ESTIMATING VARIANCE COMPONENTS IN
A MULTI-WAY CLASSIFICATION
by
Norman Bush
and
R. L. Anderson

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CHAPTER I

INTRODUCTION

The primary purpose of this dissertation is to describe and illustrate an analytic procedure for obtaining variances of estimates of variance components in a general multi-way classification, when there are unequal numbers of replications in the subclasses. The variance component model is always taken to be Eisenhart's (1947) Model II, and the estimation procedure is based on equating the expected mean squares, from the analysis of variance table, to the computed mean squares, (as if it was a fixed model), and then solving a set of equations for the estimates of the variance components. For the fixed effects models we will use Eisenhart's definition of Model I, the only variance component being \( \sigma_e^2 \).

Specifically, three estimating procedures, A, B and H, are worked out for the two- and three-way classifications, and the generalization to higher-way classifications is indicated. Procedure A is based on the method of fitting constants, Yates (1934) and Henderson (1953); Procedure B, the method of weighted squares of means, Yates (1934); and Procedure H, the unadjusted sums of squares, Henderson (1953) and Le Roy and Gluckowski (1961). For the two-way classification, the three procedures are computed for a number of experimental designs and parameter values (true variance components) with the use of a UNIVAC 1105 digital computer.
The computer comparison was designed to compare experimental designs, for estimating variance components, as well as to compare the three estimation procedures. A summary of the computer results is given in Section 4.5, and the actual results in Appendix B. The computer program is shown in Appendix A.

A simplified computational method for Procedure A, in the two-way classification, is given in Chapter V, and the extensions to multi-way and nested classifications in Chapter VI.

All the experimental designs that are considered in this dissertation are connected designs. A connected design is defined as follows:

"A treatment and a block are said to be associated if the treatment occurs in that particular block. Two treatments, two blocks or a treatment and a block are said to be connected if it is possible to pass from one to the other by means of a chain, consisting alternately of treatments or blocks such that any two consecutive members are associated. Thus if treatments \( i_0 \), and \( i_n \) are connected, we must have a chain of treatments and blocks say,

\[ i_0, j_1, i_1, j_2, \ldots, i_{n-1}, j_n, i_n, \]

where \( i_0, i_1, \ldots, i_n \) are treatments, and \( j_1, j_2, \ldots, j_n \) are blocks, and \( i_{p-1}, i_p \) occur in block \( j_p \). A design is said to be connected if every block and treatment of the design is connected to every other block and treatment."

---

\(^1\)R. C. Bose gives this definition in his class notes of Analysis of Variance with application to experimental designs, University of North Carolina, Chapel Hill, North Carolina.
CHAPTER II

REVIEW OF LITERATURE

For the completely random model, in the multi-way crossed classification with unequal numbers in the subclasses, very little work has been done in the area of what is the best procedure and best experimental design for simultaneous estimation of variance components. Crump [1946, 1951] and Anderson and Bancroft [1952] discuss the estimation problem for the balanced case, and mention the difficulties in the unbalanced case. Henderson [1953] and Henderson et al. [1957] describe, with some examples, Procedures A and H for a two-way classification (these are Methods 3 and 1, respectively, in the article), but do not evaluate the estimation procedures. Searle [1956, 1958] develops a matrix procedure by which the variances of the estimated variance components for Procedure H can be obtained in the two-way classification. These results are extended to the three-stage nested design by Searle [1961]. Estimating procedure H is extended to the three-way classification by Le Roy and Gluckowski [1961]. None of the above articles attempts to compare or evaluate the various estimation procedures, except for Searle's work on Procedure H.

The area of design for estimating variance components is even more recent than the work in the estimation procedures. The earliest work, to the author's knowledge, on designs for the two-way classification is by Gaylor [1960]. Anderson [1961] reviews the problems
In this dissertation the L-design will be compared to some other designs; however, the BDR-design will not be considered, since it is not a connected design.

Instead we have been led to two connected rectangles design which are denoted as the S- and C-designs. For the case of six rows and six columns and \( n_{ij} = 0 \) or 1, these designs appear as follows:

\[
\begin{array}{cccccc}
S\text{-design} & & & & & \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 \\
\end{array}
\quad
\begin{array}{cccccc}
C\text{-design} & & & & & \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 \\
\end{array}
\]
of design in the nested classification and comments on Gaylor's 1960 work for the two-way classification model.

Gaylor points out that, when the experimenter is interested in simultaneously estimating both the row and column variance component (\( \sigma_r^2 \) and \( \sigma_c^2 \)), the optimum design depends upon the estimator used and the values of the parameters. He studied the simultaneous estimation of \( \sigma_r^2 \) and \( \sigma_c^2 \), when \( \sigma_r^2 = \sigma_c^2 \) and \( \text{Var}(\hat{\sigma}_r^2) = \text{Var}(\hat{\sigma}_c^2) \), for two designs which are denoted as the L-design and the balanced disjointed rectangles design (BDR-design). Examples of the L and BDR-designs are shown below for a two-way classification with six rows and six columns, in terms of the number of observations in each subclass.

<table>
<thead>
<tr>
<th>L-design</th>
<th>BDR-design</th>
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<tr>
<td>1 1 0 0 0 0</td>
<td>1 1 1 0 0 0</td>
</tr>
<tr>
<td>1 1 0 0 0 0</td>
<td>1 1 1 0 0 0</td>
</tr>
<tr>
<td>1 1 0 0 0 0</td>
<td>1 1 1 0 0 0</td>
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<td>1 1 0 0 0 0</td>
<td>0 0 0 1 1 1</td>
</tr>
<tr>
<td>1 1 1 1 1 1</td>
<td>0 0 0 1 1 1</td>
</tr>
<tr>
<td>1 1 1 1 1 1</td>
<td>0 0 0 1 1 1</td>
</tr>
</tbody>
</table>

Gaylor's general conclusions were based on changes in \( \rho \), where

\[
\rho = \frac{\sigma_r^2}{\sigma_{rc}^2 + \sigma_e^2} = \frac{\sigma_c^2}{\sigma_{rc}^2 + \sigma_e^2},
\]

and \( \sigma_r^2, \sigma_c^2, \sigma_{rc}^2 \), and \( \sigma_e^2 \) represent the row, column, interaction and error components in the random model. As \( \rho \) increases, the L-design is better than the BDR design, and as \( \rho \) decreases the BDR-design is better. The variances of the estimates of \( \sigma_r^2 \) (and \( \sigma_c^2 \)) for the two designs are about equal when \( \rho \) is approximately 2.
CHAPTER III

THEORETICAL DEVELOPMENT

3.1 Introduction

The main purpose of this chapter is to illustrate, in a general way, how to calculate the variances of the estimates of the variance components for different estimation procedures. Thus it becomes possible to compare and study a class of estimation procedures over a suitable range of parameter values and unequal subclass numbers. In order to understand the development of this analytic procedure, it is necessary to discuss first the calculation of sums of squares in Model I, through the cell means rather than the more conventional row, column and interaction effects. Then, as it will be shown for Model II, it is relatively easy to obtain the variance-covariance matrix of a vector of sum of squares, which is the key computation in obtaining the variances of the estimates of the variance components.

3.2 Analysis of Variance in Model I

The mathematical model for a multi-way classification can be written as

\[ y_{ijk...} = \mu + \alpha_i + \beta_j + \delta_k + \ldots + (\alpha \beta)_{ij} + (\alpha \delta)_{ik} + \ldots \\
+ (\alpha \beta \delta)_{ijk} + \ldots + e_{ijk...}; \]

(3.1)

\[ y_{ijk...} \] - the observed value for cell \( ijk... \),

\[ \mu \] - the population mean,
\( \alpha, \beta, \delta, \) etc. - the population main effects,

\((\alpha \beta), (\alpha \delta), (\alpha \beta \delta), \) etc. - the population two factor and higher interaction effects,

\( \epsilon_{ijk...} \) - a random residual assumed normally independently distributed with a mean zero and variance

\( \sigma_e^2; \text{NID}(0, \sigma_e^2). \)

In matrix notation (3.1) can be expressed as

\[
\begin{align*}
\mathbf{y} & = \mathbf{A} \mathbf{\xi} + \mathbf{\epsilon}, \\
\text{n \times l} & \quad \text{n \times m} \quad \text{m \times l} \quad \text{n \times l}
\end{align*}
\]

(3.2)

where

\( \mathbf{y} \) : column vector of observed values,

\( \mathbf{A} \) : known matrix of ones and zeroes for a particular design, where \( \text{Rank}(\mathbf{A}) = t \leq m, \) and \( n > t, \)

\( \mathbf{\xi} \) : vector of parameters, \( \mu, \alpha_i, \beta_j, ..., \)

\( \mathbf{\epsilon} \) : error vector, where \( \text{E}(\mathbf{\epsilon}) = 0, \text{E}(\mathbf{\epsilon} \mathbf{\epsilon}') = \sigma_e^2 \mathbf{I}, \)

\( \mathbf{I} \) being the identity matrix.

In the multi-way classification, the mathematical model can also be written in terms of subclass parameters, i.e., the expected values of the means of each non-empty cell. This model would then appear as

\[ \text{Underscored Roman or Greek small letters will be column vectors; the prime will denote transpose. Thus } \mathbf{y} \text{ is a column vector of } \mathbf{n \times l} \text{ elements, while } \mathbf{y}' \text{ is a row vector. Capital Roman letters will be used for matrices, e.g. } \mathbf{A} \text{ is a matrix of } \mathbf{n \times m} \text{ rows and } \mathbf{m \times m} \text{ columns.} \]
\[ \gamma = A \gamma + \epsilon \quad (3.3) \]

where now \( A \) is of full rank (=t), and \( \gamma \) is the column vector of the cell mean parameters. Note that \( t \) equals the number of non-empty subclasses in the experimental design.

To illustrate the formulation of (3.2) and (3.3), consider a two-way (2 x 3) classification.

\[ \gamma_{ijk} = \mu + \alpha_i + \beta_j + (\alpha \beta)_{ij} + \epsilon_{ijk} \quad (3.4) \]

where \( i = 1, 2; j = 1, 2, 3; k = 1, 2, \ldots, n_{ij}, \) i.e., two rows and three columns, with (possibly) unequal numbers in each subclass.

In (3.2),

\[ \mathbf{\hat{x}} = \begin{bmatrix} \mu, \alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, (\alpha \beta)_{11}, (\alpha \beta)_{12}, (\alpha \beta)_{13}, \\ (\alpha \beta)_{21}, (\alpha \beta)_{22}, (\alpha \beta)_{23} \end{bmatrix}, \]

which has 12 parameters, among which only 6 independent linear contrasts are estimable, if all \( n_{ij} > 0 \); and in (3.3)

\[ \mathbf{\gamma}' = (\gamma_{11}, \gamma_{12}, \gamma_{13}, \gamma_{21}, \gamma_{22}, \gamma_{23}) \]

which has 6 estimable parameters if all \( n_{ij} > 0 \). Any hypothesis that can be expressed as some linear function of the parameters in \( \mathbf{\hat{x}} \), can also be expressed as a linear function of the \( \gamma \)'s. A complete and detailed description of testing hypotheses in univariate analysis of variance, in terms of (3.2) can be found in a book by Roy [1957] and an article by Roy and Gnanadesikan [1952]. Some comments on testing for (3.3) are in a paper by Elston and Bush [1967].
For the remainder of this dissertation the development of the necessary mathematics depends on computing sum of squares through the estimates of the parameters in (3.3); thus, (3.1) and (3.2) will only be used for illustrative purposes.

In model I, any hypothesis can be expressed as

\[ H_0: \mathbf{C'}x = 0, \quad \text{vs} \quad H_1: \mathbf{C'}x \neq 0, \]

where the \( \mathbf{C'} \)-matrix in terms of its coefficients reflects the particular hypothesis to be tested and is of rank \( s \). Since \( \mathbf{A} \) is of full rank in (3.3), the customary \( F \) statistic for testing \( H_0 \) as shown by Roy \( \int_{1957} \) is

\[ F = \frac{\left( \mathbf{y}'(\mathbf{A}'\mathbf{A})^{-1}\mathbf{C'}(\mathbf{A}'\mathbf{A})^{-1}\mathbf{C'y} \right)^2}{\left( \mathbf{y}'\mathbf{y} - \mathbf{y}'(\mathbf{A}'\mathbf{A})^{-1}\mathbf{A'y} \right)/n_e}, \]

where \( s \) is the hypothesis degrees of freedom (d.f.) and \( n_e = n-t \) is the error d.f.

The sum of squares for hypothesis, shown between the curly brackets of the numerator of (3.5) can be written as

\[ \text{SSH} = \mathbf{y}'\mathbf{C} \sum_{\mathbf{C'}(\mathbf{A}'\mathbf{A})^{-1}\mathbf{C'y}} \mathbf{s}, \text{ with } s \text{ d.f.,} \]

where \( \mathbf{y}' = \mathbf{y}'(\mathbf{A}'\mathbf{A})^{-1} \), the sample estimate of \( \mathbf{y} \), and \( \mathbf{B}(\mathbf{y}) = \mathbf{y} \).

Note that \( \mathbf{y} \) is simply a vector of the mean values from each non-empty cell of the design. Defining

\[ \mathbf{Q} = \mathbf{C} \sum_{\mathbf{C'}(\mathbf{A}'\mathbf{A})^{-1}\mathbf{C'y}} \mathbf{y}, \]

it is easy to see for different hypotheses, say \( H_0: \mathbf{C'}\gamma = 0 \), \( j = 1, 2, \ldots, \) that in

\[ \text{SS}_j = \mathbf{y}'\mathbf{Q}_j \mathbf{y}, \]
as \( j \) varies only the \( Q \) matrix is altered. Regardless of the total number of observations in the experiment, the dimension of the \( Q \) matrix depends only on the number of non-empty cells in the experiment.

### 3.3 Constructing \( C' \)-matrices for Model I

The main purpose of this section is to show that all the sums of squares of interest, which come from a connected design, can be found either directly in terms of (3.6), or through some linear combination of SSH's. Although some of the SSH's to be described below are computationally very simple without the use of matrix algebra, it is the associated \( C' \)-matrices that are necessary for the theoretical development.

For illustrative purposes, consider the \( 2 \times 3 \) classification shown in (3.4). Using the convention that a dot instead of a subscript denotes a total over that subscript \( \sum_{j=1}^{3} y_{1j} = y_{1.} \), the observations for the experiment are:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>Totals</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rows</td>
<td>y_{11.}</td>
<td>y_{12.}</td>
<td>y_{13.}</td>
<td>y_{1..}</td>
</tr>
<tr>
<td></td>
<td>y_{21.}</td>
<td>y_{22.}</td>
<td>y_{23.}</td>
<td>y_{2..}</td>
</tr>
<tr>
<td>Totals</td>
<td>y_{.1}</td>
<td>y_{.2}</td>
<td>y_{.3}</td>
<td>y_{..} = G</td>
</tr>
</tbody>
</table>
Note that in this example and throughout the dissertation the order of the parameters, for the purpose of constructing matrices, will always be $\gamma_{11}, \gamma_{12}, \ldots, \gamma_{21}, \gamma_{22}, \ldots$. The numbers in each subclass, and the corresponding parameters are shown below:

<table>
<thead>
<tr>
<th>Columns</th>
<th>Totals</th>
<th>Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_{11}$</td>
<td>$n_{12}$</td>
<td>$n_{13}$</td>
</tr>
<tr>
<td>$n_{21}$</td>
<td>$n_{22}$</td>
<td>$n_{23}$</td>
</tr>
<tr>
<td>Totals</td>
<td>$n_{.1}$</td>
<td>$n_{.2}$</td>
</tr>
</tbody>
</table>

In (3.3), $t = 6$, and the $A'$ and $(A'A)$ matrices appear as

$$
A' = \begin{bmatrix}
1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 1 & 1 & \ldots & 1
\end{bmatrix}
$$

$$
(A'A) = \begin{bmatrix}
n_{11} & 0 & 0 & 0 & 0 & 0 \\
0 & n_{12} & 0 & 0 & 0 & 0 \\
0 & 0 & n_{13} & 0 & 0 & 0 \\
0 & 0 & 0 & n_{21} & 0 & 0 \\
0 & 0 & 0 & 0 & n_{22} & 0 \\
0 & 0 & 0 & 0 & 0 & n_{23}
\end{bmatrix}
$$

In the general two-way classification (expressed as the model in (3.3)), $(A'A)$ will be a text diagonal matrix with $n_{ij}$'s on the diagonal.
The sum of squares with one d.f. due to the mean, denoted as \( M \), is
\[
M = \frac{G^2}{n}.
\]

In terms of (3.6) if
\[
C'_m = (n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23}),
\]
then
\[
\sum C'_m (A'A)^{-1} C'_m^{-1} = \frac{1}{n}; \quad \hat{\chi}_m C'_m = G
\]
and
\[
M = \hat{\chi}_m \sum C'_m (A'A)^{-1} C'_m^{-1} \hat{\chi}_m = \frac{G^2}{n}.
\]

As usual the sum of squares due to rows (R), with two d.f., uncorrected for the mean and unadjusted for columns, will be
\[
R = \frac{(y_{1..})^2}{n_{1.}} + \frac{(y_{2..})^2}{n_{2.}}.
\]

The \( C' \)-matrix for (3.6) that will yield \( R \) is
\[
C'_{ru} = \begin{bmatrix}
n_{11} & n_{12} & n_{13} & 0 & 0 & 0 \\
0 & 0 & 0 & n_{21} & n_{22} & n_{23}
\end{bmatrix}. \tag{3.10}
\]

Also the \( C' \)-matrix for the sum of squares due to columns (C), with three d.f., uncorrected for the mean and unadjusted for rows is
\[
C'_{cu} = \begin{bmatrix}
n_{11} & 0 & 0 & n_{21} & 0 & 0 \\
0 & n_{12} & 0 & 0 & n_{22} & 0 \\
0 & 0 & n_{13} & 0 & 0 & n_{23}
\end{bmatrix}. \tag{3.11}
\]

The uncorrected sum of squares for subclasses is
\[
S = \sum_{i=1}^{2} \sum_{j=1}^{3} \frac{y_{ij}^2}{n_{ij}}, \text{ with six d.f. where the } C' \text{-matrix is}\]
\[
\begin{bmatrix} I_{6\times6} \end{bmatrix}.
\]
The interaction sum of squares, I*, with two d.f., which is adjusted for row and column effects as obtained from the method of fitting constants or the weighted squares of means Yates [1934], can be found by using

$$C'_{rc} = \begin{bmatrix} 1 & -1 & 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & -1 & 0 & 1 \end{bmatrix}.$$  \hspace{1cm} (3.12)

The general construction of the $C'_{rc}$-matrices will be explained at the end of this section.

With linear combinations of M, R, C, S, and I*, it is possible to obtain other meaningful sums of squares for the two-way classification. For example denote R* as the sum of squares for rows, adjusted for columns, with one d.f., when interaction is not included in the model. This is what would be obtained for adjusted rows on the method of fitting constants. Then for the $2 \times 3$ classification,

$$R^* = S - C - I^* .$$

Corrected sums of squares for R, C and S are obtained by just subtracting M from each.

It is possible to calculate row and column sums of squares as weighted squares of the means (see Yates [1934] and Snedecor [1946]) in terms of the parameters in (3.3). The $C'$-matrices for the rows and columns in the $2 \times 3$ classification are shown below and a proof of the identity of the sum of squares by means of (3.6) with the weighted squares of means will be given after the development of the $C'$-matrices for the $r \times c$ classification.
Testing the hypothesis that all the row means are equal is

$$H_0: \gamma_{11} + \gamma_{12} + \gamma_{13} = \gamma_{21} + \gamma_{22} + \gamma_{23}$$

or

$$H_0: (\gamma_{11} + \gamma_{12} + \gamma_{13}) - (\gamma_{21} + \gamma_{22} + \gamma_{23}) = 0 .$$

For the weighted squares of means for rows (R** with one d.f.) the C'-matrix is

$$C'_r = \begin{pmatrix} 1 & 1 & 1 & -1 & -1 & -1 \end{pmatrix}.$$ (3.13)

Testing the hypothesis that all column means are equal is

$$H_0: \gamma_{11} + \gamma_{21} = \gamma_{12} + \gamma_{22} = \gamma_{13} + \gamma_{23}$$

or

$$H_0: (\gamma_{11} + \gamma_{21}) - (\gamma_{13} + \gamma_{23}) = 0 ,$$

$$H_0: (\gamma_{12} + \gamma_{22}) - (\gamma_{13} + \gamma_{23}) = 0 .$$

For the weighted squares of means for columns (C**, with two d.f.) the C'-matrix is

$$C'_c = \begin{bmatrix} 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 1 & -1 \end{bmatrix}.$$ (3.14)

Although the sums of squares described above are for a 2 x 3 classification, it is easy to see the generalization of the C'-matrices in (3.9), (3.10), (3.11), (3.13), and (3.14) to a r x c classification. For example in a r x c classification (all n_{ij}'s > 0; here rc = t):

$$C'_m = \begin{pmatrix} n_{11}, n_{12}, \ldots, n_{1c} ; n_{21}, n_{22}, \ldots, n_{2c} ; \ldots ; n_{r1}, n_{r2}, \ldots, n_{rc} \end{pmatrix} \text{ l} \times \text{rc}$$
\[
C_{ru}^{'} = \begin{bmatrix}
n_{11} & n_{12} & \ldots & n_{1c} & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & n_{21} & n_{22} & \ldots & n_{2c} & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & \ldots & n_{r1} & n_{r2} & \ldots & n_{rc}
\end{bmatrix}
\]

\[
C_{cu}^{'} = \begin{bmatrix}
n_{11} & 0 & \ldots & 0 & n_{21} & 0 & \ldots & 0 & \ldots & n_{r1} & 0 & \ldots & 0 \\
0 & n_{12} & 0 & 0 & n_{22} & 0 & \ldots & 0 & \ldots & 0 & n_{r2} & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & n_{1c} & 0 & 0 & \ldots & n_{2c} & \ldots & 0 & 0 & \ldots & n_{rc}
\end{bmatrix}
\]

\[
C_{r}^{'} = \begin{bmatrix}
1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & -1 & -1 & \ldots & -1 \\
0 & 0 & \ldots & 0 & 1 & 1 & \ldots & 1 & 0 & \ldots & 0 & -1 & -1 & \ldots & -1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 1 & 1 & \ldots & 1 & -1 & -1 & \ldots & -1
\end{bmatrix}
\]

\[
C_{c}^{'} = \begin{bmatrix}
1 & 0 & \ldots & 0 & -1 & 1 & 0 & \ldots & 0 & -1 & 1 & 0 & \ldots & 0 & -1 \\
0 & 1 & \ldots & 0 & -1 & 0 & 1 & \ldots & 0 & -1 & 0 & 1 & \ldots & 0 & -1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & -1 & 0 & 0 & \ldots & 1 & -1 & 0 & 0 & \ldots & 1 & -1
\end{bmatrix}
\]

The corresponding sums of squares would have respective d.f. of 1, r, c, r-1, c-1.

If any \( n_{ij} = 0 \), the column associated with that \( n_{ij} \) is ignored and the above matrices are compressed. Note that for \( C_{r}^{'} \) and \( C_{c}^{'} \), one must be careful that the orthogonality property with respect to the other factor is preserved. That is, the elements in
each row must always add to zero. In this way it is possible to construct \( C'_r \) - and \( C'_c \) - matrices for cases where some \( n_{ij} = 0 \), and thereby extend the weighted squares of means procedure.

Now it will be shown that in a two-way classification when all \( n_{ij} > 0 \), the sums of squares from (3.6) with the \( C'_r \) -matrix \( C'_r \) is identical to the weighted squares of means for rows. Defining

\[
W_i = \frac{1}{c} \sum_{j=1}^{c} \left( \frac{1}{n_{ij}} \right), \quad i = 1, 2, \ldots, r
\]

\[
j = 1, 2, \ldots, c
\]

then the weighted squares of means for rows is

\[
\sum_{i=1}^{r} W_i \left[ \sum_{j=1}^{c} \gamma_{ij} \right]^{2} - \left[ \sum_{i=1}^{r} W_i \left( \sum_{j=1}^{c} \gamma_{ij} \right) \right]^{2} \left/ \sum_{i=1}^{r} W_i \right.
\]

with \( r-1 \) d.f. Since the weighted squares of means is in terms of row totals of subclass means, denote \( t_i = \sum_{j=1}^{c} \gamma_{ij} \). Thus for (3.6),

\[
\sum_{i=1}^{r} C'_r = t_1 - t_r, \ldots, t_2 - t_r, \ldots, \sum_{i=1}^{r} t_r - t_r - \gamma_{ij}
\]

and

\[
\sum_{i=1}^{r} \left( A'A \right)^{-1} C'_r - \gamma_{ij} = D(1/W_i) - (1/W_i) \gamma_{ij} - 1
\]

where \( D(1/W_i) \) is a diagonal matrix with \( 1/W_i, i = 1, 2, \ldots, r-1 \), on the diagonal and \( j \) is a vector of ones. Using Roy's 1956 work on inverting patterned matrices, and defining \( W' = (W_1, W_2, \ldots, W_{r-1}) \), it is seen that

\[
\sum_{i=1}^{r} \left( A'A \right)^{-1} C'_r - \gamma_{ij} = D(W_i) - (1/W_i) \gamma_{ij} \gamma_{ij} - 1
\]

\[W' \]
Now by combining terms

\[
\sum_i C_{ij} \left( W_i - \frac{1}{\sum_i W_i} \sum_i W_i \right) \cdot \sum_{ij} C_{ij} \cdot \sum_{ij}
\]

\[
= \frac{1}{\sum_i W_i} \sum_i W_i^2 \cdot \sum_i W_i - \frac{1}{\sum_i W_i} \sum_i W_i \sum_i W_i - \frac{1}{\sum_i W_i} \sum_i W_i \sum_i W_i + \cdots
\]

\[
+ \sum_i W_i \sum_j \sum_{i \neq j} t_i t_j W_i W_j
\]

\[
= \sum_i W_i t_i^2 - \sum_{i \neq j} W_i W_j t_i t_j - \frac{1}{\sum_i W_i} \sum_i W_i t_i^2 - \frac{1}{\sum_i W_i} \sum_i W_i t_i^2
\]

which is the weighted squares of means for rows. An analogous development can be shown for the weighted squares of means for columns.

It is possible to write down a set of rules for constructing interaction matrices, such as (3.12) for a connected design. Basically it consists of writing down all possible one d.f. interaction contrasts and then selecting a basis which is denoted as the \( C' \)-matrix. Any basis will yield the same \( Q \) matrix as calculated by (3.7). For example in the 2 x 3 classification, there are three possible one d.f. interaction contrasts. They are shown as \( l'_1, l'_2, l'_3 \):
Note that \( \nu^i_2 - \nu^i_1 = \nu^i_3 \); hence, one basis is \((\nu^i_1, \nu^i_2)\), which is given in (3.12). However \((\nu^i_2, \nu^i_3)\) and \((\nu^i_1, \nu^i_3)\) are also suitable for interaction \(G^i_{rc}\) matrices. In a general \(r \times c\) classification if all the possible one d.f. contrasts are listed as \(\nu^i_1, \nu^i_2, \ldots, \nu^i_c\), then it is known for a connected design that there will be \(n_{in} = t - r - c + 1\) linearly independent \(\nu^i\)'s that can be selected as a basis. One way to find \(n_{in}\) such \(\nu^i\)'s is to first write down all the \(\nu^i\)'s in the design. The general \(r \times c\) case would appear as

\[
\begin{array}{cccccccc}
\text{Cell} & 11 & 12 & 13 & 21 & 22 & 23 \\
\nu^i_1: & 1 & -1 & 0 & -1 & 1 & 0 \\
\nu^i_2: & 1 & 0 & -1 & -1 & 0 & 1 \\
\nu^i_3: & 0 & 1 & -1 & 0 & -1 & 1 \\
\end{array}
\]

First take any \(\nu^i\) as your first row in \(G^i_{rc}\), say \(\nu^i_1\); select another \(\nu^i\), say \(\nu^i_j\), and compute \(\nu^i_1 + \nu^i_j\), \(\nu^i_1 - \nu^i_j\) and \(\nu^i_j - \nu^i_1\). Now eliminate any \(\nu^i\) in the set that is equal to any of above three combinations. Then select a third \(\nu^i\), say \(\nu^i_k\), \(i \neq j \neq k\), and take all possible additions and differences of \((\nu^i_1, \nu^i_j)\) with \(\nu^i_k\). If any of these additions or differences appear as any of the remaining \(\nu^i\)'s, then they are to be eliminated from the set.
Continue this process until \( n_{\text{in}} \) s are found. Call this set \( C'_{rc} \), the C'-matrix for interaction. Since all the \( \mathcal{I}' \) s consist only of 0, 1, and -1, the addition and subtraction of \( \mathcal{I}' \) s will yield all other linearly dependent \( \mathcal{I}' \) s. If some \( n_{ij} = 0 \), then those cells cannot be used in constructing an interaction contrast.

Another way to find the contrasts in \( C'_{rc} \) is through the \( C'_r \) and \( C'_c \) matrices. Denote the elements of the \( m \)-th row vector in \( C'_r \) and the \( k \)-th row vector in \( C'_c \) as \( (C'_r)_m \mathcal{I}' \) and \( (C'_c)_k \mathcal{I}' \) respectively; \( \mathcal{I}' = 1, 2, \ldots, t \). Then for the case where \( n_{ij} > 0 \) (all \( i,j \)), the row vector

\[
\sum (C'_r)_m (C'_c)_k \mathcal{I}'_i, (C'_r)_m (C'_c)_k \mathcal{I}'_2, \ldots, (C'_r)_m (C'_c)_k \mathcal{I}'_t,
\]

given any \( m \) and \( k \), is an interaction contrast. By taking all the \((r-1)(c-1)\) combinations of \( m \) and \( k \), all the rows of \( C'_{rc} \) will be formed. When some \( n_{ij} = 0 \), all that is required is to eliminate those combinations of \( m \) and \( k \) which do not make up an interaction contrast. Note that in order for the above multiplication to work, the contrasts in \( C'_r \) and \( C'_c \) must be linear forms.

3.4 Variance Components in Model II

Model II will be the same as that proposed by Eisenhart. In the two-way classification, the mathematical model is

\[
Y_{ijk} = \mu + r_i + c_j + (rc)_{ij} + e_{ijk}
\]

where \( \mu \) is a constant, \( r_i \), \( c_j \), \( (rc)_{ij} \) and \( e_{ijk} \) are all NID with means zero and respective variances \( \sigma^2_r \), \( \sigma^2_c \), \( \sigma^2_{rc} \) and \( \sigma^2_e \). If (3.15)
is rewritten in terms of subclass random variables, and subclass means \((\bar{\gamma}_{ij} = \gamma_{ij})\) the mathematical model is
\[
\gamma_{ij} = \gamma_{ij} + \bar{e}_{ij},
\]
where \(\bar{\gamma}_{ij}\) are the subclass random variables \(\gamma_{ij} = \mu + r_i + c_j + (rc)_{ij}\); \(\bar{e}_{ij}\) is the average error for the \(i, j\)-th subclass, and \(\gamma_{ij}\) has a multivariate normal distribution with variance-covariance matrix which shall be denoted by \(V\). Now \(\bar{e}_{ij} \sim NID(0, \sigma_e^2/n_{ij})\).

It is shown in Section 3.2 that all sums of squares for (3.3) can be written in the quadratic form of (3.8). Using results obtained from Whittle \(1952\), Lancaster \(1954\) and Searle \(1958\),
\[
E(SS_j) = tr(V Q_j) \tag{3.16}
\]
\[
\text{Var}(SS_j) = 2tr(V Q_j)^2 \tag{3.17}
\]
\[
\text{Cov}(SS_j, SS_k) = 2tr(V Q_j)(V Q_k) \tag{3.18}
\]
where \(tr(A)\) is the trace of matrix \(A\).

For the general multi-way classification, say \(k\) classifications, there will be \(2^k\) variance components to be estimated (including \(\sigma_e^2\)). However since the within cells sum of squares is independent of all other between cells sums of squares, \(\sigma_e^2\) need not be included in the set of variance components to be estimated. The sum of squares for error (SSE) divided by \(\sigma_e^2\) in the analysis of variance has a \(\chi^2\) distribution with \(n_e = n - t\) d.f.; hence, the variance of the estimate of \(\sigma_e^2\) is
\[
\text{Var}(\sigma_e^2) = 2\sigma_e^4/n_e.
\]
One procedure for estimating variance components is to select a set of sums of squares, such that their expected values are linear in terms of the parameters (the variance components), and that the produced set of linear equations uniquely determine the estimates.

Let \( t^* = H_t, \quad q \geq p, \) denote such a set of sums of squares. Thus \( t^* \) can also be expressed as a linear combination of another set of sums of squares, through the matrix \( H \). If \( H = I \) \((p = q)\), then \( t^* = \hat{t} \).

Define the expected value of \( t^* \) as

\[
\mathbb{E}(t^*) = \begin{bmatrix} P & \tilde{v} \\ \tilde{v}^T & \sigma^2_e \end{bmatrix}_{px(p+1)(p+1)x1} \tag{3.19}
\]

where \( p = 2^k - 1 \), \( \tilde{v} \) contains the \( p+1 \) true variance components (including \( \sigma^2_e \)) and \( \tilde{P} \) is the coefficient matrix that satisfies (3.19). If \( \sigma^2_e \) is considered as the last element in \( \tilde{v} \), then (3.19) can be rewritten as

\[
\mathbb{E}(t^*) = \begin{bmatrix} P & m \\ m^T & 0 \end{bmatrix}_{px(p+1)x1} \begin{bmatrix} \tilde{v} \\ \sigma^2_e \end{bmatrix}_{1x1} \tag{3.20}
\]

\[
= P \tilde{v} + \sigma^2_e m, \quad pxp \quad pxl \quad pxp \quad pxl
\]

where \( \tilde{P} \) is partitioned, \( \tilde{P} = \begin{bmatrix} P & m \end{bmatrix} \). Substituting \( H_t \) for \( t^* \), (3.20) becomes

\[
H \mathbb{E}(t) = P \tilde{v} + \sigma^2_e m.
\]

Now denoting \( \tilde{v} \) as the vector of estimates of \( \sqrt{\mathbb{E}(v)} = \tilde{v} \).
and replacing $E(t)$ by its sample values, the estimating equation

$$Ht = P\bar{y} + \frac{\Lambda^2}{\sigma^2_e} \bar{m},$$

is obtained. Solving for $\bar{y}$ yields

$$\bar{y} = P^{-1} Ht - \frac{\Lambda^2}{\sigma^2_e} \bar{m}. \quad (3.21)$$

However there is more than one set of sums of squares that will satisfy the above conditions. Thus the question becomes, which set should be used for estimating the variance components?

### 3.5 Variances of Estimates of the Variance Components

One way of evaluating which set of sums of squares will yield the best estimates of $\bar{y}$, would be to select the set for which $\text{var}(\bar{y})$ is smallest. Taking the variance of both sides of (3.21) gives

$$\text{var}(\bar{y}) = P^{-1}_{p\times p} \sum_{p \times p} \text{var}(t) H' + \left(\frac{\Lambda^2}{\sigma^2_e/n_e}\right) P_{p \times p} \bar{m}' P^{-1}_{p \times p}. \quad (3.22)$$

Since $\text{cov}(\sigma^2_e, t) = 0$, there is no cross product term on the right hand side of (3.22). It is always assumed that $t$ never contains any within subclass sum of squares.

In any given analysis of data following a multi-way classification, $P$ and $H$ are fixed matrices, and $\bar{m}$ is a fixed vector. The difficult computation is to find $\text{var}(t)$ (a $q \times q$ matrix). However if it is possible to express the sums of squares in $t$ in terms of the $C'$-matrices in (3.6), then by using (3.17) and (3.18), it is easy to construct $\text{var}(t)$.

One of the many advantages, computationally, in the use of (3.6) with (3.17) and (3.18) in finding $\text{var}(t)$, is that the largest
matrix, $V$ has the order of the number of non-empty subclasses $(t)$ in the multi-way classification. Also note that $Q$ is of the general form $C B C'$; hence, since usually $s < t$ using the identity $\text{tr}(AB) = \text{tr}(BA)$, equations (3.17) and (3.18) become computationally simpler.

It should be noted that $\text{var}(y)$, for a particular experiment, will be a function of the true magnitude of the variance components and the arrangement of the subclass numbers.

Searle (1958) considers a special case of (3.22) for a two-way classification. This will be discussed in more detail in Chapter IV.
CHAPTER IV

COMPARISON OF THREE PROCEDURES FOR ESTIMATING VARIANCE COMPONENTS

4.1 Introduction

As mentioned in Chapter III, there are many different sets of sums of squares that could be used to determine \( y \), and hence, \( \text{var}(y) \). This chapter will consider only three such sets for the general two-way classification. The first set is

\[
\begin{pmatrix}
R - M \\
C - M \\
S - R - C + M
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 & -1 \\
0 & 1 & 0 & -1 \\
-1 & -1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
R \\
C \\
S \\
M
\end{pmatrix}
\]

where

\[
H_h = \begin{pmatrix}
1 & 0 & 0 & -1 \\
0 & 1 & 0 & -1 \\
-1 & -1 & 1 & 1
\end{pmatrix}, \quad t_h = \begin{pmatrix}
R \\
C \\
S \\
M
\end{pmatrix}.
\]

This set is described by Henderson \(^{1953}\) and Searle \(^{1958}\) in the Method 1 procedure and is comprised of unadjusted sums of squares. For this dissertation the estimating procedure will be denoted as Procedure H.

The next set is

\[
\begin{pmatrix}
R^* \\
C^* \\
I^*
\end{pmatrix}
= \begin{pmatrix}
0 & -1 & 1 & -1 \\
-1 & 0 & 1 & -1 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
R \\
C \\
S \\
I^*
\end{pmatrix}
\]
where \( R^* = S - C - I^* \) and \( C^* = S - R - I^* \), from Section 3.3 and

\[
H_a = \begin{bmatrix}
0 & -1 & 1 & -1 & \\
-1 & 0 & 1 & -1 & \\
0 & 0 & 0 & 1 & \\
\end{bmatrix}, \quad t_a = \begin{bmatrix}
R \\
C \\
S \\
I^* \\
\end{bmatrix}
\]

This set of sums of squares (\( t^*_a \)) has been used at the Department of Experimental Statistics, North Carolina State College for a number of years and is based on the method of fitting constants, Yates \([1934]\). It is also described as Method 3 in Henderson \([1953]\). In this dissertation it will be denoted as Procedure A.

The last set to be considered is

\[
t^*_b = t_b = \begin{bmatrix}
R^{**} \\
C^{**} \\
I^{**} \\
\end{bmatrix}
\]

where \( H_b = I \). These are the weighted squares of means described by Yates \([1934]\). The estimating procedure from \( t_b \) will be denoted as Procedure B.

The main distinction between \( t^*_a \) and \( t_b \), is that in \( t^*_a \) interaction is ignored in computing \( R^* \), and \( C^* \), while in \( t_b \), \( R^{**} \), and \( C^{**} \) are obtained with interaction in the model.
4.2 Estimating Equations for $y$ and $\text{var}(y)$

For Procedure A, we can let

$$E(t_{a}^{\star}) = \begin{bmatrix} a_{1} & 0 & a_{2} & \mathbf{r}-1 \cr 0 & a_{3} & a_{4} & \mathbf{c}-1 \cr 0 & 0 & a_{5} & n_{in} \end{bmatrix} \begin{bmatrix} \sigma^{2}_{r} \\ \sigma^{2}_{c} \\ \sigma^{2}_{rc} \\ \sigma^{2}_{e} \end{bmatrix},$$

or in matrix notation

$$E(t_{a}^{\star}) = P_{a} y + m \sigma^{2}_{e}.$$  

Solving for the estimates of $y$, as in (3.21)

$$y_{a} = P_{a}^{-1} \sum_{a} y_{a} - \frac{\sigma^{2}_{e}}{m} m' J P_{a},$$

and from (3.22)

$$\text{var}(y_{a}) = P_{a}^{-1} \sum_{a} \text{var}(y_{a}) H_{a}' H_{a} = \text{var}(\hat{y}_{a}) m' m^{-1} P_{a}.$$  

(4.1)

where

$$\text{var}(t_{a}) = \begin{bmatrix} \text{var}(R) & \text{cov}(R,C) & \text{cov}(R,S) & \text{cov}(R,I^{\star}) \\ \text{cov}(C) & \text{cov}(C,S) & \text{cov}(C,I^{\star}) \\ \text{var}(S) & \text{cov}(S,I^{\star}) \\ \text{var}(I^{\star}) \end{bmatrix},$$

(4.2)
For Procedure B, we can let

\[
E(t_b) = \begin{bmatrix} b_1 & 0 & b_2 \\ 0 & b_3 & b_4 \\ 0 & 0 & b_5 \end{bmatrix} \begin{bmatrix} \sigma_r^2 \\ \sigma_c^2 \\ \sigma_{rc} \end{bmatrix} + \begin{bmatrix} r-1 \\ c-1 \\ n_{in} \end{bmatrix} \sigma_e^2 ,
\]

or in matrix notation

\[
E(t_b) = P_b \varphi + m \sigma_e^2 .
\]

Again solving for the estimates of \( \varphi \), i.e. \( \hat{\varphi}_b \)

\[
\hat{\varphi}_b = P_b^{-1} \left( t_b - \frac{\sigma_e^2}{m} \right)
\]

and

\[
\text{var}(\hat{\varphi}_b) = P_b^{-1} \left( \text{var}(t_b) + \text{var}(\sigma_e^2) \right) m \left( m' \right) P_b^{-1}
\]

where

\[
\text{var}(t_b) = \begin{bmatrix} \text{var}(R^{**}) & \text{cov}(R^{**},C^{**}) & \text{cov}(R^{**},I^{**}) \\ \text{cov}(R^{**},C^{**}) & \text{var}(C^{**}) & \text{cov}(C^{**},I^{**}) \\ \text{cov}(R^{**},I^{**}) & \text{cov}(C^{**},I^{**}) & \text{var}(I^{**}) \end{bmatrix}.
\] (4.3)

For Procedure H, using results on Searles [1958], we can let

\[
E(t_h^*) = \begin{bmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ -h_4 & -h_2 & h_7 \end{bmatrix} \begin{bmatrix} \sigma_r^2 \\ \sigma_c^2 \\ \sigma_{rc} \end{bmatrix} + \begin{bmatrix} r-1 \\ c-1 \\ n_{in} \end{bmatrix} \sigma_e^2
\]

or in matrix notation

\[
E(t_h^*) = P_h \varphi + m \sigma_e^2 .
\]
Solving for $v_h$, 

$$
v_h = P_h^{-1} \left[ H_h t_h - \frac{\Lambda_0}{\sigma_e} \right] m^T
$$

(4.4)

and

$$
\text{var}(v_h) = P_h^{-1} \left[ H_h \text{var}(t_h) H_h' + \frac{\Lambda_0}{\sigma_e} \right] m m^T P_h^{-1} \text{.}
$$

(4.5)

where

$$
\text{var}(t_h) = \begin{bmatrix}
\text{var}(R) & \text{cov}(R,C) & \text{cov}(R,S) & \text{cov}(R,M) \\
\text{var}(C) & \text{cov}(C,S) & \text{cov}(C,M) & \\
\text{var}(S) & \text{cov}(S,M) & \\
\text{var}(M)
\end{bmatrix} 
$$

(4.6)

The computational details for obtaining $P_a$, $P_b$, $P_h$, $\text{var}(t_a)$, $\text{var}(t_b)$ and $\text{var}(t_h)$ are given in Section 4.3 and 4.4. Although Searle [1958] has worked out all the computational details for (4.4) and (4.5), they will be repeated in this dissertation as an illustration of one special case of a more general procedure. Besides some of the matrices in Procedure H are needed in Procedure A.

4.3 Calculating $P_h$, $P_a$ and $P_b$

As stated in (3.16), $E(SS_{ij}) = \text{tr}(VQ_{ij})$, where $V$ is the variance-covariance matrix of $Z$ for Model II. For the two-way classification, $V$ is comprised of four parameters $\sigma_r^2$, $\sigma_c^2$, $\sigma_{rc}^2$ and $\sigma_e^2$. Since the four parameters appear additively in $V$, it is possible to separate $V$ into four matrices, where each matrix is only associated with one of the parameters. Then

$$
V = \sigma_r^2 V_r + \sigma_c^2 V_c + \sigma_{rc}^2 V_{rc} + \sigma_e^2 V_e \text{.}
$$

(4.7)
For the case where all \( n_{ij} > 0 \) (\( t = rc \)), it is easy to see that

\[
V_r = \begin{bmatrix} J & 0 & \cdots & 0 \\ cxc & cxc & \cdots & cxc \\ 0 & J & \cdots & 0 \\ cxc & cxc & \cdots & cxc \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J \\ cxc & cxc & \cdots & cxc \end{bmatrix}, \quad V_e = \begin{bmatrix} I & I & \cdots & I \\ cxc & cxc & \cdots & cxc \\ I & I & \cdots & I \\ cxc & cxc & \cdots & cxc \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ I & I & \cdots & I \\ cxc & cxc & \cdots & cxc \end{bmatrix}
\]

\( V_{rc} = I, \quad V_e = (A'A)^{-1}, \quad J \) is a matrix of all ones, \( 0 \) is a matrix of all zeroes and \( I \) is an identity matrix. Therefore (4.7) becomes

\[
V = \sigma^2_{vr} V_r + \sigma^2_{vc} V_e + \sigma^2_{rc} I + \sigma^2_{e}(A'A)^{-1}.
\]  

(4.8)

When some cells have \( n_{ij} = 0 \), the associated rows and columns of \( V \) will not appear for those cells, and the \( J \) and \( I \) matrices in \( V_r \) and \( V_e \) will have different dimensions; however, the pattern shown will be the same.

Now in Procedure B, for \( SS_j = \hat{Z}' Q_j \hat{Z} \),

- \( R \) has \( Q_{ru} = C_{ru} \sum_{ru} (A'A)^{-1} C_{ru} \sum_{ru} C_{ru} = C_{ru} B_{ru} C'_{ru} \);
- \( C \) has \( Q_{cu} = C_{cu} \sum_{cu} (A'A)^{-1} C_{cu} \sum_{cu} C_{cu} = C_{cu} B_{cu} C'_{cu} \);
- \( S \) has \( Q_s = (A'A) \);
- \( M \) has \( Q_m = (C_{m} C'_{m})/n \)

where these matrices are defined in Section 3.3.
For R-M, corrected rows,

\[
E(R) - E(M) = \sum \text{tr}(V_{ru}Q_{ru}) - \text{tr}(V_{ru}Q_{rn})\sigma_{r}^{2} + \sum \text{tr}(V_{ru}Q_{ru}) - \text{tr}(V_{ru}Q_{rn})\sigma_{c}^{2} \\
+ \sum \text{tr}Q_{ru} - \text{tr}Q_{rn}\sigma_{c}^{2} + \sum \text{tr}(A'A)^{-1}Q_{ru} - \text{tr}(A'A)^{-1}Q_{rn}\sigma_{e}^{2}
\]

(4.9)

Simplifying (4.9), it is seen that the following relationships hold

<table>
<thead>
<tr>
<th>Identities</th>
<th>Searle's Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{tr}(V_{ru}Q_{ru}) = \text{tr}(C'<em>{ru} V</em>{c} B_{ru}) = n )</td>
<td>k_{12}</td>
</tr>
<tr>
<td>( \text{tr}(V_{ru}Q_{rn}) = \text{tr}(C'<em>{ru} V</em>{c} B_{ru}) = \Sigma (\Sigma n_{ij}/n) )</td>
<td>k_{12}</td>
</tr>
<tr>
<td>( \text{tr}(V_{ru}Q_{rn}) = \text{tr}(C'<em>{ru} V</em>{c} B_{ru}) = \Sigma n_{i}/n )</td>
<td>k_{1}</td>
</tr>
<tr>
<td>( \text{tr}(V_{ru}Q_{rn}) = \text{tr}(C'<em>{ru} V</em>{c} B_{ru}) = n )</td>
<td>k_{2}</td>
</tr>
<tr>
<td>( \text{tr}Q_{ru} = \text{tr}(C'<em>{ru} C</em>{ru} B_{ru}) = k_{12} )</td>
<td>k_{12}</td>
</tr>
<tr>
<td>( \text{tr}Q_{rn} = \text{tr}(C'<em>{rn} C</em>{rn} B_{rn}) = \Sigma \Sigma n_{ij}/n )</td>
<td>k_{12}</td>
</tr>
<tr>
<td>( \text{tr}(A'A)^{-1}Q_{ru} = \text{tr}C'<em>{ru} (A'A)^{-1} C</em>{ru} B_{ru} - \text{tr}I = r )</td>
<td>k_{12}</td>
</tr>
<tr>
<td>( \text{tr}(A'A)^{-1}Q_{rn} = \text{tr}(C'<em>{rn} (A'A)^{-1} C</em>{rn})/n = 1 )</td>
<td>k_{12}</td>
</tr>
<tr>
<td>( \text{tr}Q_{e} = \text{tr}(A'A) = n )</td>
<td>k_{12}</td>
</tr>
</tbody>
</table>

Therefore

\[
E(R) - E(M) = (n-k_{1})\sigma_{r}^{2} + (k_{2} - k_{1})\sigma_{c}^{2} + (k_{12} - k_{2})\sigma_{c}^{2} + \Sigma (\sigma_{e}^{2} + (r-1)\sigma_{e}^{2})
\]
For C-M, the additional expressions in \( E(C) - E(M) \) are

\[
\begin{align*}
\text{Identities} & \quad \text{Searle's } 1958 \text{ Notation} \\
\text{tr}(V_{rc}Q_{cu}) &= \text{tr}(C'_{cu}V_{rc}C_{cu}B_{cu}) = \sum \left( \sum \frac{n_{ij}^2}{n_{ij}} \right) j_i j_j k_{21} \\
\text{tr}(V_cQ_{cu}) &= \text{tr}(C'_{cu}V_cC_{cu}B_{cu}) = n \\
\text{tr}(Q_{cu}) &= \text{tr}(C'_{cu}C_{cu}B_{cu}) = k_{21} k_{21} \\
\text{tr}(A'A)^{-1}Q_{cu} &= \text{tr}(C'_{cu}(A'A)^{-1}C_{cu}B_{cu}) \\
&= \text{tr} I = c \\
\end{align*}
\]

Therefore

\[
E(C) - E(M) = (k_{21} - k_1)\sigma_r^2 + (n - k_2)\sigma_c^2 + (k_{21} - k_3)\sigma_{rc}^2 + (c - 1)\sigma_e^2.
\]

For S-R-C+M, the only unknown quantity is the coefficient of \( \sigma_{rc}^2 \) and it is just \( n-k_2-k_3+k_5 \). Finally then, as is also shown by Searle 1958.

\[
P_h = \begin{bmatrix}
    n - k_1 & k_{12} - k_2 & k_{12} - k_3 \\
    k_{21} - k_1 & n - k_2 & k_{21} - k_3 \\
    k_1 - k_{21} & k_2 - k_{12} & n - k_{12} - k_{21} + k_3
\end{bmatrix}
\]

For Procedure A, \( I^# = \hat{Z}' Q_{rc} \hat{Z} \), where

\[
Q_{rc} = C_{rc}C'_{rc}(A'A)^{-1}C'_{rc}C_{rc} = C_{rc}B_{rc}C_{rc}
\]
For rows,

\[ E(R^*) = E(S) - E(C) - E(I^*) \]

\[ = \sum_{r} \text{tr}(V_r Q_s) - \text{tr}(V_r Q_{cu}) - \text{tr}(V_r Q_{rc}) \sum_{r} \sigma^2_r + \sum_{s} \text{tr}Q_s - \text{tr}Q_{cu} - \text{tr}Q_{rc} \sum_{r} \sigma^2 \]

\[ + \sum_{A} \text{tr}(A'A)^{-1} Q_s - \text{tr}(A'A)^{-1} Q_{cu} - \text{tr}(A'A)^{-1} Q_{rc} \sum_{s} \sigma^2 \]

\[ = (n-k_{21})\sigma^2_r + \sum_{s} (n-k_{21}) \text{tr}(C^t c B \text{ rc}) \sum_{r} \sigma^2 + (r-1)\sigma_e^2. \]

Note that \( \text{tr}(V_r Q_{rc}) = 0 \), and that \( r-1=t-c-n_{in} \).

For columns,

\[ E(C^*) = E(S) - E(R) - E(I^*) \]

\[ = \sum_{c} \text{tr}(V_c Q_s) - \text{tr}(V_c Q_{ru}) - \text{tr}(V_c Q_{rc}) \sum_{c} \sigma^2_c + \sum_{s} \text{tr}Q_s - \text{tr}Q_{ru} - \text{tr}Q_{rc} \sum_{c} \sigma^2 \]

\[ + \sum_{A} \text{tr}(A'A)^{-1} Q_s - \text{tr}(A'A)^{-1} Q_{ru} - \text{tr}(A'A)^{-1} Q_{rc} \sum_{c} \sigma^2 \]

\[ = (n-k_{12})\sigma^2_c + \sum_{s} (n-k_{12}) \text{tr}(C^t c B \text{ rc}) \sum_{c} \sigma^2 + (c-1)\sigma_e^2. \]

In \( E(C^*) \), note also that \( \text{tr}(V_c Q_{rc}) = 0 \).

For Interaction

\[ E(I^*) = (\text{tr}Q_{rc})\sigma^2_{rc} + \sum_{A} \text{tr}(A'A)^{-1} Q_{rc} \sum_{s} \sigma^2 \]

\[ = \text{tr}(C^t c B \text{ rc}) \sigma^2_{rc} + n_{in} \sigma_e^2. \]

Therefore, defining \( k_{30} = \text{tr}(C^t c B \text{ rc}) \)

\[ P_a = \begin{bmatrix} n-k_{21} & 0 & n-k_{21} - k_{30} \\ 0 & n-k_{12} & n-k_{12} - k_{30} \\ 0 & 0 & k_{30} \end{bmatrix} \]
Searle and Henderson [1961] give a simplified procedure that can be used for finding $P_a$ but would not be useful in this general matrix approach for finding $\text{var}(v_a)$. For Procedure B,

$$R^{**} \text{ has } Q_r = C_r \int C_r'(A'A)^{-1} C_r \int^{-1} C_r' \equiv C_r B_r C_r'$$

$$C^{**} \text{ has } Q_c = C_c \int C_c'(A'A)^{-1} C_c \int^{-1} C_c' \equiv C_c B_c C_c'$$

Then

$$E(R^{**}) = \int \text{tr}(V_r Q_r) \sigma_r^2 + (\text{tr}Q_r) \sigma^2_{rc} + \int \text{tr}(A'A)^{-1} Q_r \sigma_e^2$$

$$= \int \text{tr}(C_r'V_r C_r B_r) \sigma_r^2 + \int \text{tr}(C_r' C_r B_r) \sigma^2_{rc} + (r-1) \sigma_e^2$$

and

$$E(C^{**}) = \int \text{tr}(V_c Q_c) \sigma_c^2 + (\text{tr}Q_c) \sigma^2_{rc} + \int \text{tr}(A'A)^{-1} Q_c \sigma_e^2$$

$$= \int \text{tr}(C_c'V_c C_c B_c) \sigma_c^2 + \int \text{tr}(C_c' C_c B_c) \sigma^2_{rc} + (c-1) \sigma_e^2.$$

Since $E(I^*)$ is the same as (4.10),

$$P_b = \begin{bmatrix} \text{tr}(C_r' V_r C_r B_r) & 0 & \text{tr}(C_r' C_r B_r) \\ 0 & \text{tr}(C_c' V_c C_c B_c) & \text{tr}(C_c' C_c B_c) \\ 0 & 0 & \text{tr}(C_c' C_c B_c) \end{bmatrix}$$

Henderson et al. [1957] gives the elements of $P_b$ for the special case of $n_{ij} > 0$ (all i, j). $E(R^{**})$ and $E(C^{**})$ are simple functions of the $n_{ij}$'s.

In all three procedures $m' = (r-1, c-1, n_{in})$. 
4.4 Calculating var(t_h), var(t_a) and var(t_b)

Using the V-matrix and the Q-matrices from Section 4.3 in equations (3.17), and (3.18), it is an easy matter to write down all the elements of the three sums of squares variance-covariance matrices (4.2), (4.3), and (4.6). Again it should be mentioned that for Procedure II, Searls [1958] has worked out explicit expressions for the elements in var(t_h), (4.6) in terms of the n_ij's and the four parameters \( \sigma^2_r, \sigma^2_c, \sigma^2_{rc} \) and \( \sigma^2_e \). The equivalent expressions are shown below in terms of the general development described in Section 3.4.

For Procedure II, the ten unique elements of var(t_h) are

\[
\begin{align*}
\text{var}(R) & = 2\text{tr}(VQ_{ru})^2 = 2\text{tr}(C'_{ru} VC_{ru} B_{ru})^2, \\
\text{var}(C) & = 2\text{tr}(VQ_{cu})^2 = 2\text{tr}(C'_{cu} VC_{cu} B_{cu})^2, \\
\text{cov}(R,C) & = 2\text{tr}(VQ_{ru} VQ_{cu}) = 2\text{tr}(C'_{cu} VC_{ru} B_{ru} C'_{cu} VC_{cu} B_{cu}), \\
\text{var}(S) & = 2\text{tr}(VQ_s)^2 = 2\text{tr}(VA'A)^2, \\
\text{cov}(R,S) & = 2\text{tr}(VQ_{ru} VQ_s) = 2\text{tr}(C'_{ru} VA'A VC_{ru} B_{ru}), \\
\text{cov}(C,S) & = 2\text{tr}(VQ_{cu} VQ_s) = 2\text{tr}(C'_{cu} VA'A VC_{cu} B_{cu}), \\
\text{var}(M) & = 2\text{tr}(VQ_m)^2 = 2(C'_{-m} V C_{-m})/n^2, \\
\text{cov}(R,M) & = 2\text{tr}(VQ_{ru} V Q_m) = 2(C'_{-m} VC_{ru} B_{ru} C'_{cu} VC_{cu} B_{cu})/n, \\
\text{cov}(C,M) & = 2\text{tr}(VQ_{cu} V Q_m) = 2(C'_{-m} V C_{cu} B_{cu} C'_{cu} V C_{-m})/n, \\
\text{cov}(S,M) & = 2\text{tr}(VQ_s V Q_m) = 2(C'_{-m} VA'A V C_{-m})/n.
\end{align*}
\]
Now in Procedure A, the first six elements shown for var(\(t_a\)) are the same as above and the new last four elements are:

\[
\text{var}(I^*) = 2\text{tr}(VQ_{rc})^2 = 2\text{tr}(c^i_{rc}VC_{rc}B_{rc})^2
\]

\[
\text{cov}(R,I^*) = 2\text{tr}(VQ_{ru}VQ_{rc}) = 2\text{tr}(c^i_{rc}VC_{rc}B_{rc}c^i_{ru}VQ_{rc})
\]

\[
\text{cov}(C,I^*) = 2\text{tr}(VQ_{cu}VQ_{rc}) = 2\text{tr}(c^i_{rc}VC_{rc}B_{rc}c^i_{cu}VQ_{rc})
\]

\[
\text{cov}(S,I^*) = 2\text{tr}(VQ_{s}VQ_{rc}) = 2\text{tr}(c^i_{rc}VA'AVC_{rc}B_{rc})
\]

Finally for Procedure B, only var(I*) need not be repeated.

The remaining five elements are:

\[
\text{var}(R^{**}) = 2\text{tr}(VQ_{r})^2 = 2\text{tr}(c^i_{r}VC_{r}B_{r})^2
\]

\[
\text{var}(C^{**}) = 2\text{tr}(VQ_{c})^2 = 2\text{tr}(c^i_{c}VC_{c}B_{c})^2
\]

\[
\text{cov}(R^{**},C^{**}) = 2\text{tr}(VQ_{r}VQ_{c}) = 2\text{tr}(c^i_{r}VC_{r}B_{r}c^i_{c}VC_{c}B_{c})
\]

\[
\text{cov}(R^{**},I^*) = 2\text{tr}(VQ_{r}VQ_{rc}) = 2\text{tr}(c^i_{rc}VC_{rc}B_{rc}c^i_{r}VC_{rc}B_{rc})
\]

\[
\text{cov}(C^{**},I^*) = 2\text{tr}(VQ_{c}VQ_{rc}) = 2\text{tr}(c^i_{rc}VC_{rc}B_{rc}c^i_{c}VC_{rc}B_{rc})
\]

Note that the matrices have been arranged so that there is a minimum of computation.

The computer program described in Appendix A evaluates var(\(v_{nh}\)), var(\(v_{a}\)) and var(\(v_{b}\)), for combinations of different sets of \(\sigma_{1j}^2\) and parameter values \((\sigma_{r}^2, \sigma_{c}^2, \sigma_{rc}^2, \sigma_{e}^2)\). The program was written in order to compare the three procedures described above, and also to investigate experimental designs for their efficiency in simultaneously estimating \(\sigma_{r}^2\) and \(\sigma_{c}^2\).
4.5 A Computer Comparison

Using an UNIVAC 1105, a computer program was written to compute the variances of the estimates of the variance components, in a two-way classification, for procedures A, B and H. The computer program listing is given in Appendix A, and the results from the various computer runs, by design, is given in Appendix B. The two main purposes of making a computer comparison are:

i) To compare the three estimating procedures,

ii) To compare some experimental designs for estimating variance components.

Because the results did not indicate any general type of invariance over the changing \( n_{ij} \) patterns, or over changing parameter sets \((\sigma_r^2, \sigma_c^2, \sigma_{rc}^2, \sigma_e^2 = 1)\), a limited number of computer runs were made. However, the data from these runs are sufficient to indicate the difficulties encountered in attempting to choose an optimum procedure or design.

The computer runs were divided into two groups. The purpose of one group was to compare possible experimental designs for the use of estimating variance components. For these designs there were always six rows and six columns (the largest experimental design that the program would accommodate). The three basic types of designs, S, C and L, are given in Table 4.1. The S and C type of design are variations of Gaylor's \( \frac{1}{\sqrt{1960}} \) balanced disjointed rectangles design. Gaylor \( \frac{1}{\sqrt{1960}} \) has suggested and analyzed the L design. The complete tabulation of computer results are given in Appendix B, for each of the designs shown in Table 4.1.
Note that for the parameter values in Appendix B, $\sigma_r^2 \geq \sigma_c^2$.

Because of the symmetry in most of the designs it was decided not to do the redundant combinations where $\sigma_r^2 > \sigma_c^2$. Also for designs with replication, $\sigma_c^2 = 1$. The other major group is a set of $3 \times 3$ designs which are shown in Table 4.2.

Table 4.1. The $n_{ij}$ arrangements for the $6 \times 6$ designs

<table>
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<tr>
<th>Equal</th>
<th>S 16</th>
<th>S 22</th>
</tr>
</thead>
<tbody>
<tr>
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<td>1 1 0 0 0 0</td>
<td>2 1 0 0 0 0</td>
</tr>
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</table>

<table>
<thead>
<tr>
<th>C 18</th>
<th>C 24</th>
</tr>
</thead>
<tbody>
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</tr>
<tr>
<td>1 1 1 0 0 0</td>
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<tr>
<td>0 0 0 1 1 1</td>
<td>0 0 0 1 1 2</td>
</tr>
<tr>
<td>0 0 0 1 1 1</td>
<td>0 0 0 1 1 2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>L 20</th>
<th>L 24</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 1 0 0 0 0</td>
<td>1 1 0 0 0 0</td>
</tr>
<tr>
<td>1 1 0 0 0 0</td>
<td>1 1 0 0 0 0</td>
</tr>
<tr>
<td>1 1 0 0 0 0</td>
<td>2 1 0 0 0 0</td>
</tr>
<tr>
<td>1 1 0 0 0 0</td>
<td>1 2 0 0 0 0</td>
</tr>
<tr>
<td>1 1 1 1 1 1</td>
<td>1 1 2 1 1 1</td>
</tr>
<tr>
<td>1 1 1 1 1 1</td>
<td>1 1 1 2 1 1</td>
</tr>
</tbody>
</table>
Table 4.2. The \( n_{ij} \) arrangements for the 3 x 3 designs

<table>
<thead>
<tr>
<th></th>
<th>( D_{18-1} )</th>
<th>( D_{18-2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 2 2</td>
<td>1 1 1</td>
<td>1 2 3</td>
</tr>
<tr>
<td>2 2 2</td>
<td>2 2 2</td>
<td>2 3 1</td>
</tr>
<tr>
<td>2 2 2</td>
<td>3 3 3</td>
<td>3 1 2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( D_{18-3} )</th>
<th>( D_{18-4} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 1 4</td>
<td>1 1 1</td>
</tr>
<tr>
<td>1 4 1</td>
<td>1 2 3</td>
</tr>
<tr>
<td>4 1 1</td>
<td>1 3 5</td>
</tr>
</tbody>
</table>

For these designs the total number of observations was held fixed and the \( n_{ij} \) patterns were changed. In all instances, only connected designs are considered.

Some of the more striking results of Appendix B are shown as relative variances of variance components, for a fixed parameter set, in Tables 4.4 through 4.8. In all cases, the variances for the 6 x 6-equal designs are used as the base of the comparison. First all the variances were multiplied by \( (n_t/36) \), where \( n_t \) is the total number of observations for the design, then the adjusted variances were divided by the variances for the 6 x 6-equal design. If any relative variance in Tables 4.4 through 4.8, is less than one, the design for that particular variance component is more efficient than the 6 x 6-equal design; and if greater than one, it is a less efficient design.
From the comparison-of-designs viewpoint, the most information per sample occurs with S, C or L designs whenever \( \sigma_r^2 \) and \( \sigma_c^2 \) dominate \( \sigma_{rc}^2 \), and the least information when \( \sigma_{rc}^2 \) dominates \( \sigma_r^2 \) and \( \sigma_c^2 \). In other words, and S, C or L design can be much better than the balanced design for purposes of estimating variance components when \( \sigma_{rc}^2 \) is smaller than \( \sigma_r^2 \) and \( \sigma_c^2 \).

Although Tables 4.4 through 4.8 are constructed primarily to make comparisons between designs, it is also easy to compare the three estimating procedures. From all the data in Appendix B, Table 4.3 summarizes the major results for the comparisons of designs S, C and L, and estimating procedures A, B and H.

Table 4.3. General results from the 6x6 designs S, C and L

<table>
<thead>
<tr>
<th>Parameter Combinations</th>
<th>Best Design</th>
<th>Best Estimating Procedure</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_r^2 ) ( \sigma_c^2 )</td>
<td>L</td>
<td>H</td>
</tr>
<tr>
<td>( \sigma_{rc}^2 )</td>
<td></td>
<td>A</td>
</tr>
<tr>
<td>( \sigma_{rc}^2 ) ( \sigma_r^2 )</td>
<td>{S or C for rows } ( \sigma_{rc}^2 ) ( \sigma_{rc}^2 ) &gt;&gt;</td>
<td>A ( ) ^(^\text{a}^)</td>
</tr>
</tbody>
</table>

\(^\text{a}^\)In this case H appears to be slightly better for rows than A; however, since H gives such a poor estimate for columns, procedure A is generally recommended. See Table 4.7 as an example.
Note that Tables 4.4 through 4.8 illustrate each of the four parameter combinations in Table 4.3. The analysis of the computer results will be mainly in terms of jointly estimating $\sigma_r^2$ and $\sigma_c^2$. Procedures A and B have the same $\text{var}(\sigma_{rc}^2)$, since interaction is computed the same way in both systems. In most situations Procedures A and B give a smaller $\text{var}(\sigma_{rc}^2)$ than Procedure H. The exception occurs when $\sigma_{rc}^2$ dominates $\sigma_r^2$ and $\sigma_c^2$. (See Table 4.8)

As noted in Table 4.3, Procedure H is also best for estimating $\sigma_r^2$ and $\sigma_c^2$ when $\sigma_{rc}^2$ is much larger than $\sigma_r^2$ and $\sigma_c^2$. However, since Procedure H can be extremely erratic under different $n_{ij}$ arrangements and parameter combinations, it cannot be a recommended procedure unless the experimenter has some a priori information of the magnitude of the variance components. The differences between Procedures A and B are not very great, and in general, Procedure A has variances which are smaller than Procedure B, for designs S, C or L.

The results for the 3x3 designs are also given in Tables 4.4 through 4.8. As with the 6x6 designs, there are small differences between the variances for Procedures A and B. However in this case Procedure B has slightly lower variances. The exceptions seem to occur when $\sigma_r^2$ and $\sigma_c^2$ are larger than $\sigma_{rc}^2$. (See Tables 4.4 and 4.5). It is interesting to note that changes in the $n_{ij}$ arrangement do not drastically affect $\text{var}(\sigma_r^2)$, and $\text{var}(\sigma_c^2)$ for A and B. However, as in the 6x6 designs, Procedure H is again erratic.
An interesting comparison between the 3x3 and 6x6 designs is that when $\sigma^2_{rc}$ dominates $\sigma^2_r$ and $\sigma^2_c$, H is best, and B is poorest for 6x6 designs, while B is best and H is poorest for 3x3 designs.

Procedures A and B have the desirable characteristic that \( \text{var}(\sigma^2_r) \) remains invariant over any change in $\sigma^2_c$, and vice versa. However, since Procedure H depends upon unadjusted sums of squares, it can be seen, from Appendix B, that, if $\sigma^2_r$, $\sigma^2_{rc}$ and $\sigma^2_e$ are held fixed, an increasing $\sigma^2_c$ causes an increasing \( \text{Var}(\sigma^2_r) \). The same thing is true for \( \text{var}(\sigma^2_c) \), when $\sigma^2_e$, $\sigma^2_{rc}$ and $\sigma^2_r$ are held fixed and $\sigma^2_r$ is increased. This is one of the objections to procedure H, in that it is so easily affected by changes in $\sigma^2_r$ and $\sigma^2_c$.

A somewhat analogous situation with procedures A and B is that for a fixed $\sigma^2_r$ and $\sigma^2_e$, an increasing $\sigma^2_{rc}$ causes \( \text{var}(\sigma^2_r) \) to increase. Also for a fixed $\sigma^2_c$ and $\sigma^2_e$, an increasing $\sigma^2_{rc}$ causes \( \text{var}(\sigma^2_c) \) to increase. However this effect (in procedures A and B) is also noticed in the balanced case.

The computer program can also print out the covariances between the row, column and interaction variance components. However, a preliminary analysis of these covariances, in terms of correlations, did not indicate any consistent behavior, so no further analysis was attempted.

Summarizing the results, it seems that if the experimenter does not have any a priori information concerning the magnitudes of the variance components, then an S-design would probably be the
best first choice. When there are more rows than columns, or vice versa, it is still possible to work out some type of S-design. When the number of rows equals the number of columns, say $r = c = k$ an S-design has the desirable feature that it will always have $k-1$ d.f. for rows, columns and interaction. The interaction contrasts matrix $C_{rc}$ is easy to construct since there are only $k-1$ interaction contrasts. For any form of an S, C or L design, it would probably be best to use Procedure A for estimating the variance components and then switch to procedure B, if $\sigma_{rc}^2$ is much greater than $\sigma_r^2$ and $\sigma_c^2$. Chapter V of this dissertation discusses the computational aspects of Procedure A.
<table>
<thead>
<tr>
<th>Type of design and arrangement</th>
<th>Procedure</th>
<th>Columns</th>
<th>Interaction</th>
<th>Columns</th>
<th>Procedure</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rows</td>
<td>Procedure</td>
<td>H</td>
<td>A</td>
<td>B</td>
<td>H A B</td>
</tr>
<tr>
<td>6x6b = 816</td>
<td>H</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td></td>
<td>A</td>
<td>1.03</td>
<td>1.03</td>
<td>1.03</td>
<td>1.03</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>6x6b = 822</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.67</td>
<td>0.67</td>
<td>0.67</td>
<td>0.67</td>
</tr>
<tr>
<td>6x6b = 824</td>
<td></td>
<td>0.67</td>
<td>0.67</td>
<td>0.67</td>
<td>0.67</td>
</tr>
<tr>
<td>6x6b = 814</td>
<td></td>
<td>0.67</td>
<td>0.67</td>
<td>0.67</td>
<td>0.67</td>
</tr>
<tr>
<td>3x3 = 120</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>A</td>
<td>1.03</td>
<td>1.03</td>
<td>1.03</td>
<td>1.03</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
</tbody>
</table>

Each variance was divided by the variance for 6x6 = equal, then adjusted for the variance for 8x8 = equal, they were multiplied by one.

For these designs σ^2 = 0, and σ^2 was increased by one.
Table 4.5. Relative variances\(^a\) of estimates of variance components for 
\[ \sigma_r^2 = 16, \quad \sigma_c^2 = 2, \quad \sigma_{rc}^2 = 1/4 \text{ and } \sigma_e^2 = 1 \]

| Type of design and \( n_{ij} \) arrangement | Rows | | Columns | | Interaction |
|-----------------|-----------------|-----------------|-----------------|-----------------|
|                 | Procedure | H | A | B | Procedure | H | A | B | Procedure |
| 6x6 - equal     | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 6x6\(^b\) - S16 | 0.51 | 0.55 | 0.56 | 7.29 | 0.73 | 0.77 | 109. | 2.22 |
| 6x6 - S22      | 0.50 | 0.78 | 0.79 | 13.00 | 0.95 | 0.98 | 199. | 2.99 |
| 6x6\(^b\) - C18 | 0.55 | 0.58 | 0.61 | 7.40 | 0.76 | 0.79 | 109. | 1.79 |
| 6x6 - C24      | 0.73 | 0.79 | 0.81 | 11.30 | 0.92 | 0.97 | 172. | 3.00 |
| 6x6\(^b\) - L20 | 0.73 | 0.75 | 0.79 | 4.92 | 0.84 | 0.88 | 67.2 | 1.54 |
| 6x6 - L24      | 0.88 | 0.87 | 0.93 | 7.20 | 0.97 | 1.02 | 104. | 3.69 |
| 3x3 - D18-1    | 1.26 | 1.26 | 1.26 | 1.30 | 1.30 | 1.30 | 1.35 | 1.35 |
| 3x3 - D18-2    | 1.32 | 1.26 | 1.26 | 1.33 | 1.33 | 1.34 | 1.55 | 1.55 |
| 3x3 - D18-3    | 1.32 | 1.27 | 1.27 | 4.55 | 1.34 | 1.34 | 52.9 | 1.67 |
| 3x3 - D18-4    | 1.32 | 1.41 | 1.32 | 9.79 | 1.39 | 1.40 | 1.38 | 2.88 |
| 3x3 - D18-5    | 1.41 | 1.41 | 1.32 | 2.35 | 1.48 | 1.42 | 16.1 | 1.81 |

\(^a\) Each variance was divided by the variance for 6x6-equal, then adjusted for different numbers of observations; e.g., after the variances for S16 were divided by those for 6x6-equal, they were multiplied by 16/36.

\(^b\) For these designs \( \sigma_e^2 = 0 \) and \( \sigma_{rc}^2 \) was increased by one.
Table 4.6. Relative variances\(^a\) of estimates of variance components for
\[ \sigma_r^2 = 1, \sigma_c^2 = 1, \sigma_{rc}^2 = 1 \text{ and } \sigma_e^2 = 1 \]

<table>
<thead>
<tr>
<th>Type of design and (n_{ij}) arrangement</th>
<th>Rows Procedure</th>
<th>Columns Procedure</th>
<th>Interaction Procedure</th>
</tr>
</thead>
<tbody>
<tr>
<td>6x6 - equal</td>
<td>H 1.00 A 1.00 B 1.00</td>
<td>H 1.00 A 1.00 B 1.00</td>
<td>H 1.00 A &amp; B 1.00</td>
</tr>
<tr>
<td>6x6(^b) - S16</td>
<td>1.20 1.28 1.42</td>
<td>1.20 1.28 1.42</td>
<td>2.89 2.22</td>
</tr>
<tr>
<td>6x6 - S22</td>
<td>1.70 1.56 1.67</td>
<td>1.70 1.56 1.57</td>
<td>4.52 2.77</td>
</tr>
<tr>
<td>6x6(^b) - C18</td>
<td>1.13 1.10 1.20</td>
<td>1.18 1.13 1.23</td>
<td>2.53 1.78</td>
</tr>
<tr>
<td>6x6 - C24</td>
<td>1.57 1.40 1.49</td>
<td>1.59 1.39 1.51</td>
<td>3.98 2.48</td>
</tr>
<tr>
<td>6x6(^b) - L20</td>
<td>1.18 1.09 1.15</td>
<td>1.18 1.09 1.15</td>
<td>2.10 1.54</td>
</tr>
<tr>
<td>6x6 - L24</td>
<td>1.49 1.27 1.31</td>
<td>1.49 1.27 1.31</td>
<td>3.54 2.50</td>
</tr>
<tr>
<td>3x3 - equal</td>
<td>1.65 1.65 1.65</td>
<td>1.65 1.65 1.65</td>
<td>1.84 1.84</td>
</tr>
<tr>
<td>3x3 - D18-1</td>
<td>1.83 1.83 1.83</td>
<td>1.80 1.80 1.73</td>
<td>2.04 2.04</td>
</tr>
<tr>
<td>3x3 - D18-2</td>
<td>2.23 1.78 1.73</td>
<td>2.23 1.78 1.73</td>
<td>3.57 2.10</td>
</tr>
<tr>
<td>3x3 - D18-3</td>
<td>3.27 1.90 1.85</td>
<td>3.27 1.90 1.85</td>
<td>6.27 2.50</td>
</tr>
<tr>
<td>3x3 - D18-4</td>
<td>2.13 2.01 1.85</td>
<td>2.13 2.01 1.85</td>
<td>2.65 2.30</td>
</tr>
</tbody>
</table>

\(^a\) Each variance was divided by the variance for 6x6-equal, then adjusted for different numbers of observations; e.g., after the variances for S16 were divided by those for 6x6-equal, they were multiplied by 16/36.

\(^b\) For these designs \(\sigma_e^2 = 0\) and \(\sigma_{rc}^2\) was increased by one.
Table 4.7. Relative variances\(^{a}\) of estimates of variance components for
\[ \sigma_r^2 = 16, \sigma_c^2 = 0, \sigma_{rc}^2 = 4 \text{ and } \sigma_e^2 = 1 \]

<table>
<thead>
<tr>
<th>Type of design and ( n_{ij} ) arrangement</th>
<th>Rows</th>
<th></th>
<th></th>
<th></th>
<th>Columns</th>
<th></th>
<th></th>
<th></th>
<th>Interaction</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>H</td>
<td>A</td>
<td>B</td>
<td></td>
<td>H</td>
<td>A</td>
<td>B</td>
<td></td>
<td>H A ( \neq ) B</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6x6 - equal</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td></td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td></td>
<td>1.00</td>
<td>1.00</td>
<td></td>
</tr>
<tr>
<td>6x6(^b) - S16</td>
<td>.32</td>
<td>.62</td>
<td>.64</td>
<td></td>
<td>4.20</td>
<td>6.67</td>
<td>7.64</td>
<td></td>
<td>8.13</td>
<td>2.22</td>
<td></td>
</tr>
<tr>
<td>6x6 - S22</td>
<td>.71</td>
<td>.89</td>
<td>.90</td>
<td></td>
<td>77.0</td>
<td>7.94</td>
<td>8.86</td>
<td></td>
<td>14.4</td>
<td>2.87</td>
<td></td>
</tr>
<tr>
<td>6x6(^b) - C18</td>
<td>.54</td>
<td>.64</td>
<td>.68</td>
<td></td>
<td>4.07</td>
<td>1.47</td>
<td>5.10</td>
<td></td>
<td>7.85</td>
<td>1.79</td>
<td></td>
</tr>
<tr>
<td>6x6 - C24</td>
<td>.75</td>
<td>.87</td>
<td>.91</td>
<td></td>
<td>65.6</td>
<td>6.04</td>
<td>6.80</td>
<td></td>
<td>12.3</td>
<td>2.34</td>
<td></td>
</tr>
<tr>
<td>6x6(^b) - L20</td>
<td>.73</td>
<td>.79</td>
<td>.83</td>
<td></td>
<td>2.32</td>
<td>3.30</td>
<td>3.47</td>
<td></td>
<td>4.83</td>
<td>1.54</td>
<td></td>
</tr>
<tr>
<td>6x6 - L24</td>
<td>.89</td>
<td>.93</td>
<td>.97</td>
<td></td>
<td>35.2</td>
<td>4.10</td>
<td>4.08</td>
<td></td>
<td>7.13</td>
<td>1.93</td>
<td></td>
</tr>
<tr>
<td>3x3 - equal</td>
<td>1.35</td>
<td>1.35</td>
<td>1.35</td>
<td></td>
<td>5.07</td>
<td>5.07</td>
<td>5.07</td>
<td></td>
<td>2.55</td>
<td>2.55</td>
<td></td>
</tr>
<tr>
<td>3x3 - DL8-1</td>
<td>1.50</td>
<td>1.50</td>
<td>1.50</td>
<td></td>
<td>6.49</td>
<td>6.49</td>
<td>5.42</td>
<td></td>
<td>2.83</td>
<td>2.83</td>
<td></td>
</tr>
<tr>
<td>3x3 - DL8-2</td>
<td>1.10</td>
<td>1.38</td>
<td>1.36</td>
<td></td>
<td>20.7</td>
<td>6.08</td>
<td>5.37</td>
<td></td>
<td>5.28</td>
<td>2.78</td>
<td></td>
</tr>
<tr>
<td>3x3 - DL8-3</td>
<td>1.51</td>
<td>1.39</td>
<td>1.36</td>
<td></td>
<td>4.71</td>
<td>6.44</td>
<td>5.68</td>
<td></td>
<td>9.74</td>
<td>2.93</td>
<td></td>
</tr>
<tr>
<td>3x3 - DL8-4</td>
<td>1.54</td>
<td>1.55</td>
<td>1.42</td>
<td></td>
<td>11.7</td>
<td>7.30</td>
<td>5.63</td>
<td></td>
<td>3.85</td>
<td>3.11</td>
<td></td>
</tr>
</tbody>
</table>

\(^a\) Each variance was divided by the variance for 6x6-equal, then adjusted for different numbers of observations; e.g., after the variances for S16 were divided by those for 6x6-equal, they were multiplied by 16/36.

\(^b\) For these designs \( \sigma_e^2 = 0 \) and \( \sigma_{rc}^2 \) was increased by one.
Table 4.8. Relative variances\(^a\) of estimates of variance components for 
\[ \sigma_r^2 = 1, \sigma_c^2 = 1, \sigma_{rc}^2 = 16 \text{ and } \sigma_e^2 = 1 \]

<table>
<thead>
<tr>
<th>Type of design and (n_{ij}) arrangement</th>
<th>Rows Procedure</th>
<th>Columns Procedure</th>
<th>Interaction Procedure</th>
</tr>
</thead>
<tbody>
<tr>
<td>6x6 - equal</td>
<td>H   A   B</td>
<td>H   A   B</td>
<td>H   A   B</td>
</tr>
<tr>
<td>6x6(^b) - s16</td>
<td>2.28 4.44 5.07</td>
<td>2.28 4.44 5.07</td>
<td>1.70 2.22</td>
</tr>
<tr>
<td>6x6 - s22</td>
<td>3.68 5.69 6.35</td>
<td>3.68 5.69 6.35</td>
<td>2.49 3.02</td>
</tr>
<tr>
<td>6x6(^b) - c18</td>
<td>1.94 3.11 3.42</td>
<td>2.00 3.12 3.54</td>
<td>1.53 1.79</td>
</tr>
<tr>
<td>6x6 - c24</td>
<td>3.01 4.29 4.59</td>
<td>3.28 4.45 5.00</td>
<td>2.19 2.41</td>
</tr>
<tr>
<td>6x6(^b) - l20</td>
<td>1.93 2.41 2.52</td>
<td>1.93 2.41 2.52</td>
<td>1.43 1.54</td>
</tr>
<tr>
<td>6x6 - l24</td>
<td>2.70 3.10 3.07</td>
<td>2.70 3.10 3.07</td>
<td>1.77 1.91</td>
</tr>
<tr>
<td>3x3 - equal</td>
<td>4.40 4.40 4.40</td>
<td>4.40 4.40 4.40</td>
<td>2.94 2.94</td>
</tr>
<tr>
<td>3x3 - dl8-1</td>
<td>4.88 4.88 4.88</td>
<td>5.54 5.54 4.58</td>
<td>3.26 3.26</td>
</tr>
<tr>
<td>3x3 - dl8-2</td>
<td>6.12 5.15 4.53</td>
<td>6.12 5.15 4.53</td>
<td>3.50 3.15</td>
</tr>
<tr>
<td>3x3 - dl8-3</td>
<td>9.42 5.24 4.58</td>
<td>9.42 5.24 4.58</td>
<td>4.23 3.20</td>
</tr>
<tr>
<td>3x3 - dl8-4</td>
<td>6.56 6.25 4.84</td>
<td>6.56 6.25 4.84</td>
<td>3.70 3.59</td>
</tr>
</tbody>
</table>

\(^a\) Each variance was divided by the variance for 6x6-equal, then adjusted for different numbers of observations; e.g., after the variances for S16 were divided by those for 6x6-equal, they were multiplied by 16/36.

\(^b\) For these designs \(\sigma_e^2 = 0\) and \(\sigma_{rc}^2\) was increased by one.
CHAPTER V

A SIMPLIFIED COMPUTATIONAL METHOD FOR PROCEDURE A

5.1 Introduction

The main purpose of this chapter is to show an easy computational procedure for obtaining the $4 \times 4$ variance-covariance matrix $\text{var}(t_a)$, shown in (4.2). In general, the major portion of the computation in evaluating $\text{var}(y)$ is in finding $\text{var}(t)$. Searle [1958] and Henderson et al. [1957] show a simplified set of equations for evaluating $\text{var}(t_h)$, in Procedure H. Searle was able to reduce the matrix products to simple combinations of sums of the $n_{ij}$'s and the true variance components. Since six of the ten unique elements in $\text{var}(t_h)$ and $\text{var}(t_a)$ are identical, Searle's results will be repeated in this chapter. The other work in this chapter is concerned with obtaining a computational simplification of the remaining four elements in $\text{var}(t_a)$, (variances and covariances with $I^*$).

5.2 Searle's Results for $\text{var}(t_h)$

Because there is no consistent notation between the Searle [1958] and Henderson et al. [1957] papers, the notation used in this section for defining sums and products of the $n_{ij}$'s will be different from that of Searle in some instances. Define

$$
\begin{align*}
n &= \sum_i \sum_j n_{ij}; \quad n_i = \sum_j n_{ij}; \quad n_{.j} = \sum_i n_{ij}; \\
r &= \text{number of rows in the experimental design}; \\
c &= \text{number of columns in the experimental design}; \\
t &= \text{number of non-empty subclasses}.
\end{align*}
$$
Also we have the following $k_i$:

<table>
<thead>
<tr>
<th>$k_i$</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_1$</td>
<td>$\Sigma \frac{n_{1i}^2}{n}$</td>
</tr>
<tr>
<td>$k_2$</td>
<td>$\Sigma \frac{n_{j}^2}{n}$</td>
</tr>
<tr>
<td>$k_{12}$</td>
<td>$\Sigma \frac{\Sigma n_{ij}^2}{n_{1i} n_{j}}$</td>
</tr>
<tr>
<td>$k_{21}$</td>
<td>$\Sigma \frac{\Sigma n_{ij}^2}{n_{1j} n_{i}}$</td>
</tr>
<tr>
<td>$k_3$</td>
<td>$\Sigma \Sigma n_{ij}^2$</td>
</tr>
<tr>
<td>$k_{26}$</td>
<td>$\Sigma \Sigma (n_{ij} n_{i} n_{j})$</td>
</tr>
<tr>
<td>$k_5$</td>
<td>$\Sigma \Sigma n_{ij}^3$</td>
</tr>
<tr>
<td>$k_8$</td>
<td>$\Sigma \Sigma n_{ij}^3$</td>
</tr>
</tbody>
</table>

$\Sigma$ denotes the summation over the respective indices.

<table>
<thead>
<tr>
<th>$k_i$</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_{14}$</td>
<td>$\Sigma \frac{\Sigma (n_{ij}^2 n_{i})}{n_{j}}$</td>
</tr>
<tr>
<td>$k_{23}$</td>
<td>$\Sigma \frac{\Sigma (n_{ij}^2 n_{j})}{n_{i}}$</td>
</tr>
<tr>
<td>$k_{27}$</td>
<td>$\Sigma \frac{\Sigma n_{ij}^3}{n_{i} n_{j}}$</td>
</tr>
<tr>
<td>$k_{28}$</td>
<td>$\Sigma \frac{\Sigma n_{ij}^4}{n_{i} n_{j}}$</td>
</tr>
<tr>
<td>$k_7$</td>
<td>$\Sigma \Sigma n_{ij}^2 / n_{i}^2$</td>
</tr>
<tr>
<td>$k_{10}$</td>
<td>$\Sigma \Sigma n_{ij}^2 / n_{j}^2$</td>
</tr>
<tr>
<td>$k_{24}$</td>
<td>$\Sigma \Sigma (\Sigma n_{ij} n_{i} n_{j})^2 / n_{i} n_{i'}$</td>
</tr>
<tr>
<td>$k_{25}$</td>
<td>$\Sigma \Sigma (\Sigma n_{ij} n_{i} n_{j})^2 / n_{j} n_{j'}$</td>
</tr>
</tbody>
</table>

where $k_6$, $k_{14}$, $k_{23}$, $k_{24}$, $k_{25}$, $k_{26}$, $k_{27}$ and $k_{28}$ are the same as in Henderson et al. 1957, and that $k_1$, $k_2$, $k_3$, $k_{12}$ and $k_{21}$ have already been defined in Section 4.3. To best illustrate the computation of $k_{24}$, $k_{25}$ and the rest of the $k$'s consider the $n_{ij}$ data in Table 5.1.
Table 5.1. Subclass \( n_{ij} \) data for a two-way classification

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Rows</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Totals</td>
<td>3</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>( n_{11} )</td>
<td>( n_{12} )</td>
<td>( n_{13} )</td>
</tr>
</tbody>
</table>

Now

\[
k_{24} = 2 \left\{ \frac{\sum (0(2)+1(1)+1(1))^2}{2(4)} + \frac{\sum (0(1)+1(1)+1(2))^2}{2(4)} + \frac{\sum (1(1)+1(1)+1(2))^2}{4(4)} \right\}
\]

\[= \frac{51}{8}, \]

\[
k_{25} = 2 \left\{ \frac{\sum (0(1)+2(1)+1(1))^2}{3(3)} + \frac{16}{12} + \frac{16}{12} \right\} = \frac{22}{3}, \]

\[
k_1 = 3.6, \quad k_{14} = 15.5, \]
\[
k_2 = 3.4, \quad k_{23} = 13.75, \]
\[
k_{12} = 4, \quad k_{27} = 85/48, \]
\[
k_{21} = 25/6, \quad k_{28} = 14/48, \]
\[
k_3 = 1.4, \quad k_7 = 5.5, \]
\[
k_{26} = 57/48, \quad k_{10} = 217/38, \]
\[
k_5 = 6, \quad \]
\[
k_8 = 6.5. \]
Searle \(1957\) and Henderson et al. \(1957\) show that

\[
\text{var}(R) = nk_1(2\sigma_r^2 + k_2 \sigma_c^2 + k_7 \sigma_e^2 + k_9 \sigma_{rc}^2 + n(4\sigma_c^2 + 4\sigma_{rc}^2 + 4\sigma_e^2)) + nk_3(4\sigma_r^2 + 4\sigma_c^2 + 4\sigma_{rc}^2 + 4\sigma_{re}^2 + 4\sigma_{rc}^2 + 4\sigma_{re}^2) + n(4\sigma_r^2 + 4\sigma_c^2 + 4\sigma_{rc}^2 + 4\sigma_{re}^2 + 4\sigma_{rc}^2 + 4\sigma_{re}^2) (5.1)
\]

In Section 4.4, \(\text{var}(R) = 2\text{tr}(C'_{ru} V C'_{ru} B_{ru})^2\), which would be considerably more difficult to compute. By using the expansion of \(V\) of (4.8) in \(2\text{tr}(C'_{ru} V C'_{ru} B_{ru})^2\), it can be seen that (5.1) will be obtained. Table 5.2 at the end of Section 5.3 gives the remaining coefficients developed by Searle \(1957\) and Henderson et al. \(1957\) for \(\text{var}(C), \text{cov}(R, C), \text{var}(S), \text{cov}(R, S)\) and \(\text{cov}(C, S)\).

5.3 Computing \(\text{var}(I^*), \text{cov}(R, I^*), \text{cov}(C, I^*)\) and \(\text{cov}(S, I^*)\)

In the expansion of each of the four quantities, \(\text{var}(I^*),\)
\(\text{cov}(R, I^*), \text{cov}(C, I^*)\) and \(\text{cov}(S, I^*)\), the matrix \(V\) is next to a \(C'_{rc}\) interaction matrix. Since \(C'_{rc}\) is constructed in order to eliminate row and column effects, then \(C'_{rc} V = C'_{rc} V = 0\) in general. Thus for the considered four items, the general \(V\)-matrix can be written as

\[
V = \sigma^2_{rc} I + \sigma^2_{e}(A'A)^{-1} (5.2)
\]

Using the \(V\) in (5.2)
\[ \text{var(I*)} = 2 \text{tr} \sum C'_{rc} (\sigma^2_{rc} I + \sigma^2_e (A'A)^{-1}) C_{rc} B_{rc} \] 

\[ = 2 \text{tr} \sum (C'_{rc} C_{rc} B_{rc}) \sigma^2_{rc} + (C'_{rc} (A'A)^{-1} C_{rc} B_{rc}) \sigma^2_e \] 

\[ = 2 \text{tr} \sum (C'_{rc} C_{rc} B_{rc} \sigma^2_{rc} + I_{n \times n}) \sigma^2_e \] 

\[ = (2 \sigma^4_{rc} \text{tr}(C'_{rc} C_{rc} B_{rc})) + (4 \sigma^2_{rc} \sigma^2_e \text{tr}(C'_{rc} C_{rc} B_{rc})) \] 

\[ + (2 \sigma^4_e) n \] .

Before expanding \( \text{cov}(R, I^*) \), note that \((A'A)^{-1} C_{ru}\) is always a matrix of ones and zeroes, where the ones reflect the non-empty subclasses of rows in the experimental design. Therefore just as \( C'_{rc} V_r = 0, C'_{rc} (A'A)^{-1} C_{ru} = 0 \), since the \( C'_{rc} \)-matrix consists of contrasts which are orthogonal to rows and columns. Analogously \( C'_{rc} (A'A)^{-1} C_{cu} = 0 \). Hence

\[ \text{cov}(R, I^*) = 2 \text{tr} \left\{ C'_{rc} \sum \sigma^2_{rc} + \sigma^2_e (A'A)^{-1} C_{ru} B_{ru} C'_{rc} \right\} \]

\[ = (2 \sigma^4_{rc} \text{tr}(C'_{rc} C_{ru} B_{ru} C_{rc} B_{rc})) , \]

since every other expression contains \( C'_{rc} (A'A)^{-1} C_{ru} \), which is a null matrix. Analogously

\[ \text{cov}(C, I^*) = (2 \sigma^4_{rc} \text{tr}(C'_{rc} C_{cu} B_{cu} C'_{rc} B_{rc})) . \]
Also
\[
\text{cov}(S; I^\ast) = 2\text{tr} \left\{ \left( c'_{rc} \sigma^2_{rc} I + \sigma^2_e (A'A)^{-1} A'A \right) \left( c'_{rc} \sigma^2_{rc} I + \sigma^2_e (A'A)^{-1} c_{rc} B_{rc} \right) \right\}
\]
\[
= (2\sigma^4_{rc}) \text{tr} (c'_{rc} (A'A) c_{rc} B_{rc}) + (4\sigma^2_{rc} \sigma^2_e) \text{tr} (c'_{rc} c_{rc} B_{rc}) + (2\sigma^4_e) n_{in}
\]

Using

\[
k_{30} = \text{tr} (c'_{rc} c_{rc} B_{rc}), \text{ as previously done in Section 4.3},
\]
\[
k_{31} = \text{tr} (c'_{rc} c_{rc} B_{rc})^2,
\]
\[
k_{32} = \text{tr} (c'_{rc} c_{ru} c_{ru} c_{rc} B_{rc}),
\]
\[
k_{33} = \text{tr} (c'_{rc} c_{uc} c_{uc} c_{rc} B_{rc}), \text{ and}
\]
\[
k_{34} = \text{tr} (c'_{rc} (A'A) c_{rc} B_{rc}),
\]

\text{var}(I^\ast), \text{cov}(R; I^\ast), \text{cov}(C; I^\ast) \text{ and } \text{cov}(S; I^\ast) \text{ are represented in Table 5.2.}

When some \( n_{ij} = 0 \), there does not seem to be any way of simplifying \( k_{30}, \ldots, k_{34} \). However when all \( n_{ij} > 0 \), for any given two-way classification, it is possible to write out some of the matrix products in terms of the \( n_{ij} \)'s. This would then require a tabulation of matrices (of \( n_{ij} \)'s) for all pertinent combinations of \( r \) and \( c \).

In any event the computation of \( k_{30}, \ldots, k_{34} \) involves \( B \), which is obtained by inverting \( c_{rc} (A'A)^{-1} c_{rc} \), an \( (n_{in} \times n_{in}) \) matrix. It does not seem likely that this inversion can be bypassed in the general unbalanced case.
Finally after \( \text{var}(t_e) \) is obtained through the use of the results in Table 5.2, it is then necessary to evaluate (4.1) in order to obtain the desired variances of the estimates of the variance components. Since only the diagonal terms of the right hand side of (4.1) are required, it is not necessary to compute the covariance terms of \( \text{var}(y_a) \).
Table 5.2. Coefficients of terms in \( \text{var}(t_a) \)

<table>
<thead>
<tr>
<th>Elements of ( \text{var}(t_a) )</th>
<th>( 2\sigma_r^4 )</th>
<th>( 2\sigma_c^4 )</th>
<th>( 2\sigma_{rc}^4 )</th>
<th>( l\sigma_r^2 \sigma_c^2 )</th>
<th>( l\sigma_r^2 \sigma_{rc}^2 )</th>
<th>( l\sigma_r^2 \sigma_e^2 )</th>
<th>( l\sigma_c^2 \sigma_{rc}^2 )</th>
<th>( l\sigma_c^2 \sigma_e^2 )</th>
<th>( l\sigma_{rc}^2 \sigma_e^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{var}(R) )</td>
<td>( n_k )</td>
<td>( k_{24} + k_7 )</td>
<td>( k_7 )</td>
<td>( r )</td>
<td>( n_k )</td>
<td>( n_k )</td>
<td>( n )</td>
<td>( k_7 )</td>
<td>( k_{12} )</td>
</tr>
<tr>
<td>( \text{var}(C) )</td>
<td>( k_{25} )</td>
<td>( k_{10} )</td>
<td>( n_k )</td>
<td>( c )</td>
<td>( n_k )</td>
<td>( k_{10} )</td>
<td>( k_{21} )</td>
<td>( n_k )</td>
<td>( n )</td>
</tr>
<tr>
<td>( \text{cov}(R,C) )</td>
<td>( k_{14} )</td>
<td>( k_{23} )</td>
<td>( k_{28} )</td>
<td>( k_{26} )</td>
<td>( n_k )</td>
<td>( k_8 )</td>
<td>( k_{21} )</td>
<td>( k_5 )</td>
<td>( k_{12} )</td>
</tr>
<tr>
<td>( \text{var}(S) )</td>
<td>( n_k )</td>
<td>( n_k )</td>
<td>( n_k )</td>
<td>( t )</td>
<td>( n_k )</td>
<td>( n_k )</td>
<td>( n )</td>
<td>( n_k )</td>
<td>( n )</td>
</tr>
<tr>
<td>( \text{cov}(R,S) )</td>
<td>( n_k )</td>
<td>( k_{23} )</td>
<td>( k_5 )</td>
<td>( r )</td>
<td>( n_k )</td>
<td>( n_k )</td>
<td>( n )</td>
<td>( k_5 )</td>
<td>( k_{12} )</td>
</tr>
<tr>
<td>( \text{cov}(C,S) )</td>
<td>( k_{14} )</td>
<td>( n_k )</td>
<td>( k_8 )</td>
<td>( c )</td>
<td>( n_k )</td>
<td>( k_8 )</td>
<td>( k_{21} )</td>
<td>( n_k )</td>
<td>( n )</td>
</tr>
<tr>
<td>( \text{var}(I*) )</td>
<td>( _ )</td>
<td>( _ )</td>
<td>( k_{31} )</td>
<td>( n^{__} )</td>
<td>( _ )</td>
<td>( _ )</td>
<td>( _ )</td>
<td>( _ )</td>
<td>( k_{30} )</td>
</tr>
<tr>
<td>( \text{cov}(R,I*) )</td>
<td>( _ )</td>
<td>( _ )</td>
<td>( k_{32} )</td>
<td>( _ )</td>
<td>( _ )</td>
<td>( _ )</td>
<td>( _ )</td>
<td>( _ )</td>
<td>( _ )</td>
</tr>
<tr>
<td>( \text{cov}(C,I*) )</td>
<td>( _ )</td>
<td>( _ )</td>
<td>( k_{33} )</td>
<td>( _ )</td>
<td>( _ )</td>
<td>( _ )</td>
<td>( _ )</td>
<td>( _ )</td>
<td>( _ )</td>
</tr>
<tr>
<td>( \text{cov}(S,I*) )</td>
<td>( _ )</td>
<td>( _ )</td>
<td>( k_{34} )</td>
<td>( n^{__} )</td>
<td>( _ )</td>
<td>( _ )</td>
<td>( _ )</td>
<td>( _ )</td>
<td>( k_{30} )</td>
</tr>
</tbody>
</table>
CHAPTER VI

EXTENSION OF RESULTS TO THE MULTI-WAY CLASSIFICATION

6.1 Introduction

In the previous three chapters the general two-way classification was thoroughly investigated with respect to the three estimating procedures A, B and H. It is the specific purpose of this chapter to show how each of these estimating procedures can be extended to the three-way classification, and also to indicate general extensions to four- and higher-way classifications. A $2 \times 2 \times 3$ experimental design is used to illustrate the necessary matrices for each estimating procedure.

6.2 Sums of Squares in the $2 \times 2 \times 3$ Classification

Consider the mathematical model as

$$y_{ijk} = \gamma_{ijk} + e_{ijk}$$  

$$= \mu + a_i + b_j + c_k + (ab)_{ij} + (ac)_{ik} + (bc)_{jk} + (abc)_{ijk} + e_{ijk}$$

(i=1,2, j=1,2, k=1,2,3, $f=1,2,\ldots,n_{ijk}$)

where in Model I, there are all fixed effects except $e_{ijk}$ which is normally distributed with mean zero and variance $\sigma^2$. The experimental design in terms of the $n_{ijk}$'s is shown in Table 6.1.
Table 6.1. Three factor design in terms of $n_{ijk}$'s

<table>
<thead>
<tr>
<th></th>
<th>$A_1$</th>
<th></th>
<th>$A_2$</th>
<th></th>
<th></th>
<th>Totals</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$B_1$</td>
<td>$B_2$</td>
<td>$B_1$</td>
<td>$B_2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$C_1$</td>
<td>$n_{111}$</td>
<td>$n_{121}$</td>
<td>$n_{211}$</td>
<td>$n_{221}$</td>
<td>$n_{.1}$</td>
<td></td>
</tr>
<tr>
<td>$C_2$</td>
<td>$n_{112}$</td>
<td>$n_{122}$</td>
<td>$n_{212}$</td>
<td>$n_{222}$</td>
<td>$n_{.2}$</td>
<td></td>
</tr>
<tr>
<td>$C_3$</td>
<td>$n_{113}$</td>
<td>$n_{123}$</td>
<td>$n_{213}$</td>
<td>$n_{223}$</td>
<td>$n_{.3}$</td>
<td></td>
</tr>
<tr>
<td>Totals</td>
<td>$n_{11.}$</td>
<td>$n_{12.}$</td>
<td>$n_{21.}$</td>
<td>$n_{22.}$</td>
<td>$n_{..}$</td>
<td></td>
</tr>
</tbody>
</table>

where $\sum_{i} n_{ijk} = n_{i..}$; $\sum_{j} n_{ijk} = n_{.ik}$; $\sum_{k} n_{ijk} = n_{ij.}$.

As before, the parameters $\gamma_{ijk}$ of the model shown in (6.1) are the true subclass means of each cell and $\gamma_{ijk}^A$ are the observed cell means. Now as in the two-way classification, it is important to be able to express the relevant sums of squares through the $C'$-matrices, as shown in (3.6). Because there will have to be a large number of sums of squares defined in the three-way classification, the convention will be adopted to use $SS(j)$ as the sum of squares associated with the $C_{j}$ matrix.

First assuming that all $n_{ijk} > 0$, the relevant $C'$-matrices in the 2x2x3 design for obtaining all the necessary sums of squares are as follows:

The general mean:

$$C'_m = \begin{pmatrix} n_{111}, n_{121}, n_{211}, n_{221}, n_{112}, n_{122}, n_{212}, n_{222}, n_{113}, n_{123}, n_{213}, n_{223} \end{pmatrix}$$
Factor A, unadjusted and uncorrected:

\[
C'_{\text{au}} = \begin{bmatrix}
    n_{111} & n_{121} & 0 & 0 & n_{112} & n_{122} & 0 & 0 & n_{113} & n_{123} & 0 & 0 \\
    0 & 0 & n_{211} & n_{221} & 0 & 0 & n_{212} & n_{222} & 0 & 0 & n_{213} & n_{223}
\end{bmatrix}
\]

Factor B, unadjusted and uncorrected:

\[
C'_{\text{bu}} = \begin{bmatrix}
    n_{111} & 0 & n_{211} & 0 & n_{112} & 0 & n_{212} & 0 & n_{113} & 0 & n_{213} & 0 \\
    0 & n_{121} & 0 & n_{221} & 0 & n_{122} & 0 & n_{222} & 0 & n_{123} & 0 & n_{223}
\end{bmatrix}
\]

Factor C, unadjusted and uncorrected:

\[
C'_{\text{cu}} = \begin{bmatrix}
    n_{111} & n_{121} & n_{122} & n_{121} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & n_{112} & n_{122} & n_{212} & n_{222} & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & n_{113} & n_{123} & n_{213} & n_{223}
\end{bmatrix}
\]

For the uncorrected subclasses, \(C'_{\text{abu}} = I_{12x12}\). Now it is also necessary to be able to express the set of sums of squares which would be obtained from equation (6.1) if it were reduced to all the possible two-way classifications.

For the two-way subclasses AB(uncorrected),

\[
C'_{\text{abu}} = \begin{bmatrix}
    \frac{n_{11}}{n_{11}} & 0 & 0 & 0 & \frac{n_{12}}{n_{11}} & 0 & 0 & 0 & \frac{n_{13}}{n_{11}} & 0 & 0 & 0 \\
    0 & \frac{n_{12}}{n_{12}} & 0 & 0 & 0 & \frac{n_{12}}{n_{12}} & 0 & 0 & 0 & \frac{n_{13}}{n_{12}} & 0 & 0 & 0 \\
    0 & 0 & \frac{n_{21}}{n_{21}} & 0 & 0 & 0 & \frac{n_{21}}{n_{21}} & 0 & 0 & 0 & \frac{n_{23}}{n_{21}} & 0 & 0 \\
    0 & 0 & 0 & \frac{n_{22}}{n_{22}} & 0 & 0 & 0 & \frac{n_{22}}{n_{22}} & 0 & 0 & 0 & \frac{n_{23}}{n_{22}}
\end{bmatrix}
\]
For the two-way subclasses AC (uncorrected),

\[
\begin{bmatrix}
\frac{n_{111}}{n_{1.1}} & \frac{n_{121}}{n_{1.1}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{n_{211}}{n_{2.1}} & \frac{n_{221}}{n_{2.1}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{n_{112}}{n_{1.2}} & \frac{n_{122}}{n_{1.2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{n_{212}}{n_{2.2}} & \frac{n_{222}}{n_{2.2}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{n_{113}}{n_{1.3}} & \frac{n_{123}}{n_{1.3}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{n_{213}}{n_{2.3}} & \frac{n_{223}}{n_{2.3}} & 0 & 0 \\
\end{bmatrix}
\]

For the two-way subclasses BC (uncorrected),

\[
\begin{bmatrix}
\frac{n_{111}}{n_{1.1}} & 0 & \frac{n_{211}}{n_{1.1}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{n_{121}}{n_{2.1}} & 0 & \frac{n_{221}}{n_{2.1}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{n_{112}}{n_{1.2}} & \frac{n_{122}}{n_{1.2}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{n_{212}}{n_{2.2}} & \frac{n_{222}}{n_{2.2}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{n_{113}}{n_{1.3}} & \frac{n_{123}}{n_{1.3}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{n_{213}}{n_{2.3}} & \frac{n_{223}}{n_{2.3}} & 0 & 0 \\
\end{bmatrix}
\]
For the two-way interaction AB (adjusted for A and B, ignoring C), matrix $C'_{ab(c)}$ is

$$
\begin{bmatrix}
-n_{11} & -n_{12} & n_{12} & n_{22} & n_{12} & n_{12} & n_{12} & n_{22} & n_{12} & n_{12} & n_{22} & n_{12} & n_{22} \\
-n_{11} & n_{21} & n_{21} & n_{12} & n_{12} & n_{12} & n_{12} & n_{12} & n_{12} & n_{22} & n_{22} & n_{22}
\end{bmatrix}.
$$

For the two-way interaction AC (adjusted for A and C, ignoring B), matrix $C'_{ac(b)}$ is

$$
\begin{bmatrix}
n_{11} & n_{12} & -n_{21} & n_{22} & -n_{12} & n_{12} & n_{22} & -n_{12} & n_{22} & 0 & 0 & 0 & 0 \\
n_{11} & n_{11} & n_{21} & n_{21} & n_{12} & n_{22} & n_{22} & n_{12} & n_{22} & 0 & 0 & 0 & 0 \\
n_{11} & n_{11} & n_{21} & n_{21} & 0 & 0 & 0 & 0 & n_{11} & n_{13} & n_{23} & n_{23} & n_{23}
\end{bmatrix}.
$$

For the two-way interaction BC (adjusted for B and C, ignoring A), matrix $C'_{bc(a)}$ is

$$
\begin{bmatrix}
n_{11} & -n_{12} & n_{21} & n_{22} & -n_{12} & n_{12} & n_{22} & -n_{12} & n_{22} & 0 & 0 & 0 & 0 \\
n_{11} & n_{21} & n_{11} & n_{11} & n_{12} & n_{22} & n_{12} & n_{12} & n_{22} & 0 & 0 & 0 & 0 \\
n_{11} & n_{21} & n_{11} & n_{11} & 0 & 0 & 0 & 0 & n_{11} & n_{13} & n_{23} & n_{23} & n_{23}
\end{bmatrix}.
$$

(6.4)

Now again in terms of the three factor model:

Factor A (weighted squares of means)

$$C'_{aw} = \begin{bmatrix} 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 \end{bmatrix}.$$

Factor B (weighted squares of means)

$$C'_{bw} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}.$$
Factor C (weighted squares of means)

\[
C'_{cw} = \begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

Interaction AB

\[
C'_{abw} = (1, -1, -1, 1, 1, -1, 1, 1, -1, 1, -1, 1).
\]

Interaction AC

\[
C'_{acw} = \begin{bmatrix}
1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

Interaction BC

\[
C'_{bcw} = \begin{bmatrix}
1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

Interaction ABC

\[
C'_{abcw} = \begin{bmatrix}
1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 \\
1 & -1 & -1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

(6.5)

Note that as in Section (3.3), the above four interaction matrices can be constructed by multiplying the proper contrasts. For example, \(C'_{abcw}\) can be obtained by multiplying the columns of \(C'\) with \(C'_{bcw}\).

As before when some \(n_{ijk} = 0\), the interaction matrices are constructed by eliminating those contrasts which are not interaction contrasts.
The necessary remaining sums of squares are obtained by combining some of the above $C'$-matrices and taking linear combinations of them.

All interactions: $C'_{3w}^{1w} = \begin{bmatrix}
C_{abw} \\
C_{acw} \\
C_{bcw} \\
C_{abcw}
\end{bmatrix}$

AB, ABC: $C'_{abw} = \begin{bmatrix}
C_{abw} \\
C_{abcw}
\end{bmatrix}$

AC, ABC: $C'_{acw} = \begin{bmatrix}
C_{acw} \\
C_{abcw}
\end{bmatrix}$

BC, ABC: $C'_{bcw} = \begin{bmatrix}
C_{bcw} \\
C_{abcw}
\end{bmatrix}$

Let us use the convention that $R(\mu, A, B, \text{etc.})$ denotes the total sum of squares for a model involving the parameters shown inside the parentheses. Those items that do not appear inside the parentheses are not in the model. For example, $R(\mu, A)$ refers to a model, $Y_{ij} = \mu + A_i + e_{ij}$.

In order to obtain the remaining sums of squares for the general three-way classification, a proof will be presented on the partitioning of the sums of squares due to regression into two or more independent
components. The particular distinction in finding these independent components is that it is obtained from two mathematical models. For the three-way classification, the two models are (6.1), and (6.2). However, to be more general, consider Model $M_2$ as (3.2), where a basis of $A$ is $X$. Model $M_2$ is the cell mean reparameterization, and is shown as (3.3). The coefficient matrix $A$, is already a full rank. Thus the sum of squares due to total regression with t d.f. is

$$SS(Reg) = Y'X (X'X)^{-1}X'Y = Y'A(A'A)^{-1}A'Y$$

Model $M_2$ Model $M_3$

Consider a subset of the parameters in Model $M_2$ which are involved in a part of the total regression sum of squares, say $R(\mu, A, B, \text{etc.}) = Y'Q_{a,b}Y$, with t-q d.f. Also consider a $C'$-matrix, with $q$ contrasts in Model $M_3$, involving another part of the total regression sum of squares, say $Y'Q_{a,b}Y$ with q d.f. Therefore by applying Cochran's theorem, as shown in Graybill [1961], on the independence of quadratic forms, if $Q_aQ_b = 0$, then the sums of squares associated with $Q_a$ and $Q_b$ are independent under Model I.\footnote{There is no assurance that the sums of squares components which are independent under Model I, are also independent under Model II. See an example from the nested design by Prairie [1962].}

Roy [1957] shows that quadratic forms like $Y'Q_{a,b}Y$ and $Y'Q_{b,a}Y$ are orthogonal to the error sum of squares in his chapter on least squares. Therefore since $Y'Q_{a,b}Y$ and $Y'Q_{b,a}Y$ are subsets of the total regression sum of squares and the sum of their d.f.
(t - q + q = t) equals the total regression d.f., it follows that

\[ SS(\text{reg}) = y'Q_a y + y'Q_b y \]

Now the key requirement is that \( Q_a Q_b = 0 \). Consider \( X_a \) as the partitioning of \( X \) so that

\[ X_a (X_a'X_a)^{-1}X_a = Q_a \]

and \( C'_b \) is the \( C' \)-matrix such that, as in the development of the F statistic in (3.5),

\[ A(A'A)^{-1}C_b \sum C'_b (A'A)^{-1}C_b^{-1}C'_b (A'A)^{-1}A' = Q_b \]

It can be shown, for the proper contrasts in \( C'_b \), that \( X'_a A(A'A)^{-1}C_b = 0 \), which implies \( Q_a Q_b = 0 \). Regardless of how \( X_a \) is constructed it is possible to show that for any \( n_{ij} \) arrangement, \( \sum_{(t-q)x} X'A(A'A)^{-1}A' \) is independent of the \( n_{ij} \)'s.

To illustrate this with an example consider the three-way classification shown in Table 6.1. For this example \( t = 12 \), \( n = \sum_{i} \sum_{j} n_{ijk} \), and \( q = 4 \). The matrix \( X_a \) will be taken to represent \( R(\mu, A, B, C, AB, AC) \), which in this example will have 8 d.f. The convention will be used that \( \mathbf{d} \) is a column vector of all ones, \( \mathbf{0} \) is a column vector of all zeroes, and that the length of \( \mathbf{d} \) or \( \mathbf{0} \) is shown on the top of each column of \( X'_a \), and at the right of each row of \( A \). Thus
\[
\begin{bmatrix}
\begin{array}{cccccccccccc}
\bar{d} & \bar{d} & \bar{d} & \bar{d} & \bar{d} & \bar{d} & \bar{d} & \bar{d} & \bar{d} & \bar{d} & \bar{d} & \bar{d} \\
\bar{d} & \bar{d} & 0 & 0 & \bar{d} & \bar{d} & \bar{d} & 0 & \bar{d} & \bar{d} & 0 & 0 \\
\bar{d} & 0 & \bar{d} & \bar{d} & \bar{d} & \bar{d} & \bar{d} & \bar{d} & \bar{d} & \bar{d} & \bar{d} & 0 \\
\bar{d} & \bar{d} & \bar{d} & \bar{d} & \bar{d} & 0 & 0 & \bar{d} & \bar{d} & \bar{d} & \bar{d} & \bar{d} \\
0 & 0 & \bar{d} & \bar{d} & \bar{d} & \bar{d} & \bar{d} & \bar{d} & \bar{d} & \bar{d} & \bar{d} & \bar{d} \\
0 & 0 & 0 & \bar{d} & \bar{d} & \bar{d} & \bar{d} & \bar{d} & \bar{d} & \bar{d} & \bar{d} & \bar{d} \\
0 & 0 & 0 & 0 & \bar{d} & \bar{d} & \bar{d} & \bar{d} & \bar{d} & \bar{d} & \bar{d} & \bar{d} \\
0 & 0 & 0 & 0 & 0 & \bar{d} & \bar{d} & \bar{d} & \bar{d} & \bar{d} & \bar{d} & \bar{d} \\
0 & 0 & 0 & 0 & 0 & 0 & \bar{d} & \bar{d} & \bar{d} & \bar{d} & \bar{d} & \bar{d} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{d} & \bar{d} & \bar{d} & \bar{d} & \bar{d} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{d} & \bar{d} & \bar{d} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{d} & \bar{d} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{d} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{d} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{d} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{d} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{d} \\
\end{array}
\end{bmatrix}
\begin{array}{c}
\mu \\
A \\
B \\
C \\
AB \\
AC
\end{array}
\]

\[
\begin{bmatrix}
\bar{d} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \bar{d} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \bar{d} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \bar{d} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \bar{d} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \bar{d} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \bar{d} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{d} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{d} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{d} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{d} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{d} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{d} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{d} \\
\end{bmatrix}
\begin{array}{c}
\bar{n}_{111} \\
\bar{n}_{121} \\
\bar{n}_{211} \\
\bar{n}_{112} \\
\bar{n}_{212} \\
\bar{n}_{222} \\
\bar{n}_{113} \\
\bar{n}_{123} \\
\bar{n}_{213} \\
\bar{n}_{223}
\end{array}
\]

\[
A = nx12
\]
\[(A' A)^{-1} = D(1/n_{i,j,k})\]
\[12 \times 12 \quad 12 \times 12\]
where \(D(1/n_{i,j,k})\) is a diagonal matrix with \(1/n_{i,j,k}\) on the diagonal in the same order as the columns in \(A\). Hence \(X'_a (A' A)^{-1}\) is

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\(\mu\) 
\(A\) 
\(B\) 
\(C\) 
\(C_{AB}\) 
\(C_{AC}\)

Note in \(X'_a\) and \(X'_a (A' A)^{-1}\) that the column vectors are identified on the right-hand side of the matrix by their respective effects.

Thus because the \(n_{i,j,k}\)'s cancel out in \(X'_a (A' A)^{-1}\), all that is necessary is to find a \(C'_b\)-matrix which is orthogonal to \(X'_a (A' A)^{-1}\). For the example if \(C'_b\) is taken to be \(C_{bcw}'\), then

\[X'_a (A' A)^{-1} C_{bcw}' = 0\]

Generalizing again, if Model \(M_2\) is set up, so that all \(n_{i,j,k}\)'s, it can be seen that the interaction contrasts, from the weighted squares of means, are orthogonal to all columns of the coefficient matrix \(X'_a (A' A)^{-1}\) which represent main effects, or interactions of the same order or a smaller order. For example, in a three-way
classification $C'_{abcw}$ is orthogonal to the column vectors for $\mu$, A, B, C, AB, AC, and BC, while $C'_{bcw}$ is orthogonal to the column vectors for $\mu$, A, B, C, AB and AC. Using the previous developed notation, this implies that for the three-way classification

$$SS(\text{REG}) = R(\mu, A, B, C, AB, AC, BC) + SS(C_{abcw})$$

$$SS(\text{REG}) = R(\mu, A, B, C, AB, AC) + SS(C_{bcw})$$

Thus the sum of squares for interaction BC, when ABC is not in the model, is

$$R(\mu, A, B, C, AB, AC, BC) - R(\mu, A, B, C, AB, AC)$$

or, from the above identity,

$$SS(C_{bcw}) - SS(C_{abcw})$$

Note the substantially reduced amount of calculation involved in obtaining the BC interaction from $SS(C_{bcw}) - SS(C_{abcw})$.

The extensions of the above concepts to multi-way classifications and also to regression sets which contain more than two sums of squares should be straightforward.

Thus from the generalized theorem on partitioning sums of squares, the following relationships hold for the 2x2x3 example.

$$R(\mu, A, B, C) = SS(C_{abcu}) - SS(C_{iw}), \text{ with 5 d.f.} \quad (6.6)$$

$$R(\mu, A, B, C, AB, AC, BC) - R(\mu, A, B, C, AC, BC)$$

$$= SS(C_{abw}) - SS(C_{abcw}), \text{ with 1 d.f.} \quad (6.7)$$

$$R(\mu, A, B, C, AB, AC, BC) - R(\mu, A, B, C, AB, BC)$$

$$= SS(C_{acw}) - SS(C_{abcw}), \text{ with 2 d.f.} \quad (6.8)$$
\[ R(\mu, A, B, C, AB, AC, BC) - R(\mu, A, B, C, AB, AC) \]

\[ = SS(C_{bcw}) - SS(C_{abcw}), \text{with 2 d.f.} \quad (6.9) \]

Now that all the pertinent sums of squares are defined, it is possible to take linear combinations of them to obtain the analogous sets of sums of squares in the three-factor experiment. Procedure H as before, will use all unadjusted sums of squares; Procedure B, all weighted squares of means. However the extension to Procedure A is not so obvious. For the main effects, only factor A will be illustrated. We want the sum of squares for A, adjusted for B and C, when all the interactions are ignored, i.e.,

\[ SS(A|B,C) = R(\mu, A, B, C) - R(\mu, B, C) \]

Note that \( R(\mu, B, C) \) is the total regression sum of squares of a two-factor experiment with B and C. Hence, using (6.3) and (6.4),

\[ R(\mu, B, C) = SS(C_{bcu}) - SS(C_{bc(a)}) \]

With the result from (6.6), it is seen that

\[ SS(A|B,C) = SS(C_{abcu}) - SS(C_{lw}) - SS(C_{bcu}) + SS(C_{bc(a)}) \]

Analogous expressions can be obtained for \( SS(B|A,C) \), and \( SS(C|A,B) \). The interaction sums of squares for Procedure B are (6.7), (6.8), (6.9) and (6.5).

In summary, Table 6.2 shows all the sums of squares that can be associated with a main effect and an interaction for Procedure A, B and H.
Table 6.2. Sums of squares for estimation procedures

A, B, and H in a three-way classification

<table>
<thead>
<tr>
<th>Source of Variation</th>
<th>A</th>
<th>Estimation Procedures</th>
<th>B</th>
<th>M</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>R(μ, A, B, C) - R(μ, B, C)</td>
<td>SS(c_{aw})</td>
<td>SS(c_{aw}) - SS(c_{m})</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>R(μ, A, B, C) - R(μ, A, C)</td>
<td>SS(c_{bw})</td>
<td>SS(c_{bw}) - SS(c_{m})</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>R(μ, A, B, C) - R(μ, A, B)</td>
<td>SS(c_{cw})</td>
<td>SS(c_{cw}) - SS(c_{m})</td>
<td></td>
</tr>
<tr>
<td>AB</td>
<td>SS(c_{abw}) - SS(c_{abcw})</td>
<td>SS(c_{abw})</td>
<td>SS(c_{abw}) - SS(c_{m})</td>
<td></td>
</tr>
<tr>
<td>AC</td>
<td>SS(c_{acw}) - SS(c_{abcw})</td>
<td>SS(c_{acw})</td>
<td>SS(c_{acw}) - SS(c_{m})</td>
<td></td>
</tr>
<tr>
<td>BC</td>
<td>SS(c_{bcw}) - SS(c_{abcw})</td>
<td>SS(c_{bcw})</td>
<td>SS(c_{bcw}) - SS(c_{m})</td>
<td></td>
</tr>
<tr>
<td>ABC</td>
<td>SS(c_{abcw})</td>
<td>SS(c_{abcw})</td>
<td>SS(c_{abcw}) - SS(c_{m})</td>
<td></td>
</tr>
<tr>
<td>Error</td>
<td>S_e</td>
<td>S_e</td>
<td>S_e</td>
<td></td>
</tr>
</tbody>
</table>

For procedure A in the 2x2x3 experiment the two factor interactions (6.7), (6.8) and (6.9) are easy to compute in terms of the model (6.1) since the largest matrix to be inverted is only a 4x4. A slightly different arrangement of the sums of squares for Procedure H is given by Le Roy and Gluckowski [1961] for the three-way classification.

It is seen that any three-way classification will yield the exact arrangement of the sums of squares shown in Table 6.2. The 2x2x3 example which was worked out above can be generalized to any other three-way classification. When there are some $n_{ijk} = 0$, it is just a matter of constructing suitable contrasts on the main effects of Procedure B and obtaining the interaction matrices by the multiplication rule.
Extension of the above results to the general multi-way classification is not too difficult to write out although it may involve considerable computation. Procedure B is the easiest to extend, since one needs only to write out the main effect contrasts; all the interaction contrasts can be obtained by multiplication as noted in the two- and three-way classifications. For Procedure II, the extension is also not too difficult. The only problem might be the construction of the subclass matrices.

To illustrate the extension, say in a four-way-classification, a typical term for the two-way subclass is \( n_{ijk} / n_{ij} \)', and for the three-way subclass \( n_{ijk} / n_{ijk} \). The highest order subclass always involve an identity matrix which is the order of the number of non-empty cells.

Procedure A can also be extended. Again using the four-way classification as an illustration, the sum of squares for main effect A, adjusted for all other main effects, ignoring interactions, is computed by

\[
R(\mu, A, B, C, D) - R(\mu, B, C, D)
\]

or

\[
\sum SS(c_{abcd}) - SS(c_{\mu}) - \sum SS(c_{bcd}) - SS(c_{bcd(a)})
\]

where the notation is an extension of that used in the three-way classification.
A two factor interaction can be found by

$$SS(C_{abw}) - SS(C_{i3w})$$

where $C_{abw}$ includes the AB contrasts and all higher order interactions while $C_{i3w}$ includes contrasts from all three factor and higher order interactions.

Of course if all the interactions are grouped together as in $C_{iw}$, all three procedures have considerably less computation.

6.3 The Completely Random Situation

Consider the mathematical model as

$$\gamma_{ijk} = \mu + A_i + B_j + C_k + (AB)_{ij} + (AC)_{ik} + (BC)_{jk} + (ABC)_{ijk} + \epsilon_{ijk}$$  \hspace{1cm} (6.10)

\begin{align*}
(1 = 1, 2, \ldots , a; & \quad j = 1, 2, \ldots , b; \quad k = 1, 2, \ldots , c; \quad i = 1, 2, \ldots , n_{ijk})
\end{align*}

where $\mu$ is a constant, $A_i$, $B_j$, $C_k$, $(AB)_{ij}$, $(AC)_{ik}$, $(BC)_{jk}$, $(ABC)_{ijk}$, $\epsilon_{ijk}$ are all NID with means zero, and respective variances, $\sigma^2_a$, $\sigma^2_b$, $\sigma^2_c$, $\sigma^2_{ab}$, $\sigma^2_{ac}$, $\sigma^2_{bc}$, $\sigma^2_{abc}$ and $\sigma^2$.

As in the two-way classification, if (6.10) is rewritten in terms of subclass random variables and subclass means ($\overline{\gamma}_{ijk} = \gamma_{ijk}$), the mathematical model is

$$\overline{\gamma}_{ijk} = \gamma_{ijk} + \epsilon_{ijk}.$$  \hspace{1cm} (6.11)

where $\overline{\gamma}_{ijk}$ are the subclass random variables,

$$\overline{\gamma}_{ijk} = \mu + A_i + B_j + C_k + (AB)_{ij} + (AC)_{ik} + (BC)_{jk} + (ABC)_{ijk} + \epsilon_{ijk}.$$

$\overline{\gamma}_{ijk}$
is the average error for the (ijk) subclass, and \( \gamma_{ijk} \) has a multivariate normal distribution with variance covariance matrix \( V \).

Using the general results described in Section 3.4 on the sums of squares in Table 6.2, it is seen that for procedure A, the first seven items will make up the vector \( t_{a}^* \), and

\[
\begin{bmatrix}
1 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
SS(c_{abcu}) \\
SS(c_{lw}) \\
SS(c_{ab}(c)) \\
SS(c_{ac}(b)) \\
SS(c_{bc}(a)) \\
SS(c_{abu}) \\
SS(c_{acu}) \\
SS(c_{bcu}) \\
SS(c_{abw}) \\
SS(c_{acw}) \\
SS(c_{bcw}) \\
SS(c_{abcw})
\end{bmatrix}
\]

For Procedure B, the first seven items under B, in Table 6.2, are \( t_{b}^* = t_{b} \), since there are no linear combinations.
Finally, for Procedure H, it is seen that

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
\end{bmatrix}
\begin{bmatrix}
SS(c_{au}) \\
SS(c_{bu}) \\
SS(c_{cu}) \\
SS(c_{abu}) \\
SS(c_{acu}) \\
SS(c_{bcu}) \\
SS(c_{abcu}) \\
SS(c_m) \\
\end{bmatrix}
\]

In the three-way classification there are eight variance components to be estimated. However since the sum of squares for error is independent of all the other sums of squares, \( \overset{\wedge}\sigma^2_e \) and \( \text{var}(\overset{\wedge}\sigma^2_e) \) can be estimated separately. Thus the main problem is to estimate and find the variances of the estimates for the seven variance components \( \sigma^2_a, \sigma^2_b, \sigma^2_c, \sigma^2_{ab}, \sigma^2_{ac}, \sigma^2_{bc} \) and \( \sigma^2_{abc} \). Again as for the two-way classification the variance-covariance matrix of the subclass means \( \gamma_{ijk} \) is required. However now it is desired to express \( V \) as a sum of eight matrices, each associated with one parameter. That is

\[
V = \sigma^2_{aV} + \sigma^2_{bV} + \sigma^2_{cV} + \sigma^2_{abV} + \sigma^2_{acV} + \sigma^2_{bcV} + \sigma^2_{abcV} + \sigma^2_{eV},
\]

where the size of \( V \) depends upon the number of non-zero subclasses in the experiment. Now \( E(t^*_a), E(t^*_b) \) and \( E(t^*_h) \) can be obtained so that the seven variance components can be estimated for each procedure. With (3.21) expressed for each procedure, the variances of the estimates of the variance components can be found by use of (3.22). Although it is not done here, it is expected that
many of the matrix products can be reduced to simple functions of the \( n_{ijk} \)'s and the \( \sigma^2 \)'s through the use of the expansion of \( V \) in (6.11). From a computational viewpoint, note that \( \text{var}(t_a) \) is a 12x12, \( \text{var}(t_b) \) is a 7x7 and \( \text{var}(t_h) \) is an 8x8 matrix.

If the breakdown of the interaction into separate components is not required, then the analysis of three-way classification data can be somewhat simplified. The total interaction component for Procedure A and B would be obtained from \( SS(C_{1w}) \). for Procedure H, from \( SS(C_{abcu}) \). The new \( \text{var}(t_a) \) is an 8x8, \( \text{var}(t_b) \) is a 4x4 and \( \text{var}(t_h) \) is a 5x5 matrix.

6.4 Estimating Variances of the Estimates of the Variance Components for the Nested Design

Recent work by Searle \( \left[1961\right] \) and Prairie \( \left[1962\right] \) have considered the nested model

\[
Y_{ijk} = \mu + A_i + B_{ij} + C_{ijk},
\]

\((i=1,2,\ldots,a; \ j=1,2,\ldots,b_i; \ k=1,2,\ldots,n_{ij})\)

where \( \mu \) is a constant and \( A_i, B_{ij} \) and \( C_{ijk} \) are all NID with means zero and respective variances \( \sigma^2_A, \sigma^2_B \) and \( \sigma^2_C \). Since the nested design can be considered as a special case of the multi-way classification, (6.12) can be obtained from the two-way classification model (3.15), by equating \( r_i = A_i, (rc)_{ij} = B_{ij}, e_{ijk} = C_{ijk} \) and \( c_j = 0 \).

For the parameters \( \sigma^2_r = \sigma^2_A, \sigma^2_{rc} = \sigma^2_B, \sigma^2_e = \sigma^2_C \) and \( \sigma^2_c = 0 \). Then with the development shown for Procedure H, in the two-way classification and with the above changes under model (6.12), the variances of the
estimated variance components, \( \hat{\sigma}_A^2, \hat{\sigma}_B^2 \) and \( \hat{\sigma}_C^2 \) can be obtained. These results are exactly the same as those shown by Searle \( \text{[1961]} \) or Prairie \( \text{[1962]} \). Extension to the analysis of higher order nested models is not difficult.

As long as it is possible to write out the sums of squares for Procedure H, it will be possible to obtain a linear combination of these sums of squares to represent the nested sums of squares. As an additional example, consider the nested model

\[
\gamma_{ijk} = \mu + A_i + B_{ij} + C_{ijk} + D_{ijk},
\]

\((i=1,2,\ldots,a; j=1,2,\ldots,b_i; k=1,2,\ldots,c_{ij}; \ell=1,2,\ldots,n_{ijk})\)

where \( \mu \) is a constant and \( A_i, B_{ij}, C_{ijk} \) and \( D_{ijk} \) are all \( \text{NID} \) with means zero and respective variances \( \sigma_A^2, \sigma_B^2, \sigma_C^2 \) and \( \sigma_D^2 \).

Again we use results worked out for the model in (6.2) and equate

\( A_i = A_i' \), \( (AB)_{ij} = B_{ij} \), \( (ABC)_{ijk} = C_{ijk} \), \( c_{ijk} = D_{ijk} \) and \( B_j = C_k = (AC)_{ij} = (BC)_{jk} = 0 \). For the variances, \( \sigma_B^2 = \sigma_C^2 = \sigma_{ac}^2 = \sigma_{bc}^2 = 0 \).

The appropriate sums of squares for (6.13) are shown in Table 6.3.

### Table 6.3. Sums of squares for the four stage nested classification

<table>
<thead>
<tr>
<th>Source</th>
<th>d.f.</th>
<th>Sum of squares</th>
</tr>
</thead>
<tbody>
<tr>
<td>A classes</td>
<td>( a - l )</td>
<td>( SS(c_{au}) - SS(c_m) )</td>
</tr>
<tr>
<td>B in A classes</td>
<td>( \Sigma_i b_i - a )</td>
<td>( SS(c_{aub}) - SS(c_{au}) )</td>
</tr>
<tr>
<td>C in B classes</td>
<td>( \Sigma_i \Sigma_j c_{ij} - \Sigma_i b_i )</td>
<td>( SS(c_{abcu}) - SS(c_{abu}) )</td>
</tr>
<tr>
<td>D in C classes</td>
<td>( n - \Sigma_i \Sigma_j c_{ij} )</td>
<td>( S_c )</td>
</tr>
<tr>
<td>Total</td>
<td>( n - l )</td>
<td></td>
</tr>
</tbody>
</table>
It is anticipated that an algebraic simplification can be worked out for any nested classification so that the variances of the estimates of the variance components appear explicitly in terms of linear combinations of the population variance components.

6.5 Other Mathematical Models

Now that a general procedure has been illustrated and described for the multi-way and nested classifications, it should not be too difficult to extend these results to combinations of multi-way and nested designs and possibly to the mixed model. In fact for any experimental design where the number of sums of squares in \( t^* \) is the same as the number of variance components to be estimated (excluding \( \sigma_e^2 \)), the general results of Section 3.5 can be applied. The important criterion is to be able to find the sums of squares in \( t^* \) either directly, or through some linear combination of other sums of squares, where all necessary sums of squares are expressible in terms of a \( G' \)-matrix for the model under consideration.
CHAPTER VII

SUMMARY AND CONCLUSIONS

7.1 Introduction

The basic problem was to develop a procedure by which variances of estimates of the variance components in Model II could be computed for multi-way classifications, when there are unequal numbers in the subclasses. Subsequently it was possible to make comparisons of estimation procedures and experimental designs. As a by-product of this development the extension to nested designs was worked out.

In all cases the variances of estimates of the variance components involved the true parameters (the variance components) and the number of observations in the subclasses \( (n_{ij}) \). A computer program was written on the UNIVAC 1105 for evaluating the variances of the estimates of the variance components for three estimation procedures (A, B and H) in the two-way classification. The computer program is given in Appendix A. The computer results, for 18 sets of parameter values, a number of different designs and for each estimation procedures, is shown in Appendix B.

7.2 The Two-Way Classification

The analytic results, and examples of the construction of the necessary matrices, for the two-way crossed classification are given in Chapters III and IV. The essential requirement, based on the fixed effects model (Model I), is to be able to express the sums of
squares for a particular estimation procedure in terms of linear combinations of quadratic forms such as

\[
SS_j = \gamma' Q_j \gamma_{1xt} \gamma_{txt} \gamma_{ttxl} ,
\]

where \( \gamma \) is a vector of the t cell means, and \( Q_j \) is a matrix of constants. Then by the analysis of variance procedure of computing mean squares based on Model I, and equating these mean squares to their expectations under Model II, a set of equations, linear in the parameters, is obtained. For this discussion we will assume that only one mean square is computed for each source of variation in the analysis of variance, e.g., main effect A, main effect B, ..., interaction AB, .... In this case there are as many equations as unknowns (including \( \sigma^2_e \); which can be estimated independently of the other variance components). Thus the variance-covariance matrix of the estimates (\( \gamma \)) of the variance components is

\[
\text{var}(\gamma) = F^{-1} \sum H \var(t) H' + \left( \frac{2\sigma^4_e}{n_e} \right) \ m m' F^{-1} \quad (7.1)
\]

where the above notation is explained in Section 3.4.

This is a generalization of Searle's 1958 work for the estimation procedure using unadjusted sums of squares (Procedure H). In Model II, the variance-covariance matrix of the cell means \( \gamma_{ij} \) is \( V \), and the variances and the covariances of the sums of squares, ttx
\( \text{var}(t) \), can be found through

\[
\text{var}(SS_j) = 2\text{tr}(VQ_j)^2,
\]

\[
\text{cov}(SS_j, SS_k) = 2\text{tr}(VQ_j)(VQ_k).
\]

The computational details, in matrix notation, for the estimation procedures,

- **A** - based on method of fitting constants;
- **B** - based on weighted squares of means;
- **H** - based on unadjusted sums of squares,

are described in Chapter IV. Searle \( 1958^{1} \) shows a simplified set of equations for obtaining \( \text{var}(y) \) using Procedure **H**, and Chapter V gives an analogous simplified set of equations for obtaining \( \text{var}(y) \) using Procedure **A**.

The computer comparison evaluates Procedures **A**, **B** and **H** for the following experimental designs:

<table>
<thead>
<tr>
<th>6x6-equal</th>
<th>6x6 - S</th>
<th>6x6 - C</th>
<th>6x6 - L</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 1 1 1 1 1</td>
<td>1 1 1 1 1 0</td>
<td>0 0 1 1 1 0</td>
<td>1 1 1 1 1 1</td>
</tr>
<tr>
<td>1 1 1 1 1 0</td>
<td>1 1 1 1 0 0</td>
<td>0 0 1 1 1 0</td>
<td>0 0 0 1 1 1</td>
</tr>
<tr>
<td>0 1 1 1 0 0</td>
<td>0 0 0 1 1 1</td>
<td>0 0 0 1 1 0</td>
<td>1 1 1 1 1 1</td>
</tr>
<tr>
<td>1 1 1 1 1 1</td>
<td>1 1 1 0 0 0</td>
<td>0 0 1 1 1 1</td>
<td>1 1 1 1 1 1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>3x3 - equal</th>
<th>3x3 - D</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 2 2</td>
<td>( n_{11} ) ( n_{12} ) ( n_{13} ) ( 3 3 ) ( \Sigma \Sigma n_{ij} = 18 )</td>
</tr>
<tr>
<td>2 2 2</td>
<td>( n_{21} ) ( n_{22} ) ( n_{23} )</td>
</tr>
<tr>
<td>2 2 2</td>
<td>( n_{31} ) ( n_{32} ) ( n_{33} )</td>
</tr>
</tbody>
</table>
A summary of the computer results, for the 6x6 designs showed that when \( \sigma_r^2 \) and \( \sigma_c^2 \) dominate \( \sigma_{rc}^2 \), the best design is S or C, and the best estimation procedure is A. When \( \sigma_{rc}^2 \) dominates \( \sigma_r^2 \) and \( \sigma_c^2 \), the best design is L, and the best estimation Procedure is H. If \( \sigma_{rc}^2 \) falls between \( \sigma_r^2 \) and \( \sigma_c^2 \), then depending upon whether \( \sigma_r^2 \) or \( \sigma_c^2 \) is more important, either S, C or L would be used; however, Procedure A is the best estimation procedure. For the 3x3 designs, Procedure B is slightly better than Procedure A; however since the differences appear to be negligible and Procedure A is easier to calculate, it is recommended.

Based on these results, it is suggested that if the experimenter has no prior knowledge of the relative magnitude of the variance components in a two-way classification, and if the observations are either costly or time consuming, then he should use the S type of design with estimation based on Procedure A. If Procedure A indicates a large \( \sigma_{rc}^2 \), in contrast to \( \sigma_r^2 \) and \( \sigma_c^2 \), a recomputation of the data using Procedure H might be an improvement. The variances of the estimates of the variance components can be estimated by using the sample estimates for the parameters in (7.1).

The general matrix development for Procedures A, B and H in the three-way and higher-way classification is outlined in Chapter VI.

Although only three procedures of estimation are discussed in this dissertation, the general matrix set-up is suitable for
other procedures, as well as other experimental designs. It is almost impossible to consider a comparison of estimation procedures or experimental designs without the use of a digital computer. The computer program in Appendix A is written in the Remington Rand UNICODE language which is very similar to the FORTRAN language. It should not be a difficult task for those who are interested, to rewrite the UNICODE program into a different compiler language.

7.3 Suggested Future Research

1. Compare experimental designs and estimation procedures for two-way classifications with the larger core digital computers in order to handle more than six rows and columns.

2. Compare experimental designs and estimation procedures for three-way classification models using a digital computer.

3. Investigate the problem of what to do with negative estimates of variance components.

4. Develop an analytic procedure for the purpose of putting confidence bounds on the variance components.

5. Obtain the distribution of the estimates of the variance components for the above procedures, analytically, or by empirical sampling, especially when the random variables do not follow a normal distribution.

6. There is a need for investigating and evaluating procedures, other than the analysis of variance technique, by which we could obtain estimates of the variance components.
LIST OF REFERENCES


Elston, R. C. and Bush, N. 1963. The hypotheses tested by different sums of squares when there are interactions in the model. Submitted for publication to Biometrics.


APPENDIX A

COMPUTER PROGRAM FOR VAR(\(v_n\)), VAR(\(v_a\)) AND VAR(\(v_b\)) IN THE TWO-WAY CLASSIFICATION

The computer program evaluates variances of estimates of variance components for Procedures H, A and B in the two-way classification. For each procedure

\[
\text{var}(v) = P^{-1/2} \text{var}(t) H' \times \left(2c_i^j n_e \right) m/m' P^{-1/2}
\]

is computed and printed out. The program was written in the UNICODE language for the UNIVAC 1105 digital computer (with an 8K core and 32K drum). A description of the input, output, notation and flow diagram follows:

**INPUT**

<table>
<thead>
<tr>
<th>Program</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>h1</td>
<td>(H_h) matrix for Procedure H</td>
</tr>
<tr>
<td>h2</td>
<td>(H_a) matrix for Procedure A</td>
</tr>
<tr>
<td>const(0) = irow</td>
<td>Number of rows in design</td>
</tr>
<tr>
<td>const(1) = icol</td>
<td>Number of columns in design</td>
</tr>
<tr>
<td>const(2) = indfi</td>
<td>d.f. for interaction</td>
</tr>
<tr>
<td>const(3) = inco</td>
<td>Number of non-empty cells (=t)</td>
</tr>
<tr>
<td>const(4)</td>
<td>{ = 0 No print out } &gt; 0 Print out constants for Procedure H</td>
</tr>
<tr>
<td>const(5)</td>
<td>{ = 0 No print out } &gt; 0 Print out (B_{rc}) matrix</td>
</tr>
<tr>
<td>const(6)</td>
<td>{ = 0 No print out } &gt; 0 Print out (P_h), (P_a) and (P_b) matrices</td>
</tr>
<tr>
<td>const(7)</td>
<td>{ = 0 No print out } &gt; 0 Print out (V) and (VB_s) matrices</td>
</tr>
</tbody>
</table>
const(8) - \begin{cases} 
= 0 & \text{No print out} \\
> 0 & \text{Print out } \var{y_n}, \var{y_a} \text{ and } \var{y_b} \text{ matrices}
\end{cases}

const(9) - \begin{cases} 
= 0 & \sigma_e^2 = 1, \text{ and some } n_{ij} > 1 \\
> 0 & \sigma_e^2 = 0, \text{ all } n_{ij} = 1, \text{ or } 0
\end{cases}

cr - C^{!}_r \text{-matrix for rows in Procedure B}

cc - C^{!}_c \text{-matrix for columns in Procedure B}

cl - C^{!}_{rc} \text{-matrix for interaction in Procedures A and B}

cofr - \text{tr}(C^{!}_r V_r C_r B_r)\), that is the expected value for \sigma_r^2 in the estimating equations of Procedure B

cofc - \text{tr}(C^{!}_c V_c C_c B_c)\), that is the expected value for \sigma_c^2 in the estimating equations of Procedure B

cre - C^{!}_{ru} \text{-matrix where 1 represents a non-empty cell.}
  \text{Note the program places the correct value of } n_{ij}
  \text{ in the } C^{!}_{ru} \text{-matrix}

ccu - C^{!}_{cu} \text{-matrix where 1 represents a non-empty cell.}
  \text{Note the program places the correct value of } n_{ij}
  \text{ in the } C^{!}_{cu} \text{-matrix}

\sigma_{0} - \text{Value of } \sigma_r^2

\sigma_{1} - \text{Value of } \sigma_c^2

\sigma_{2} - \text{Value of } \sigma_{rc}^2

\sigma_{3} - \text{Value of } \sigma_e^2, \text{ which is always one in a parameter set}

The program can handle 36 parameter sets for a particular \(n_{ij}\) arrangement. A parameter set is \((\sigma_r^2, \sigma_c^2, \sigma_{rc}^2, \sigma_e^2)\). The parameters are entered through the sigma vector in groups of four numbers. The stopping rule is to put \(\sigma_e^2 = 0\) after the last parameter set.
For example the sigma vector

\[
\begin{bmatrix}
1,1,1,1, & 1,1,1,1, & 1,1,1,1, & 0,0,0,0, & \ldots, & 0
\end{bmatrix}
\]

will compute three parameter sets, for a particular design, before continuing on with the next \(n_{ij}\) design in the program.

\[n = n_{ij}\] arrangement for a particular design

The program is set up for an unspecified number of \(n_{ij}\) matrices. Once the zero cells of an \(n_{ij}\) matrix are located, these cells remain zero for the computer run. However any cell having \(n_{ij} > 0\) can be changed to any non-zero value. For example if

\[
\begin{bmatrix}
0 & n_{12} & n_{13} \\
n_{21} & n_{22} & n_{23}
\end{bmatrix}
\]

any arrangement of values for the \(n_{ij}\)'s in the 2 x 3 classification would be acceptable.

The program is designed to handle any two-way classification, up to six rows, six columns and 20 non-empty cells. Because of the amount of storage space needed for program and data, it was not feasible, in terms of computer time, to consider a program to handle more rows, columns or non-empty cells.

Thus, for a particular set of \(n_{ij}\) arrangements, the program will compute the variances of the estimates of the variance components for every combination of \(n_{ij}\) arrangements and parameter sets. The \(n_{ij}\) arrangements for which the zero cell pattern are different constitute different computer runs.
OUTPUT

The output is partially governed by the const( ) vector. Const(4) through const(8) provide for an optional output. The outputs which are not optional are:

1- The $n_{ij}$ arrangement
2- The parameter set
3- The variances of the estimates of the variance components

NOTATION

<table>
<thead>
<tr>
<th>Constants in program</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>indfr</td>
<td>d.f. for rows</td>
</tr>
<tr>
<td>indfc</td>
<td>d.f. for columns</td>
</tr>
<tr>
<td>ne</td>
<td>d.f. for error</td>
</tr>
<tr>
<td>ns</td>
<td>$n$</td>
</tr>
<tr>
<td>sigr</td>
<td>$\sigma_r^2$</td>
</tr>
<tr>
<td>sigc</td>
<td>$\sigma_c^2$</td>
</tr>
<tr>
<td>sigrc</td>
<td>$\sigma_r^2 + \sigma_c^2 + \sigma_{rc}^2$</td>
</tr>
<tr>
<td>nk15</td>
<td>$n(k_1)$</td>
</tr>
<tr>
<td>nk1</td>
<td>$n(k_2)$</td>
</tr>
<tr>
<td>nk3</td>
<td>$n(k_3)$</td>
</tr>
<tr>
<td>nk17</td>
<td>$k_{12}$</td>
</tr>
<tr>
<td>nk5</td>
<td>$k_{21}$</td>
</tr>
<tr>
<td>$P_2(2,2)$</td>
<td>$k_{30}$</td>
</tr>
<tr>
<td>Matrices in program</td>
<td>Explanation</td>
</tr>
<tr>
<td>---------------------</td>
<td>-------------</td>
</tr>
<tr>
<td>cf</td>
<td>$C_m$</td>
</tr>
<tr>
<td>p2</td>
<td>$P_a$</td>
</tr>
<tr>
<td>p3</td>
<td>$P_b$</td>
</tr>
<tr>
<td>pl</td>
<td>$P_h$</td>
</tr>
<tr>
<td>br</td>
<td>$B_r$</td>
</tr>
<tr>
<td>bc</td>
<td>$B_c$</td>
</tr>
<tr>
<td>bi</td>
<td>$B_{rc}$</td>
</tr>
<tr>
<td>bru</td>
<td>$B_{ru}$</td>
</tr>
<tr>
<td>bcu</td>
<td>$B_{cu}$</td>
</tr>
<tr>
<td>brc</td>
<td>$B_{rC'}$</td>
</tr>
<tr>
<td>bcc</td>
<td>$B_{cC'}$</td>
</tr>
<tr>
<td>bic</td>
<td>$B_{rC'C'}$</td>
</tr>
<tr>
<td>bruc</td>
<td>$B_{ruC'}$</td>
</tr>
<tr>
<td>bcuc</td>
<td>$B_{cuC'}$</td>
</tr>
<tr>
<td>v</td>
<td>$V$</td>
</tr>
<tr>
<td>vbs</td>
<td>$V_{bs}$</td>
</tr>
<tr>
<td>cfv</td>
<td>$C_{mV}$</td>
</tr>
<tr>
<td>crv</td>
<td>$C_{rV}$</td>
</tr>
<tr>
<td>ccv</td>
<td>$C_{cV}$</td>
</tr>
<tr>
<td>civ</td>
<td>$C_{rC'}V$</td>
</tr>
<tr>
<td>cruv</td>
<td>$C_{ruV}$</td>
</tr>
<tr>
<td>ccuv</td>
<td>$C_{cuV}$</td>
</tr>
<tr>
<td>$f_{mm}$</td>
<td>$m m^t$</td>
</tr>
<tr>
<td>$f_{mmme}$</td>
<td>$(2/n_e) m m^t$, for $c_e^2 = 1$</td>
</tr>
</tbody>
</table>
FLOW DIAGRAM

Sentences

1

3, 20, 30
40, 160, 344
47, 48, 106, 122, 3
165, 312

4 - 6
7 - 16
30.1 - 30.9
31 - 46
50 - 58
61 - 63
70 - 72.1
74 - 76.1
80 - 87
88 - 95
96 - 103
107 - 122

Description

maximum storage allocation for matrices and vectors
input
fixed output
optional output

\[ m \text{ m'} \]

make floating point numbers fixed point integers

C_\text{m'}

constants for Procedure H

\( \frac{2}{n_\theta} m \text{ m' }, \) since \( \sigma^2_\theta = 1 \)

C_{ru}
C_{cu}
B_r
B_c
B_{rc}
P_a and P_b
shift parameter sets
V and VBs
var(t_n), var(t_a) and var(t_0)

var(v_n)
var(v_a)
var(v_0)

Subroutines

400 - 404
400 - 404.6

800 - 800.6

2tr(A B)
pxq qxp

899 - 899.22

G = A B^t
pxn pxq pxn

997 - 997.18

invert matrix in place

998 - 998.22

C = A B
pxn pxq pxn

999 - 999.22

c^t = a^t B
lxn lxq pxn
unicode program.
exact variances of variance components.
norman bush jan 1962.

1  dimension n(6,6), const(10), cru(6,20), ccu(6,20), cf(20),
    cr(5,20), cc(5,20), ci(9,20), sigma(150), v(20,20),
    vbs(20,20), cru(6,20), bruc(6,20), ccuv(6,20), bcuc(6,20),
    bru(6), bcu(6), cfv(20), ttt(9,20), tt2(9,20), civ(9,20),
    bic(9,20), crv(5,20), brc(5,20), ccv(5,20), bcc(5,20), ssh(4,4),
    ssa(4,4), ssb(3,3), fmm(27,27), fmm(3,3), vch(3,3), vca(3,3),
    vcb(3,3), h1(3,4), h2(3,4), p1(3,3), p2(3,3), p3(3,3), br(5,5),
    bc(5,5), bi(9,9), cofr(5,5), cofc(5,5), tk5(6), tk17(6),
    tvl(20), tv2(20).

2  start.
3  read h1, h2, const, cr, cc, ci, cofr, cofc.
3.11 cru(0,0) = 0.
3.12 ccuv(0,0) = 0.
3.13 cfv(0) = 0.
3.14 tt2(0,0) = 0.
3.15 civ(0,0) = 0.
3.16 bic(0,0) = 0.
3.17 crv(0,0) = 0.
3.18 brc(0,0) = 0.
3.19 ccv(0,0) = 0.
3.2  bcc(0,0) = 0.
3.21 tvl(0) = 0.
3.22 tv2(0) = 0.
3.3  if const(9) < 0 jump to sentence 7.
3.4  vary i 0(1)2 sentences 5 thru 6.
3.5  vary j 0(1)2 sentence 6.
3.6  fmm(i,j) = const(i)Xconst(j).
3.7  compute fix(const(0)).
3.8  irow = kk.
3.9  indfr = irow - 1.
3.10 compute fix(const(1)).
3.11 icol = kk.
3.12 indfc = icol - 1.
3.13 compute fix(const(2)).
3.14 indf1 = kk-1.
3.15 compute fix(const(3)).
3.16 insc = kk-1.
20  read cru, ccu, sigma.
20.1 mm = -1.
20.2 m1 = 0.
21  vary i 0(1)8 sentences 22 thru 23.2.
22  bcu(i)=0.
23  bru(i)=0.
23.1 tk5(i)=0.
23.2 tk17(i)=0.
24    vary i 0(1)2 sentences 25 thru 27.
25    vary j 0(1)2 sentences 26 thru 27.
25.1  vch(i,j)=0.
25.2  vca(i,j)=0.
25.3  vcb(i,j)=0.
26    p2(i,j)=0.
27    p3(i,j)=0.
28    ns=0.
28.1  nk3=0.
28.2  nk1=0.
28.3  nk15=0.
28.4  nk5=0.
28.5  nk17=0.
30    read n, if end of data, jump to sentence 399.
30.1  k = 0.
30.2  i = 0.
30.3  j = 0.
30.4  if n(i,j) = 0 jump to sentence 30.7.
30.5  cf(k) = n(i,j).
30.6  k = k+1.
30.7  j = j+1.
30.8  if j = icol jump to sentence 30.4.
30.9  i = i+1.
30.10 if i = icol jump to sentence 30.3.
31    vary i 0(1)insc sentence 31.7.
31.1  ns = ns + cf(i).
33    vary i 0(1)irow sentences 34 thru 40.
34    vary j 0(1)icol sentences 35 thru 40.
35    nk3 = nk3 + n(i,j) X n(i,j).
36    bru(i)=bru(i) + n(i,j).
37    bcu(j)=bcu(j) + n(i,j).
38    tkn5(j)=tkn5(j) + n(i,j) X n(i,j).
39    tkn17(i)=tkn17(i) + n(i,j) X n(i,j).
40    list i, j, n(i,j), tape 5, ((subclass matrix)).
41    vary i 0(1) irow sentences 42 thru 43.
42    nk15=nk15 + bru(i) X bru(i).
43    nk17=nk17+tkn17(i)/bru(i).
44    vary j 0(1) icol sentences 45 thru 46.
45    nk1=nk1+bcu(j) X bcu(j).
46    nk5=nk5+(tkn5(j)/bcu(j)).
46.1  if const(4) = 0 jump to sentence 50.
47    list nk1, nk3, nk15, nk5, nk17, tape 5.
48    list ns, tape 5.
p1(0,0) = ns-(nk15/ns).
p1(0,1) = nk17-(nk1/ns).
p1(0,2) = nk17-(nk3/ns).
p1(1,0) = nk5-(nk15/ns).
p1(1,1) = ns-(nk1/ns).
p1(1,2) = nk5-(nk3/ns).
p1(2,0) = -p1(1,0).
p1(2,1) = -p1(0,1).
p1(2,2) = ns-nk17-nk5+(nk3/ns).

if const(9) G 0 jump to sentence 70.
ne = ns-const(3).
vary i 0(1)2 sentences 62 thru 63.
vary j 0(1)2 sentence 63.

fmm(i,j) = (2.0/ne) X fmm(i,j).
vary i 0(1)irrow sentences 71 thru 72.1.
vary j 0(1)insc sentences 72 thru 72.1.
cru(i,j) = cru(i,j) X cf(j).
bru(i,j) = cru(i,j)/bru(i).
vary i 0(1)icol sentences 75 thru 76.1.
vary j 0(1)insc sentences 76 thru 76.1.
ccu(i,j) = ccu(i,j) X cf(j).
bcuc(i,j) = ccu(i,j)/bcu(i).

vary i 0(1)indfr sentences 81 thru 82.
vary j 0(1)insc sentence 82.
ttl(i,j) = cr(i,j)/cf(j).
compute prod(t(br(0,0), ttl(0,0), indfr, insc, cr(0,0), indfr)).
if indfr = 0 jump to sentence 87.
compute invert(br(0,indfr)).
jump to sentence 88.
br(0,0) = 1.0/br(0,0).
vary i 0(1)indfc sentences 89 thru 90.
vary j 0(1)insc sentence 90.
ttl(i,j) = cc(i,j)/cf(j).
compute prod(t(bc(0,0), ttl(0,0), indfc, insc, cc(0,0), indfc)).
if indfc = 0 jump to sentence 95.
compute invert(bc(0,indfc)).
jump to sentence 96.
bc(0,0) = 1.0/bc(0,0).
vary i 0(1)indfi sentences 97 thru 98.
vary j 0(1)insc sentence 98.
ttl(i,j) = ci(i,j)/cf(j).
compute prod(t(bi(0,0), ttl(0,0), indfi, insc, ci(0,0), indfi)).
if indfi = 0 jump to sentence 103.
compute invert(bi(0,indfi)).
jump to sentence 103.1.
bi(0,0) = 1.0/bi(0,0).
103.1 if const(5) = 0 jump to sentence 107.
104 vary i 0(1) indfi sentences 105 thru 106.
105 vary j 0(1)1 sentence 106.
106 list i, j, bi(i,j), tape 5.
107 compute prodt(ttl(0,0), ci(0,0), indfi, insc, ci(0,0), indfi).
108 compute tr(ttl(0,0), bi(0,0), indfi, indfi).
109 p2(2,2) = trace/2.0.
110 p3(2,2) = p2(2,2).
111 compute prodt(ttl(0,0), cr(0,0), indfr, insc, cr(0,0), indfr).
112 compute tr(ttl(0,0), br(0,0), indfr, indfr).
112.1 p3(1,2) = trace/2.0.
113 compute tr(cofr(0,0), br(0,0), indfr, indfr).
114 p3(0,0) = trace/2.0.
115 compute prodt(ttl(0,0), cc(0,0), indfc, insc, cc(0,0), indfc).
116 compute tr(ttl(0,0), bc(0,0), indfc, indfc).
116.1 p3(1,2) = trace/2.0.
117 compute tr(cofc(0,0), bc(0,0), indfc, indfc).
118 p3(1,1) = trace/2.0.
119 p2(0,0) = ns - nk5.
120 p2(0,2) = p2(0,0) - p2(2,2).
121 p2(1,1) = ns - nk7.
122 p2(1,2) = p2(1,1) - p2(2,2).
122.05 if const(6) = 0 jump to sentence 123.
122.1 vary i 0(1)2 sentences 122.2 thru 122.3.
122.2 vary j 0(1)2 sentence 122.3.
122.3 list i, j, pi(i,j), p2(i,j), p3(i,j), tape 5.
123 compute invert(p1(0,2)).
124 compute invert(p2(0,2)).
125 compute invert(p3(0,2)).
130 compute prod(brc(0,0), br(0,0), indfr, indfr, cr(0,0), insc).
131 compute prod(bcc(0,0), bc(0,0), indfc, indfc, cc(0,0), insc).
132 compute prod(bic(0,0), bi(0,0), indfi, indfi, ci(0,0), insc).
150.1 mm = mmm+1.
150.2 m1 = 4×mm.
150.3 m2 = m1 + 3.
150.4 vary i 0(1)3 with j m1(1)m2 sentence 150.5.
150.5 sigma(i) = sigma(j).
150.6 if sigma(3) = 0 jump to sentence 20.
151 cfvc=0.
152 trts=0.
153 trtf=0.
153.1 tcts=0.
154 tctf=0.
155 tstf=0.
156 tsti=0.
160 list sigma(0), sigma(1), sigma(2), sigma(3), tape 5,
((scale factors for theoretical variance components)), (row s.f.),
(column s.f.), (interaction s.f.), (error s.f.).
160.1 sigr = sigma(0) .
160.1 sigc = sigma(i) .
160.1 sigrc = sigma(2) + sigr + sigc .
160.1 i = 0 .
160.1 iv = 0 .
160.1 j = -1 .
160.1 jv = iv .
160.17 i v G insc jump to sentence 161 .
160.18 j = j+1 .
160.19 if j G icol jump to sentence 160.6 .
160.2 k = 1 .
160.21 m = j .
160.22 if n(i,j) = 0 jump to sentence 160.18 .
160.23 v(iv,jv) = sigrc .
160.24 jv = jv+1 .
160.25 m = m+1 .
160.26 if m G icol jump to sentence 160.31 .
160.27 if n(k,m) = 0 jump to sentence 160.25 .
160.28 v(iv,jv) = sigr .
160.29 v(jv,iv) = sigr .
160.3 jump to sentence 160.24 .
160.31 m = 0 .
160.32 k = k+1 .
160.33 if k G irow jump to sentence 160.5 .
160.34 if m G icol jump to sentence 160.31 .
160.35 if n(k,m) = 0 jump to sentence 160.4 .
160.36 if m = j jump to sentence 160.42 .
160.37 v(iv,jv) = 0 .
160.38 v(jv,iv) = 0 .
160.39 jv = jv+1 .
160.4 m = m+1 .
160.41 jump to sentence 160.34 .
160.42 v(iv,jv) = sigc .
160.43 v(jv,iv) = sigc .
160.44 jv = jv+1 .
160.45 m = m+1 .
160.46 jump to sentence 160.34 .
160.47 iv = iv+1 .
160.49 jump to sentence 160.16 .
160.5 j = 0 .
160.51 i = i+1 .
160.6 jump to sentence 160.2 .
160.7 if const(9) G 0 jump to sentence 160.91 .
160.8 sigee = 1.0 .
160.9 jump to sentence 161 .
160.91 sigee = 0 .
161 vary i 0(1) insc sentences 162 thru 164 .
162 v(i,i) = v(i,i) + (sigee/cf(i)) .
163 vary j 0(1) insc sentence 164 .
164 vary v(i,j) = v(i,j)Xcf(j) .
164.1 if const(7) = 0 jump to sentence 170 .
164.2 vary i 0(1) insc sentences 164.3 thru 165 .
164.3 vary j 0(1) insc sentence 165 .
165 list i,j, v(i,j), vbs(i,j), tape 5 .
compute prod(cr2v(0,0), cru(0,0), irow, insc, v(0,0), insc).
compute prod(ccuv(0,0), ccu(0,0), icol, insc, v(0,0), insc).
compute vtpod(cf(0), cfv(0), insc, v(0,0), insc).
compute prod(tt1(0,0), cruv(0,0), irow, insc, bruc(0,0), irow).
compute tr(tt1(0,0), tt1(0,0), irow, irow).
s(0,0) = trace.
s(0,0) = trace.
compute prod(tt1(0,0), ccuv(0,0), icol, insc, bcuc(0,0), icol).
compute tr(tt1(0,0), tt1(0,0), icol, icol).
s(1,1) = trace.
s(1,1) = trace.
compute tr(vbs(0,0), vbs(0,0), insc, insc).
s(2,2) = trace.
s(2,2) = trace.
vary i 0(1) insc sentence 192.
cfv = cfv + cfv(i) X cf(i).
s(3,3) = 2.0 X (cfv/ins) X (cfv/ins).
compute prod(tt1(0,0), ccuv(0,0), icol, insc, bruc(0,0), irow).
compute prod(tt2(0,0), cruv(0,0), irow, insc, bcuc(0,0), icol).
compute tr(tt1(0,0), tt2(0,0), icol, irow).
s(0,1) = trace.
s(1,0) = trace.
s(0,1) = trace.
s(1,0) = trace.
compute prod(tt1(0,0), cruv(0,0), irow, insc, vbs(0,0), insc).
compute prod(tt2(0,0), tt1(0,0), irow, insc, bruc(0,0), irow).
vary i 0(1) irow sentence 205.
trt = trt + tt1(i,i).
trt = 2.0 X trts.
ssh(0,2) = trts.
ssh(2,0) = trts.
ssh(0,2) = trts.
ssh(2,0) = trts.
compute vtpod(tv1(0), cfv(0), insc, bruc(0,0), irow).
compute vtpod(tv2(0), cf(0), insc, cruv(0,0), irow).
vary i 0(1) irow sentence 214.
trtf = trtf + tv1(1) X tv2(1).
ssh(0,3) = 2.0 X (trtf/ins).
ssh(3,0) = 2.0 X (trtf/ins).
compute prod(tt1(0,0), ccuv(0,0), icol, insc, vbs(0,0), insc).
compute prod(tt2(0,0), tt1(0,0), icol, insc, bcuc(0,0), icol).
vary i 0(1) icol sentence 220.
tct = tct + tt2(i,i).
tct = 2.0 X tct.
ssh(1,2) = tct.
ssh(2,1) = tct.
ssh(1,2) = tct.
ssh(2,1) = tct.
compute vtpod(tv1(0), cfv(0), insc, bcuc(0,0), icol).
compute vtpod(tv2(0), cf(0), insc, ccuv(0,0), icol).
vary i 0(1) icol sentence 229.
tctf = tctf + tv1(1) X tv2(1).
ssh(1,3) = 2.0 X (tctf/ins).
ssh(3,1) = 2.0 X (tctf/ins).
compute vtpod(tvl(0), cfv(0), insc, vbs(0,0), insc).
 vary i 0(1) insc sentence 234.
 tsti=tsti + tvl(i) X cf(i).
 ssh(2,3) = 2.0 X (tsti/hs).
 ssh(3,2) = 2.0 X (tsti/hs).
 compute prod(civ(0,0), ci(0,0), indfi, insc, v(0,0), insc).
 compute prod(tt1(0,0), civ(0,0), indfi, insc, bic(0,0), indfi).
 compute tr(tt1(0,0), tt1(0,0), indfi, indfi).
 ssh(3,3)=trace.
 ssh(2,2)=trace.
 compute prod(tt1(0,0), civ(0,0), indfi, insc, bruc(0,0), irow).
 compute prod(tt2(0,0), cruv(0,0), irow, insc, bic(0,0), indfi).
 compute tr(tt1(0,0), tt2(0,0), indfi, irow).
 ssh(0,3)=trace.
 ssh(3,0)=trace.
 compute prod(tt1(0,0), civ(0,0), indfi, insc, bcuc(0,0), icol).
 compute prod(tt2(0,0), ccuv(0,0), icol, insc, bic(0,0), indfi).
 compute tr(tt1(0,0), tt2(0,0), indfi, icol).
 ssh(1,3)=trace.
 ssh(3,1)=trace.
 compute prod(tt2(0,0), civ(0,0), indfi, insc, vbs(0,0), insc).
 compute prod(tt2(0,0), tt1(0,0), indfi, insc, bic(0,0), indfi).
 vary i 0(1) indfi sentence 268.
 tsti=tsti + tt2(1,1).
 ssh(2,3) = 2.0 X tsti.
 ssh(3,2) = 2.0 X tsti.
 compute prod(crv(0,0), cr(0,0), indfr, insc, v(0,0), insc).
 compute prod(ccv(0,0), co(0,0), indfr, insc, v(0,0), insc).
 compute prod(tt1(0,0), crv(0,0), indfr, insc, brc(0,0), indfr).
 compute tr(tt1(0,0), tt1(0,0), indfr, indfr).
 ssh(0,0)=trace.
 compute prod(tt1(0,0), ccv(0,0), indfc, insc, bcc(0,0), indfc).
 compute tr(tt1(0,0), tt1(0,0), indfc, indfc).
 ssh(1,1)=trace.
 compute prod(tt2(0,0), ccv(0,0), indfc, insc, bcc(0,0), indfc).
 compute prod(tt2(0,0), crv(0,0), indfr, insc, bcc(0,0), indfr).
 compute tr(tt1(0,0), tt2(0,0), indfc, indfr).
 ssh(0,1)=trace.
 ssh(1,0)=trace.
 compute prod(tt1(0,0), civ(0,0), indfi, insc, brc(0,0), indfr).
 compute prod(tt2(0,0), crv(0,0), indfr, insc, bic(0,0), indfi).
 compute tr(tt1(0,0), tt2(0,0), indfi, indfr).
 ssh(0,2)=trace.
 ssh(2,0)=trace.
 compute prod(tt1(0,0), civ(0,0), indfi, insc, bcc(0,0), indfc).
 compute prod(tt2(0,0), ccv(0,0), indfc, insc, bic(0,0), indfi).
 compute tr(tt1(0,0), tt2(0,0), indfi, indfc).
 ssh(1,2)=trace.
 ssh(2,1)=trace.
 if const(8) = 0 jump to sentence 313.
 vary i 0(1)3 sentences 511 thru 512.
 vary j 0(1)3 sentence 512.
 list i, j, ssh(i,j), ssa(i,j), ssh(1,j), tape 5, (( sums of squares matrix ))).
compute \prod(ttl(0,0), h1(0,0), 2, 3, ssh(0,0), 3).
compute \prod(ttl(0,0), h2(0,0), 2, 3, h1(0,0), 2).
compute \prod(ttl(0,0), h2(0,0), 2, 3, ssa(0,0), 3).
compute \prod(ttl(0,0), h2(0,0), 2, 3, h2(0,0), 2).
if const(9) \leq 0 jump to sentence 330.
vary i 0(1)2 sentences 321 thru 324.
vary j 0(1)2 sentences 322 thru 324.
ssh_i(j) = ssh_i(j) + fmm_e(i,j).
ssa_i(j) = ssa_i(j) + fmm_e(i,j).
ssb_i(j) = ssb_i(j) + fmm_e(i,j).
compute \prod(ttl(0,0), p_i(0,0), 2, 2, ssh(0,0), 2).
compute \prod(ttl(0,0), p_i(0,0), 2, 2, p_i(0,0), 2).
compute \prod(ttl(0,0), p_i(0,0), 2, 2, ssa(0,0), 2).
compute \prod(ttl(0,0), p_i(0,0), 2, 2, p_i(0,0), 2).
compute \prod(ttl(0,0), p_i(0,0), 2, 2, ssa(0,0), 2).
compute \prod(ttl(0,0), p_i(0,0), 2, 2, p_i(0,0), 2).
vary i 0(1)2 sentence 344.
list i, vch(i,i), vca(i,i), vcb(i,i), tape 5,
(( variance of estimates for rows, columns, and interactions )),
( ), (h est.), (a est.), (b est.).
jump to sentence 150.1.
stop.

fix(q(ka)).
zz = q(ka).
vary kk 0(1)9999 with zz zz(-1)-2 sentence 403.
if zz L 0.5 jump to sentence 404.
exit.

tr(aal(igmin,kgmin),bbi(kfmin,ifmin),itr, itc ).
trace = 0.
vary ig igmin(1)itr sentences 800.3 thru 800.4.
vary kg kgmin(1)itr sentence 800.4.
trace = trace + aal(ig,kg)Xbbi(kg,ig).
trace = 2.0Xtrace.
igmin = ifmin.
kgmin = igmin.
kfmin = kgmin.
exit.
899 prod(cg(igmin, jgmin), ag(ifmin, kgmin), ifmax, kgmax, 
bg(kfmin, jfmin), jfmax).
899.03 jg = jgmin.
899.04 jf = jfmin.
899.05 produc = 0.
899.06 kg = kgmin.
899.07 kf = kfmin.
899.08 produc = produc + ag(ifmin, kg)Xbg(jf, kf).
899.1 kg = kg + 1.
899.11 kf = kf + 1.
899.12 if kg L= kgmax jump to sentence 899.08.
899.13 cg(igmin, Jg) = produc.
899.15 jf = jf + 1.
899.16 jg = jg + 1.
899.17 if jf L= jfmax jump to sentence 899.05.
899.18 ifmin = ifmin + 1.
899.2 igmin = igmin + 1.
899.21 if ifmin L= ifmax jump to sentence 899.03.
899.22 exit.

997 invert(array(ie, maxi)).
997.01 factor = 1/array(ie, ie).
997.02 je = maxi.
997.03 array(ie, je) = array(ie, je)Xfactor.
997.04 je = je + 1.
997.05 if je=0, jump to sentence 997.03.
997.06 array(ie, ie) = factor.
997.07 je = je + 1.
997.08 if je=ie, jump to sentence 997.15.
997.09 factor = -array(ie, ie).
997.1 ke = 0.
997.11 array(je, ke) = array(je, ke) + factorXarray(ie, ke).
997.12 ke = ke + 1.
997.13 if ke = maxi, jump to sentence 997.11.
997.14 array(je, ie) = array(ie, ie)Xfactor.
997.15 if jeImaxi, jump to sentence 997.07.
997.16 ie = ie + 1.
997.17 if ie = maxi, jump to sentence 997.01.
997.18 exit.
prod(cg(i_min, j_min), ag(i_fmin, k_min), i_max, k_max, 
    bg(k_fmin, j_fmin), j_max).
996.03 jg = j_min.
996.04 jf = j_min.
996.05 produc = 0.
996.06 kg = k_min.
996.07 kf = k_fmin.
996.08 produc = produc + ag(i_fmin, k)Xbg(kf, jf).
996.1 kg = kg + 1.
996.11 kf = kf + 1.
996.12 if kgl = k_max, jump to sentence 996.08.
996.13 cg(i_min, jg) = produc.
996.15 jf = jf + 1.
996.16 jg = jg + 1.
996.17 if jf L = j_max, jump to sentence 996.05.
996.19 if min = i_min + 1.
996.20 igmin = igmin + 1.
996.21 if fminL = i_max, jump to sentence 996.03.
996.22 exit.

vprod(cg(j_min), ag(k_min), k_max, bg(k_fmin, j_fmin), j_max).
999.03 jg = j_min.
999.04 jf = j_min.
999.05 produc = 0.
999.06 kg = k_min.
999.07 kf = k_fmin.
999.08 produc = produc + ag(k)Xbg(jf, kf).
999.1 kg = kg + 1.
999.11 kf = kf + 1.
999.12 if kgl = k_max, jump to sentence 999.08.
999.13 cg(jg) = produc.
999.15 jf = jf + 1.
999.16 jg = jg + 1.
999.17 if jf L = j_max, jump to sentence 999.05.
999.22 exit.

zzzzzz end of tape.
APPENDIX B

COMPUTER RESULTS FOR VARIANCES OF ESTIMATES
OF VARIANCE COMPONENTS

The computer results for each design in Tables 4.1 and 4.2 are shown on the following pages. The variances of the estimates of the variance components are given for estimation procedures A, B and H and for 18 different parameter sets.
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292. Bhattacharya, P. K. Some properties of the least square estimator in regression analysis when the 'independent' variables are stochastic. June, 1961.