'j} = r s_{i} - c_{i'i'} = \frac{1}{k} \lambda_{ii'} \]

where \(\lambda_{ii'}\) represents the number of blocks in which the i-th and i'-th treatments occur together in the same block. For a BIB design, we have by definition

\[(95) \quad \lambda_{ii'} = \begin{cases} r & \text{if } i = i' \\ \lambda & \text{if } i \neq i' \end{cases} \]

hence, the matrix \(C\) given in (99) simplifies to

\[(96) \quad C = r I_v - \frac{1}{k} [(r - \lambda) I_v + \lambda J_v] = (r - \frac{r - \lambda}{k}) I_v - \frac{\lambda}{k} J_v \]

where \(J_v\) is a \((v \times v)\) matrix of ones. From (70), recall that \(C_{i} J_v = 0\); thus the following relations must hold in a BIB design

\[(97) \quad (r - \frac{r - \lambda}{k}) = \frac{\lambda v}{k} \quad \text{or} \quad r(k - 1) + \lambda = \lambda v \quad \text{or} \quad r(k - 1) = \lambda(v - 1) \]

The equations (97) could also have been deduced by combinatorial considerations since both \(r(k - 1)\) and \(\lambda(v - 1)\) represent the total number of times other treatments occur in the same blocks as some specified treatment does. Substituting (97) into (96), we obtain

\[(98) \quad C = \frac{\lambda v}{k} (I_v - \frac{1}{v} J_v) \]
The adjusted normal equations (34) now reduce to

\[ \frac{\lambda V}{k} (I_v - \frac{1}{V} J_v) t = Q \quad \text{or} \]

\[ t = \frac{k}{\lambda V} Q + \frac{1}{V} J_v t \]

If we use the non-estimable restriction \( J_v t = 0 \) in (99), we may write

\[ (100) \quad \hat{t} = \frac{k}{\lambda V} Q \]

From (100), it follows that a conditional inverse of (98) is

\[ (101) \quad C^G = \frac{k}{\lambda V} I_v \]

By appropriate substitution into (45), the variance of the estimated difference between two treatment effects is found to be

\[ (102) \quad \text{Var}(\hat{t}_i - \hat{t}_{i'},) = (c_{ii}^G - c_{ii'}^G, c_{i'i}^G + c_{i'i'}^G) \sigma^2 = 2\left(\frac{k}{\lambda V}\right)\sigma^2. \]

If we define the constant \( E \) by

\[ (103) \quad E = \frac{\lambda V}{r k} , \]

then (102) may be re-written as

\[ (104) \quad \text{Var}(\hat{t}_i - \hat{t}_{i'},) = \frac{1}{E} \left( \frac{2\sigma^2}{x} \right) . \]

The constant \( E \) is called the efficiency factor of a BID design relative to a randomized blocks (RB) design in which every treatment occurrence in each of \( r \) blocks.

**Theorem 9:** \( E \leq 1 \)

**Proof:** \( E - 1 = \frac{\lambda V}{r k} - 1 = \frac{\lambda V - rk}{r k} \)

\[ = \frac{\lambda - r}{r k} \quad \text{because of (97)} \]

\[ \leq 0 \quad \text{because} \quad \lambda \leq r. \]
Theorem 8 does not, however, imply that a BIB design is poorer than the corresponding RB design because the error variances associated with the experiments are generally not the same. In fact because the blocks in BIB designs are smaller and more homogeneous, the associated error variance is likely to be more or less smaller than that associated with the RB design. Thus, for any given situation, a comparison of the two designs depends on the general homogeneity of the experimental units.

Finally, we note that for a BIB design, the adjusted treatment sum of squares and the error sum of squares have the forms

\[(105) \quad S_t^2 = \frac{1}{k} Q = \frac{k}{N} Q'Q = \frac{1}{rE} Q'Q \quad \text{and} \]

\[(106) \quad S_e^2 = E'VE - \frac{1}{k} B'EB - \frac{1}{rE} Q'Q \quad ; \]

also, we have

\[(107-a) \quad \epsilon(S_t^2) = (v-1)\sigma^2 + \frac{1}{v} \Sigma t_i = (v-1)\sigma^2 + \frac{v}{rE} \sum_{i=1}^{V} \left( t_i - \frac{1}{v} \right)^2 \]

\[(107-b) \quad \epsilon(S_e^2) = n_e \sigma^2 \]

With the above modifications, the same analysis of variance as given in Table I may now be formed.
Previously, we have assumed that the block effects \( \{b_j\} \) in the model (5) were fixed constants.

However, if either

i. the blocks in the experiment are selected at random from a large population of blocks or

ii. sets of treatments which are to represent blocks are assigned at random to the blocks to be used in the experiment,

then the block effects \( \{b_j\} \) may be regarded as uncorrelated random variables with expected values equal to zero and variances equal to a constant \( \sigma_b^2 \); also the \( \{b_j\} \) and the \( \{e_u\} \) may be regarded as uncorrelated sets of random variables. We will now consider two procedures by which this additional information can be used to improve on the estimates, given in Theorem 4, for certain classes of treatment contrasts. They are

i. the recovery of inter-block information and

ii. the generalized least squares.

For the sake of convenience, we shall suppose the observations in the vector (10) to be so arranged that the first \( k_1 \) elements come from the first block; the next \( k_2 \) elements come from the second block; etc. The model (11) can now be modified to

\[
\varepsilon(y) = \mathbf{a} + \mathbf{H} b, \\
\tag{108}
\]

\[
\text{Var} (y) = \varepsilon((y - \varepsilon(y))(y - \varepsilon(y))') = I_n \sigma^2 + L'L \sigma_b^2 = \Sigma.
\]

First let us observe that with respect to the model (108), the vectors \( \mathbf{Q} \) and \( \mathbf{B} \), as defined by (62) and (16) respectively, are uncorrelated since
(109) \[ \text{Cov}(Q, B) = K \Sigma L' \]
\[ = (H - ND_k^{-1} L)(I_n \sigma^2 + L' L \sigma_b^2) L' \]
\[ = (H - ND_k^{-1} L)(L') (I_n \sigma^2 + D_k \sigma_b^2) \]
\[ = (H - ND_k^{-1} D_k) (\sigma^2 I_n + D_k \sigma_b^2) \]
\[ = 0_{vb} \]

on the use of (24) and (27); in fact, Q and B are also orthogonal since

(110) \[ KL' = (H - ND_k^{-1} L) L' = N - ND_k^{-1} D_k = 0_{vb}. \]

Thus, from (109) and (110) we see that Q is a vector of linear functions of within-block contrasts each of which is uncorrelated with the vector B of block totals. Thus, we call the estimates of treatment contrasts, given in Theorem 4, intra-block estimates because they are linear functions of the Q_1.

Previously, block totals were not useful in the estimation of treatment contrasts because their expected values involved the fixed unknown block effects. However, with respect to the model (108), we observe that

\[ \epsilon(B) = \epsilon(L Y) = L_4 g + LH't = kg + N't \]

(111)

\[ \text{Var}(B) = LL' \sigma^2 + L L' \sigma_b^2 = D_k \sigma^2 + D_k \sigma_b^2 \]

we call (111) the inter-block model. If the method of weighted least squares is applied to the model (111), then best estimates of linear functions which are estimable with respect to it may be constructed; these are called inter-block
estimates. The weighted least squares (normal) equations of estimation associated with the model (111) are

\[(112) \begin{bmatrix} k' \begin{bmatrix} D_k^{-1} D_w \end{bmatrix} [X \ N'] \begin{bmatrix} e \\ t \end{bmatrix} = \begin{bmatrix} k' \\ t \end{bmatrix} D_k^{-1} D_w \begin{bmatrix} \tilde{w} \\ \begin{bmatrix} \tilde{w'}_{\text{k}} \end{bmatrix} \end{bmatrix} \begin{bmatrix} L N \end{bmatrix} \end{bmatrix} \]

where \( D_w \) is a diagonal matrix whose diagonal elements are weights \( \tilde{w}_j \) defined by

\[(113) \tilde{w}_j = \frac{1}{\sigma^2 + k_j \sigma^2} \quad j = 1, 2, \ldots, b.\]

More exactly, the equations (112) are

\[(114) \begin{bmatrix} D_w^{-1} D_w [X \ N'] \begin{bmatrix} e \\ t \end{bmatrix} = \begin{bmatrix} D_w^{-1} D_w \begin{bmatrix} \tilde{w} \\ \begin{bmatrix} \tilde{w'}_{\text{k}} \end{bmatrix} \end{bmatrix} \begin{bmatrix} L N \end{bmatrix} \end{bmatrix} \]

Eliminating \( e \) from (114), we obtain the adjusted equations

\[(115) \begin{bmatrix} N D_k^{-1} D_w [X \ N'] \begin{bmatrix} \tilde{w} \\ \begin{bmatrix} \tilde{w'}_{\text{k}} \end{bmatrix} \end{bmatrix} \begin{bmatrix} L N \end{bmatrix} \end{bmatrix} \begin{bmatrix} t \end{bmatrix} = N D_k^{-1} D_w \begin{bmatrix} \tilde{w} \\ \begin{bmatrix} \tilde{w'}_{\text{k}} \end{bmatrix} \end{bmatrix} \begin{bmatrix} L N \end{bmatrix} \]

or

\[(116) \begin{bmatrix} N D_k^{-1} D_w [X \ N'] \begin{bmatrix} \tilde{w} \\ \begin{bmatrix} \tilde{w'}_{\text{k}} \end{bmatrix} \end{bmatrix} \begin{bmatrix} L N \end{bmatrix} \end{bmatrix} \begin{bmatrix} t \end{bmatrix} = N D_k^{-1} D_w \begin{bmatrix} \tilde{w} \\ \begin{bmatrix} \tilde{w'}_{\text{k}} \end{bmatrix} \end{bmatrix} \begin{bmatrix} L N \end{bmatrix} \]

or

\[(117) N[D_k^{-1} D_w - \begin{bmatrix} \tilde{w} \\ \begin{bmatrix} \tilde{w'}_{\text{k}} \end{bmatrix} \end{bmatrix} \begin{bmatrix} L N \end{bmatrix} \end{bmatrix} \begin{bmatrix} t \end{bmatrix} = N[D_k^{-1} D_w - \begin{bmatrix} \tilde{w} \\ \begin{bmatrix} \tilde{w'}_{\text{k}} \end{bmatrix} \end{bmatrix} \begin{bmatrix} L N \end{bmatrix} \]

where \( \tilde{w} \) is a column vector whose elements are the \( \tilde{w}_j \) defined in (113). Let

\[(118) M = N[D_k^{-1} D_w - \begin{bmatrix} \tilde{w} \\ \begin{bmatrix} \tilde{w'}_{\text{k}} \end{bmatrix} \end{bmatrix} \begin{bmatrix} L N \end{bmatrix} \]

and
\begin{equation}
(119) \quad m = N [ D^{-1}_k \frac{\bar{W}}{W} - \frac{\bar{W}}{W} \frac{\bar{W}}{W} ] \Lambda W \ ;
\end{equation}

then the adjusted inter-block equations may be written

\begin{equation}
(120) \quad M_b = m \ .
\end{equation}

A solution to (120) is given by

\begin{equation}
(121) \quad \hat{\theta} = M^g m
\end{equation}

where $M^g$ is a conditional inverse of $M$. Hence, if a linear function $h' \tau$ of treatment effects is estimable with respect to the model (111), then

\begin{equation}
(122) \quad \text{best inter-block estimate of } h' \tau = h' \hat{\tau}
\end{equation}

where $\hat{\tau}$ is given by (121). In addition, the variance of the best inter-block estimate of estimable $h' \tau$ is given by

\begin{equation}
(123) \quad \text{Var}(h' \hat{\tau}) = h' M^g h \ .
\end{equation}

The question of which linear functions of treatment effects are estimable with respect to the inter-block model now arises. We now consider several theorems which provide a partial answer.

\textbf{Theorem 8:} If $h' \tau$ is estimable with respect to the model (111), then it must be a treatment contrast.

\textbf{Proof:} If $h' \tau$ is estimable with respect to the model (111), then there exists a row vector $c' = c'(1 \times b)$ such that

\begin{equation}
(124) \quad c(c'B) = h' \tau \text{ or}
\end{equation}
(125) \( c' \mathbf{kg} + c' \mathbf{N'} t = h' t \).

But (125) implies that

(126) \( c' k = 0, \ c' \mathbf{N'} = h' \).

Hence, using (4-b) and (126), we find

(127) \( h' \mathbf{d}_v = c' \mathbf{N'} \mathbf{d}_v = c' k = 0 \);

thus \( h' t \) must be a treatment contrast.

Theorem 2: All treatment contrasts are estimable with respect to the model (111) if and only if \( \text{Rank} \ (\mathbf{N}) = v \).

Proof: Let \( h' t \) be any treatment contrast. If \( \text{Rank} \ (\mathbf{N}) = v \), then there exists \( c' = c'(1 \times \mathbf{b}) \) such that

(128) \( c' \mathbf{N'} = h' \).

But \( h' t \) is a contrast and thus

(129) \( 0 = h' \mathbf{d}_v = c' \mathbf{N'} \mathbf{d}_v = c' k \);

combining (128) and (129), we obtain

(130) \( \varepsilon[c' \mathbf{B}] = c' \mathbf{kg} + c' \mathbf{N'} t = h' t \);

hence \( h' t \) is estimable.

On the other hand, suppose that all treatment contrasts are estimable. Then \( \mathbf{r}_t \), where \( \mathbf{r} \) is given by (71), is a set of estimable functions and hence there exists a matrix \( \mathbf{U} = \mathbf{U}(v-1) \times \mathbf{b} \) such that
(131) \( \varepsilon(U B) = U k g + U N' t = \Gamma t \).

Again, (125) implies

(132) \( U k = 0 \_{v-1} \), \( U N' = \Gamma \).

In addition

(133) \( A' N' = r' \).

Since

(134) \[
\begin{bmatrix}
A' \\
B' \\
U'
\end{bmatrix}
\]
\[
N' = \begin{bmatrix}
r' \\
r
\end{bmatrix}
\]

we have that

(135) \( v \geq \text{Rank} (N) \geq \text{Rank} \begin{bmatrix}
A' \\
B' \\
U'
\end{bmatrix} N' = \text{Rank} \begin{bmatrix}
r' \\
r
\end{bmatrix} = v \)

since \( \Gamma \) is a matrix of rank \((v-1)\) such that \( \Gamma _{1v} = 0 \) and \( r' _{1v} = n \).

\underline{Theorem 10:} If \( \text{Rank} (N) = \tilde{n}_0 \), then at most \((\tilde{n}_0 - 1)\) independent treatment contrasts are estimable with respect to the model (111).

\underline{Proof:} Since \( \text{Rank} (N) = \tilde{n}_0 \), at most \( \tilde{n}_0 \) independent linear combinations of the columns of \( N \) can be formed. Because one of these is given by (133) where \( r' \) is independent of the coefficient vector of any contrast, it follows that at most \((\tilde{n}_0 - 1)\) independent linear combinations of the columns of \( N \) which would represent coefficient vectors of contrasts could be formed.

\underline{Corollary:} If \( b \leq v \), then at most \((b-1)\) independent treatment contrasts are estimable with respect to the model (111).
The corollary is of interest because there are a number of connected designs, namely certain partially balanced incomplete block designs (PBIB designs), in which \( b \leq v \).

Thus, as the preceding results indicate, the existence of an inter-block estimate of any given treatment contrast depends on the properties of the experimental design and in particular on the rank of the incidence matrix \( N \).

We now investigate some of the properties of \( M \) and \( m \). First we note that

\[
(136) \quad \left[ D_k^{-1} \begin{array}{c} \tilde{w} \\ \tilde{w}' \end{array} \right] N' \begin{array}{c} \tilde{w} \\ \tilde{w}' \end{array} = \left[ D_k^{-1} \begin{array}{c} \tilde{w} \\ \tilde{w}' \end{array} \right] k
\]

\[
= [D_k^{-1} \begin{array}{c} k \\ \tilde{w} \end{array}]
\]

\[
= [D_k^{-1} \begin{array}{c} 1 \\ \tilde{w} \end{array}]
\]

\[
= 0_b
\]

From (136), it follows that

\[
(137) \quad M^i v = 0_v \quad \text{and}
\]

\[
(138) \quad m^i v = 0
\]

Thus, \( M \) is a matrix of coefficient vectors of treatment contrasts.

Hence,

\[
(139) \quad \text{Rank}(M) = \text{number of independent treatment contrasts estimable in inter-block model} \leq v - 1
\]

because (121) and (122) imply that the inter-block estimate of any treatment contrast, which is estimable with respect to (111), is a linear combination of the estimates of \( M \), namely \( m \). Furthermore, since
(140) \[ e(m) = N \left[ D_k^{-1} D_w^{-1} - \frac{\bar{w}}{\bar{w}' k} \right] e(1_k) \]

\[ = N \left[ D_k^{-1} D_w^{-1} - \frac{\bar{w}}{\bar{w}' k} \right] \left[ k g + N' t \right] \]

\[ = M t \]

each element in the vector \( M t \) is estimable with respect to the model (111) and its interblock estimate is the corresponding element of \( m \). The variance-covariance matrix for the elements of \( m \) is given by

(141) \[ \text{Var}(m) = N \left[ D_k^{-1} D_w^{-1} - \frac{\bar{w}}{\bar{w}' k} \right] \left[ D_k \sigma^2 + D_k \sigma^2 b \right] \left[ D_k^{-1} D_w^{-1} - \frac{\bar{w} \bar{w}'}{\bar{w}' k} \right] N' \]

\[ = N \left[ D_k^{-1} D_w^{-1} - \frac{\bar{w}}{\bar{w}' k} \right] \left[ D_k D_w^{-1} \right] \left[ D_k^{-1} D_w^{-1} - \frac{\bar{w} \bar{w}'}{\bar{w}' k} \right] N' \]

\[ = N \left[ \bar{w} - \frac{\bar{w}'}{k} \right] \left[ D_k^{-1} D_w^{-1} - \frac{\bar{w} \bar{w}'}{\bar{w}' k} \right] N' \]

\[ = N \left[ (D_k^{-1} D_w^{-1} - \frac{\bar{w} \bar{w}'}{\bar{w}' k}) - \frac{1}{\bar{w}' k} (\bar{w} \bar{w} - \bar{w} \bar{w}') \right] N' \]

\[ = N \left[ D_k^{-1} D_w^{-1} - \frac{\bar{w} \bar{w}'}{\bar{w}' k} \right] N' \]

\[ = M \]

In addition, since the elements of \( m \) are linear functions of the elements of \( B \), we have
(142) \[
\text{Cov} (\mathbf{q}, \mathbf{m}) = 0_{vv}
\]

in the model (108). Finally, the variance-covariance matrix of the elements of \( \mathbf{q} \) with respect to the model (108) is found to be

(143) \[
\text{Var} (\mathbf{q}) = K \Sigma K'
\]

\[
= K[I_n \sigma^2 + L'L \sigma_b^2]K'
\]

\[
= KK' \sigma^2
\]

\[= \Sigma \sigma^2
\]

by use of (110) and Theorem 6. Thus, we may write

(144) \[
\text{Var} \begin{bmatrix} \mathbf{q} \\ \mathbf{m} \end{bmatrix} = \begin{bmatrix} \Sigma \sigma^2 & 0_{vv} \\ 0_{vv} & M \end{bmatrix}
\]

With the preceding results in mind, we now observe that if a treatment contrast is estimable in the inter-block model, then its intra-block estimate and its inter-block estimate can be combined according to the inverses of their variances to form a new estimate for which the variance is at least as small as the variances of either of the separate estimates. This method of improving on the intra-block estimate is known as the recovery of inter-block information. Although this procedure is not a difficult one theoretically, it can become computationally difficult if \( C \) and \( M \) do not have simple forms.

So far we have avoided the question of what is to be done if \( \sigma_b^2 \) and \( \sigma^2 \) are unknown. Since the \( \tilde{w}_j \) play an important role in the construction of the inter-block estimates, it is necessary that we be able to estimate \( \sigma_b^2 \) and \( \sigma^2 \) and hence the \( \tilde{w}_j \) from the data. This can be achieved by finding the expected values of \( S_b^2 \) and \( S_e^2 \) with respect to the model (108), and then estimating \( \sigma_b^2 \) and \( \sigma^2 \) by
the appropriate solution to the equations obtained by setting $S_b^2$ and $S_e^2$ equal to their expected values. As before

\[(145)\] \[c(S_e^2) = n_e \sigma^2\]

since $S_e^2$ is a function of the $\{e_u\}$ only and thus is free of block effects. Instead of using (84) to define $S_b^2$, we reverse the roles of $b$ and $t$ in (1) - (107) and write

\[(146)\] \[S_b^2 = \hat{b}^* \cdot Q_b\]

where $\hat{b}^*$ is a solution to the equations

\[(147)\] \[C_b \cdot b^* = Q_b\]

with

\[(148)\] \[C_b = D_k - N' D_r^{-1} N\]

and

\[(149)\] \[Q_b = B - N' D_r^{-1} T\]

\[= (L - N' D_r^{-1} H)X\]

\[= K_b X\]

With respect to (108),

\[(150)\] \[c(Q_b) = (L - N' D_r^{-1} H)(\Delta_h g + H't)\]

\[= (k g + N't - N' D_r^{-1} r g - N't)\]

\[= (k g - N' \Delta_r g)\]

\[= Q_b\]

32
and
\begin{equation}
\text{Var}(\mathcal{Q}_b) = (L - N' D_r^{-1} H)(I_n \sigma_b^2 + L' L \sigma_b^2)(L' - H' D_r^{-1} N) \\
= (D_k - N' D_r^{-1} N - N' D_r^{-1} N + N' D_r^{-1} N) \sigma_b^2 + (D_k - N' D_r^{-1} N) \sigma_b^2 \\
= (D_k - N' D_r^{-1} N) \sigma_b^2 + (D_k - N' D_r^{-1} N) \sigma_b^2 \\
= c_b \sigma_b^2 + c_b^2 \sigma_b^2.
\end{equation}

Hence, we have
\begin{equation}
\mathcal{E} \{ \mathcal{S}_b^2 \} = \mathcal{E} \{ \mathcal{Q}_b^2 \} = \mathcal{E} \{ c_b^2 \mathcal{Q}_b \} \\
= \mathcal{E} (\text{tr}(c_b^2 \mathcal{Q}_b \mathcal{Q}_b)) \\
= \text{tr}(c_b^2 \mathcal{E}(\mathcal{Q}_b \mathcal{Q}_b)) \\
= \text{tr}(c_b^2 c_b \sigma_b^2) + \text{tr}(c_b^2 c_b \sigma_b^2) \\
= \text{tr}(K_b' (K_b')^2 K_b \sigma_b^2) + \{ \text{tr} c_b \} \sigma_b^2,
\end{equation}

since $K_b' (K_b')^2 K_b$ is idempotent and has rank $(b-1)$, (152) may be re-written in the form
\begin{equation}
\mathcal{E} \{ \mathcal{S}_b^2 \} = (b-1) \sigma_b^2 + \text{tr}(D_k - N' D_r^{-1} N) \sigma_b^2 \\
= (b-1) \sigma_b^2 + \{ n - \text{tr}(D_r^{-1} N' N) \} \sigma_b^2.
\end{equation}

If the elements of $N$ are either 0's or 1's with
\begin{equation}
n_{ij} = \begin{cases} 
1 & \text{if the } i\text{-th treatment occurs in the } j\text{-th block} \\
0 & \text{otherwise}
\end{cases},
\end{equation}

then the diagonal elements of $NN'$ are the corresponding elements of $\mathbf{r}$. Hence,
for this special case

\[(155) \quad \text{tr}(D_x^{-1} N N') = \text{tr}(I_v) = v\]

and (153) simplifies to

\[(156) \quad \varepsilon(s_b^2) = (b - 1)s^2 + (n - v)s_b^2.\]

In any event, from (145) and (153) we may form the following estimates of \( \sigma^2 \) and \( \sigma_b^2 \)

\[(157) \quad \hat{\sigma}^2 = \frac{1}{n_e} s_e^2 = s_e^2\]

\[(158) \quad \hat{\sigma}_b^2 = \left\{ \frac{b - 1}{n - \text{tr}(D_x^{-1} N N')} \right\} \left\{ \frac{s_b^2}{b - 1} - s_e^2 \right\} = \frac{(b - 1)(s_b^2 - s_e^2)}{n - \text{tr}(D_x^{-1} N N')} .\]

If (157) and (158) are substituted into (113), the estimates of the \( \tilde{w}_j \) may be formed. With this done, the inter-block estimates and their estimated variances can be obtained.

It is worthwhile to note, at this time, that if all the blocks are of the same size \( k \) (i.e.,

\[(159) \quad k = k_i\]

then a number of important simplifications appear in the inter-block analysis. First of all, the model (111) becomes

\[(160) \quad \varepsilon(g) = \varepsilon(Iy) = k_i g + N't\]

\[\text{Var}(g) = (k \sigma^2 + k^2 \sigma_b^2) I_b\]

As a result, we may apply the ordinary least squares procedure. If we let
(161)  \( \tilde{\omega} = \frac{1}{\sigma^2 + k\sigma_b^2} \)

denote the common value of the \( \tilde{\omega}_j \) in (113), then \( M \) and \( m \) may be written

\[
M = N \left[ \frac{\tilde{\omega}}{k} I_b - \frac{\tilde{\omega}^2}{k} J_b \right] N' = \tilde{\omega} \left[ \frac{1}{k} N N' - \frac{r r'}{n} \right]
\]

\[m = N \left[ \frac{\tilde{\omega}}{k} I_b - \frac{\tilde{\omega}^2}{k} J_b \right] \lambda_Y = \tilde{\omega} \left[ \frac{1}{k} N B - \frac{r}{n} G \right].\]

Now define \( \tilde{C} = \tilde{\omega}(v \times v) \) and \( \tilde{G} = \tilde{\omega}(v \times 1) \) by

\[
(163) \quad \tilde{C} = \frac{1}{\tilde{\omega}} M = \frac{1}{k} N N' - \frac{r r'}{n}, \quad \tilde{G} = \frac{1}{\tilde{\omega}} m = \frac{1}{k} N B - \frac{r}{n} G;^4
\]

then the adjusted inter-block equations may be written

\[
(164) \quad \tilde{C} \tilde{t} = \tilde{G}
\]

and solved similarly to (120).

With regard to estimability in the inter-block model, the same remarks as contained in Theorems 8, 9, and 10 apply equally well here. Similarly, it follows from (139) that

\[
(165) \quad \text{Rank} \tilde{C} = \text{number of independent treatment contrasts} \leq v - 1 \quad \text{estimable in inter-block model}
\]

Finally, (140)-(144) may be modified to

\[
(166) \quad \mathcal{E}(\tilde{G}) = \tilde{C} \tilde{t} \quad \text{and}
\]

---

^4 Note that the \( \tilde{G}_i \) represent a weighted sum of the excesses which the blocks containing the \( i \)-th treatment have over the general average, where the weight corresponding to the \( j \)-th block is the number \( n_{ij} \) of occurrences of the \( i \)-th treatment in the \( j \)-th block.
\[
(167) \quad \text{var} \begin{bmatrix}
\bar{\rho} \\
\bar{\sigma}
\end{bmatrix} = \begin{bmatrix}
\sigma^2 & 0 \\
0 & \sigma^2 + k \sigma_b^2
\end{bmatrix}.
\]

One should note that we now no longer need estimates of \(\sigma_b^2\) and \(\sigma^2\) to obtain the inter-block estimates. However, these estimates are needed to estimate the variances of the intra-block estimates and the inter-block estimates before the combined estimate in the recovery of inter-block information can be formed. Thus, (157) and (158) still are useful.
Alternatively, we can improve on the intra-block estimates if we apply the generalized least squares procedure to the model (108). This approach will be seen to be more desirable because it leads to best estimates of all treatment contrasts with a minimum of computations. From the assumed arrangement of \( \mathbf{Y} \) it follows that \( \Sigma = \Sigma(n \times n) \) may be written as

\[
\Sigma = \begin{bmatrix}
W_1 & 0_{k_1k_2} & \cdots & 0_{k_1k_b} \\
0_{k_2k_1} & W_2 & \cdots & 0_{k_2k_b} \\
\cdots & \cdots & \cdots & \cdots \\
0_{k_bk_1} & 0_{k_bk_2} & \cdots & W_b
\end{bmatrix}
\]

(168)

where \( W_j = W_j(k_j \times k_j) \) is given by

\[
W_j = I_{k_j} \sigma^2 + J_{k_j} \sigma_b^2 
\]

(169)

\[ j = 1, 2, \ldots, b \]

since

\[
\sum_{j=1}^{b} f_{ju} f_{ju'} = \begin{cases} 
1 & \text{if the } u\text{-th and } u'\text{'th observations are both in same block.} \\
0 & \text{otherwise}
\end{cases}
\]

If we suppose that \( W_j^{-1} \) has the form

\[
W_j^{-1} = \omega I_{k_j} + \beta_j J_{k_j}
\]

(171)

then from the relation,

\[
W_j W_j^{-1} = \sigma_w^2 I_{k_j} + (\sigma_b^2 \omega + \sigma^2 \beta_j + k_j \beta_j \sigma_b^2) J_{k_j} = I_{k_j}
\]

(172)

we obtain

\[
\omega = \frac{1}{\sigma_w^2} , \quad \beta_j = -\frac{\sigma_b^2 \omega}{\sigma^2 + k_j \sigma_b^2} = -\frac{\sigma_b^2 \omega}{\tilde{W}_j}
\]

(173)
where \( \tilde{w}_j \) is defined by (113). Substituting (173) into (171), we find

\[
W_j^{-1} = \frac{1}{\sigma^2} \left\{ I_{k_j} - \frac{\sigma_b^2}{\sigma^2 + k_j \sigma_b^2} J_{k_j} \right\} \quad j = 1, 2, \ldots, b
\]

\[
= w \left( \frac{w - \tilde{w}_j}{k_j} \right) J_{k_j}
\]

From (168) and (174), it follows that

\[
\Sigma^{-1} = \begin{bmatrix}
W_1^{-1} & 0_{k_1 k_2} & \cdots & 0_{k_1 k_b} \\
0_{k_2 k_1} & W_2^{-1} & \cdots & 0_{k_2 k_b} \\
\vdots & \ddots & \ddots & \vdots \\
0_{k_b k_1} & 0_{k_b k_2} & \cdots & W_b^{-1}
\end{bmatrix}
\]

\[
= w \mathbf{I}_n - wL D_k^{-1} L + L' D_k^{-1} D \tilde{w} L
\]

on the application of an argument implicit in (170). The generalized least squares (normal) equations of estimation associated with the model (108) are

\[
\begin{bmatrix}
i_n' \\
\mathbf{H}
\end{bmatrix}
\Sigma^{-1} \begin{bmatrix}
i_n & \mathbf{H}'
\end{bmatrix}
\begin{bmatrix}
g \\
t
\end{bmatrix} = \begin{bmatrix}
i_n' \\
\mathbf{H}
\end{bmatrix} \Sigma^{-1} \mathbf{Y} \quad \text{or}
\]

\[
\begin{bmatrix}
i_n' \\
\mathbf{H}
\end{bmatrix} \Sigma^{-1} \begin{bmatrix}
i_n & \mathbf{H}'
\end{bmatrix}
\begin{bmatrix}
g \\
t
\end{bmatrix} = \begin{bmatrix}
i_n' \Sigma^{-1} \mathbf{Y} \\
\mathbf{H} \Sigma^{-1} \mathbf{Y}
\end{bmatrix} \quad .
\]

Eliminating \( g \) from (119), we obtain the adjusted equations

\[
\left[ \mathbf{H} \Sigma^{-1} \mathbf{H}' - \frac{1}{i_n' \Sigma^{-1} i_n} \mathbf{H} \Sigma^{-1} J_n \Sigma^{-1} \mathbf{H}' \right] \mathbf{t} = \mathbf{H} \Sigma^{-1} \mathbf{Y} - \frac{1}{i_n' \Sigma^{-1} i_n} \mathbf{H} \Sigma^{-1} J_n \Sigma^{-1} \mathbf{Y} \quad \text{or}
\]

\[
\mathbf{H} \left[ \Sigma^{-1} - \frac{1}{i_n' \Sigma^{-1} i_n} \Sigma^{-1} J_n \Sigma^{-1} \right] \mathbf{H}' \mathbf{t} = \mathbf{H} \left[ \Sigma^{-1} - \frac{1}{i_n' \Sigma^{-1} i_n} \Sigma^{-1} J_n \Sigma^{-1} \right] \mathbf{Y} .
\]
From (117), it follows that

\[(180) \quad H[\Sigma_{\text{r}n}^{-1} J_n \Sigma_{\text{r}n}^{-1}] = H[w I_n - w L' D_k^{-1} L + L' D_k^{-1} D_w L] J_n [w I_n - w L' D_k^{-1} L + L' D_k^{-1} D_w L] \]

\[= [w H - w N D_k^{-1} L + N D_k^{-1} D_w L] [w J_n - w \bar{a}_n^{-1} k L + \bar{a}_n^{-1} k D_k^{-1} D_w L] \]

\[= [w x - w N D_k^{-1} L + N D_k^{-1} D_w \bar{x} k] [w \bar{a}_n^{-1} - w \bar{a}_b^{-1} L + \bar{a}_b^{-1} D_w L] \]

\[= [w x - w N \bar{a}_b^{-1} + N D_w \bar{a}_b^{-1}] [w \bar{a}_n^{-1} - w \bar{a}_n^{-1} + \bar{a}_b^{-1} D_w L] \]

\[= [w x - w x + N D_w \bar{a}_b^{-1}] [\bar{a}_n^{-1} D_w L] \]

\[= N \tilde{w} \tilde{w}' L \]

and that

\[(181) \quad HE_{\text{r}n}^{-1} = H[w I_n - w L' D_k^{-1} L + L' D_k^{-1} D_w L] = [w H - w N D_k^{-1} L + N D_k^{-1} D_w L]; \]

also,

\[(182) \quad \bar{a}_n^{-1} \Sigma_{\text{r}n}^{-1} \bar{a}_n^{-1} = \bar{a}_n^{-1} [w I_n - w L' D_k^{-1} L + L' D_k^{-1} D_w L] \bar{a}_n^{-1} \]

\[= [w n - w k' D_k^{-1} k + k' D_k^{-1} D_w k] \]

\[= [w n - w \bar{a}_b^{-1} k + \bar{a}_b^{-1} D_w k] \]

\[= [w n - w n + \tilde{w}' k] \]

\[= \tilde{w}' k \]

Hence

\[(183) \quad H[\Sigma_{\text{r}n}^{-1} - \frac{1}{\bar{a}_n^{-1} \Sigma_{\text{r}n}^{-1} \bar{a}_n^{-1}} \Sigma_{\text{r}n}^{-1} J_n \Sigma_{\text{r}n}^{-1}] H' = [w H - w N D_k^{-1} L + N D_k^{-1} D_w L - \frac{1}{\tilde{w}' k} \tilde{w} L] H' \]
\[ = [wD_r - w N D_k^{-1} N' + N D_k^{-1} D_w N' - \frac{1}{\tilde{\nu}' k} N \tilde{\nu} \tilde{\nu}' N'] \]

\[ = [w C + M] \]

and

\[ H[\Sigma^{-1} - \frac{1}{d_n} \Sigma^{-1} J_n \Sigma^{-1}]Y = [wH Y - w N D_k^{-1} L Y + N D_k^{-1} D_w Y - \frac{1}{\tilde{\nu}' k} N \tilde{\nu} \tilde{\nu}' Y] \]

\[ = [w(\Psi - N D_k^{-1} B) + (N D_k^{-1} D_w B - \frac{1}{\tilde{\nu}' k} N \tilde{\nu} \tilde{\nu}' B)] \]

\[ = w Q + m \]

where C, Q, M, and m have been defined previously in (32), (33), (118), (119).

If we substitute (183) and (184) into (179), then we may write the adjusted equations as

\[ (185) \quad [wC + M]_t = wQ + m \]

As can be seen, the equations (185) represent a synthesis of the intra-block equations (34) and the inter-block equations (120). Using (39), (110), (140) and (144), we have

\[ (186) \quad \varepsilon(wQ + m) = [wC + M]_t \]

and

\[ (187) \quad \text{Var}[wQ + m] = wC + M \]

Also, from (70) and (137), we have

\[ (188) \quad [wC + M]_t = 0 \]

hence the elements of \([wC + M]_t\) are treatment contrasts. Since all contrasts
have intra-block estimates, all are estimable with respect to the model (108) and their best estimates are linear functions of the elements of \((w Q + m)\). From arguments similar to ones used before, it follows that

(189) \[ \text{Rank} [wC + M] = v - 1. \]

A solution to (185) is

(190) \[ \overline{t} = [wC + M]^G [wQ + m] \]

where \([wC + M]^G\) is a conditional inverse of \(wC + M\). Thus, we have the following theorem

\[ \text{Theorem II: For a connected incomplete blocks design in which block effects may be regarded as random, any treatment contrast} \ h' \ t \ \text{is estimable and} \]

i. best estimate of \(h' \overline{t}\) is \(h' \overline{t}\)

ii. \(\text{Var} \ \text{best estimate of} \ h' \overline{t} = h'[wC + M]^G h\).

To apply Theorem II, \(w\) and the \(\widehat{w}_j\) must be known; if this is not the case, then (157) and (158) may again be used to construct estimates of them.

To test the hypothesis

(191) \[ H_0: \ t_1 = t_2 = \ldots = t_v \]

in the model (108), let us consider

(192) \[ S_t^2 = [wQ + m]' \overline{t}. \]

If \(R = R(v \times v)\) denotes an orthogonal matrix of the form

(193) \[ R = \begin{bmatrix} \frac{1}{\sqrt{v}} & i_v' \\ \sqrt{v} & P \end{bmatrix} \]
such that

\[
(194) \quad R[wC + M]R' = \begin{bmatrix} 0 & 0 \gamma_v-1 \\ 0 \gamma_v-1 & P(wC + M)P' \end{bmatrix} = \begin{bmatrix} 0 & 0 \gamma_v-1 \\ 0 \gamma_v-1 & D\lambda \end{bmatrix}
\]

where \( \lambda_1, \lambda_2, \ldots, \lambda_{v-1} > 0 \) because \((wC + M)\) is symmetric and has rank \((v-1)\), then

\[
(195) \quad \frac{S^2_t}{\sigma^2} = [wQ + m]' \frac{I}{\sigma^2}
\]

\[
= [wQ + m]'R'R \frac{I}{\sigma^2}
\]

\[
= [wQ + m]' \left[ \frac{1}{v} J_v + P'P \right] \frac{I}{\sigma^2}
\]

\[
= [wQ + m]' \frac{P'P}{\sigma^2}
\]

Now \(Pt\) is a vector of linearly independent treatment contrasts whose best estimates \(P\) are the solution of the equations

\[
(196) \quad P[wC + M] P'P = P(wQ + m)
\]

\[
D\lambda P = P(wQ + m)
\]

which are obtained by pre-multiplying \((185)\) by \(R\) and replacing \(I_v\) by \(R'R\) and then simplifying the result through the use of \((194)\). Solving \((196)\) and substituting into \((195)\), we obtain

\[
(197) \quad \frac{S^2_t}{\sigma^2} = [wQ + m]'P'D^{-1}P[wQ + m]
\]

If \(H_0\), as given in \((191)\), is true, then

\[
(198) \quad Pt = \Theta_{v-1}
\]

and hence
\[ (199) \quad \mathcal{E}(P[w \mathbf{Q} + \mathbf{M}]) = P[w \mathbf{C} + \mathbf{M}] \mathbf{t} \]
\[ = P[w \mathbf{C} + \mathbf{M}] \mathbf{P}' \mathbf{P} = D_\lambda \mathbf{C}_{v-1} \]
\[ = \mathbf{C}_{v-1} \]

also

\[ (200) \quad \text{Var}(P[w \mathbf{Q} + \mathbf{M}]) = P[w \mathbf{C} + \mathbf{M}] \mathbf{P}' = D_\lambda . \]

Thus, if we assume that \( \mathbf{y} \) in (106) is normally distributed, then it follows from (199) and (200) that \( S_t^2 \) has a chi square distribution with \((v-1)\) degrees of freedom when \( H_0 \) is true.

Finally, it should be noted here that if \( w \) and the \( \tilde{w}_j \) have to be estimated from the data, then \( S_t^2 \) does not have a chi square distribution. However, if \( \sigma^2 \) and \( \sigma^2_b \) are replaced by consistent estimates like (157) and (158) and \( w \) and the \( \tilde{w}_j \) are estimated from such, then \( S_t^2 \) has the chi square distribution with \((v-1)\) degrees of freedom asymptotically for large \( n \).

If we again assume that all blocks are of the same size \( k \), then the same simplifications as indicated in (159)-(167) arise here. In particular we now have

\[ (201) \quad \Sigma^{-1} = w I_n - \frac{w - \tilde{w}}{k} L' L \]

thus (185) now simplifies to

\[ (202) \quad [w \mathbf{C} + \tilde{w} \tilde{\mathbf{C}}] \mathbf{t} = w \mathbf{Q} + \tilde{w} \tilde{\mathbf{Q}} \]

where \( \tilde{\mathbf{C}} \) and \( \tilde{\mathbf{Q}} \) are defined by (163). The results given in (186)-(200) and Theorem 11 apply equally here with appropriate modifications.

Finally, we want to consider the relationship between the generalized least
squares procedure (which always leads to best estimates of treatment contrasts in the model (108)) and the procedure of combining estimates associated with the recovery of inter-block information. This leads to the following theorem.

**Theorem 12:** For a connected incomplete blocks design in which all blocks are of the same size $k$ and block effects may be regarded as random, the generalized least squares estimate of any treatment contrast which is estimable in the inter-block model is the same as the estimate obtained by combining the intra-block estimate and the inter-block estimate according to the inverses of their variances.

**Proof:** So that the basic argument may be seen clearly, let us assume, for the present, that each treatment is replicated $r$ times. For this case, the identity

\[
(203) \quad r \nu = b k = n
\]

will be useful.

Now the intra-block equations are

\[
(204) \quad C \bar{t} = \bar{Q}
\]

where

\[
(205) \quad C = r I_v - \frac{1}{k} N N', \quad Q = T - \frac{1}{k} N B
\]

while the inter-block equations are

\[
(206) \quad \bar{C} \bar{t} = \bar{Q} \quad \text{where}
\]

\[
(207) \quad \bar{C} = \frac{1}{k} N N' - \frac{r^2}{n} J_v, \quad \bar{Q} = \frac{1}{k} N B - \frac{rQ}{n} \mathbf{1}.
\]

Let $R = R(\nu \times \nu)$ be an orthogonal matrix of the form

\[
(208) \quad R = \left[ \begin{array}{c}
\frac{1}{\sqrt{\nu}} \mathbf{1}_v' \\
\mathbf{p}
\end{array} \right]
\]
such that

\[(209) \quad R \ C \ R' = \begin{bmatrix} 0 & 0'_{v-1} \\ 0'_{v-1} & P \ C \ P' \end{bmatrix} = \begin{bmatrix} 0 & 0'_{v-1} \\ 0'_{v-1} & D_{\alpha} \end{bmatrix} \]

where \(\alpha_1, \alpha_2, \ldots, \alpha_{v-1} > 0\) because \(C\) is symmetric and has rank \((v-1)\). Also, let us define

\[(210) \quad \bar{u} = P_t, \quad S = PQ\]

and note that \(\bar{u}\) is a vector of \((v-1)\) independent treatment contrasts. Then the intra-block equations may be transformed to

\[(211) \quad R \ C \ R' \ R \ t = RQ\]

or, more simply,

\[(212) \quad D_{\alpha} \bar{u} = \bar{S}\]

because of \((208), (209),\) and \((210)\). Since

\[(213) \quad D_{\alpha} = P \ C \ P' = P[r \ M_v - \frac{1}{k} \ N \ N'] \ P' = r \ M_{v-1} - \frac{1}{k} \ P \ N \ N' \ P' \quad \text{and}\]

\[(214) \quad P \ J_v \ P' = 0_{v-1}, \ v-1, \]

we have

\[(215) \quad R \tilde{C} \ R' = \begin{bmatrix} 0 & 0'_{v-1} \\ 0'_{v-1} & P(\frac{1}{k} \ N \ N' - \frac{r^2}{n} \ J_v) \ P' \end{bmatrix} = \begin{bmatrix} 0 & 0'_{v-1} \\ 0'_{v-1} & r \ M_{v-1} - D_{\alpha} \end{bmatrix}\]

where \(0 < \alpha_1, \alpha_2, \ldots, \alpha_{v-1} \leq r\) because \(\tilde{C}\) is symmetric. If we define

\[(216) \quad \tilde{S} = P \tilde{C}, \]

45
then the inter-block equations may be transformed to

\[(217) \quad P \tilde{C} P' \bar{x} = P \tilde{s} \quad \text{or} \quad (r I - D_{\alpha}) \bar{u} = \tilde{s}\]

by arguments similar to those used before. We now observe that

\[(218) \quad e(\tilde{s}) = P C t = D_{\alpha} \bar{u}, \quad \text{Var}(\tilde{s}) = P C P' \sigma^2 = \frac{1}{w} D_{\alpha} \quad \text{and} \]
\[(219) \quad e(\tilde{s}) = P \tilde{C} \tilde{t} = (r I - D_{\alpha}) \bar{u}, \quad \text{Var}(\tilde{s}) = P \tilde{C} P' (\sigma^2 + k \sigma^2) = \frac{1}{w} (r I_{v-1} - D_{\alpha}).\]

Because variances are necessarily non-negative, (218) and (219) provide an alternative proof that \(0 < \alpha_1, \alpha_2, \ldots, \alpha_{v-1} \leq r\). Suppose now that \(\alpha_i < r\); then \(u_i\) is estimable in the inter-block model and

\[(220) \quad \text{intra-block estimate of } u_i = \frac{1}{\alpha_i} s_i\]
\[(221) \quad \text{inter-block estimate of } u_i = \frac{1}{r - \alpha_i} \tilde{s}_i\]

Since

\[(221) \quad \text{Var(intra-block estimate of } u_i) = \frac{1}{\alpha_i^2} \quad \text{Var}(s_i) = \frac{1}{\alpha_i}\]

\[(222) \quad \text{Var(inter-block estimate of } u_i) = \frac{1}{(r - \alpha_i)^2} \quad \text{Var} (\tilde{s}_i) = \frac{1}{\tilde{w}(r - \alpha_i)}\]

we find that

\[(222) \quad \text{combined estimate of } u_i = ((w \alpha_i)^{s_i = \tilde{s}_i}) \left[ \frac{1}{\alpha_i} + \tilde{w}(r - \alpha_i) \right] \left[ \frac{1}{(r - \alpha_i)^2} \right] (w \alpha_i + \tilde{w}(r - \alpha_i))^{-1}
\]
\[= \left[ w s_i + \tilde{w} \tilde{s}_i \right] (w \alpha_i + \tilde{w}(r - \alpha_i))^{-1} .\]

The generalized least squares equations are

\[(223) \quad [w C + \tilde{w} \tilde{C}] \bar{x} = w \tilde{Q} + \tilde{w} \tilde{s} .\]
these may be transformed to

\[(224) \quad P[w \ C + \tilde{w} \ C] P'P_t = P[w \ Q + \tilde{w} \ Q] \quad \text{or} \quad [v \ D_{\alpha} + \tilde{w}(rI_{V-1} - D_{\alpha})] u = w \ S + \tilde{w} \ S'] \]

Solving (223) for the estimate of \( u_i \), we obtain

\[(225) \quad \bar{u}_i = (v \ S_i + \tilde{w} \ S_i)(v \ \alpha_i + w(r - \alpha_i))^{-1} \]

= combined estimate of \( u_i \).

Hence, we see that if a treatment contrast can be written as a linear combination of \( u \)'s which are estimable in the inter-block model (i.e., \( u \)'s for which the corresponding \( \alpha \)'s are less than \( r \)), then the two methods of estimation lead to the same estimator.

Suppose now that \( \alpha_i' = r \) for some \( i' \). Since

\[(226) \quad \bar{S}_i = 0 \cdot u_i', = 0, \]

the generalized least squares estimate of \( u_i \), degenerates to

\[(227) \quad \bar{u}_i', = \frac{w}{v \alpha_i}, \quad S_i', = \frac{1}{\alpha_i}, \quad S_i', \]

= intra-block estimate of \( u_i' \).

Thus, if a treatment contrast can be written as a linear combination of \( u \)'s for which the corresponding \( \alpha \)'s are equal to \( r \), then its intra-block estimate cannot be improved upon.

Most important, however, if a treatment contrast does not belong to either of the two distinct classes previously considered, then it does not have an inter-block estimate, but its generalized least squares estimate is better than
its intra-block estimate. In fact, such contrasts can be partitioned as the sum of two orthogonal components; namely, a component in the space generated by the u's whose α's are less than r and a component in the space generated by the u's whose α's equal r. The best estimate is the sum of the best estimates of the components; but the best estimate of the one component is better than the corresponding intra-block estimate. Hence, the best estimate of such a treatment contrast must be better than the intra-block estimate. As a result, we see that the generalized least squares procedures can be applied to more general estimation situations than the combined estimator procedure involved in the recovery of inter-block information.

Let us now turn to the proof of Theorem 12 when the restrictive assumptions on the \( r_i \) are removed. In this case

\[
C = D_r - \frac{1}{k} N N', \quad \tilde{C} = \frac{1}{k} N N - \frac{1}{n} D_r J D_r.
\]

Let \( \sqrt{r} \) denote a column vector in which the elements are \( \sqrt{r_1}, \sqrt{r_2}, \ldots, \sqrt{r_v} \), and let \( D_{\sqrt{r}} \) denote a diagonal matrix in which the diagonal elements are \( \sqrt{r_1}, \sqrt{r_2}, \ldots, \sqrt{r_v} \). Let

\[
C^* = D_{\sqrt{r}}^{-1} C D_{\sqrt{r}}^{-1} = I_v - \frac{1}{k} D_{\sqrt{r}}^{-1} N N' D_{\sqrt{r}}^{-1},
\]

\[
\tilde{C}^* = D_{\sqrt{r}}^{-1} \tilde{C} D_{\sqrt{r}}^{-1} = \frac{1}{k} D_{\sqrt{r}}^{-1} N N' D_{\sqrt{r}}^{-1} - \frac{1}{n} D_{\sqrt{r}} J D_{\sqrt{r}}.
\]

Let \( R = R(v \times v) \) be an orthogonal matrix of the form

\[
R = \begin{bmatrix}
\frac{1}{\sqrt{n}} & \sqrt{n}' \\
\sqrt{n}' & p
\end{bmatrix}
\]

such that
(231) \[
R \begin{bmatrix} C * R' \\
\begin{bmatrix} 0 & 0'_{-1} \\
0_{-1} & \begin{bmatrix} C * P' \\
P_{-1} & \end{bmatrix}
\end{bmatrix}
\end{bmatrix} = \begin{bmatrix} 0 & 0'_{-1} \\
0_{-1} & \begin{bmatrix} D_{2} \\
D_{1}
\end{bmatrix}
\end{bmatrix}
\]

where \( \alpha_1, \alpha_2, \ldots, \alpha_{v-1} > 0 \) because \( C^* \) is symmetric and has rank \((v-1)\). From (229) and (231), we obtain

(232) \[
D_{\alpha} = P C^* P' = P[I_{v} - \frac{1}{k} D^{-1}_{R} N N' D^{-1}_{R}] P' = I_{v-1} - \frac{1}{k} P D^{-1}_{R} N N' D^{-1}_{R} P'.
\]

Since

(233) \[
\frac{1}{n} P D^{-1}_{R} J D^{-1}_{R} P' = \frac{1}{n} P D^{-1}_{R} \sqrt{\frac{1}{n}} P' = 0_{v-1}, \quad v-1
\]

by definition of \( R \), we have, on using (232) and (233),

(234) \[
R \begin{bmatrix} C^* R' \\
\begin{bmatrix} 0 & 0'_{-1} \\
0_{-1} & \begin{bmatrix} C^* P' \\
P_{-1} & \end{bmatrix}
\end{bmatrix}
\end{bmatrix} = \begin{bmatrix} 0 & 0'_{-1} \\
0_{-1} & \begin{bmatrix} I_{v-1} - D_{\alpha} \\
D_{\alpha}
\end{bmatrix}
\end{bmatrix};
\]

hence \( 0 < \alpha_1, \alpha_2, \ldots, \alpha_{v-1} \leq 1 \) because \( \tilde{C}^* \) is symmetric. On letting

(235) \[
u = P D^{-1}_{R} t
\]

and

(236) \[
S = P D^{-1}_{R} Q, \quad \tilde{S} = P D^{-1}_{R} \tilde{Q}
\]

and on applying arguments similar to ones used before we may transform the intra-block equations and the inter-block equations to the forms

(237) \[
D_{\alpha} u = S, \quad (I_{v-1} - D_{\alpha}) u = \tilde{S}
\]

respectively. Note that \( u \) is a vector of treatment contrasts since

(238) \[
P D^{-1}_{R} 1 = P \sqrt{\lambda} = 0.
\]

49
The remainder of the proof follows as indicated in (218)-(227), the only visible difference is that "r" there is replaced by "l" here. The comments about the partitioning of the space of estimable functions and the relationships between the two methods of estimation apply equally well here.

It is worthwhile to note now that although it seems reasonable that Theorem 12 might be valid when the blocks are not of the same size, the method of proof indicated above does not generalize to this situation because of the structure of $M$. For the present, we shall let the above remain an open question.

Finally, we consider the previous results for incomplete blocks experiments in which block effects may be regarded as random for the special case of a BIB design. The intra-block analysis is given in (87)-(107). For the inter-block analysis, we have

$$\tilde{c} = \frac{1}{k} N N' - \frac{r^2}{n} J_v$$

$$= \frac{1}{k} \left[ (r - \lambda) I_v + \lambda J_v \right] - \frac{r^2}{n} J_v$$

because of (95). Hence, the inter-block equations are

$$\left[ \left( \frac{r-\lambda}{k} \right) I_v + \left( \frac{\lambda}{k} - \frac{r^2}{n} \right) J_v \right] t = \tilde{c} \quad \text{or}$$

$$\left( \frac{r-\lambda}{k} \right) t = \tilde{c} + \left( \frac{r^2}{n} - \frac{\lambda}{k} \right) J_v t$$

If we use the non-estimable restriction $J_v t = 0$ in (240) and note from (97) and (103) that

$$\frac{r-\lambda}{k} = r - \frac{\lambda N}{k} = r - rE = r(1-E),$$

$$241$$
then we may write

\[(242) \quad \tilde{\alpha} = \frac{1}{r(1-E)} \tilde{\alpha}.\]

From (242), it follows that a conditional inverse of (239) is

\[(243) \quad \tilde{\mathcal{C}} = \frac{1}{r(1-E)} I_v; \]

hence the variance of the estimated difference between two treatment effects is found to be

\[(244) \quad \text{Var}(\tilde{\alpha}_1 - \tilde{\alpha}_1') = \frac{2}{r(1-E)} (\sigma^2 + k\alpha_b^2) = \frac{2}{r(1-E)\tilde{\alpha}} \]

in the inter-block model.

Previously, we found the intra-block estimates to be

\[(245) \quad \hat{\alpha} = \frac{k}{\lambda} \bar{Q} = \frac{1}{rE} \bar{Q}; \]

from (104), we recall,

\[(246) \quad \text{Var}(\hat{\alpha}_1 - \hat{\alpha}_1') = \frac{2\sigma^2}{rE} = \frac{2}{rEV}. \]

From (241), (244), (245), and (246), it follows that

\[(247) \quad \text{combined estimate of } t_1 - t_1' = \frac{(w)(Q_1 - Q_1') + \tilde{w}(\tilde{Q}_1 - \tilde{Q}_1')}{rEw + r(1-E)\tilde{w}} = \frac{w(Q_1 - Q_1') + \tilde{w}(\tilde{Q}_1 - \tilde{Q}_1')}{r(Ew + (1-E)\tilde{w})}. \]

The generalized least squares equations are
\[(248) \quad [w \mathbf{C} + \tilde{w} \mathbf{C}] \mathbf{t} = w \mathbf{Q} + \tilde{w} \mathbf{Q} \quad \text{or} \quad [w \mathbf{R} \mathbf{E}(I \mathbf{l}_v - \frac{1}{N} J \mathbf{l}_v) + \tilde{w}(r(1 - \mathbf{E})) I \mathbf{l}_v + \sqrt{\mathbf{n}} \left( \frac{\lambda}{k} - \frac{1}{n} \right) J \mathbf{l}_v] \mathbf{t} = w \mathbf{Q} + \tilde{w} \mathbf{Q} \cdot \]

Using the restriction \( i_v^t \mathbf{t} = 0 \) to solve (248), we obtain

\[(249) \quad \bar{\mathbf{t}} = \frac{w \mathbf{Q} + \tilde{w} \mathbf{Q}}{r(w E + \tilde{w}(1 - E))} \cdot \]

From (249), it follows that the best estimate of the difference between two treatment effects is

\[(250) \quad \bar{t}_i - \bar{t}_i^r = \frac{w(Q_i - Q_i^r) + \tilde{w}(\tilde{Q}_i - \tilde{Q}_i^r)}{r(w E + \tilde{w}(1 - E))} \]

\[= \text{combined estimate of } t_i - t_i^r. \]