ON MAXIMAL AND MINIMAL SUB-FIELDS OF CERTAIN TYPES

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D. Basu

University of North Carolina
and
Indian Statistical Institute

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DEPARTMENT OF STATISTICS
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Let \((\mathcal{L}, \mathcal{A}, \mathbb{P})\) be a given statistical model. We concern ourselves with particular families of sub-fields of \(\mathcal{A}\), and, for each such family we ask ourselves whether the family has maximal and minimal elements with respect to the natural partial order of inclusion relation. For example the family \(\mathcal{S}\) of sub-fields that are sufficient for \(\mathcal{A}\) has received a great deal of attention from theoretical statisticians. Clearly, \(\mathcal{A} \subseteq \mathcal{S}\) and is the maximum element of \(\mathcal{S}\). It is known [4] that, in general, \(\mathcal{S}\) has no minimal element. However, if we assume that \(\mathbb{P}\) is dominated by a \(\sigma\)-finite measure then it can be shown [1] that \(\mathcal{S}\) has an essentially minimum element \(\mathcal{S}_0\), i.e., given any \(\mathcal{S}_1 \subseteq \mathcal{S}\) and any \(\mathcal{A} \subseteq \mathcal{S}_0\), there exists \(\mathcal{B} \subseteq \mathcal{S}_1\) such that

\[ \mathbb{P}(\mathcal{A} \Delta \mathcal{B}) = 0 \quad \text{for all} \quad \mathcal{B} \subseteq \mathcal{P}, \]

where \(\Delta\) stands for the operation of symmetric difference.

Let \(\mathcal{B}\) be a fixed sub-field of \(\mathcal{A}\) and let \(\mathcal{C}\) be the family of all sub-fields that are independent of \(\mathcal{B}\), i.e., for any such \(\mathcal{C}\) it is true that

\[ \mathbb{P}(\mathcal{B}\mathcal{C}) = \mathbb{P}(\mathcal{B})\mathbb{P}(\mathcal{C}) \quad \text{for all} \quad \mathcal{B} \subseteq \mathcal{B}, \ \mathcal{C} \subseteq \mathcal{C}, \ \text{and} \ \mathcal{P} \subseteq \mathcal{P}. \]

The minimum element of \(\mathcal{C}\) is the trivial sub-field consisting of the null-set and the whole space. We have the following

**Theorem 1:** For any sub-field \(\mathcal{C}\) that is independent of \(\mathcal{B}\), there exists at least one maximal subfield \(\mathcal{C}^*\) such that \(\mathcal{C} \subseteq \mathcal{C}^*\).

The proof of Theorem 1 is given in the next section.

For the next problem, let us suppose that the family \(\mathcal{P}\) is indexed by
two parameters \( \theta \) and \( \varphi \), i.e.,

\[
P = \{ P_{\theta, \varphi} \}, \quad (\theta, \varphi) \in \Theta \times \Phi.
\]

Consider all sub-fields \( \mathcal{D} \) that are generated by statistics whose probability distributions do not involve the parameter \( \varphi \), i.e., each \( \mathcal{D} \) is a sub-field such that, for every \( D \in \mathcal{D} \), \( P_{\theta, \varphi}(D) \) is a function of \( \theta \) only.

The trivial sub-field is again the minimum element of \( \{ \mathcal{D} \} \); but, does this family have maximal elements? We have

**Theorem 2:** Given any sub-field \( \mathcal{D} \) such that the restriction of \( P_{\theta, \varphi} \) to \( \mathcal{D} \) does not involve \( \varphi \), there exists a maximal such sub-field \( \mathcal{D}^* \) that contains \( \mathcal{D} \).

The above Theorem is an immediate generalization of Theorem 1 in [2].

In the next section we give the proof of the above two theorems. In the final section we comment on some further problems of the same kind.

2. Proofs of Theorems 1 and 2

We need the following well-known lemmas.

**Lemma 1 (Zorn's Lemma):** If for a partially ordered set it is true that every linearly ordered sub-set has an upper (lower) bound, then given any element \( x \) of the set there exists a maximal (minimal) element \( x^* \) in the set such that \( x \) is less (greater) than \( x^* \).

[The terms that are underlined are defined in terms of the partial order relation.]

**Lemma 2 (Extension of Measures):** Given a measure \( \mu \) defined on a field \( \mathcal{F} \) of sets there exists one and only one extension \( \mu^* \) of \( \mu \) to the Borel-extension \( \mathcal{F}^* \) of \( \mathcal{F} \).
Corollary: If the two measures $\mu$ and $\nu$ agree on a field $F$ of sets they necessarily agree on the Borel-extension $F^*$ of $F$.

Lemma 3: If the family $\{B_\alpha\}$ of sub-fields of $A$ be linearly ordered with respect to the inclusion relation then

$$F = \bigcup B_\alpha$$

is a field of sub-sets.

We omit the proofs of the lemmas.

Now let $F$ be a given sub-field of $A$ and let $E$ be the class of all sets $E \in A$ such that $E$ is independent of $F$, i.e.

$$P(EB) = P(E)P(B) \text{ for all } B \in B, \ E \in E \text{ and } P \in P.$$ 

It is easy to check that $F$ contains the null-set and the whole space and further that $F$ is closed for complementation and countable disjoint unions. In case $F$ is a $\sigma$-field there is nothing to prove in Theorem 1, as $E$ is then the maximal sub-field for which we are searching. However, $E$ is usually not a $\sigma$-field (see example 1).

Let $\{C_\alpha\}$ be a family of sub-fields in $F$ and be linearly ordered with respect to the inclusion relation and let

$$C_\alpha = \bigcup C_\alpha.$$ 

Now, from Lemma 3, $C_\alpha$ is a field of sub-sets of $A$ and, since $C_\alpha \subseteq E$, every member of $C_\alpha$ is independent of $B$. Choose and fix $B \in B$ and $P \in P$.

Consider the two measures $P(AB)$ and $P(A)P(B)$ defined for all sets $A \in A$. These two measures agree over the field $C_\alpha$ and hence, from the corollary to Lemma 2, they agree over the Borel-extension $C_\alpha^*$ of $C_\alpha$.

Remembering that $B$ and $P$ were arbitrary members of $B$ and $P$ respectively, we now have
\[ P(\text{AB}) = P(A)P(B) \text{ for all } A \in \mathcal{C}^*, \ B \in \mathcal{D} \text{ and } P \in \mathcal{P}. \]

Thus, \( \mathcal{C}^* \) includes every \( \mathcal{C}_\alpha \) and is independent of \( \mathcal{D} \). The conditions of Lemma 1 are satisfied and hence the proof of Theorem 1 is complete.

We now turn our attention to Theorem 2. Let \( \mathcal{F} \) be the class of all sets \( F \in \mathcal{A} \) such that

\[ P_{\theta, \varphi}(F) \text{ is a function of } \theta \text{ only.} \]

Again, it is easy to check that \( \mathcal{F} \) contains the null-set and the whole space and is closed for complementation and countable disjoint unions. In example 2 we shall see that \( \mathcal{F} \) is usually not a sub-field of \( \mathcal{A} \).

Let \( (\mathcal{F}_\alpha) \) be a family of sub-fields in \( \mathcal{F} \) and be linearly ordered with respect to the inclusion relation and let

\[ \mathcal{L}_0 = \prod_{\alpha} \mathcal{F}_\alpha. \]

As before \( \mathcal{L}_0 \) is a field of sub-sets of \( X \). Let \( \mathcal{L}_0^* \) be the Borel-extension of \( \mathcal{L}_0 \).

We define the measure \( Q_\theta \) on \( \mathcal{L}_0 \) as the restriction of \( P_{\theta, \varphi} \) on \( \mathcal{L}_0 \). [Since \( \mathcal{L}_0 \subseteq \mathcal{F} \), the measure \( Q_\theta \) on \( \mathcal{L}_0 \) must be independent of \( \varphi \).]

From Lemma 2, for each \( \theta \in \Theta \), the extension \( Q_\theta^* \) of \( Q_\theta \) from \( \mathcal{L}_0 \) to \( \mathcal{L}_0^* \) is unique. From the corollary to Lemma 2, the two measures \( Q_\theta^* \) and \( P_{\theta, \varphi} \) must agree on \( \mathcal{L}_0^* \).

In other words, the restriction of \( P_{\theta, \varphi} \) to the sub-field \( \mathcal{L}_0^* \) is independent of \( \varphi \). Also, \( \mathcal{L}_0^* \) includes every \( \mathcal{L}_\alpha \). The conditions of Lemma 1 are satisfied and hence the proof of Theorem 2 is complete.
3. Examples and Comments.

The following two examples demonstrate that the maximal element is usually not unique.

Example 1: Let $X$ consist of the four points $a, b, c$ and $d$ and let $F$ consist of just one probability distribution namely the uniform distribution over the four points. Consider the three sub-fields $B, C_1$ and $C_2$ each consisting of four sub-sets of $X$:

- $B$ consists of $X$, $(a,b)$ and their complements,
- $C_1$ consists of $X$, $(a,c)$
- $C_2$ consists of $X$, $(a,d)$

and each of them is a maximal such sub-field. Incidentally, $C_1$ and $C_2$ are also independent of each other.

Example 2: Let $X$ consist of the five points $a, b, c, d, e$ and let the probability distribution over the five points be:

Points: $a, b, c, d, e$

Probs: $1 - \theta, \varphi, \varphi, \theta(\frac{1}{2} \varphi), \theta(\frac{1}{2} \varphi)$

where $0 < \theta < 1$ and $0 < \varphi < \frac{1}{2}$.

The family $F$ of all sets whose probability does not involve $\varphi$ consists of 12 sets, that is:

$X, \{a\}, \{b,d\}, \{b,e\}, \{c,d\}, \{c,e\}$

and their complements.

Note that $F$ does not constitute a sub-field. There are two different maximal sub-fields in $F$, namely:

$A_1$: consisting of $X, \{a\}, \{b,d\}, \{c,e\}$

and their complements.
and \( \mathcal{E}_2 \) consisting of \( x \), \{a\}, \{b,e\}, \{c,d\}
and their complements.

Theorems 1 and 2 only establish the existence of maximal sub-fields
in \( E \) and \( F \) respectively. It would be of some interest to develop general
methods for proving the maximality of certain given sub-fields of \( E \) and \( F \).
One such method, with very limited application, is given in Theorem 7 of [2].

Consider the problem where we have \( n \) independent observations
\( x_1, x_2, \ldots, x_n \) on a real random variable \( x \) with cumulative distribution
function of the form
\[
F\left(\frac{x - \varphi}{\theta}\right), \quad -\infty < \varphi < \infty, \quad 0 < \theta < \infty.
\]
where the function \( F \) is known and \( \theta \) and \( \varphi \) are the so-called scale and
location parameters.

If \( y \) stand for the vector-valued statistic
\[
(x_1 - x_n, x_2 - x_n, \ldots, x_{n-1} - x_n)
\]
then the distribution of \( y \) does not involve the location parameter \( \varphi \). Is
\( y \) a maximal such statistic? In the language of sub-fields, if \( \mathcal{S}_y \) be the
sub-field generated by \( y \) then is it true that \( \mathcal{S}_y \) is maximal in the sense
of Theorem 2? The author does not expect the answer to be 'yes' for all \( F \).

4. Some further problems

The sub-field \( B \subset A \) is said to be sufficient for the sub-field
\( C \subset A \) if for every \( C \in C \) there exists a \( B \)-measurable function \( f(x ; C) \)
mapping \( X \) into the real line such that
\[
P(BC) = \int_B f(x ; C) \, dP(x) \quad \text{for all} \quad P \in \mathcal{P}
\]
and \( B \in \mathcal{B} \).
In other words, \( B \) is sufficient for \( C \) if, for every \( C \in \mathcal{C} \), there exists a choice for the conditional probability (function) of \( C \) given \( B \) that serves for all \( P \in \mathcal{P} \).

Now, for a fixed \( B \), let us enquire about the family \( \{C\} \) of all subfields \( C \) such that \( B \) is sufficient for \( C \). Clearly, the minimum element of \( \{C\} \) is the trivial sub-field consisting of only the null-set and the whole space. Do there exist maximal elements in \( \{C\} \)?

Let \( G \) be the class of all sets \( G \subseteq \mathcal{A} \) such that \( B \) is sufficient for \( G \) in the sense mentioned above, namely, for every \( G \in \mathcal{G} \) there exists a \( B \)-measurable \( f(x;G) \) such that

\[
P(GB) = \int f(x;G) dP(x) \quad \text{for all } P \in \mathcal{P}
\]

and \( B \in \mathcal{B} \).

The class \( \mathcal{G} \) is similar to the classes \( \mathcal{E} \) and \( \mathcal{F} \) considered before in that \( \mathcal{G} \) contains the null-set and the whole space and is closed for complementation and countable disjoint unions. As before, \( \mathcal{G} \) is usually not a sub-field. The rest of the arguments in Theorems 1 and 2 will apply if we could prove a result of the following type:

"If \( B \) is sufficient for each member of a field \( \mathcal{C} \) of sets in \( \mathcal{A} \), then \( B \) is sufficient for the Borel extension \( \mathcal{C}^* \) of \( \mathcal{C} \)."

The above statement does not seem to be true in the generality stated above.

In the particular case where \( B \) is the trivial sub-field, the question posed above has a definite answer. For, in this case, \( B \) can be sufficient for \( G \) if and only if

\[
P(G) \text{ is the same for all } P \in \mathcal{P},
\]

and therefore, Theorem 2, or rather a particular case of it, namely, Theorem 1

Fraser in [3] introduced the notation of partial sufficiency in the following manner:

If \( \mathcal{P} = \{ P_{\theta, \varphi} \} \), \( \theta \in \Theta \), \( \varphi \in \Phi \), be a family of probability measures indexed by the two independent parameters \( \theta \) and \( \varphi \), then a sub-field \( B \subset A \) will be called \( \theta \)-sufficient for \( A \) (or simply \( \theta \)-sufficient) if

1) the restriction of \( P_{\theta, \varphi} \) to \( B \) does not depend on \( \varphi \) [i.e., \( B \) is a sub-field of the type considered in Theorem 2.],

and

2) given any \( A \in A \), there exists a choice (of the conditional probability (function) of \( A \) given \( B \) that does not depend on \( \theta \), i.e., for each \( \theta \in \Theta \) there exists a \( B \)-measurable function \( f_\theta (x; A) \) such that

\[
P_{\theta, \varphi}(AB) = \int_B f_\theta (x; A) \, dP_{\theta, \varphi} \quad \text{for all } B \subset B \\
\text{and all } (\theta, \varphi).
\]

Under what conditions does a \( \theta \)-sufficient sub-field exist? Does there exist an essentially minimum such sub-field?

As a final problem on the existence of minimal sub-fields consider the following:

Given two sub-fields \( B \) and \( C \), let \( B \lor C \) stand for the smallest sub-field that contains both \( B \) and \( C \).

Now, for a fixed \( B \subset A \), let us consider the family \( [C] \) of all sub-fields \( C \subset A \) such that

\[
B \lor C = A.
\]

Every \( C \in [C] \) may be called a complement of \( B \). The family \( [C] \) has \( A \) as its maximum element. Does \( [C] \) have minimal elements? The author expects the answer to be yes. It is easy to construct examples where there are several minimal complements to \( B \).
References


