ON CONSTRUCTING BALANCED INCOMPLETE BLOCK DESIGNS FROM ASSOCIATION MATRICES WITH SPECIAL REFERENCE TO ASSOCIATION SCHEMES OF TWO AND THREE CLASSES[1]

by

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Institute of Statistics Mimeo Series No. 496

November 1966

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This investigation was supported in part by a Public Health Service training grant (number GM-38) from the Institute of General Medical Sciences, Public Health Service, and in part by the Air Force Office of Scientific Research Contract Number AF-AFOSR-760-65.

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1. Introduction

1.1 Balanced Incomplete Block Designs

Balanced incomplete block (BIB) designs were introduced by Bose [1] and have since been studied extensively. Connor [6] introduced the concept of an incidence matrix for a BIB design; in this paper such designs will be examined via their incidence matrices. Suppose we have a BIB (v, b, r, k, λ), where v is the number of treatments, b is the number of blocks, r is the number of replications, k is the block size, and λ is the number of blocks in which any pair of treatments occur together. The $v \times b$ matrix N is the incidence matrix for the BIB design, where

(1.1) $N = (n_{ij})$, and

(1.2) $n_{ij} = 1$ if the ith treatment occurs in the jth block,
$= 0$ otherwise.

A necessary and sufficient condition for a $v \times b$ matrix N of 0's and 1's to be an incidence matrix for a BIB (v, b, r, k, λ) is that the following relations hold:

(1.3) $0 < \lambda < r$;

(1.4) $N' J_v = K J_{b,v}$;

(1.5) $N N' = r I_v + \lambda(J_v - I_v)$ [10].

By J_v we mean the $v \times v$ matrix of 1's, by $J_{b,v}$ the $b \times v$ matrix of 1's, and by I_v the $v \times v$ identity matrix.

Now for any $m \times n$ matrix A of 0's and 1's, let us call the matrix $J_{m,n} - A$ the complement of A. Then if N is an incidence matrix for a BIB (v, b, r, k, λ), and if $k < v - 1$, the complement of N is an incidence matrix for a BIB $(v, b, r^*, k^*, \lambda^*)$, where $r^* = b - r$, $k^* = v - k$, and $\lambda^* = b - 2r + \lambda$.

1.2 Association Schemes [2], [4]

An Association scheme in m associate classes is a set of v elements (objects, treatments, varieties) which satisfies the following conditions:

(i) any two treatments are either 1st, 2nd, ..., or mth associates, and the relation of association is symmetrical;

(ii) each element has exactly n_i ith associates ($i = 1, 2, \ldots, m$), where the number n_i is independent of the element chosen;

(iii) if α and β are ith associates, then the number of elements which are jth associates of α and kth associates of β is p^i_{jk}, and p^i_{jk} is independent of the pair of ith associates chosen ($i, j, k = 1, 2, \ldots, m$).

The numbers

$$(1.6) \qquad v, \; n_i, \; p^i_{jk}$$

are called the parameters of the association scheme; all must be positive integers.

The following relations among the parameters are easily shown:

$$(1.7) \qquad p^i_{jk} = p^i_{kj} \; ;$$

$$(1.8) \qquad \sum_{i=1}^{m} n_i = v-1 \; ;$$

$$(1.9) \qquad \sum_{k=1}^{m} p^i_{jk} = n_j \text{ if } i \neq j, \qquad = n_j - 1 \text{ if } i = j;$$

$$(1.10) \qquad n_i \, p^i_{jk} = n_j \, p^j_{ik} = n_k \, p^k_{ij}$$

It is useful to make the convention that each element is the zero-th associate of itself and of no other elements. Then we must have

(1.11) $n_o = 1$;

(1.12) $p^o_{ij} = p^o_{ji} = 0$ if $i \neq j$,

$= n_j$ if $i = j$;

(1.13) $p^i_{ko} = p^i_{ok} = 0$ if $i \neq k$,

$= 1$ if $i = k$.

Then (1.8) and (1.9) become

(1.14) $$\sum_{i=o}^{m} n_i = v,$$

(1.15) $$\sum_{k=o}^{m} p^i_{jk} = n_j .$$

For the case m=2, it is sufficient to specify v, n_1, p^1_{11} , and p^2_{11} , and the other parameters are then determined; see, for example, [2]

Given an m-class association scheme, we call the matrices B_i ($i = 0, 1, \ldots, m$) the association matrices of the scheme, where

(1.16)
$$B_i = (b^\beta_{\alpha i}) = \begin{bmatrix} b^1_{1i} & b^2_{1i} & \cdots & b^v_{1i} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ b^1_{vi} & b^2_{vi} & \cdots & b^v_{vi} \end{bmatrix}$$ and

(1.17) $b^\beta_{\alpha i} = 1$ if α and β are ith associates,

$= 0$ otherwise

Clearly, we have

(1.18) $B_o = I_v$

and

(1.19) $B_o + B_1 + \dots + B_m = J_v$.

Also, the linear form $c_o B_o + c_1 B_1 + \dots + c_m B_m$ is equal to the zero matrix if and only if $c_o = c_1 = \dots = c_m = 0$; i.e., the B_i's are linearly independent. The association matirces satisfy the relation

(1.20) $B_k B_j = B_j B_k = p^o_{jk} B_o + p^1_{jk} B_1 + \dots + p^m_{jk} B_m$
$(j, k = 0, 1, \dots, m)$.

The result (1.20), along with the linear independence of the B_i's, will be quite important in the proofs of the theorems of Sections 3 and 4 of this paper.

Association schemes were introduced by Bose and Shimamoto [5] to aid in the classification and analysis of partially balanced designs. In the present paper association schemes will be used in the construction of BIB designs by the following two methods, where B_i denotes an association matrix:

(i) obtaining a matrix of the form $B_{i_1} + B_{i_2} + \dots + B_{i_t}$ which will be a BIB incidence matrix, and (ii) obtaining a matrix of the form $[B_{i_1} \vdots B_{i_2} \vdots \dots \vdots B_{i_s}]$ which will be a BIB incidence matrix.

First, however, let us define a number of known types of association schemes of two and three classes.

2. Types of Association Schemes

2.1 Association Schemes of Two Classes

(a) Group Divisible (GD) Scheme [5]

Suppose, for integers $\ell \geq 2$ and $n \geq 2$, there is a set of $v = \ell n$ elements. Let the elements be arranged in a rectangular array with ℓ rows and n columns. Call any two elements which appear together in a row first associates; if two elements are in different rows they are second associates. Then the rectangular array gives us a two-class association scheme with the following parameters.

$$v = \ell n$$

$$n_1 = n-1$$

$$n_2 = n(\ell-1)$$

$$(2.1) \qquad p^1_{11} = n-2 \qquad p^2_{11} = 0$$

$$p^1_{12} = 0 \qquad p^2_{12} = n-1$$

$$p^1_{22} = n(\ell-1) \qquad p^2_{22} = n(\ell-2)$$

(b) Triangular Association Scheme [5]

Suppose, for some positive integer n, there is a set of $v = \binom{n}{2} = \frac{n(n-1)}{2}$ elements. Arrange the v elements in an nxn array as follows: leave the leading diagonal positions blank, and fill the $\frac{n(n-1)}{2}$ positions so as to make the array symmetric with respect to the diagonal. Define first associates as two elements which appear in the same row (equivalently, the same column) of the resulting array; if two treatments do not appear in the same row, they are second associates.

The v elements might also be considered as unordered pairs (i,j), where $i \neq j$ and $i, j = 0, 1, \ldots, n-1$. Then two elements are first associates if they differ in exactly one coordinate; otherwise they are second associates.

Such an array is an association scheme, called a triangular association scheme. The parameters of the scheme are as follows.

(2.2)
$$v = \frac{n(n-1)}{2}$$
$$n_1 = 2n - 4$$
$$n_2 = \frac{(n-2)(n-3)}{2}$$
$$p^1_{11} = n-2 \qquad p^2_{11} = 4$$
$$p^1_{12} - n-3 \qquad p^2_{12} = 2n - 8$$
$$p^1_{22} = \frac{(n-3)(n-4)}{2} \qquad p^2_{22} = \frac{(n-4)(n-5)}{2}$$

From the parameter values, we see that $n \geq 4$.

(c) Pseudo-Cyclic Association Scheme [5], [7]

Suppose there is a set of v elements; denote them by the integers 1, 2, ..., v. Suppose there is a set of integers $(d_1, d_2, \ldots, d_{n_1})$ satisfying the following conditions:

(i) the d's are distinct, and $0 < d_j < v$ $(j = 1, 2, \ldots, n_1)$;

(ii) among the $n_1(n_1-1)$ differences $d_i - d_j (i \neq j;\ i, j=1, 2, \ldots, n_1)$ reduced (mod v), each of the numbers $d_1, d_2, \ldots, d_{n_1}$ occurs α times and each of the numbers $e_1, e_2, \ldots, e_{n_2}$ occurs β times, where $d_1, d_2, \ldots, d_{n_1}, e_1, e_2, \ldots, e_{n_2}$ are all the integers 1, 2, ..., v-1. Clearly, $n_1 \alpha + n_2 \beta = n_1(n_1-1)$.

Given the element k (k = 1, 2, ..., v), define its first associates as the elements $k+d_1, k+d_2, \ldots, k+d_{n_1}$ (mod v); the remaining $(v-n_1-1)$ elements are the second associates of k. Then we have an association scheme, called a cyclic association scheme, with the following parameters.

$$
\begin{aligned}
&v \\
&n_1 \\
(2.3)\qquad &n_2 = v-n_1-1 \\
&p^1_{11} = \alpha \qquad && p^2_{11} = \beta \\
&p^1_{12} = n_1 - \alpha-1 \qquad && p^2_{12} = n_1 - \beta \\
&p^1_{22} = n_2 - n_1 + \alpha+1 \qquad && p^2_{22} = n_2 - n_1 + \beta - 1
\end{aligned}
$$

We see that, given v, the set of d's completely determines such a scheme. A few examples of cyclic association schemes are given below.

Some Cyclic Association Schemes

v	n_1	n_2	Set of d's
13	6	6	2,5,6,7,8, 11
17	8	8	3,5,6,7,10,11, 12, 14
29	14	14	1,4,5,6,7,9, 13, 16, 20,22,23,24,25,28

All the known cyclic association schemes are such that $v= 4u+1$, $n_1 = n_2 = 2u$, and $\alpha = u-1$, for some positive integer u. Then the association scheme has the following parameters.

$$
\begin{aligned}
&v = 4u + 1 \\
&n_1 = n_2 = 2u \\
(2.4)\qquad &p^1_{11} = u-1 \qquad && p^2_{11} = u \\
&p^1_{12} = u \qquad && p^2_{12} = u \\
&p^1_{22} = u \qquad && p^2_{22} = u-1
\end{aligned}
$$

Following the nomenclature in [7], we will call any association scheme satisfying the parameters (2.4) pseudo-cyclic, whether or not it is obtainable by the cyclic method described in [5].

(d) Singly Linked Block (SLB) Association Scheme [2], [5]

Suppose N' is an incidence matrix for a BIB design with b treatments, v blocks, k replications, block size r, and $\lambda = 1$; i.e. every pair of treatments occurs together in exactly one block. Then bk = vr and b-1 = k(r-1); this gives us $v = \frac{k(rk-k+1)}{r}$ and b=rk-k+1.

It has been shown that in this case N is an incidence matrix for a partially balanced incomplete block (PBIB) design with v treatments, b blocks, r replications, k plots per block, $\lambda_1 = 1$, and $\lambda_2 = 0$. Defining first associates as two treatments which appear together in some block of the derived PBIB design, we get a two-class association scheme called a singly linked block (SLB) association scheme, with the following parameters.

$$v = \frac{k(rk-k+1)}{r}$$

$$n_1 = r(k-1)$$

$$n_2 = \frac{(k-r)(r-1)(k-1)}{r} \tag{2.5}$$

$$p^1_{11} = k-2+(r-1)^2 \qquad p^2_{11} = r^2$$

$$p^1_{12} = (r-1)(k-r) \qquad p^2_{12} = r(k-r-1)$$

$$p^1_{22} = \frac{(r-1)(k-r)(k-r-1)}{r} \qquad p^2_{22} = (k-r)^2 + 2(r-1) - \frac{k(k-1)}{r}$$

For r=2 the SLB scheme is the same as the triangular scheme with m = k+1.

(e) Latin Square ($L_g(n)$) and Pseudo-Latin Square Association Schemes [2], [5], [7]

Suppose we have a set of $v = n^2$ elements, arranged in an nxn array. Letting two elements which appear in the same row or the same column be first associates and two elements which do not appear together in a row or column be second associates, we can define an $L_2(n)$ association scheme.

For $3 \leq g \leq n+1$, if a set of (g-2) mutually orthogonal nxn Latin squares exists, we can define a Latin square ($L_g(n)$) association scheme from the nxn array of the v elements in the following manner. If two elements appear in the same row or column of the array, or if they correspond to the same symbol in one of the (g-2) Latin squares, they are first associates; otherwise the two elements are second associates.

For the case g=4, n=4, we can take the Latin squares LS_1 and LS_2, where

LS_1 =

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

and

LS_2 =

1	2	3	4
3	4	1	2
4	3	2	1
2	1	4	3

If the 16 elements are arranged in the array

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

then the first associates of the element 8 are 5, 6, 7, 4, 12, 16, 3, 9, 14, 2, 11, and 13; for 8 corresponds to the symbol 3 in LS_1 and to the symbol 2 in LS_2.

For $2 \leq g \leq n + 1$, the Latin square association scheme has the following parameters.

$$v = n^2$$

$$n_1 = g(n-1)$$

$$n_2 = (n-g+1)(n-1)$$

(2.6) $$p^1_{11} = (g-1)(g-2) + n-2 \qquad p^2_{11} = g(g-1)$$

$$p^1_{12} = (n-g+1)(g-1) \qquad p^2_{12} = g(n-g)$$

$$p^1_{22} = (n-g+1)(n-g) \qquad p^2_{22} = (n-g)^2 + g-2$$

Following the nomenclature in [7], let us call an association scheme with the parameters (2.6) a pseudo-Latin square association scheme, whether or not it is obtainable from a set of (g-2) mutually orthogonal Latin squares. For example, an L_3 (6) scheme can be obtained from any 6 x 6 Latin square; its complement, or the scheme obtained by interchanging first and second associate classes, has the parameters of an L_4 (6) scheme, but no pair of mutually orthogonal 6 x 6 Latin squares exists.

(f) Negative Latin Square ($NL_g(n)$) Association Scheme [7]

It has been found that in many cases negative values of g and n will result in non-negative integers for the $L_g(n)$ parameters (2.6).

The simplest case is for $g = -1$ and $n = -4$; the resulting scheme has the following parameters.

$$v = 16$$

$$n_1 = 5$$

$$n_2 = 10$$

$$p^1_{11} = 0 \qquad p^2_{11} = 2$$

$$p^1_{12} = 4 \qquad p^2_{12} = 3$$

$p_{22}^1 = 6$ $\qquad$ $p_{22}^2 = 6$

Substituting -g for g and -n for n in (2.6), we get the following set of parameters.

$$v = 16$$
$$n_1 = 5$$
$$n_2 = 10$$
$$p_{11}^1 = 0 \qquad p_{11}^2 = 2$$
$$p_{12}^1 = 4 \qquad p_{12}^2 = 3$$
$$p_{22}^1 = 6 \qquad p_{22}^2 = 6$$

Substituting -g for g and -n for n in (2.6), we get the following set of parameters.

$$\begin{aligned} v &= n^2 \\ n_1 &= g(n+1) \\ n_2 &= (n-g-1)(n+1) \\ p_{11}^1 &= (g+1)(g+2)-n-2 \qquad & p_{11}^2 &= g(g+1) \\ p_{12}^1 &= (n-g-1)(g+1) \qquad & p_{12}^2 &= g(n-g) \\ p_{22}^1 &= (n-g-1)(n-g) \qquad & p_{22}^2 &= (n-g)^2-(g+2) \end{aligned} \tag{2.7}$$

An association scheme with the parameters (2.7) is called a <u>negative Latin Square</u> ($NL_g(n)$) association scheme.

(g) <u>Pseudo-Geometric Association Scheme</u> [2]

A <u>Partial geometry</u> (r, k, t) is a system of points and lines, and a relation of incidence which satisfies the following axioms:

(i) any two distinct points are incident with not more than one line;

(ii) each point is incident with r lines;

(iii) each line is incident with k points;

(iv) if the point P is not incident with the line ℓ, then there are exactly t lines ($t \geq 1$) which are incident with P and also incident with some point incident with ℓ .

Clearly, we have

(2.8) $$1 \leq t \leq k,\ 1 \leq t \leq r, \text{ where } r \text{ and } k \text{ are } \geq 2.$$

It is easily seen from an examination of the four axioms above that given a partial geometry (r, k, t), we can obtain a dual partial geometry (k, r, t) by changing points to lines and lines to points.

The number of points v and the number of lines b in a partial geometry (r, k, t) satisfy the relations

(2.9) $$v = \frac{k[(r-1)(k-1) + t]}{t}$$

and

(2.10) $$b = \frac{r[(r-1)(k-1)+t]}{t} .$$

For convenience, we may use the ordinary geometric language when referring to partial geometries. Thus if a point and line are incident, we say that the point lies on the line (is contained in the line) and that the line passes through the point. A line which contains two points P and Q joins P and Q. If a point P lies on two lines ℓ and m, we say that ℓ and m intersect at P.

Let us call the points of a partial geometry treatments and call the lines blocks. The relation of incidence will then be that of a treatment's being contained in a block. Call two treatments first associates if they occur together in a block; otherwise they are second associates. Thus we see that a partial geometry (r, k, t) is equivalent to a PBIB design with parameters

$$(2.11)\qquad v,\ b,\ r,\ k,\ \lambda_1 = 1,\ \lambda_2 = 0,$$

where v and b are given by (2.9) and (2.10). The parameters of the corresponding association scheme are the following.

$$v = \frac{k[(r-1)(k-1)+t]}{t}$$

$$n_1 = r(k-1)$$

$$n_2 = \frac{(r-1)(k-1)(k-t)}{t}$$

$$(2.12)\qquad p^1_{11} = (t-1)(r-1)+k-2 \qquad p^2_{11} = rt$$

$$p^1_{12} = (r-1)(k-t) \qquad p^2_{12} = r(k-t-1)$$

$$p^1_{22} = \frac{(r-1)(k-t)(k-t-1)}{t} \qquad p^2_{22} = \frac{(r-1)(k-1)(k-t)}{t} - r(k-t-1)-1$$

We will call any association scheme with the parameters (2.12) and for which (2.8) holds a pseudo-geometric association scheme, since such a scheme may exist without being derived from a partial geometry (r, k, t). However, if a pseudo-geometric scheme is a scheme derived from a partial geometry, we will call it a geometric association scheme.

Several of the association schemes mentioned earlier in this chapter are special cases of pseudo-geometric schemes. In particular, a partial geometry (r, k, r-1) gives rise to an $L_r(k)$ scheme. Thus a pseudo-Latin square scheme is just a special case of a pseudo-geometric scheme. Also, a partial geometry (r, k, r) gives us an SLB association scheme; thus we might introduce the term pseudo-SLB scheme, corresponding to a pseudo-geometric scheme with the appropriate parameters. It has been noted previously that a triangular scheme is a special case of an SLB scheme; hence we might speak of a pseudo-triangular scheme as a special case of a pseudo-geometric scheme.

2.2 Association Schemes of Three Classes

(a) Group Divisible (GD) m-Associate Scheme [8]

Suppose we have the number of elements

$$v = N_1 N_2 \ldots N_m, \text{ where all the } N_i\text{'s are} \geq 2.$$

We can denote an element by an ordered m-triple $(i_1, i_2, \ldots, i_m)$, where $i_j \in (0, 1, \ldots, N_j-1)$ for $j = 1, 2, \ldots, m$. Let two elements which have only the first $(m-j)$ coordinates in common be jth associates $(j=1, 2, \ldots, m)$. Then we have an association scheme, called a group divisible m-associate scheme, with the following parameters: for $i=1, 2, \ldots, m$, we have

$$v = N_1 N_2 \ldots N_m,$$

$$n_i = N_m N_{m-1} \ldots N_{m-i+2} (N_{m-i+1} - 1),$$

(2.13)

$$p^i = (p^i_{jk}) = \left[\begin{array}{c|c|c} 0_{(i-1)\times(i-1)} & \underline{x}_{i-1} & 0_{(i-1)\times(m-i)} \\ \hline \underline{x}'_{i-1} & & \\ \cline{1-1} 0_{(m-i)\times(i-1)} & \multicolumn{2}{c}{D_{(m-i+1)\times(m-i+1)}} \end{array}\right] \quad (j, k=1, 2, 3),$$

where 0_{sxt} is the sxt matrix of zeros, $\underline{x}_{i-1}$ is a column vector of order $(i-1)$ with elements $n_1, n_2, \ldots, n_{i-1}$, respectively, and $D_{(m-i+1)\times(m-i+1)}$ is a diagonal matrix with diagonal elements

$$[N_m N_{m-1} \ldots N_{m-i+2}(N_{m-i+1} - 2)],\ n_{i+1},\ n_{i+2},\ \ldots,\ n_m, \text{ respectively.}$$

For a 3-class GD scheme, we have $N_1, N_2, N_3 \geq 2$, with the parameters given below.

$$v = N_1 N_2 N_3$$

$$n_1 = N_3 - 1$$

$$n_2 = N_3(N_2 - 1)$$

$$n_3 = N_3 N_2 (N_1 - 1)$$

(2.14)

$$p^1_{11} = N_3 - 2 \qquad p^2_{11} = 0 \qquad p^3_{11} = 0$$

$$p^1_{12} = 0 \qquad p^2_{12} = N_3 - 1 \qquad p^3_{12} = 0$$

$$p^1_{13} = 0 \qquad p^2_{13} = 0 \qquad p^3_{13} = N_3 - 1$$

$$p^1_{22} = N_3(N_2 - 1) \qquad p^2_{22} = N_3(N_2 - 2) \qquad p^3_{22} = 0$$

$$p^1_{23} = 0 \qquad p^2_{23} = 0 \qquad p^3_{23} = N_3(N_2 - 1)$$

$$p^1_{33} = N_3 N_2 (N_1 - 1) \qquad p^2_{33} = N_3 N_2 (N_1 - 1) \qquad p^3_{33} = N_3 N_2 (N_1 - 2)$$

(b) Tetrahedral Association Scheme [3]

A three-class association scheme, called a tetrahedral scheme, can be defined in a manner analogous to the definition of the two-class triangular scheme. Suppose there exists a set of $v = \binom{n}{3}$ elements, for some positive integer n; we can denote the v elements by unordered triples (x_1, x_2, x_3), where $x_1 \neq x_2 \neq x_3$ and x_1, x_2, and x_3 range from 0 to n-1. The elements can then be considered as points in three-dimensional Euclidean space; two elements with the same coordinates, in any order, will be considered the same. For i=1, 2, 3, call two elements ith associates if they differ in exactly i coordinates; for example, the elements (1, 2, 3) and (1, 4, 2) are first associates. This definition of association gives us an association scheme in three classes, with the following parameters.

$$v = \frac{n(n-1)(n-2)}{6}$$

$$n_1 = 3(n-3)$$

$$n_2 = \frac{3(n-3)(n-4)}{2}$$

$$n_3 = \frac{(n-3)(n-4)(n-5)}{6}$$

(2.15)

$$p_{11}^1 = n-2 \qquad p_{11}^2 = 4 \qquad p_{11}^3 = 0$$

$$p_{12}^1 = 2(n-4) \qquad p_{12}^2 = 2(n-4) \qquad p_{12}^3 = 9$$

$$p_{13}^1 = 0 \qquad p_{13}^2 = n-5 \qquad p_{13}^3 = 3(n-6)$$

$$p_{22}^1 = (n-4)^2 \qquad p_{22}^2 = \frac{(n-5)(n+2)}{2} \qquad p_{22}^3 = 9(n-6)$$

$$p_{23}^1 = \frac{(n-4)(n-5)}{2} \qquad p_{23}^2 = (n-5)(n-6) \qquad p_{23}^3 = \frac{3(n-6)(n-7)}{2}$$

$$p_{33}^1 = \frac{(n-4)(n-5)(n-6)}{6} \qquad p_{33}^2 = \frac{(n-5)(n-6)(n-7)}{6}$$

$$p_{33}^3 = \frac{(n-6)(n-7)(n-8)}{6}$$

We see that for such a scheme we must have $n \geq 6$.

(c) Cubic Association Scheme [9]

Suppose we have a set of $v = n^3$ elements, for some integer $n \geq 2$. Consider the v elements as ordered triples (x_1, x_2, x_3), where x_1, x_2, and x_3 range from 0 to n-1. Call two treatments ith associates if they have exactly i coordinates different (i = 1, 2, 3). Equivalently, we can consider the elements as points in three-dimensional Euclidean space. Then the first associates of a point α are those points lying on the three lines through α which are perpendicular to the coordinate planes; the second associates of α are the remaining points lying in the three planes determined by the first associates of α; other points are third associates of α.

The resulting scheme is an association scheme in three classes, called a cubic scheme. The parameters are the following.

$$v = n^3$$

$$n_1 = 3(n-1)$$

$$n_2 = 3(n-1)^2$$

$$n_3 = (n-1)^3$$

(2.16) $p_{11}^1 = n-2$ $\quad p_{11}^2 = 2$ $\quad p_{11}^3 = 0$

$p_{12}^1 = 2(n-1)$ $\quad p_{12}^2 = 2(n-2)$ $\quad p_{12}^3 = 3$

$p_{13}^1 = 0$ $\quad p_{13}^2 = n-1$ $\quad p_{13}^3 = 3(n-2)$

$p_{22}^1 = 2(n-1)(n-2)$ $\quad p_{22}^2 = n^2-2n+2$ $\quad p_{22}^3 = 6(n-2)$

$p_{23}^1 = (n-1)^2$ $\quad p_{23}^2 = 2(n-2)(n-1)$ $\quad p_{23}^3 = 3(n-2)^2$

$p_{33}^1 = (n-1)^2(n-2)$ $\quad p_{33}^2 = (n-1)(n-2)^2$ $\quad p_{33}^3 = (n-2)^3$

(d) Rectangular Association Scheme [13]

Suppose we have a set of $v = \ell n$ elements for some integers ℓ, $n \geq 2$. Then we can arrange the v elements in a rectangular array with ℓ rows and n columns. If two elements appear in the same row, call them first associates; if they appear in the same column, call them second associates; otherwise call them third associates. Then we have a three-class association scheme, called a rectangular scheme, with the following parameters.

$$v = \ell n$$

$$n_1 = n-1$$

$$n_2 = \ell - 1$$

$$n_3 = (\ell - 1)(n-1)$$

(2.17) $p_{11}^1 = n-2$ $\quad p_{11}^2 = 0$ $\quad p_{11}^3 = 0$

$p_{12}^1 = 0$ $\quad p_{12}^2 = 0$ $\quad p_{12}^3 = 1$

$p^1_{13} = 0$ $\quad p^2_{13} = n-1$ $\quad p^3_{13} = n-2$

$p^1_{22} = 0$ $\quad p^2_{22} = \ell-2$ $\quad p^3_{22} = 0$

$p^1_{23} = \ell-1$ $\quad p^2_{23} = 0$ $\quad p^3_{23} = \ell-2$

$p^1_{33} = (\ell-1)(n-2)$ $\quad p^2_{33} = (\ell-2)(n-1)$ $\quad p^3_{33} = (\ell-2)(n-2)$

(e) Three-Class Association Scheme from an Orthogonal Array [12]

An orthogonal array (N, m, s, t) is an mxN rectangular array of N assembles, with m constraints, in s symbols (e. g., the elements of the array may be the integers 0, 1, ..., s-1), such that in any t-rowed submatrix of the array each of the s^t possible column vectors appears exactly λ times, where $\lambda s^t = N$. λ is called the index of the array.

Suppose we have an orthogonal array $(n^2, \beta_1 + \beta_2, n, 2)$, where n, β_1, and β_2 are positive integers such that $n \geq 2$ and $\beta_1 + \beta_2 \leq n$. We see that the index $\lambda = 1$ in this case. Consider the n^2 assemblies as treatments. Define two treatments α and β as first associates if the columns corresponding to α and β are alike in exactly one position in the first β_1 rows; let α and β be second associates if the columns corresponding to them coincide in exactly one position in the remaining β_2 rows; otherwise α and β will be third associates. Since the array has strength 2 and index 1, we see that the definition of association is unambiguous; for two columns of the array can be alike in at most one position. Then we have a three-class association scheme with the following parameters.

$$v = n^2$$

$$n_1 = \beta_1(n-1)$$

$$n_2 = \beta_2(n-1)$$

$$n_3 = \beta_3(n-1), \text{ where } \beta_3 = n+1 - (\beta_1 + \beta_2)$$

(2.18)

$$p_{11}^1 = n-2+(\beta_1-1)(\beta_1-2) \qquad p_{11}^2 = \beta_1(\beta_1-1) \qquad p_{11}^3 = \beta_1(\beta_1-1)$$

$$p_{12}^1 = \beta_2(\beta_1-1) \qquad p_{12}^2 = \beta_1(\beta_2-1) \qquad p_{12}^3 = \beta_1\beta_2$$

$$p_{13}^1 = \beta_3(\beta_1-1) \qquad p_{13}^2 = \beta_1\beta_3 \qquad p_{13}^3 = \beta_1(\beta_3-1)$$

$$p_{22}^1 = \beta_2(\beta_2-1) \qquad p_{22}^2 = n-2+(\beta_2-1)(\beta_2-2) \qquad p_{22}^3 = \beta_2(\beta_2-1)$$

$$p_{23}^1 = \beta_2\beta_3 \qquad p_{23}^2 = \beta_3(\beta_2-1) \qquad p_{23}^3 = \beta_2(\beta_3-1)$$

$$p_{33}^1 = \beta_3(\beta_3-1) \qquad p_{33}^2 = \beta_3(\beta_3-1) \qquad p_{33}^3 = n-2+(\beta_3-1)(\beta_3-2)$$

3. Construction of BIB Designs from Linear Combinations of Association Matrices

In this section we wish to construct BIB designs by obtaining matrices of the form $B_{i_1} + B_{i_2} + \dots + B_{i_t}$ which will be BIB incidence matrices, where the B_i's are association matrices. The method is quite similar to a method suggested by Shrikhande and Singh [11]. We prove the following theorem.

Theorem 3.1

Suppose we have an m-class association scheme in v elements, with association matrices $B_o = I_v, B_1, B_2, \dots, B_m$. Suppose $i_1, i_2, \dots, i_t$ are distinct integers such that $i_j \in (0, 1, 2, \dots, m)$ for $j = 1, 2, \dots, t \leq m$. Then the necessary and sufficient condition for $C = B_{i_1} + B_{i_2} + \dots + B_{i_t}$ to be an incidence matrix for a BIB

$(v, v, r, r, \lambda^{(i_1, i_2, \dots, i_t)})$, where

$$r = n_{i_1} + n_{i_2} + \dots + n_{i_t}$$

and

$$\lambda^{(i_1, i_2, \dots, i_t)} = \frac{r(r-1)}{v-1} ,$$

is that $\lambda^{(i_1, i_2, \dots, i_t)}$ be a positive integer and $\sum_{j=1}^{t} p^k_{i_j i_j} + 2 \sum_{j<\ell}^{t} p^k_{i_j i_\ell} = \lambda^{(i_1, i_2, \dots, i_t)}$ for $k = 1, 2, \dots, m$.

Proof: (For convenience, we shall denote $\lambda^{(i_1, i_2, \dots, i_t)}$ by λ during the proof.) Suppose λ is a positive integer. Then the required necessary and sufficient condition is that

$$(1) \quad C J_v = r J_v$$

and

$$(2) \qquad CC' = (r-\lambda)I_v + \lambda J_v = r I_v + \lambda (J_v - I_v).$$

Now

$$C J_v = (B_{i_1} + B_{i_2} + \dots + B_{i_t}) J_v$$
$$= n_{i_1} J_v + n_{i_2} J_v + \dots + n_{i_t} J_v$$
$$= r J_v.$$

Then (1) is always satisfied when the B_i's are association matrices.

$$CC' = (B_{i_1} + B_{i_2} + \dots + B_{i_t}) (B_{i_1} + \dots + B_{i_t})'$$
$$= B_{i_1} B_{i_1} + B_{i_2} B_{i_2} + \dots + B_{i_t} B_{i_t} + 2 \sum_{j<\ell}^{t} B_{i_j} B_{i_\ell}$$
$$= \sum_{k=o}^{m} p_{i_1 i_1}^{k} B_k + \sum_{k=o}^{m} p_{i_2 i_2}^{k} B_k + \dots + \sum_{k=o}^{m} p_{i_t i_t}^{k} B_k$$
$$+ 2 \sum_{k=o}^{m} \sum_{j<\ell}^{t} p_{i_j i_\ell}^{k} B_k .$$

Since $p_{ii}^{o} = n_i$ and $p_{ij}^{o} = 0$ for $i \neq j$, we have

$$CC' = r I_v + \left(\sum_{j=1}^{t} p_{i_j i_j}^{1} + 2 \sum_{j<\ell}^{t} p_{i_j i_\ell}^{1} \right) B_1$$
$$+ \left(\sum_{j=1}^{t} p_{i_j i_j}^{2} + 2 \sum_{j<\ell}^{t} p_{i_j i_\ell}^{2} \right) B_2$$
$$+ \dots + \left(\sum_{j=1}^{t} p_{i_j i_j}^{m} + 2 \sum_{j<\ell}^{t} p_{i_j i_\ell}^{m} \right) B_m .$$

Now

$$\lambda (B_1 + B_2 + \dots + B_m) = \lambda (J_v - I_v).$$

Then, since the B_i's are linearly independent, the necessary and sufficient condition for (2) to hold is that

$$\sum_{j=1}^{t} p^k_{i_j i_j} + 2\sum_{j<\ell}^{t} p^k_{i_j i_\ell} = \lambda\,,\ k = 1, 2, \ldots, m, \text{ and the theorem is}$$

proved.

We now apply Theorem 3.1 to the specific schemes discussed in sections 2.1 and 2.2 to determine what designs can be constructed.

Two-Class Schemes

(a) GD scheme: no BIB designs can be constructed using the method of Theorem 3.1.

(b) Triangular scheme: for n=6, the parameters (2.2) are such that the matrix B_1 is an incidence matrix for a BIB (15, 15, 8, 8, 4).

(c) Pseudo-cyclic scheme: we get no designs.

(d) SLB scheme: for k=2r + 1 in (2.5), the matrix B_1 is an incidence matrix for a BIB ($4r^2 - 1$, $4r^2 - 1$, $2r^2$, $2r^2$, r^2), if the corresponding SLB scheme exists.

(e) $L_g(n)$ scheme: for the case n = 2g in (2.6), if the scheme exists, then B_1 is an incidence matrix for a BIB ($4g^2$, $4g^2$, g(2g-1), g(2g-1), g(g-1)).

(f) NL_g (n) scheme: if n = 2g in (2.7) and the corresponding NL_g(n) scheme exists, then B_1 is an incidence matrix for a BIB

$(4g^2,\ 4g^2,\ g(2g+1),\ g(2g+1),\ g(g+1)\ )$.

(g) Pseudo-geometric scheme: if a scheme with parameters (2.12) is such that $t = k-r-1$, then B_1 is an incidence matrix for a BIB

$$\left(\frac{rk(k-2)}{k-r-1}, \frac{rk(k-2)}{k-r-1}, r(k-1), r(k-1), r(k-r-1)\right);$$

for the case $t = k-r+1$, the scheme is such that B_2 is an incidence matrix for a BIB

$$\left(\frac{k[r(k-2)+2]}{k-r+1}, \frac{k[r(k-2)+2]}{k-r+1}, \frac{(r-1)^2(k-1)}{k-r+1}, \frac{(r-1)^2(k-1)}{k-r+1}, \frac{(r-1)^2(r-2)}{k-r+1}\right)$$

Three-Class Schemes

(a) GD scheme: no designs are obtainable from Theorem 3.1.

(b) Tetrahedral scheme: if $n = 7$ in (2.15), then the scheme is such that B_2 is an incidence matrix for a BIB (35, 35, 18, 18, 9).

(c) Cubic scheme: for the case $n = 4$ in (2.16), the matrix $(B_1 + B_3)$ is an incidence matrix for a BIB (64, 64, 36, 36, 20).

(d) Rectangular scheme: for $\ell = n = 4$ in (2.17), the matrix $(B_1 + B_2)$ is an incidence matrix for a BIB (16, 16, 6, 6, 2).

(e) Scheme from an orthogonal array: in (2.18), if n is even and we take $\beta_1 = \frac{n}{2}$, then if the scheme exists B_1 is an incidence matrix for a BIB $(n^2, n^2, \frac{n}{2}(n-1), \frac{n}{2}(n-1), \frac{n}{4}(n-2))$; if n is even and we have $\beta_1 + \beta_2 = \frac{n}{2}$, then $(B_1 + B_2)$ is an incidence matrix for a BIB$(n^2, n^2, \frac{n}{2}(n-1), \frac{n}{2}(n-1), \frac{n(n-2)}{4})$.

We may be able to construct a design in the latter case which is not obtainable in the former case, e.g., for $n = 6$.

4. Construction of BIB Designs from Juxtapositions of Association Matrices

Suppose we wish to find a matrix of the form $\left[B_{i_1} \vdots B_{i_2} \vdots \cdots \vdots B_{i_s} \right]$, where the B_i's are association matrices, which will be a BIB incidence matrix; we call such a matrix a juxtaposition of association matrices. We can prove the following theorem.

Theorem 4.1

Suppose we have an m-class association scheme in v elements, with association matrices $B_o = I_v, B_1, B_2, \ldots, B_m$. Suppose $i_1, i_2, \ldots, i_t$ are distinct integers such that $i_j \in (1, 2, \ldots, m)$ for $j = 1, 2, \ldots, t \leq m$. Then the necessary and sufficient condition for $D = [B_{i_1} \vdots B_{i_2} \vdots \cdots \vdots B_{i_t}]$ to be an incidence matrix for a BIB $(v, tv, tk, k, \lambda_{i_1 i_2 \ldots i_t})$, where

$$\lambda_{i_1 i_2 \ldots i_t} = \frac{tk(k-1)}{v-1} \quad , \text{ is that}$$

(i) $\lambda_{i_1 i_2 \ldots i_t}$ be a positive integer ,

(ii) $n_{i_1} = n_{i_2} = \ldots = n_{i_t} = k$,

(iii) $\sum_{j=1}^{t} p^1_{i_j i_j} = \sum_{j=1}^{t} p^2_{i_j i_j} = \ldots = \sum_{j=1}^{t} p^m_{i_j i_j} = \lambda_{i_1 i_2 \ldots i_t}$.

Proof: (For convenience, we shall denote $\lambda_{i_1 i_2 \ldots i_t}$ by λ.) Suppose λ is a positive integer, and suppose $n_{i_1} = n_{i_2} = \ldots = n_{i_t} = k$. (It is obvious that (i) and (ii) must be satisfied for D to be a BIB incidence matrix.) Then the required necessary and sufficient condition is that

(1) $D' J_v = k J_{tv,v}$

and

$$(2) \quad DD' = (tk - \lambda)\, I_v + \lambda J_v$$

$$= tk\, I_v + \lambda(J_v - I_v).$$

Now

$$D'\, J_v = \begin{vmatrix} B_{i_1} \\ \cdots \\ B_{i_2} \\ \cdots \\ \vdots \\ \cdots \\ B_{i_t} \end{vmatrix} \quad J_v = \begin{vmatrix} kJ_v \\ \cdots \\ kJ_v \\ \vdots \\ \cdots \\ kJ_v \end{vmatrix} = k\, J_{tv,v} \quad .$$

Then (1) is always satisfied.

$$DD' = \left[B_{i_1} \vdots B_{i_2} \vdots \cdots \vdots B_{i_t} \right] \begin{vmatrix} B_{i_1} \\ \cdots \\ B_{i_2} \\ \cdots \\ \vdots \\ \cdots \\ B_{i_t} \end{vmatrix}$$

$$= B_{i_1} B_{i_1} + B_{i_2} B_{i_2} + \cdots + B_{i_t} B_{i_t} = tk\, I_v$$

$$+ \sum_{c=1}^{m} p^c_{i_1 i_1} B_c + \sum_{c=1}^{m} p^c_{i_2 i_2} B_c + \cdots + \sum_{c=1}^{m} p^c_{i_t i_t} B_c$$

$$= tk\, I_v + \left(\sum_{j=1}^{t} p^1_{i_j i_j} \right) B_1 + \left(\sum_{j=1}^{t} p^2_{i_j i_j} \right) B_2 + \cdots + \left(\sum_{j=1}^{t} p^m_{i_j i_j} \right) B_m .$$

Now

$$\lambda(B_1 + B_2 + \cdots + B_m) = \lambda(J_v - I_v).$$

Then, by the linear independence of the B_i's, the necessary and sufficient condition for (2) to hold is that

$$\sum_{j=1}^{t} p^{1}_{i_j i_j} = \sum_{j=1}^{t} p^{2}_{i_j i_j} = \dots = \sum_{j=1}^{t} p^{m}_{i_j i_j} = \lambda ;$$

thus the theorem is proved.

Let us now apply Theorem 4.1 to the schemes of sections 2.1 and 2.2.

Two-class schemes

(a) GD scheme: we get no BIB designs.

(b) Triangular scheme: we get no designs.

(c) Pseudo-cyclic scheme: for any scheme with parameters (2.4), the matrix $[B_1 \vdots B_2]$ is an incidence matrix for a BIB $(4u + 1, 2(4u+1), 4u, 2u, 2u-1)$.

(d) SLB scheme: we get no designs.

(e) $L_g(n)$ scheme: if a scheme with parameters (2.6) is such that $n = 2g-1$, then $[B_1 \vdots B_2]$ is an incidence matrix for a BIB $((2g-1)^2, 2(2g-1)^2, 4g(g-1), 2g(g-1), 2g^2-2g-1)$.

(f) $NL_g(n)$ scheme: if a scheme with parameters (2.7) has $n = 2g+1$, then $[B_1 \vdots B_2]$ is an incidence matrix for a BIB$((2g+1)^2, 2(2g+1)^2, 4g(g+1), 2g(g+1), 2g^2 + 2g-1)$.

Note that such a design is also obtainable from a scheme with $L_{g+1}(2g+1)$ parameters, if such a scheme exists.

(g) Pseudo-geometric scheme: if a scheme with parameters (2.12) is such that $k = 2r-1$ and $t= r-1$, then $[B_1 \vdots B_2]$ is an incidence matrix for a BIB $((2r-1)^2, 2(2r-1)^2, 4r(r-1), 2r(r-1), 2r^2-2r-1)$.

Three-class Schemes

(a) GD scheme: we get no designs.

(b) Tetrahedral scheme: we get no designs.

(c) Cubic scheme: we get no designs.

(d) Rectangular scheme: we get no designs.

(e) Scheme from an orthogonal array: in (2.18), if $n > 2$ is even and the orthogonal array $(n^2, n, n, 2)$ exists, then we take $\beta_1 = \beta_2 = \frac{n}{2}$, and $[B_1 \vdots B_2]$ is an incidence matrix for a BIB $(n^2, 2n^2, n(n-1), \frac{n(n-1)}{2}, \frac{n(n-2)}{2})$; if $n > 2$ and $n \equiv 2 \pmod 3$ and the orthogonal array $(n^2, \frac{2(n+1)}{3}, n, 2)$ exists, then we take $\beta_1 = \beta_2 = \beta_3 = \frac{n+1}{3}$, and $[B_1 \vdots B_2 \vdots B_3]$ is an incidence matrix for a BIB $(n^2, 3n^2, n^2-1, \frac{n^2-1}{3}, \frac{n^2-4}{3})$.

5. Acknowledgment

The author expresses his appreciation to Dr. I. M. Chakravarti for directing this research and going through the manuscript.

REFERENCES

[1] Bose, R. C. (1939), "On the construction of balanced incomplete block designs." Annals of Eugenics, 9, 353-399.

[2] Bose, R. C. (1963), "Combinatorial properties of partially balanced designs and association schemes." Sankhyā, Series A, 25, 109-136.

[3] Bose, R. C. and Laskar, Renu (1966), "A characterization of tetrahedral graphs." Unpublished paper.

[4] Bose, R. C. and Mesner, D. M. (1959), "On linear associative alegebras corresponding to association schemes of partially balanced designs." Annals of Mathematical Statistics, 30, 21-38.

[5] Bose, R. C. and Shimamoto, T. (1952), "Classification and analysis of partially balanced incomplete block designs with two associate classes." Journal of the American Statistical Association, 47, 151-184.

[6] Connor, W. S. (1952), "On the structure of balanced incomplete block designs." Annals of Mathematical Statistics, 23, 57-71.

[7] Mesner, D. M. (1964), "Negative Latin square designs." Institute of Statistics, Mimeo Series No. 410.

[8] Raghavarao, D. (1960), "A generalization of group divisible designs." Annals of Mathematical Statistics, 31, 756-771.

[9] Raghavarao, D. and Chandrasekhararao, K. (1964), "Cubic designs." Annals of Mathematical Statistics, 35, 389-397.

[10] Ryser, H. J. (1963), Combinatorial Mathematics, Syracuse: Mathematical Association of America.

[11] Shrikhande, S. S. and Singh, N. K. (1962), "On a method of constructing symmetrical balanced incomplete block designs." Sankhyā, Series A, 24, 25-32.

[12] Singh, N. K. and Shukla, G. C. (1963), "The non-existence of some partially balanced incomplete block designs with three associate classes." Journal of the Indian Statistical Association, 1, 71-77.

[13] Vartak, M. N. (1959), "The non-existence of certain partially balanced incomplete block designs." Annals of Mathematical Statistics, 30, 1051-1062.