PERCENTAGE POINTS OF WILKS' L_{mvc} AND L_{vc} CRITERIA

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1. Introduction and summary.

In connection with a p-variate normal population Wilks [5] proposed the following hypotheses: (i) H_{mvc}: that the means are equal, the variances are equal and the covariances are equal and (ii) H_{vc}: that the variances are equal and the covariances are equal. These hypotheses are of great importance in psychometry, especially in the theory of mental tests [1]. The concept of parallel tests, for instance, leads to an examination of the hypothesis H_{mvc}.

In a random sample of size n from a p-variate normal population, let $x_{i\lambda}$ denote the value of the i-th variate for the λth sample $i=1,2,\ldots,p$ $\lambda=1,2,\ldots,n$. Let

$$\bar{x}_i = \frac{1}{n}\sum_{\lambda=1}^{n} x_{i\lambda}$$

$$S_{ij} = \frac{1}{n}\sum_{\lambda=1}^{n} (x_{i\lambda} - \bar{x}_i)(x_{j\lambda} - \bar{x}_j)$$

$$T = \frac{1}{p}\sum_{i=1}^{p} S_{ii}$$

$$U = \frac{1}{p(p-1)}\sum_{i\neq j=1}^{p}\sum S_{ij}$$

$$\bar{x} = \frac{1}{p}\sum_{i=1}^{p} \bar{x}_i$$

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Wilks [5] showed that the likelihood-ratio principle gives the following test procedure:

"At the level of significance α, $0 < \alpha < 1$, reject the hypothesis H_{mvc} (H_{vc}) if the test-criterion $L_{mvc}(L_{vc})$ falls short of the constant L_α the lower 100α o/o point of the distribution of $L_{mvc}(L_{vc})$; otherwise accept the hypothesis $H_{mvc}(H_{vc})$."

And the test-criteria are defined by

$$(1.1) \qquad L_{mvc} = \frac{|s_{ij}|}{[T+(p-1)U]\left[T-U+\frac{1}{p-1}\sum_{i=1}^{p}(\bar{x}_i-\bar{x})^2\right]^{p-1}}$$

and

$$(1.2) \qquad L_{vc} = \frac{|s_{ij}|}{[T+(p-1)U](T-U)^{p-1}}$$

respectively. Wilks [5] computed the moments of these criteria and showed that asymptotically, for large n, the transforms $-n \log_e L_{mvc}$ and $-n \log_e L_{vc}$ are distributed as chi-squares with $\frac{1}{2}p(p+3)-3$ and $\frac{1}{2}p(p+1)-2$ degrees of freedom respectively. He also worked out the exact distribution of these criteria in the cases $p=2$ and $p=3$.

The chi-square approximation, however, is not good enough for moderately large values of n as it considerably over estimates significance. Approximation by a beta-variable has been suggested by Wilks and Tukey [6], but since this requires double interpolation in available tables, it is not very convenient. Verma [4] derived the exact distribution in a series form, which, however is not easy to tackle.

In view of the special importance of these criteria in psychometric work, it seems useful to derive the exact distribution in a convenient form and compute the percentage points for ready use. In this paper an asymptotic series expansion derived by one of the authors [2] [3] is used to evaluate 5 o/o and 1 o/o points of the distribution of L_{mvc} and L_{vc} criteria for p=4,5,6,7 and n=25(5)60(10)100. For higher values of n a correction-factor a is provided such that to a high degree of accuracy $-(n-a)\log_e L_{mvc}$ or $-(n-a)\log_e L_{vc}$ is distributed as a chi-square. The use of the tables is illustrated with two numerical examples. A simple non-parametric alternative procedure is suggested for testing a generalization of the H_{mvc} hypothesis.

2. The method for computing the percentage points.

It has been shown [2] [3] that for a properly chosen constant a, writing

(2.1) $$X = - N \log_e L$$

where

(2.2) $$N = n - a$$

and L is Wilks' L_{mvc} or L_{vc} criterion based on a sample of size n from a p-variate normal population, the probability density function $f_N(x)$ of X can be expressed as

(2.3) $$f_N(x) = C_N \left[p_r(x) + \frac{a_2}{N^2} p_{r+4}(x) + \frac{a_3}{N^3} p_{r+6}(x) + \frac{a_4}{N^4} p_{r+8}(x) + \dots \dots \right]$$

where $p_r(x)$ is the chi-square density function with r degrees of freedom.

(2.4) $$p_r(x) = \frac{1}{2^{\frac{r}{2}}\Gamma(\frac{r}{2})} \quad e^{-\frac{1}{2}x} \quad x^{\frac{1}{2}r-1}$$

C_N is a function of N that can be expanded in an asymptotic series

(2.5) $$1/C_N = 1 + \frac{a_2}{N^2} + \frac{a_3}{N^3} + \frac{a_4}{N^4} + \dots \dots$$

and r, a, a_2, a_3, a_4 are constants independent of N. The values of these constants are tabulated in Table 3.1 and Table 3.2 for p=4,5,6,7 and 8.

Table 2.1

Values of the constants in the asymptotic expansion of the distribution of Wilks' L_{mvc} criterion.

p	r	a	a_2	a_3	a_4
4	11	2.21718	3.48140	0.55608	9.33616
5	17	2.48530	8.24908	2.49784	50.72240
6	24	2.77500	16.29624	7.05128	180.95904
7	32	3.07638	28.78664	15.92480	530.00624
8	41	3.38502	47.05200	31.38352	1352.19984

Table 2.2

Values of the constants in the asymptotic expansion of the distribution of Wilks' L_{vc} criterion.

p	r	a	a_2	a_3	a_4
4	8	2.73611	1.47184	0.26324	60.83198
5	13	3.01923	4.24880	1.33136	187.00837
6	19	3.32105	9.41040	4.06681	456.04610
7	26	3.63248	17.95537	9.73818	978.32878
8	34	3.94958	31.04982	20.09536	1925.08274

For any given number α, $0 < \alpha < 1$, to compute L_α the lower 100 α o/o point of the distribution L we proceed as follows:

Obviously,

$$L_\alpha = \exp\,(-\,X_\alpha/N) \tag{2.6}$$

where X_α is defined by

$$\mathrm{Prob}(X > X_\alpha) = \alpha$$

Let x_o be the upper 100 α o/o point of the chi-square distribution with r degrees of freedom, that is

$$Q_r(x_o) = \alpha \tag{2.7}$$

where

$$Q_r(x) = \int_x^\infty p_r(\,\xi\,)\,d\xi \quad . \tag{2.8}$$

Since for large N, $C_N \sim 1$ and the second and succeeding terms in (3.3) are negligible, we have as a first approximation

$$X_\alpha \doteq x_o \quad . \tag{2.9}$$

Let δ be the additive correction to be made to x_o so that

$$X_\alpha = x_o + \delta \quad . \tag{2.10}$$

Retaining only the first two terms, we have

$$\alpha\left(1 + \frac{a^2}{N^2}\right) = Q_r(x_o + \delta\,) + \frac{a^2}{N^2}\,Q_{r+4}(x_o + \delta\,) \quad . \tag{2.11}$$

Expanding the right-hand side in a Taylor series about x_o and neglecting δ^2 and higher powers of δ , we get

$$(2.12) \qquad \delta_o = \frac{2\left(\frac{x_o^2}{r(r+2)} + \frac{x_o}{r}\right)\frac{a_2}{N^2}}{1+ \frac{x_o^2}{r(r+2)} \cdot \frac{a_2}{N^2}}$$

as an approximation for δ. As a second approximation to X_α we then have

$$(2.13) \qquad x_1 = x_o + \delta_o .$$

Now, from (2.3) we have the asymptotic expansion

$$(2.14) \qquad \begin{aligned} \text{Prob}(X \geq x) &= Q_r(x) \\ &+ \frac{a_2}{N^2}\left[Q_{r+4}(x) - Q_r(x)\right] \\ &+ \frac{a_3}{N^3}\left[Q_{r+6}(x) - Q_r(x)\right] \\ &+ \left[\frac{a_4}{N^4}\left[Q_{r+8}(x) - Q_r(x)\right] - \frac{a_2}{N^2}\left[Q_{r+4}(x) - Q_r(x)\right]\right] \\ &+ 0\left(\frac{1}{N^5}\right). \end{aligned}$$

The required percentage point is in the neighborhood of x_1 and can be evaluated by inverse interpolation by first tabulating Prob($X \geq x$) for several values of x around x_1. As it happens, however, further corrections to x_1 become necessary only for very small values of N.

3. Tables of percentage points.

The 5 o/o and 1 o/o points of the distribution of Wilks' L_{mvc} and L_{vc} criteria are given to 4 places of decimals for p = 4,5,6 and 7 in Table 4.1 and Table 4.2 respectively for n = 25 [5] 60 [10] 100. For n greater than 100,

$$-(n - a) \log_e L$$

can safely be used as a chi-square with r degrees of freedom. We have not extended the tables below n=25 firstly because the asymptotic expansion (2.14) with only four terms is not good enough for n smaller than 25 and secondly because a sample of size less than 25 is not to be recommended in multivariate work with four or more variates.

Table 3.1

5 o/o and 1 o/o points of L_{mvc} criterion

	5 o/o point				1 o/o point			
n/p	4	5	6	7	4	5	6	7
25	.4206	.2920	.1923	.1196	.3366	.2251	.1427	.0854
30	.4918	.3658	.2623	.1781	.4098	.2957	.2048	.1356
35	.5482	.4273	.3229	.2339	.4698	.3570	.2623	.1859
40	.5937	.4787	.3759	.2852	.5193	.4098	.3143	.2338
45	.6311	.5222	.4220	.3314	.5607	.4552	.3606	.2783
50	.6623	.5592	.4625	.3730	.5958	.4946	.4019	.3191
55	.6887	.5911	.4979	.4102	.6258	.5290	.4387	.3654
60	.7113	.6187	.5292	.4437	.6518	.5591	.4715	.3903
70	.7480	.6645	.5815	.5011	.6943	.6096	.5274	.4494
80	.7763	.7005	.6239	.5483	.7276	.6498	.5730	.4985
90	.7992	.7296	.6586	.5876	.7545	.6826	.6108	.5403
100	.8177	.7536	.6875	.6208	.7765	.7099	.6426	.5757

For larger n, use

$$-(n-a) \log_e L_{mvc}$$

as a chi-square with r degrees of freedom.

Table 3.2

5 o/o and 1 o/o points of L_{vc} criterion

	5 o/o				1 o/o			
n/p	4	5	6	7	4	5	6	7
25	.5129	.3768	.2473	.1601	.4209	.2985	.1866	.1163
30	.5773	.4490	.3219	.2273	.4908	.3709	.2563	.1756
35	.6271	.5071	.3853	.2883	.5464	.4313	.3181	.2322
40	.6666	.5546	.4390	.3424	.5913	.4819	.3721	.2841
45	.7002	.5941	.4847	.3899	.6284	.5248	.4191	.3310
50	.7251	.6273	.5239	.4318	.6594	.5613	.4601	.3731
55	.7473	.6555	.5578	.4686	.6857	.5928	.4961	.4108
60	.7663	.6799	.5873	.5013	.7083	.6201	.5278	.4446
70	.7967	.7196	.6362	.5564	.7450	.6653	.5810	.5024
80	.8202	.7506	.6748	.6008	.7736	.7011	.6236	.5499
90	.8394	.7755	.7062	.6374	.7964	.7299	.6586	.5894
100	.8560	.7959	.7321	.6679	.8151	.7538	.6877	.6226

For larger n, use

$-(n-a) \log_e L_{vc}$ as a ^{2} with r dif.

4. Illustrative example.

The following table gives means, variances and covariances for scores in four tests based on a sample of 50 examinees.

Tests	Means	Dispersion Matrix A	B	C	D
A	14.9048	25.0704	12.4363	11.7257	20.7510
B	15.4841		28.2021	9.2281	11.9732
C	14.4444			22.7390	12.0692
D	14.3810				21.8707

(a) Can the tests be regarded as parallel? (b) If not, would additive corrections applied to the means make the tests parallel?

Tests are said to be parallel [1] if scores obtained in the tests have equal means, equal variances and equal covariances. To answer question (a), the appropriate hypothesis to be tested is H_{mvc} and to answer (b) the appropriate hypothesis to be tested is H_{vc}. The numerical procedure for carrying out these tests is given below.

In this case, p = 4 and

$$|s_{ij}| = 39750.5$$
$$T = 24.47055$$
$$U = 13.03058$$
$$\Sigma(\bar{x}_i - \bar{x})^2 = 0.78094$$
$$T + (p-1)U = 63.56229$$
$$T - U = 11.43997$$

Thus

$$L_{mvc} = \frac{39750.5}{63.56229 \times (11.43997 + 0.26031)^3}$$

$$= \frac{39750.5}{104040.6} = 0.3821 \ .$$

The 1 o/o point of L_{mvc} for p=4 and n=50 is 0.5958 and therefore the hypothesis H_{mvc} has to be rejected at the 1 o/o level of significance.

The L_{vc} statistic turns out to be

$$L_{vc} = \frac{39750.5}{63.56229 \times (11.43997)^3}$$

$$= \frac{39750.5}{95164.3} = 0.4177 \ .$$

The 1 o/o point of L_{vc} for p=4 and n=50 is 0.6594 and therefore the hypothesis H_{vc} has to be rejected at the 1 o/o level. Therefore, the tests are not parallel and even additive corrections in the means would not make them so.

5. A non-parametric test for symmetry.

The hypothesis H_{mvc} for a multivariate normal population is equivalent to the hypothesis that the joint distribution function is symmetric, that is invariant under a permutation of the variates. However, the statistic L_{mvc} is not appropriate for testing symmetry of a multivariate distributions which is not definitely known to be normal. A simple non-parametric test for symmetry appropriate for any continuous multivariate distribution is proposed here.

Let $(X_1, X_2, \ldots, X_p)$ be a random sample from a continuous p-variate distribution and let $\underline{i} \equiv (i_1, i_2, \ldots, i_p)$ be a permutation of the integers $(1,2,\ldots,p)$. We shall say the random sample is of "type $\underline{i}$" if

$$X_{i_1} < X_{i_2} < \cdots < X_{i_p}$$

holds. Thus with probability 1 every random sample belongs to one and only one of the p! types. If the hypothesis of symmetry is true, all these types are equally likely: the probability that a random sample is of any particular type being 1/p! If in n random samples, $n_{\underline{i}}$ samples are of type $\underline{i}$, we compute

$$\chi^2 = \frac{p!}{n} \Sigma \left(n_{\underline{i}} - \frac{n}{p!}\right)^2 = \frac{p!}{n} \Sigma \, n_{\underline{i}}^2 - n.$$

the summation being over all types. If this exceeds the upper 100α o/o point of the chi-square statistic with (p! - 1) degrees of freedom, the hypothesis of symmetry is rejected at the 100 α o/o level of significance.

Illustrative Example

The following data gives the scores in three tests A, B, C, based on a random sample of 50 examinees. Test the hypothesis of symmetry.

Table 5.1

Scores on the tests A, B and C

A	B	C	A	B	C	A	B	C
56	40	46	50	46	23	42	36	0
34	57	48	31	45	20	39	55	10
32	47	38	43	37	32	37	58	40
55	24	32	59	43	58	28	24	29
37	63	59	38	48	14	37	52	40
32	40	7	29	36	38	62	45	50
33	58	34	27	53	18	24	29	13
62	74	58	38	35	22	32	35	39
28	42	36	40	61	12	47	36	15
41	60	16	41	42	26	45	46	24
20	35	7	46	62	32	52	43	44
47	39	24	55	54	24	65	72	84
33	53	54	52	43	15	31	49	36
44	40	31	46	38	17	54	62	64
41	42	28	48	52	61	51	48	53
28	40	42	59	52	63	40	36	42
47	50	64	55	47	56			

In this example the 3! = 6 types of permutations are $X_1 < X_2 < X_3$; $X_1 < X_3 < X_2$; $X_2 < X_1 < X_3$; $X_2 < X_3 < X_1$; $X_3 < X_1 < X_2$ and $X_3 < X_2 < X_1$ and let us call them permutations of type 1, 2, 3, 4, 5 and 6 respectively. The following table gives the observed frequencies of the six types and the expected frequency of each type when the hypothesis of symmetry is true is 8.3333.

Table 5.2

Type i	Frequency n_i
1	8
2	8
3	5
4	5
5	14
6	10
Total	50

Here $p = 3$, $n = 50$ and

$$\chi^2 = \frac{p!}{n} \Sigma n_i^2 - n = \frac{6}{50} \times 474 - 50$$

$$= 6.88$$

The 5 o/o point of χ^2 with $p! - 1 = 5$ degrees of freedom is 11.07 and thus there is no evidence for rejecting the hypothesis of symmetry.

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