HYPOTHESES OF NO INTERACTION IN FOUR-DIMENSIONAL CONTINGENCY TABLES

V. P. Bhapkar  
University of North Carolina  
University of Poona  
and  
Gary G. Koch  
University of North Carolina

Institute of Statistics Mimeo Series No. 449

September 1965

Hypotheses of no interaction of various orders in four-dimensional contingency tables are considered appropriate to the underlying probability models. General and computationally simple test procedures based on Wald's criterion (1943) are offered and illustrated.

This paper is a follow-up of the earlier expository paper for the three-dimensional contingency tables.

This research was supported by the National Institute of Health Institute of General Medical Sciences Grant No. GM-12868-01.

DEPARTMENT OF STATISTICS  
UNIVERSITY OF NORTH CAROLINA  
Chapel Hill, N. C.
HYPOTHESES OF NO INTERACTION IN FOUR-DIMENSIONAL CONTINGENCY TABLES

1. Introduction

In a previous expository paper, the authors have discussed hypotheses of "no interaction" in three-dimensional contingency tables (See Bhapkar and Koch [1965]). There the importance of specifying the underlying model and of formulating hypotheses appropriate to it was strongly emphasized. In addition, it was demonstrated that for any given formulation, a computationally simple statistical test (based upon a criterion due to Wald [1943]) may be constructed.

In this paper, hypotheses of "no interaction" in four-dimensional contingency tables are considered from the same point of view as outlined above. First of all, there are four principal types of four-dimensional contingency tables of experimental interest. These are

(i). the "four response, no factor" tables,
(ii). the "three response, one factor" tables,
(iii). the "two response, two factor" tables,
(iv). the "one response, three factor" tables.

Given any particular one of the above situations, our object is to obtain a clearer understanding of the behavior of experimental units by investigating whether the structure of the underlying model can be simplified in a meaningful manner. We shall see that one way of accomplishing this is by considering hypotheses of "no interaction". Some of the problems arising here are natural extensions of those discussed for the three-dimensional case; others, however, have no such analogue.

Before proceeding any further, we recall that in multi-dimensional contingency tables, the term "no interaction" has been given two
interpretations. One is related to the nature of the association among responses; the other, to the nature of the way in which factors affect (the distribution of) responses and functions thereof. This distinction will be emphasized throughout this paper since it is important to the formulation of appropriate hypotheses of "no interaction" for the models cited previously. Finally, it may be pointed out here that only the case (i) has received an adequate consideration in the literature so far; reference may be made to Darroch [1962], Good [1963], Goodman [1964].

NOTATION: The subscript "c" denotes that the corresponding suffix has been summed over.

The subscript * denotes that a quantity is independent of the corresponding suffix.

The subscript "." denotes that the definition of a quantity does not involve the corresponding suffix.

2. Formulations of Hypotheses of "No Interaction"

2.1. The "Four Response, No Factor" Model

Let $p_{hijk}$ (where we assume $p_{hijk} > 0$) denote the probability that an experimental unit belongs to the response $(h, i, j, k)$ - i.e., to the $h$th category of the first response, the $i$th category of the second response, the $j$th category of the third response, and the $k$th category of the fourth response - in an $r \times s \times t \times u$ contingency table; thus, the $p$'s are subject to the constraint

$$\sum_{h,i,j,k} p_{hijk} = 1$$

We shall say that there is "no third order interaction" among the four responses if some measure of association among any three of the
responses is constant over levels of the fourth. Before proceeding any further, however, we need to clarify what we mean by a measure of association among three responses. Let us consider for a moment a $2 \times 2 \times 2 \times 2$ table. If

$$\Delta_{1jk} = \frac{p_{1jk}p_{2jk}}{p_{2jk}p_{1jk}} \quad j = 1, 2$$

$$\quad k = 1, 2$$

is taken as a measure of association between the first and second response within the $j$th category of the third response and the $k$th category of the fourth response, then

$$\lambda_{11k} = \frac{\Delta_{11k}}{\Delta_{112k}} \quad k = 1, 2$$

may be interpreted as a measure of the extent to which the data within the $k$th category of the fourth response tend to deviate from the hypothesis of no interaction among the first three responses. For this reason $\lambda_{11k}$ is a natural measure of association among the first three responses within the $k$th level of the fourth. Hence, we obtain the following formulation of the hypothesis of "no third order interaction" among four responses

$$\lambda_{11k} = \lambda_{11*} \quad k = 1, 2;$$

and this is readily seen to be a straightforward generalization of Bartlett's formulation of the hypothesis of no interaction (among three responses) in $2 \times 2 \times 2$ tables (See Bartlett [1935]). Returning now to the $r \times s \times t \times u$ tables, we let
(2.1) \[ \Delta_{hijk} = \frac{\Phi_{hijk} \Phi_{rsik}}{\Phi_{rijk} \Phi_{hsjk}} \quad h = 1, 2, \ldots, (r-1) \]
\[ i = 1, 2, \ldots, (s-1) \]
\[ j = 1, 2, \ldots, t \]
\[ k = 1, 2, \ldots, u \]

then

(2.2) \[ \lambda_{hijk} = \frac{\Delta_{hijk}}{\Delta_{hitk}} \quad h = 1, 2, \ldots, (r-1) \]
\[ i = 1, 2, \ldots, (s-1) \]
\[ j = 1, 2, \ldots, (t-1) \]

\[ = \frac{\Phi_{hijk} \Phi_{rsik}}{\Phi_{rijk} \Phi_{hsjk} \Phi_{hitk} \Phi_{rstk}} \]

can be regarded as a measure of association among the first three responses within the \( k \)th category of the fourth response where \( k = 1, 2, \ldots, u \).

The hypothesis of "no third order interaction" among four responses may then be written as

(2.3) \[ \lambda_{hijk} = \lambda_{hij*} \quad h = 1, 2, \ldots, (r-1) \]
\[ i = 1, 2, \ldots, (s-1) \]
\[ j = 1, 2, \ldots, (t-1) \]

for \( k = 1, 2, \ldots, u \).

This formulation is essentially identical to that originally given by Darroch [1962]. At this point, one should observe that (2.3) is symmetric in the sense of Simpson [1951]; i.e., if the given measure of association among (say) the FIRST three responses is independent of the level of the fourth response, then the measure of association among ANY three responses is independent of the level of the excluded response. Finally, note that a sufficient condition for (2.3) to hold is that the \( \Phi_{hijk} \) satisfy

(2.4) \[ \Phi_{hijk} = \frac{\Phi_{hick} \Phi_{hojk} \Phi_{poijk} \Phi_{poio} \Phi_oio \Phi_oook}{\Phi_{hico} \Phi_{hojo} \Phi_{hoook} \Phi_{poijo} \Phi_{pooch} \Phi_{pook}} \]
as noticed by Darroch [1962].
With the above discussion of "third order interaction" in mind, the
question naturally arises as to how lower order interactions may be
meaningfully defined. Suppose we ignore classifications according to
(say) the fourth response and consider the marginal probability \( p_{hijo} \)
of the marginal response (\( h, i, j, . \)). Thus, we are in essentially
a three response situation and the corresponding hypothesis of no interaction is

\[
\lambda_{hij} = \frac{p_{hijo} p_{rjio}}{p_{rjio} p_{hsjo}} = \lambda_{hi*} \quad i = 1, 2, \ldots, (s-1)
\]

for \( j = 1, 2, \ldots, t \).

If we let

\[
\Delta_{hij} = \frac{p_{hijo}}{p_{rjio}} \quad h = 1, 2, \ldots, (r-1)
\]

\( i = 1, 2, \ldots, s \)

for \( j = 1, 2, \ldots, t \),

then note that

\[
\lambda_{hij} = \frac{\Delta_{hij}}{\Delta_{hsj}}
\]

We call (2.5) the hypothesis of "no second order interaction" among the
first three responses. This definition is consistent in the sense that
it is valid when there is no fourth response (i.e., \( u = 1 \)). It is to be
distinguished from the formulation given by Goodman [1964] since it is
expressed in terms of sums of probabilities as opposed to sums of
natural logarithms of probabilities. Now observe that a sufficient
condition for (2.5) to hold is that the \( p_{hijo} \) satisfy

\[
p_{hijo} = \frac{p_{hojo} p_{oiio} p_{hioo}}{p_{hoio} p_{oiio} p_{ojo}}
\]
Now if (2.4) and (2.7) hold, then

\[ \frac{p_{hijk}}{p_{oook}} = \frac{p_{hi^*k}p_{oijk}}{p_{hok}p_{oick}p_{oook}} \]

holds; and if (2.8) holds, then

\[ \Delta_{hijk} = \Delta_{hi^*k} \]

\[ j = 1, 2, \ldots, t \]

\[ k = 1, 2, \ldots, u \]

where \( \Delta_{hijk} \) is defined by (2.1). But (2.9) is a formulation of the hypothesis that within each level of the fourth response, there is no interaction among the first three responses. The reader should observe here that (2.9) is really a special case of (2.3) in which \( \lambda_{hi^*k} = 1 \).

Carrying the above ideas one step further, suppose we ignore classifications according to (say) the third and fourth responses and consider the marginal probability \( p_{hico} \) of the marginal response \( (h, i, . , .) \). If we have

\[ \lambda_{hi..} \equiv \frac{p_{hico}}{p_{rioc}} = \lambda_{h^*..} \]

\[ h = 1, 2, \ldots, (r-1) \]

\[ i = 1, 2, \ldots, (s-1) \]

for \( i = 1, 2, \ldots, s \),

then we shall say that there is "no first order interaction" between the first and second response; note (2.10) holds if and only if

\[ p_{hico} = p_{hooc}p_{oico} ; \]

i.e., if and only if the first two responses are independent.
Next, observe that (2.7) and (2.11) imply

\[(2.12) \quad p_{hij} = \frac{p_{hij}p_{cij}}{p_{cij}} \quad ;\]

and if (2.12) holds, then

\[(2.13) \quad \Delta_{hij} = \Delta_{h^*j}, \quad h = 1, 2, \ldots, (r-1)\]

\[i = 1, 2, \ldots, s\]

for \(j = 1, 2, \ldots, t\)

where \(\Delta_{hij}\) is defined by (2.6); that is, within each level of the third response (with the fourth response summed over), there is no interaction between the first two responses. Finally, we note that (2.13) is a special case of (2.5) in which \(\lambda_{h^*i} = 1\).

Thus, we see that the various hypotheses of no interaction are all related to one another in a particular way and that they impose conditions on the \(p_{hijk}\) which may be interpreted in a meaningful fashion.

2.2. The "Three Response, One Factor" Model

Let \(p_{hijk}\) (where we assume \(p_{hijk} > 0\)) denote the probability that an experimental unit from the \(k\)th category of the factor belongs to the response \((h, i, j)\) in an \(r \times s \times t \times u\) contingency table; thus, the \(p\)'s are subject to the \(u\) constraints

\[(2.14) \quad \sum_{h, i, j} p_{hijk} = 1 \quad k = 1, 2, \ldots, u .\]

If we regard \(\lambda_{hijk}\) (as defined in (2.2)) as a measure of association among the three responses within the \(k\)th category of the factor, then one
possible formulation of the hypothesis of "no third order interaction" among three responses and one factor is given by (2.3) where now, however, the $p_{hijk}$ appearing there also satisfy the constraints (2.14). In the "four response, no factor" case, we observed that this formulation was symmetric in the four responses. For the given situation, this symmetry may be interpreted as saying that the nature of dependence of a measure of association between any two of the responses upon the factor level is independent of the third response; for example, if we let

$$\mu_{hijk} = \frac{\Delta_{hijk}}{\Delta_{hiju}},$$

then (2.3) holds if and only if

$$\mu_{hijk} = \mu_{hi*jk}.$$

Next we note that (2.4) as such has no meaningful interpretation here since quantities like $p_{hijo}$ involve summation over levels of the factor and hence are not probabilities. A more appropriate sufficient condition for (2.3) to hold for this model is obtained by replacing $p$'s by $\tau$'s and "o" subscripts by "s"'s; in doing this, we understand that quantities like $\tau_{hij*}$ (which replaces $p_{hijo}$) are not necessarily probabilities.

Now let us consider the nature of second order interactions in this model. We first observe that (2.5), (2.6), (2.7) are meaningless here because they involve quantities like $p_{hijo}$. On the other hand, if we ignore (say) the first response and consider the marginal probability $p_{oijk}$ of the marginal response $(., i, j)$ within the $k$th category of the factor, then
\[ \lambda_{ijk} = \frac{P_{ijkP_{stk}}}{P_{osjkP_{otk}}} \quad i = 1, 2, \ldots, (s-1) \]
\[ j = 1, 2, \ldots, (t-1) \]
for \( k = 1, 2, \ldots, u \)

may be taken as a measure of association between the second and third response (ignoring the first response) within the \( k \)th category of the factor. Hence,

\[ \lambda_{ijk} = \lambda_{ij}^* \quad i = 1, 2, \ldots, (s-1) \]
\[ j = 1, 2, \ldots, (t-1) \]
for \( k = 1, 2, \ldots, u \)

may be regarded as a formulation of the hypothesis of "no second order interaction" between the second and third responses and the factor.

Another type of hypothesis of interest is (2.9). Here, however, it is interpreted as saying that within each category of the factor, the measure of association \( \Delta_{hijk} \) (defined by (2.1)) between the first two responses does not depend on the level of the third response; i.e., within each category of the factor, there is no interaction among the three responses. Note again that (2.9) is a special case of (2.3) in which \( \lambda_{ij}^* = 1 \). Finally a special case of (2.9) which may be of additional interest is

\[ \Delta_{hijk} = \Delta_{hij}^{**} \quad h = 1, 2, \ldots, (r-1) \]
\[ i = 1, 2, \ldots, (s-1) \]
\[ j = 1, 2, \ldots, t \]
\[ k = 1, 2, \ldots, u \]

i.e., this measure of association between the first two responses is independent of the third response and the factor.

To carry these ideas one step further, we want to deal with first order interactions. Again (2.10), (2.11), (2.12), (2.13) are all meaningless here because they involve quantities like \( p_{hijo} \) in which the factor has been summed over. However, if we ignore (say) the first two responses and
consider the marginal probability $p_{oojk}$ of the marginal response $(.,.,j)$
within the k th level of the factor, then the statement

$$p_{oojk} \text{ is independent of } k \quad j = 1, 2, \ldots, (t-1)$$
$$k = 1, 2, \ldots, u$$

may be regarded as a formulation of the hypothesis of "no first order
interaction" between the third response and the factor; the hypothesis
(2.20) may also be interpreted as saying that the marginal distribution
of the third response does not depend on the factor.

If we let

$$\Delta_{ijk} = \frac{p_{oijk}}{p_{osjk}} \quad i = 1, 2, \ldots, (s-1)$$
$$j = 1, 2, \ldots, t$$
$$k = 1, 2, \ldots, u$$

then

$$\Delta_{ijk} = \Delta_{i*kJ} \quad i = 1, 2, \ldots, (s-1)$$
$$j = 1, 2, \ldots, t$$
$$k = 1, 2, \ldots, u$$

is a formulation of the hypothesis that within each of the factor categories,
there is no (first order) association between the second and third responses;
i.e., within each factor category, the second and third responses are
independent. Note that (2.22) is equivalent to the statement

$$p_{oijk} = p_{oiok}p_{oojk} \quad i = 1, 2, \ldots, (s-1)$$
$$j = 1, 2, \ldots, (t-1)$$
$$k = 1, 2, \ldots, u$$

and is also a special case of (2.18) in which $\lambda_{ij*} = 1.$
2.3. The "Two Response, Two Factor" Model

Let $p_{hijk}$ (where we assume $p_{hijk} > 0$) denote the probability that an experimental unit from the $(j, k)$ th combination of factor categories belongs to the response $(h, i)$ in an $r \times s \times t \times u$ contingency table; thus, the $p$'s are subject to the $(tu)$ constraints

$$\sum_{h,i} p_{hijk} = 1 \quad j = 1, 2, \ldots, t \quad k = 1, 2, \ldots, u.$$  

Let us regard $\Delta_{hijk}$ (as defined in (2.1)) as a measure of association between the two responses within the $(j, k)$ th combination of factor categories. We will say that there is "no third order interaction" between two responses and two factors if the nature of the dependence of $\Delta_{hijk}$ on the first factor is constant over levels of the second factor. A multiplicative formulation of such a hypothesis is

$$\Delta_{hijk} = \tau_{hij} \tau_{hik} \quad h = 1, 2, \ldots, (r-1) \quad i = 1, 2, \ldots, (s-1) \quad j = 1, 2, \ldots, t \quad k = 1, 2, \ldots, u;$$

however, (2.25) is equivalent to (2.3) where here the $p_{hijk}$ also satisfy (2.24) and where $\lambda_{hijk}$ is interpreted as a measure of the dependence of $\Delta_{hijk}$ on the first factor within the $k$ th level of the second factor.

Also of interest is the formulation (2.9). Here, this is interpreted as saying that within each category of the second factor, the measure of association $\Delta_{hijk}$ between the two responses is independent of the first factor. Similarly, we could consider (2.19) which here means that the measure of association $\Delta_{hijk}$ between the two responses is independent of both of the factors. Each of these two hypotheses are, of course, special cases of (2.25).
We could have formulated the hypothesis of "no third order interaction" in terms of another measure of association between the two responses within the (j, k) th factor combination; for example,

\[ \phi_{hijk} = \frac{p_{hijk}}{p_{hojk}p_{oijk}} \quad h = 1, 2, \ldots, (r-1) \]

\[ i = 1, 2, \ldots, (s-1) \]

\[ j = 1, 2, \ldots, t \]

\[ k = 1, 2, \ldots, u \]

The corresponding hypothesis of "no third order interaction" is

\[ \phi_{hijk} = \tau_{hij} \tau_{hi} \tau_{ik} \quad h = 1, 2, \ldots, (r-1) \]

\[ i = 1, 2, \ldots, (s-1) \]

\[ j = 1, 2, \ldots, t \]

\[ k = 1, 2, \ldots, u \]

Note that (2.27) implies (2.3) where here, of course, the \( p \)'s satisfy the constraints (2.24).

Let us now consider the nature of lower order interactions in this model. As before (2.5), (2.6), (2.7) are meaningless here for the same reasons as given in Sub-section 2.2. Suppose we ignore (say) the first response and consider the marginal probability \( p_{oijk} \) of the marginal response \( (., i) \) within the (j, k) th combination of factor categories. We are in essentially a "one response, two factor" situation; hence, an "additive" formulation of the hypothesis of "no second order interaction" between two factors and the (marginal distribution of the) second response is

\[ p_{oijk} = \tau_{i} + \tau_{ij} + \tau_{ik} \quad i = 1, 2, \ldots, (s-1) \]

\[ j = 1, 2, \ldots, t \]

\[ k = 1, 2, \ldots, u \]

or equivalently

\[ p_{oijk} - p_{oikt} = \tau_{ij} \quad i = 1, 2, \ldots, (s-1) \]

\[ j = 1, 2, \ldots, (t-1) \]

\[ k = 1, 2, \ldots, u \]
similarly, a "multiplicative" formulation would be

\begin{equation}
(2.30) \quad p_{i,j,k} = \tau_{i,j,k} \tau_{i,k} \quad i = 1, 2, \ldots, (s-1)
\end{equation}

\text{for} \quad j = 1, 2, \ldots, t
\quad k = 1, 2, \ldots, u

or equivalently

\begin{equation}
(2.31) \quad \frac{p_{i,j,k}}{p_{i,t,k}} = a_{i,j,k} \quad i = 1, 2, \ldots, (s-1)
\end{equation}

\text{for} \quad j = 1, 2, \ldots, (t-1)
\quad k = 1, 2, \ldots, u

Also, hypotheses of the above type could be considered for both the marginal distribution of the first response and the marginal distribution of the second response together. For example, a "multiplicative" formulation of such a hypothesis is given by

\begin{equation}
(2.32) \quad p_{h,j,k} = \tau_{h,j,k} \tau_{h,k} \quad h = 1, 2, \ldots, (r-1)
\end{equation}

\begin{equation}
(2.33) \quad p_{i,j,k} = \tau_{i,j,k} \tau_{i,k} \quad i = 1, 2, \ldots, (s-1)
\end{equation}

\text{for} \quad j = 1, 2, \ldots, t
\quad k = 1, 2, \ldots, u

Note that this is a hypothesis of simultaneous vanishing of two second order interactions.

For this situation, the definitions of first order interaction given in the previous sub-sections have no meaningful interpretation. We can, however, consider the statement

\begin{equation}
(2.33) \quad p_{i,j,k} \text{ is independent of } k \quad i = 1, 2, \ldots, (s-1)
\end{equation}

\text{for} \quad j = 1, 2, \ldots, t
\quad k = 1, 2, \ldots, u

i.e., within each category of the first factor, the marginal distribution of the second response does not depend on the second factor. Note that

\begin{equation}
(2.33) \text{ is a special case of (2.28) with } \tau_{i,k} = 0, \text{ a special case of (2.30) with } \tau_{i,k} = 1, \text{ or more generally a special case of any reasonable formulation of the hypothesis of "no second order interaction"}
\end{equation}
between the two factors and the second response.

Let us now assume that there is "no second order interaction" between the two factors and the second response; for example, let us assume that (2.28)(or (2.30)) holds. Then we shall say that there is "no first order interaction" between the second factor and the second response if the conditional formulation

(2.34) \[(2.33) \text{ given } (2.28)\]

(or

(2.35) \[(2.33) \text{ given } (2.30)\]

holds. This definition is consistent with those given in the previous sub-sections since the relationship between the formulations (2.28)(or (2.30)), (2.34)(or (2.35)), and (2.33) is similar to the relationship between the formulations (2.5), (2.11), and (2.12) and between the formulations (2.18), (2.20), and (2.23).

2.4. The "One Response, Three Factor" Model

Let \( p_{hijk} \) (where we assume \( p_{hijk} > 0 \)) denote the probability that an experimental unit from the \((i, j, k)\) th combination of factor categories belongs to the \(h\) th category of the response in an \(r \times s \times t \times u\) contingency table; thus the \(p\)'s are subject to the \((stu)\) constraints

\[
(2.36) \quad \sum_{h} p_{hijk} = 1 \quad \text{for } i = 1, 2, \ldots, s \quad j = 1, 2, \ldots, t \quad k = 1, 2, \ldots, u
\]

We shall say that there is "no third order interaction" between one response and three factors if the dependence of the (distribution of the) response on any two of the factors is constant over the levels of the remaining factor. An "additive" formulation of such a hypothesis is
(2.37) \[ p_{hijk} = \tau_{hij} + \tau_{hik} + \tau_{hjk} \]
\[ \text{for } j = 1, 2, \ldots, t \]
\[ \text{for } k = 1, 2, \ldots, u \]
which is equivalent to

(2.38) \[ p_{hijk} - p_{hsk} - p_{hjk} + p_{hsj} = \beta_{hij} \]
\[ \text{for } k = 1, 2, \ldots, u \]
Similarly, a "multiplicative" formulation is

(2.39) \[ p_{hijk} = \tau_{hij} \tau_{hik} \tau_{hjk} \]
\[ \text{for } j = 1, 2, \ldots, s \]
\[ \text{for } k = 1, 2, \ldots, u \]
which is equivalent to

(2.40) \[ \frac{p_{hijk} p_{hsk}}{p_{hsk} p_{hjt}} = a_{hij} \]
\[ \text{for } k = 1, 2, \ldots, u \]

Note that the formulations of hypotheses of "no third order interaction" for this model are different from (2.3) which was appropriate for each of the previous ones; in fact, (2.40) is a special case of (2.3) in which

\[ \lambda_{hijk} = \frac{a_{hij}}{a_{rij}} = \lambda_{hij} \]

For this situation, the definitions of second order interaction given in the previous sub-sections have no meaningful interpretation. However, we can consider formulations of the following type

(2.41) \[ p_{hijk} = \tau_{hij} + \tau_{hik} \]

15
or

\[ \Phi_{hijk} = \tau_{hij} \star \tau_{hi^*k} \]

Each of these is interpreted as saying that within each category of the first factor, there is "no second order interaction" between the response and the second and third factors. Also, (2.41) is a special case of (2.37) with \( \tau_{h^*jk} = 0 \) while (2.42) is a special case of (2.39) with \( \tau_{h^*jk} = 1 \).

Let us now assume that there is "no third order interaction" between the response and the three factors; for example, let us assume that (2.37) (or (2.39)) holds. Then, we shall say that there is "no second order interaction" between the response and the second and third factors if the conditional formulation

\[ (2.43) \quad (2.41) \text{ given } (2.37) \]

(or

\[ (2.44) \quad (2.42) \text{ given } (2.39) \]

holds. This definition is consistent with those given in the previous sub-sections since the relationship between the formulations (2.37) (or (2.39)), (2.43) (or (2.44)), and (2.41) (or (2.42)) is similar to the relationship between the formulations (2.3), (2.5), and (2.9) and between (2.3), (2.18), and (2.9). Hence, given that there is "no third order interaction" between the response and the three factors, there is "no second order interaction" between the response and the second and third factors if those factors affect the (distribution of the ) response independently (within each level of the first factor).

Continuing in this manner, we can consider the hypothesis

16
\[(2.45) \quad p_{hijk} \text{ is independent of } k \quad ;\]
i.e., within each combination of categories of the first two factors, the
distribution of the response does not depend on the third factor. Note
that \((2.45)\) is a special case of \((2.41)\) with \(\tau_{hi*k} = 0\) and a special case
of \((2.42)\) with \(\tau_{hi*k} = 1.\)

Let us assume that

(i). there is "no third order interaction" between the response and
the three factors,

(ii). there is "no second order interaction" between the response
and the second and third factors;

for example, let us assume that \((2.41)\) (or \((2.42)\)) holds. Then, we shall
say that there is "no first order interaction" between the response and
the third factor if the conditional formulation

\[(2.46) \quad (2.45) \text{ given } (2.41)\]

(or

\[(2.47) \quad (2.45) \text{ given } (2.42)\])

holds. As has been noted previously, this type of definition is consistent
with previous formulations of the hypothesis of "no first order interaction".
3. Test Criteria

Here, we shall use a criterion due to Wald [1943] to obtain test statistics for some of the hypotheses considered in the previous section. For a more complete discussion of Wald's procedure as it applies to multidimensional contingency tables, the reader is referred to Bhapkar [1965], Bhapkar and Koch [1965].

3.1. The "Four Response, No Factor" Model

Let $n_{hijk}$ denote the frequency of response $(h, i, j, k)$; let

$$q_{hijk} = \frac{n_{hijk}}{N}, \text{ where } N = \sum_{h,i,j,k} n_{hijk},$$

denote the unrestricted maximum likelihood estimator of $p_{hijk}$.

a. Hypothesis (2.3) of no third order interaction among four responses (with p's subject to $\sum_{h,i,j,k} p_{hijk} = 1$):

For $k = 1, 2, \ldots, u$, let

$$\chi_{hijk} = \frac{q_{hijk} q_{rsjk} q_{ritk} q_{nstk}}{q_{rijk} q_{hsjk} q_{hitk} q_{rstk}} \quad h = 1, 2, \ldots, (r-1)\quad i = 1, 2, \ldots, (s-1)\quad j = 1, 2, \ldots, (t-1)$$

$$= \frac{n_{hijk} n_{rsjk} n_{ritk} n_{nstk}}{n_{rijk} n_{hsjk} n_{hitk} n_{rstk}},$$

$$\chi_{k}^t = [\chi_{111k}^t, \chi_{211k}^t, \ldots, \chi_{(r-1)11k}^t, \ldots, \chi_{(r-1)(s-1)(t-1)k}^t]$$

and let $\chi_k$ denote the "sample covariance matrix" of $\chi_k$ (that is, the consistent estimate of the covariance matrix of $\chi_k$ obtained by replacing p's by q's in the covariance matrix of the linear parts of the Taylor expansions corresponding to their components). It can be easily verified that to the first order of approximation for large $N$, $\chi_k$ and $\chi_k^t$ are uncorrelated while we have for elements of $\chi_k$ terms of the type
\[
\text{Var}(y_{hijkl}) = y_{hijkl}^2 \left( \frac{1}{n_{hijkl}} + \frac{1}{n_{rsjk}} + \frac{1}{n_{ritk}} + \frac{1}{n_{hstk}} + \frac{1}{n_{rijk}} + \frac{1}{n_{hsjk}} + \frac{1}{n_{hitk}} + \frac{1}{n_{rstk}} \right)
\]

\[
\hat{\text{Cov}}(y_{hijkl}, y_{hij'k}) = y_{hijkl}y_{hij'k} \left( \frac{1}{n_{ritk}} + \frac{1}{n_{hstk}} + \frac{1}{n_{hitk}} + \frac{1}{n_{rstk}} \right)
\]

\[
\hat{\text{Cov}}(y_{hijkl}, y_{hi'j'k}) = y_{hijkl}y_{hi'j'k} \left( \frac{1}{n_{hstk}} + \frac{1}{n_{rstk}} \right)
\]

\[
\hat{\text{Cov}}(y_{hijkl}, y_{h'i'j'k}) = y_{hijkl}y_{h'i'j'k} \left( \frac{1}{n_{rstk}} \right)
\]

with \(j' \neq j\), \(i' \neq i\) and \(h' \neq h\) and ^ denoting the estimate. The test statistic \(\chi^2_y\) is given by

\[
\chi^2_y = \sum_{k=1}^{u} y_k^2 \left( \sum_{k=1}^{u} y_k^{-1} \right) \chi_k - \chi \left( \sum_{k=1}^{u} y_k^{-1} \right) \chi_k
\]

\[\text{d.f.} = (r-1)(s-1)(t-1)(u-1)\]

where

\[
\chi = \left( \sum_{k=1}^{u} y_k^{-1} \right)^{-1} \left( \sum_{k=1}^{u} y_k^{-1} y_k \right).
\]

Special cases:

(i) \(r = s = t = 2\)

Let \(y_k = y_{111k}\) and

\[
y_k^2 = y_k^2 \left( \frac{1}{n_{111k}} + \frac{1}{n_{112k}} + \frac{1}{n_{121k}} + \frac{1}{n_{211k}} + \frac{1}{n_{122k}} + \frac{1}{n_{212k}} + \frac{1}{n_{221k}} + \frac{1}{n_{222k}} \right)
\]

Then
\[ x^2 = \sum_{k=1}^{u} y_k^{-1} y_k^2 - \left( \sum_{k=1}^{u} y_k^{-1} y_k \right)^2 \left/ \sum_{k=1}^{u} y_k^{-1} \right. \],

d.f. = u-1.

(ii) \( r = s = t = u = 2 \)

\[ x^2 = \frac{(y_1 - y_2)^2}{y_1 + y_2} , \quad \text{d.f.} = 1. \]

Note here that the test of (2.3) can be carried out alternatively by using the Wald statistic for the formulation obtained by taking natural logarithms of both sides of (2.3). Let

\[ g_{hijk} = \log y_{hijk} , \]

\[ \mathbf{g} = \mathbf{g}_{111k} , \ldots , g_{(r-1)(s-1)(t-1)k} \]

and let \( \mathbf{V}_{\mathbf{g}} \) be the "sample covariance matrix" of \( \mathbf{g} \); it can be verified easily that \( \mathbf{V}_{\mathbf{g}} \) can be obtained from the expressions given above for elements of \( \mathbf{V}_{\mathbf{y}} \) by delating the y's. The alternative test statistic is then

\[ x^2 = \sum_{k=1}^{u} g_k \mathbf{g}^t \mathbf{g}_k^{-1} g_k - g_k \left( \sum_{k=1}^{u} g_k \mathbf{g}_k^{-1} \right) g_k , \]

d.f. = (r-1)(s-1)(t-1)(u-1)

where

\[ g_k = \left( \sum_{k=1}^{u} g_k^{-1} \right)^{-1} \left( \sum_{k=1}^{u} g_k^{-1} g_k \right) ; \]

for the special cases mentioned above we simply replace y's by g's and take

\[ \mathbf{V}_g = \mathbf{V}_y / y_k^2 . \]

\[ x^2_y \quad \text{and} \quad x^2_g , \quad \text{though not identical, are asymptotically equivalent.} \]
b. Hypothesis (2.5) of no second order interaction among the first
three responses:

This hypothesis is really associated with the "three response, no
factor" situation in a three-dimensional contingency table. Test statistics
are given, for example, in Bnupkar and Koch [1965]. These statistics have
\((r-1)(s-1)(t-1)\) degrees of freedom.

c. Hypothesis (2.9) of no second order interaction among the first three
responses for each level of the fourth response:

For \(j = 1, 2, \ldots, t\) and \(k = 1, 2, \ldots, u\), let

\[
d_{hijk} = \frac{q_{hijk} q_{rsjk}}{q_{rij} q_{sijk}} = \frac{n_{hijk} n_{rsjk}}{n_{rij} n_{sijk}} \quad h = 1, \ldots, (r-1)
\]

\[
d_{-ijk} = \begin{bmatrix} d_{1ljk} & \ldots & d_{(r-1)ljk} \\ \vdots & \ddots & \vdots \\ d_{(r-1)(s-1)jk} & \ldots & d_{(r-1)(s-1)(t-1)jk} \end{bmatrix}
\]

and let \(\tilde{\Sigma}_{d_{jk}}\) denote the "sample covariance matrix" of \(d_{ijk}\). Again it can
be easily verified that to the first order of approximation for large \(N\)
\(d_{ijk}\)'s are uncorrelated while for the elements of \(\tilde{\Sigma}_{d_{jk}}\), we have terms of
the type

\[
\hat{\text{Var}}(d_{hijk}) = d_{hijk}^2 \left( \frac{1}{n_{hijk}} + \frac{1}{n_{rsjk}} + \frac{1}{n_{rij}} + \frac{1}{n_{sijk}} \right)
\]

\[
\hat{\text{Cov}}(d_{hijk}, d_{hi'jk}) = d_{hijk} d_{hi'jk} \left( \frac{1}{n_{rsjk}} + \frac{1}{n_{sijk}} \right)
\]

\[
\hat{\text{Cov}}(d_{hijk}, d_{hi'jk}) = d_{hijk} d_{hi'jk} \left( \frac{1}{n_{sijk}} \right)
\]

with \(i' \neq i\) and \(h' \neq h\). The test statistic \(X_d^2\) is given by
\[ x_d^2 = \sum_{k=1}^{u} \sum_{j=1}^{t} d_{jk}^{-1} d_{jk}^{-1} - \sum_{k=1}^{u} d_{jk} \left( \sum_{j=1}^{t} d_{jk}^{-1} \right) \sim \chi^2_{d.f. = (r-1)(s-1)(t-1)} \]

where

\[ d_{jk} \sim \left( \sum_{j=1}^{t} d_{jk}^{-1} \right)^{-1} \left( \sum_{j=1}^{t} d_{jk}^{-1} d_{jk} \right). \]

Special cases:

(i) \( r = s = 2 \)

Let \( d_{jk} = d_{11jk} \) and

\[ d_{jk} = d_{11jk}^2 \left( \frac{1}{n_{11jk}} + \frac{1}{n_{12jk}} + \frac{1}{n_{21jk}} + \frac{1}{n_{22jk}} \right). \]

Then

\[ x_d^2 = \sum_{k=1}^{u} \sum_{j=1}^{t} d_{jk}^2 \left( \frac{1}{n_{11jk}} + \frac{1}{n_{12jk}} + \frac{1}{n_{21jk}} + \frac{1}{n_{22jk}} \right), \]

\[ d.f. = (t-1)u. \]

(ii) \( r = s = t = 2 \)

\[ x_d^2 = \frac{\sum_{k=1}^{u} (d_{1k} - d_{2k})^2}{(d_{1k} + d_{2k})}, \text{ d.f. = } u. \]

As in (a), the tests can be carried out alternatively in terms of logarithms by letting

\[ e_{hijk} = \log d_{hijk}, \]

\[ e'_{ijk} = [e_{11jk}, \ldots, e_{(r-1)(s-1)jk}]. \]
and obtaining $V_e$ from $V_d$ by deleting the $d_i's$. The alternate test statistics

$\chi^2_e$ are defined by expressions similar to those of $\chi^2_d$ obtained from

replacing $d_i's$ by $e_i's$ and $V_d$ by $V_e$. For the special case $r=s=2$, in

particular, note that $e_{jk} = \log d_{jk}$ and $V_{e_{jk}} = V_{d_{jk}} / d_{jk}^2$

Again $\chi^2_d$ and $\chi^2_e$, though not identical, are asymptotically equivalent.

Statistics for hypotheses of type (2.11) and (2.13) involving lower

order interactions are fairly well-known and hence are deleted.

3.2. The "Three Response, One Factor" Model

Let $n_{hijk}$ denote the frequency of response $(h,i,j)$ by subjects from

the factor category $k$; let $q_{hijk} = (n_{hijk} / n_{oook})$, where $n_{oook} = \sum_{h,i,j} n_{hijk}$,

de note the unrestricted maximum likelihood estimator of $p_{hijk}$.

a. Hypothesis (2.3) of no third order interaction among three responses

and one factor (with $p_i's$ here subject to $\sum_{h,i,j} p_{hijk} = 1$):

The statistics defined in (a) of Sub-section 3.1 can be used.

b. Hypothesis (2.18) of no second order interaction between the second

and third responses and the factor:

This hypothesis is really associated with the "two response, one

factor" situation in a three-dimensional contingency table. Test statistics

are given, for example, in Bhapkar and Koch [1965]; these have $(s-1)(t-1)(u-1)$

degrees of freedom.

c. Hypothesis (2.9) of no second order interaction among the three responses

at each factor level:

Statistics defined in (c) of Sub-section 3.1 can be used.

d. Hypothesis (2.19) that the association between the first two responses
is independent of the third response and the factor:

Let $d_{jk}^o$ and $V_{jk}^o \sim d_{jk}^o$ be defined as in (c) of Sub-section 3.1. The test statistic $X^2_{d_0}$ is given by

$$X^2_{d_0} = \sum_{k=1}^{u} \sum_{j=1}^{t} d_{jk}^o \sim d_{jk}^o - d_{jk}^o(\sum_{k=1}^{u} \sum_{j=1}^{t} V_{jk}^{-1}) d_{jk}^o,$$

$$d.f. = (r-1)(s-1)(t_u-1)$$

where

$$d = (\sum_{k=1}^{u} \sum_{j=1}^{t} V_{jk}^{-1})^{-1} (\sum_{k=1}^{u} \sum_{j=1}^{t} V_{jk}^{-1} d_{jk}^o).$$

Comments similar to those at the end of (c) of Sub-section 3.1 apply here.

Statistics for hypotheses of type (2.20) and (2.22) involving lower order interactions are fairly well-known and hence are deleted.

3.3. The "Two Response, Two Factor" Model

Let $n_{hijk}$ denote the frequency of response $(h,i)$ by subjects from the factor combination $(j,k)$; let $q_{hijk} = (n_{hijk}/n_{oojk})$, where $n_{oojk} = \sum_{h,i} n_{hijk}$, denote the unrestricted maximum likelihood estimator of $p_{hijk}$.

a. Hypothesis (2.25) of no third order interaction between the two responses and the two factors, or equivalently, (2.3) with $p$'s subject to

$$\sum_{h,i} p_{hijk} = 1.$$

The statistics in (a) of Sub-section 3.1 can be used.

b. Hypothesis (2.9) that within each category of the second factor, the association between the two responses is independent of the first factor:

The statistics in (c) of Sub-section 3.1 can be used.
c. Hypothesis (2.19) that the association between the two responses is independent of the factor combination:

The statistics in (d) of Sub-section 3.2 can be used.

d. Hypothesis (2.30)(or (2.28)) of no second order interaction between the two factors and the second response:

This has been discussed in the "one response, two factor" situation in the three-dimensional contingency table. Test statistic, say $\chi^2_m$, is given in Bhapkar and Koch [1965]; this has $(s-1)(t-1)(u-1)$ degrees of freedom.

e. Hypothesis (2.33) that the marginal distribution of the second response is independent of the second factor (within each category of the first factor):

The Pearson - $\chi^2$ statistic is easily seen to be

$$\sum_{k=1}^{u} \sum_{j=1}^{t} \sum_{i=1}^{s} \left( \frac{n_{oijk}}{n_{oij}} - \frac{n_{oiojk}}{n_{ooj}} \right)^2 / \left( \frac{n_{oiojk}}{n_{ooj}} \right),$$

d.f. = $(s-1)(t(u-1))$;

this is simpler to obtain than the Wald statistic (or, equivalently, the Neyman - $X^2_1$ statistic as shown by Bhapkar [1965]) which is, say $X^2_f$, given by

$$X^2_f = \sum_{k=1}^{u} \sum_{j=1}^{t} \sum_{i=1}^{s} \left( n_{oijk} - n_{oojk} \hat{p}_{oijk} \right)^2 / n_{oijk},$$

where $\hat{p}_{oijk} = (\bar{q}_{i} \bar{q}_{o} / \bar{q}_{oij})$, $\bar{q}_{oij} = \sum_{i=1}^{s} \bar{q}_{i} \bar{q}_{o}ij$

and

$$n_{ooij} = \sum_{k=1}^{u} n_{oiojk} / q_{oij}.$$

$X^2_f$ is seen to simplify to
\[
X^2_f = \sum_{j=1}^{t} \frac{n_{ooj}}{\bar{a}_{oj}} - N,
\]

where \( N = \sum_{j=1}^{t} n_{ooj} \). \( X^2_f \) also has \((s-1)t(u-1)\) degrees of freedom.

f. Hypothesis (2.35)(or (2.34)) of no first order interaction between the second factor and the second response:

The statistic is \( X^2_f - X^2_m \) with \((s-1)(u-1)\) degrees of freedom (with \( X^2_m, X^2_f \) mentioned in (d) and (e) above).

3.4. The "One Response, Three Factor" Model

Let \( n_{hijk} \) denote the frequency of response \( h \) by subjects from the factor combination \((i, j, k)\); let \( q_{hijk} = (n_{hijk}/n_{oijk}) \), where \( n_{oijk} = \sum_{h} n_{hijk} \) denote the unrestricted maximum likelihood estimator of \( p_{hijk} \).

a. Additive formulation (2.37) of the hypothesis of no third order interaction between one response and three factors (with \( p \)'s subject to \( \sum_{h} p_{hijk} = 1 \)):

For \( k = 1, 2, \ldots, u, \) let

\[
b_{hijk} = q_{hijk} - q_{hsjk} - q_{hitk} + q_{hstk}, \quad h = 1, \ldots, (r-1) \]
\[
i = 1, \ldots, (s-1) \]
\[
j = 1, \ldots, (t-1) \]

\[
b_k = [b_{11lk}, \ldots, b_{(r-1)(s-1)(t-1)k}] \]

and let \( V_k \) denote the "sample covariance matrix" of \( b_k \). Note that \( b_k \)'s are mutually uncorrelated and elements of \( V_k \) are of the type.
\[ \text{Var}(b_{hijk}) = \frac{q_{hij}(1-q_{hij})}{n_{oijk}} + \frac{q_{hsijk}(1-q_{hsijk})}{n_{osjk}} + \frac{q_{hitk}(1-q_{hitk})}{n_{oitk}} + \frac{q_{hst}(1-q_{hst})}{n_{ostk}} \]

\[ \text{Cov}(b_{hijk}, b_{h'i'j'k}) = -\left( \frac{q_{hij}q_{h'i'j'k}}{n_{oijk}} + \frac{q_{hsijk}q_{h'i'sjk}}{n_{osjk}} + \frac{q_{hitk}q_{h'i'itk}}{n_{oitk}} + \frac{q_{hst}q_{h'i'stk}}{n_{ostk}} \right) \]

\[ \text{Cov}(b_{hijk}, b_{h'i'j'k}) = \frac{q_{hsijk}(1-q_{hsijk})}{n_{osjk}} + \frac{q_{hst}(1-q_{hst})}{n_{ostk}} \]

\[ \text{Cov}(b_{hijk}, b_{h'i'j'k}) = -\left( \frac{q_{hsijk}q_{h'i'sik}}{n_{osjk}} + \frac{q_{hst}q_{h'i'stk}}{n_{ostk}} \right) \]

\[ \text{Cov}(b_{hijk}, b_{h'i'j'k}) = \frac{q_{hst}(1-q_{hst})}{n_{ostk}} \]

\[ \text{Cov}(b_{hijk}, b_{h'i'j'k}) = -\frac{q_{hst}q_{h'i'stk}}{n_{ostk}} \]

with \( h' \neq h, i' \neq i \) and \( j' \neq j \).

The test statistic is \( X^2_b \) given by

\[ X^2_b = \sum_{k=1}^{u} \frac{b_k}{\tilde{b}_k} \tilde{b}_k^{-1} b_k - b'(\sum_{k=1}^{u} \tilde{b}_k^{-1}) b \]

\[ \text{d.f.} = (r-1)(s-1)(t-1)(u-1) \]

where

\[ b = (\sum_{k=1}^{u} \tilde{b}_k^{-1})^{-1} \left( \sum_{k=1}^{u} \tilde{b}_k^{-1} b_k \right). \]
Special cases:

(i) \( r = s = t = 2 \)

Let \( b_k = b_{111k} \) and

\[
V_{b_k} = \frac{q_{111k}q_{211k}}{n_{011k}} + \frac{q_{121k}q_{221k}}{n_{021k}} + \frac{q_{112k}q_{212k}}{n_{012k}} + \frac{q_{122k}q_{222k}}{n_{022k}}.
\]

Then

\[
X_b^2 = \sum_{k=1}^{u} \frac{V_{b_k}^{-1} b_k^2}{(\sum_{k=1}^{u} V_{b_k}^{-1} b_k)^2/(\sum_{k=1}^{u} V_{b_k}^{-1})},
\]

d.f. = \( u - 1 \).

(ii) \( r = s = t = u = 2 \)

\[
X_b^2 = (b_1 - b_2)^2/(V_{b_1} + V_{b_2}), \text{ d.f. } = 1.
\]

(b) Multiplicative formulation (2.39) of the hypothesis of no third order interaction between one response and three factors:

For \( k = 1, 2, \ldots, u, \) let

\[
a_{hijk} = \frac{q_{hijk} q_{nhtk}}{q_{hsjk} q_{nhtk}} \quad h = 1, \ldots, (r-1)
\]

\[
i = 1, \ldots, (s-1)
\]

\[
j = 1, \ldots, (t-1)
\]

and let \( \tilde{V}_a \) denote the "sample covariance matrix" of

\[
\tilde{a}_k = [a_{111k}, \ldots, a_{(r-1)(s-1)(t-1)k}]^T.
\]

Note that \( \tilde{a}_k \)'s are mutually uncorrelated and elements of \( \tilde{V}_a \) are of the type

\[
\text{Var}(a_{hijk}) = a_{hijk}^2 \left( \frac{1}{n_{hijk}} + \frac{1}{n_{hstk}} + \frac{1}{n_{hsjk}} + \frac{1}{n_{hitk}} - \frac{1}{n_{oijk}} - \frac{1}{n_{ostk}} - \frac{1}{n_{osjk}} - \frac{1}{n_{oitk}} \right)
\]

28
\[
\hat{\text{Cov}}(a_{hijk}', a_{hi'jk}) = -a_{hijk}a_{hi'jk} \left( \frac{1}{n_{oijk}} + \frac{1}{n_{ostk}} + \frac{1}{n_{osjk}} + \frac{1}{n_{oictk}} \right)
\]

\[
\hat{\text{Cov}}(a_{hijk}, a_{hi'jk}) = a_{hijk}a_{hi'jk} \left( \frac{1}{n_{hstk}} + \frac{1}{n_{hsjk}} - \frac{1}{n_{ostk}} - \frac{1}{n_{osjk}} \right)
\]

\[
\hat{\text{Cov}}(a_{hijk}, a_{hi'jk}) = -a_{hijk}a_{hi'jk} \left( \frac{1}{n_{ostk}} + \frac{1}{n_{osjk}} \right)
\]

\[
\hat{\text{Cov}}(a_{hijk}, a_{hi'jk}) = a_{hijk}a_{hi'jk} \left( \frac{1}{n_{hstk}} - \frac{1}{n_{ostk}} \right)
\]

\[
\hat{\text{Cov}}(a_{hijk}, a_{hi'jk}) = -\frac{a_{hijk}a_{hi'jk}}{n_{ostk}}
\]

with \( h' \neq h, i' \neq i \) and \( j' \neq j \).

The test statistic \( X_a^2 \) with \((r-1)(s-1)(t-1)(u-1)\) d.f. is defined as \( X_b^2 \) above with \( b \)'s replaced by \( a \)'s and \( \hat{\text{Cov}}_{b_k} \) replaced by \( \hat{\text{Cov}}_{a_k} \).

\( V_a \). In particular, for the special case \( r = s = t = 2 \), we have

\[
a_k = a_{111k}
\]

and

\[
V_{a_k} = a_k^2 \left( \frac{1}{n_{111k}} + \frac{1}{n_{112k}} + \frac{1}{n_{121k}} + \frac{1}{n_{122k}} + \frac{1}{n_{c11k}} + \frac{1}{n_{c12k}} + \frac{1}{n_{c21k}} + \frac{1}{n_{c22k}} \right).
\]

An asymptotically equivalent statistic corresponding to the logarithmic formulation of (2.39) is, say \( X_c^2 \), with

\[
c_{hijk} = \log a_{hijk},
\]

\[
\tilde{c}_k = [c_{111k}, \ldots, c_{(r-1)(s-1)(t-1)k}]
\]

29
and \( V \) obtained from \( V \) by deleting a's; \( X^2_c \) is defined in terms of \( c_k \) and \( V \) just as \( X^2_b \) is in terms of \( b_k \) and \( V \). Special cases follow along similar lines.

c. Additive formulation (2.41) of the hypothesis of no second order interaction between the response and the last two factors within each category of the first factor:

For \( i = 1, 2, \ldots, s \) and \( k = 1, 2, \ldots, u, \) let

\[
\hat{w}_{ijk} = q_{hijk} - q_{hitk} \quad h = 1, \ldots, (r-1) \quad j = 1, \ldots, (t-1)
\]

\[
\hat{w}_{ik} = [w_{iilk}, \ldots, w_{(r-1)i(t-1)k}]^T
\]

and let \( \hat{V}_{ik} \) be the "sample covariance matrix" of \( \hat{w}_{ik} \).

Note again that \( \hat{w}_{ik} \)'s are mutually uncorrelated and elements of \( \hat{V}_{ik} \) are of the type

\[
\text{Var}(\hat{w}_{ijk}) = \frac{q_{hijk}(1-q_{hijk})}{n_{oijk}} + \frac{q_{hitk}(1-q_{hitk})}{n_{oitk}}
\]

\[
\text{Cov}(\hat{w}_{ijk}, \hat{w}_{ijk}') = -\left( \frac{q_{hijk}q_{h'ijk}}{n_{oijk}} + \frac{q_{hitk}q_{hitk}'}{n_{oitk}} \right)
\]

\[
\text{Cov}(\hat{w}_{ijk}, \hat{w}_{ijk}) = \frac{q_{hitk}(1-q_{hitk})}{n_{oitk}}
\]

\[
\text{Cov}(\hat{w}_{ijk}, \hat{w}_{ijk}') = -\frac{q_{hitk}q_{hitk}'}{n_{oitk}}
\]

The statistic is \( X^2_w \) given by
\[ X^2_w = \sum_{k=1}^{u} \sum_{i=1}^{r} w_{i*k}^{1} \sum_{i}^{1} \sum_{k}^{1} w_{i*k}^{1} \sum_{i}^{1} \sum_{k}^{1} (u_{i*k}^{1} v_{i*k}^{1}) w_{i*k}^{1} \]

where

\[ w_{i*k}^{1} = (\sum_{k=1}^{u} v_{i*k}^{1})^{-1} (\sum_{k=1}^{u} w_{i*k}^{1} v_{i*k}^{1}) \]

Special cases:

(i) \( r = t = 2 \)

Here let \( w_{i*} = w_{il} \) and

\[ v_{i*} = \frac{q_{il} q_{2il}}{n_{o} l} + \frac{q_{i2k} q_{2i2k}}{n_{o} 2k} \]

(ii) \( r = t = u = 2 \)

\[ X^2_w = \sum_{i=1}^{s} (w_{i*1} - w_{i*2})^2 / (v_{w_{i*1}} + v_{w_{i*2}}), \text{ d.f. = s.} \]

d. Multiplicative formulation (2.42) of the hypothesis of no second-order interaction between the response and the last two factors within each category of the first factor:

This is very similar to the case (c) above with \( w \)'s replaced by \( z \)'s defined as

\[ z_{hijk} = q_{hijk} / q_{hitk} \]

\[ z_{i*k} = [ z_{ik} , \ldots, z_{(r-1)i(t-1)k} ] \]

and with elements in \( V_{i*k} \) of the type
\[ \text{Var}(z_{hijk}) = z_{hijk}^2 \left( \frac{1}{n_{hijk}} + \frac{1}{n_{hitk}} - \frac{1}{n_{cijk}} - \frac{1}{n_{oitk}} \right) \]

\[ \text{Cov}(z_{hijk}, z_{h'ijk}) = -z_{hijk}z_{h'ijk} \left( \frac{1}{n_{cijk}} + \frac{1}{n_{oitk}} \right) \]

\[ \text{Cov}(z_{hijk}, z_{h'ij'k}) = z_{hijk}z_{h'ij'k} \left( \frac{1}{n_{hitk}} - \frac{1}{n_{oitk}} \right) \]

\[ \text{Cov}(z_{hijk}, z_{h'ij'k}) = -z_{hijk}z_{h'ij'k} \left( \frac{1}{n_{oitk}} \right) \]

The statistic \( X_z^2 \) is defined as \( X_w^2 \) and has \( (r-1)s(t-1)(u-1) \) degrees of freedom. Special cases follow along the same lines.

An asymptotically equivalent statistic \( X_\xi^2 \) corresponding to the logarithmic formulation of (2.42) is also immediate if we take

\[ \varepsilon_{hijk} = \log z_{hijk} \]

and define \( V_{\sim \varepsilon_{i*}} \) from \( V_{\sim z_{i*}} \) by deleting \( z \)'s. Again special cases follow in the same manner.

e. Additive formulation (2.43) of no second order interaction among the response and the last two factors:

Statistic is \( X_w^2 - X_b^2 \) with \( (r-1)(t-1)(u-1) \) d.f. where \( X_w^2 \) and \( X_b^2 \) are defined in (c) and (a) above.

f. Multiplicative formulation (2.44) of no second order interaction among the response and the last two factors:

Statistic is \( X_z^2 - X_a^2 \) with \( (r-1)(t-1)(u-1) \) d.f. where \( X_z^2 \) may be used in place of \( X_a^2 \) and similarly \( X_c^2 \) in place of \( X_a^2 \); these are defined in (d) and (b) above.
Hypotheses involving lower order interactions can be tested along similar lines and the details are omitted.
4. Numerical Illustrations

In this section, we consider four contingency tables from the unpublished data of Lessler. The experiment was conducted to study the nature of sexual symbolism with the basic observation being a subject's classification of an object as being either masculine or feminine when that object is shown to him at the following exposure rates: 1/1000 second, 1/500 second, 1/100 second, 1/50 second, 1/5 second.

The subjects involved in the study were classified according to

(a) sex (males and females)

(b) those who were not told the purpose of the experiment and those who were told the purpose of the experiment (Group A and Group C respectively)

while the objects involved in it had been (by a previous study Lessler (1962)) assigned

(i) a cultural meaning (M or F) related to which sex used it

(ii) an anatomical meaning (M or F) related to which sex it inately evoked

(iii) an intensity (W or S) related to the strength of the degree to which the object was representative of its classification according to (i) and (ii).

Thus, as can be seen, the experiment is of the "five response, five factor type" with the factors being (a), (b), (i), (ii), (iii) above and the responses being the classifications at 1/1000, 1/500, 1/100, 1/50, 1/5 second.

The tables below are presented only to illustrate tests of the hypotheses of "no interaction" in various situations; no complete analysis
is attempted here. Also, an earlier analysis of parts of the same data
given here appears in Bhapkar and Koch \[1965]\.

I. A "Three Response, One Factor" Situation

Table 1: Subject Type - Males

Object Type - Culturally Masculine

Anatomically Feminine

Weak Intensity

<table>
<thead>
<tr>
<th>Subject Group</th>
<th>A</th>
<th>C</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>M F Sub-Total</td>
<td>M F Sub-Total</td>
<td></td>
</tr>
<tr>
<td>M</td>
<td>171 7 178</td>
<td>184 7 191</td>
<td>369</td>
</tr>
<tr>
<td></td>
<td>6 7 13</td>
<td>10 20 30</td>
<td>43</td>
</tr>
<tr>
<td>Sub-Total</td>
<td>177 14 191</td>
<td>194 27 221</td>
<td>412</td>
</tr>
<tr>
<td>F</td>
<td>18 7 25</td>
<td>38 7 45</td>
<td>70</td>
</tr>
<tr>
<td></td>
<td>12 56 68</td>
<td>14 114 128</td>
<td>196</td>
</tr>
<tr>
<td>Sub-Total</td>
<td>30 63 93</td>
<td>52 121 173</td>
<td>266</td>
</tr>
<tr>
<td>TOTAL</td>
<td>207 77 284</td>
<td>246 148 394</td>
<td>678</td>
</tr>
</tbody>
</table>

\[d_{ma} = 28.5000 \quad e_{ma} = 3.3499 \quad V_{e_{ma}} = 0.458229 \quad V_{d_{ma}} = 372.1965\]

\[d_{mc} = 52.5714 \quad e_{mc} = 3.9621 \quad V_{e_{mc}} = 0.298292 \quad V_{d_{mc}} = 824.4051\]

\[d_{fa} = 12.0000 \quad e_{fa} = 2.4849 \quad V_{e_{fa}} = 0.299603 \quad V_{d_{fa}} = 43.1428\]

\[d_{fc} = 44.2041 \quad e_{fc} = 3.7887 \quad V_{e_{fc}} = 0.249373 \quad V_{d_{fc}} = 487.2755\]

35
\[ y_a = 2.375000 \quad g_a = 0.8650 \quad V_{g_a} = 0.757832 \quad y_{a} = 4.274646 \]

\[ y_c = 1.189288 \quad g_c = 0.1734 \quad V_{g_c} = 0.547665 \quad y_{c} = 0.774621 \]

\[ y_a - y_c = 1.185712 \quad g_a - g_c = 0.6916 \quad V_{g_a} + V_{g_c} = 1.305497 \quad V_{y_a} + V_{y_c} = 5.049267 \]

1. \[ X^2_y = 0.278 \quad X^2_g = 0.366 \quad \text{d.f.} = 1 \]

Hence, the data are consistent with the hypothesis of "no third order interaction" between three responses and one factor.

\[ d_{ma} - d_{fa} = 16.5000 \quad e_{ma} - e_{fa} = 0.8650 \quad V_{e_{ma}} + V_{e_{fa}} = 0.757832 \quad V_{d_{ma}} + V_{d_{fa}} = 415.3393 \]

\[ d_{mc} - d_{fc} = 8.3673 \quad e_{mc} - e_{fc} = 0.1734 \quad V_{e_{mc}} + V_{e_{fc}} = 0.547665 \quad V_{d_{mc}} + V_{d_{fc}} = 1311.6806 \]

2. \[ X^2_d = 0.709 \quad X^2_e = 1.042 \quad \text{d.f.} = 2 \]

Hence, the data are consistent with the hypothesis of "no second order interaction" between the three responses within each subject group.

2. \[ d = 17.4798 \quad e = 3.4217 \]

\[ X^2_{d_o} = 3.982 \quad X^2_{e_o} = 4.459 \quad \text{d.f.} = 3 \]

Hence, the data are consistent with the hypothesis that the association between response at 1/5 sec. and response at 1/100 sec. is independent of response at 1/1000 sec. and subject group.

4. If we ignore classifications according to response at 1/100 second and consider the marginal frequencies obtained by summing over this response (i.e., over i), we obtain Table 3 as given in Bhapkar and Koch [1965].

There, the test statistic for the hypothesis of "no second order interaction" between two responses and one factor was

\[ X^2_i = 0.89 \quad \text{d.f.} = 1 \]
Hence, the data are consistent with the hypothesis of "no second order interaction" between the response at 1/1000 sec. and the response at 1/5 sec. and the factor subject group.

II. A "Two Response, Two Factor" Situation

Table 2: Subject Type - Males

Object Type - Culturally Masculine

Weak Intensity

<table>
<thead>
<tr>
<th>Subject Group</th>
<th>A</th>
<th>C</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>M</td>
<td>187</td>
<td>15</td>
<td>202</td>
</tr>
<tr>
<td>F</td>
<td>42</td>
<td>40</td>
<td>82</td>
</tr>
<tr>
<td>Sub-Total</td>
<td>229</td>
<td>55</td>
<td>284</td>
</tr>
</tbody>
</table>

| M | 177 | 14 | 191 | 194 | 27 | 221 | 412 |
| F | 30 | 63 | 93 | 52 | 121 | 173 | 266 |
| Sub-Total | 207 | 77 | 284 | 246 | 148 | 394 | 678 |

TOTAL | 436 | 132 | 568 | 536 | 252 | 788 | 1356 |

\[
\begin{align*}
\frac{d_{ma}}{e_{ma}} &= 11.8730 & \frac{e_{ma}}{V_{e_{ma}}} &= 2.474 & \frac{V_{e_{ma}}}{d_{ma}} &= 0.120822 & \frac{d_{ma}}{V_{d_{ma}}} &= 17.0321 \\
\frac{d_{mc}}{e_{mc}} &= 11.1148 & \frac{e_{mc}}{V_{e_{mc}}} &= 2.408 & \frac{V_{e_{mc}}}{d_{mc}} &= 0.073255 & \frac{d_{mc}}{V_{d_{mc}}} &= 9.0498 \\
\frac{d_{fa}}{e_{fa}} &= 26.5500 & \frac{e_{fa}}{V_{e_{fa}}} &= 3.279 & \frac{V_{e_{fa}}}{d_{fa}} &= 0.126283 & \frac{d_{fa}}{V_{d_{fa}}} &= 89.0172 \\
\frac{d_{fc}}{e_{fc}} &= 16.7194 & \frac{e_{fc}}{V_{e_{fc}}} &= 2.816 & \frac{V_{e_{fc}}}{d_{fc}} &= 0.069685 & \frac{d_{fc}}{V_{d_{fc}}} &= 19.4796
\end{align*}
\]
\[ y_a = 0.4472 \quad g_a = -0.805 \quad V_{g_a} = 0.247105 \quad V_{y_a} = 0.049418 \]

\[ y_c = 0.6648 \quad g_c = -0.408 \quad V_{g_c} = 0.142940 \quad V_{y_c} = 0.063174 \]

\[ y_c - y_a = 0.2176 \quad g_c - g_a = 0.397 \quad V_{g_a} + V_{g_c} = 0.390045 \quad V_{y_a} + V_{y_c} = 0.112592 \]

1. \[ \chi^2_y = 0.421 \quad \chi^2_g = 0.404 \quad \text{d.f.} = 1 \]

Hence, the data are consistent with the hypothesis of "no third order interaction" between two responses and two factors.

\[ d_{ma} - d_{fa} = -14.6770 \quad e_{ma} - e_{fa} = -0.805 \quad V_{e_{ma} e_{fa}} = 0.247105 \quad V_{d_{ma} d_{fa}} = 106.0493 \]

\[ d_{mc} - d_{fc} = -5.6046 \quad e_{mc} - e_{fc} = -0.408 \quad V_{e_{mc} e_{fc}} = 0.142940 \quad V_{d_{mc} d_{fc}} = 28.5294 \]

2. \[ \chi^2_d = 3.132 \quad \chi^2_e = 3.787 \quad \text{d.f.} = 2 \]

Hence, the data are consistent with the hypothesis of "no second order interaction" between the two responses and anatomical meaning within each subject group; i.e., within each subject group the (Δ) association between the two responses is independent of anatomical meaning.

3. \[ d = 13.2962 \quad e = 2.7089 \]

\[ \chi^2_{d_0} = 3.220 \quad \chi^2_{e_0} = 4.431 \quad \text{d.f.} = 3 \]

Hence, the data are consistent with the hypothesis that the (Δ) association between the responses is independent of the levels of each of the factors anatomical meaning and subject group.

4. If we ignore classifications according to response at 1/5 second and consider the marginal frequencies obtained by summing over this response (i.e., over h), we obtain Table 4 as given in Bhapkar and Koch [1965]. There, the test statistic for the "multiplicative" formulation of the hypothesis of
"no second order interaction" between one response and two factors was

\[ X^2_m = 9.82, \ d.f. = 1. \]

Hence, the "multiplicative" formulation of the hypothesis of "no second order
interaction" between the response at 1/1000 second and the factors subject
group and anatomical meaning is rejected.

5.

\[ \bar{q}_{mm} = 0.736786 \quad \hat{p}_{omm} = 0.738640 \]

\[ \bar{q}_{fm} = 0.260704 \quad \bar{q}_{om} = 0.997490 \quad \hat{p}_{ofm} = 0.261360 \]

\[ \bar{q}_{mf} = 0.602823 \]

\[ \bar{q}_{ff} = 0.384226 \quad \bar{q}_{of} = 0.987049 \quad \hat{p}_{off} = 0.389267 \]

\[ \sum_{j} \frac{n_{oij}}{\bar{q}_{oj}} = 1366.6021 \quad X^2_f = 10.6021 \quad d.f. = 2 \]

Hence, the hypothesis that \( p_{oijk} \) is independent of \( k \) is rejected; that is, within each category of anatomical meaning of an object, the response of a subject at 1/1000 sec. is not independent of the subject's group.
III. A "One Response, Three Factor" Situation

Table 3: Subject Type - Males

Object Type - Weak Intensity

<table>
<thead>
<tr>
<th>(k) Subject Group</th>
<th>A</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>M F</td>
<td>M F</td>
</tr>
<tr>
<td>(j) Response at 1/1000 Sec.</td>
<td>Sub-Total</td>
<td>Sub-Total</td>
</tr>
<tr>
<td>(i) Cultural Meaning</td>
<td></td>
<td></td>
</tr>
<tr>
<td>M</td>
<td>202 82</td>
<td>284</td>
</tr>
<tr>
<td>F</td>
<td>124 160</td>
<td>284</td>
</tr>
<tr>
<td>Sub-Total</td>
<td>326 242</td>
<td>568</td>
</tr>
<tr>
<td>F</td>
<td>191 93</td>
<td>284</td>
</tr>
<tr>
<td>F</td>
<td>60 224</td>
<td>284</td>
</tr>
<tr>
<td>Sub-Total</td>
<td>251 317</td>
<td>568</td>
</tr>
</tbody>
</table>

1. \( b_a = -0.1866 \quad V_{b_a} = 0.002951 \quad b_c - b_a = 0.0851 \quad \chi^2_{b_c} = 1.44 \)

\( b_c = -0.1015 \quad V_{b_c} = 0.002073 \quad V_{b_a} + V_{b_c} = 0.005024 \quad d.f. = 1 \)

Hence, the data are consistent with the "additive" formulation of the hypothesis of "no third order interaction" between one response and three factors.

2. \( w_{m*a} = 0.27465 \quad \hat{\Var}(w_{m*a}) = 0.001589 \quad \hat{\Var}(w_{m*a}) + \hat{\Var}(w_{f*a}) = 0.002951 \)

\( w_{m*c} = 0.29442 \quad \hat{\Var}(w_{m*c}) = 0.001099 \quad \hat{\Var}(w_{m*c}) + \hat{\Var}(w_{f*c}) = 0.002073 \)

\( w_{f*a} = 0.46127 \quad \hat{\Var}(w_{f*a}) = 0.001362 \quad \hat{\Var}(w_{f*a}) + \hat{\Var}(w_{m*c}) = 0.002688 \)

\( w_{f*c} = 0.39594 \quad \hat{\Var}(w_{f*c}) = 0.00897 \quad \hat{\Var}(w_{f*a}) + \hat{\Var}(w_{f*c}) = 0.002336 \)
\[ x_{w,gr}^2 = 11.802 + 4.972 = 16.774 \quad d.f. = 2 \]

\[ x_{w,anat}^2 = 0.145 + 1.827 = 1.972 \quad d.f. = 2 \]

Hence, the data are consistent with the "additive" formulation of the hypothesis of "no second order interaction" between the response and the factors cultural meaning of object and subject group within each category of anatomical meaning of object; i.e., within each category of anatomical meaning, the factors cultural meaning and subject group affect the \( p_{hijk} \) in an independent additive way. The "additive" formulation of the hypothesis of "no second order interaction" between the response and the factors cultural meaning and anatomical meaning within each subject group is rejected; i.e., the factors anatomical meaning and cultural meaning do not affect the \( p_{hijk} \) in an independent additive way within each subject group.

\[ x_{w,gr}^2 - x_b^2 = 15.334, \quad d.f. = 1; \quad x_{w,anat}^2 - x_b^2 = .532, \quad d.f. = 1. \]

Hence, the data are consistent with the "additive formulation" of the hypothesis of "no second order interaction" between the response and the factors cultural meaning and group. The interaction between cultural meaning and anatomical meaning, however, is significant.
IV. A "One Response, Three Factor" Situation

Table 4: Subject Type - Group C Males

<table>
<thead>
<tr>
<th></th>
<th>Weak</th>
<th>Strong</th>
</tr>
</thead>
<tbody>
<tr>
<td>M</td>
<td></td>
<td></td>
</tr>
<tr>
<td>F</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sub-Total</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>M</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>M</td>
<td></td>
<td></td>
</tr>
<tr>
<td>F</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sub-Total</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

1. a. \( b_w = -0.1015 \quad V_{b_w} = 0.002073 \quad b_s - b_w = 0.1142 \quad \chi^2_b = 3.78 \)

\[ b_s = 0.0127 \quad V_{b_s} = 0.001352 \quad V_{b_s + b_w} = 0.003452 \quad d.f. = 1. \]

Since the \( \alpha = 0.05 \) critical value of the chi-square distribution with one degree of freedom is 3.84, we do not reject the "additive" formulation of "no third order interaction" between the response and the three factors;
However, the data are not completely consistent with this hypothesis either.

\[ a_w = 0.481577 \quad c_w = -0.7308 \quad V_{c_w} = 0.018607 \quad V_{a_w} = 0.004315 \]
\[ u_s = 0.324821 \quad c_s = -1.1245 \quad V_{c_s} = 0.038915 \quad V_{a_s} = 0.004106 \]
\[ a_w - a_s = 0.156756 \quad c_w - c_s = 0.3937 \quad V_{c_w} + V_{c_s} = 0.057522 \quad V_{a_w} + V_{a_s} = 0.008421 \]
\[ X^2_a = 2.92 \quad X^2_c = 2.69 \quad \text{d.f.} = 1 \]

Hence, the data are consistent with the "multiplicative" formulation of the hypothesis of "no third order interaction" between the response and the three factors. Since \( X^2_a < X^2_b \), we feel the multiplicative model (2.39) provides a better "fit" to the data. Thus, the subsequent analysis will be based on multiplicative formulations.

\[ m_{aw} = 1.6374 \quad m_{as} = 0.4931 \quad \hat{\text{Var}}(m_{aw}) = 0.003774 \quad \hat{\text{Var}}(m_{as}) = 0.010116 \]
\[ m_{sw} = 3.1250 \quad m_{sw} = 1.1394 \quad \hat{\text{Var}}(m_{sw}) = 0.005924 \quad \hat{\text{Var}}(m_{sw}) = 0.057852 \]
\[ m_{sw} = 3.4000 \quad m_{sw} = 1.2238 \quad \hat{\text{Var}}(m_{sw}) = 0.014833 \quad \hat{\text{Var}}(m_{sw}) = 0.171469 \]
\[ m_{sw} = 9.6207 \quad m_{sw} = 2.2639 \quad \hat{\text{Var}}(m_{sw}) = 0.032991 \quad \hat{\text{Var}}(m_{sw}) = 3.053577 \]

\[ X^2_{z, \text{int}} = 17.109 + 13.561 = 30.670, \quad \text{d.f.} = 2 \]
\[ X^2_{\varepsilon, \text{int}} = 28.695 + 32.494 = 61.189, \quad \text{d.f.} = 2 \]
\[ X^2_{z, \text{anat}} = 32.558 + 11.999 = 44.557, \quad \text{d.f.} = 2 \]
\[ X^2_{\varepsilon, \text{anat}} = 43.071 + 22.621 = 65.692, \quad \text{d.f.} = 2 \]
\[ X^2_{z, \text{int}} - X^2_a = 27.75, \quad \text{d.f.} = 1 \]
\[ X^2_{\varepsilon, \text{int}} - X^2_c = 58.50, \quad \text{d.f.} = 1 \]
\[ X^2_{z, \text{anat}} - X^2_a = 41.64, \quad \text{d.f.} = 1 \]
\[ X^2_{\varepsilon, \text{anat}} - X^2_c = 63.00, \quad \text{d.f.} = 1 \]
Hence, the second order interaction between the response and the factors anatomical meaning and cultural meaning is significant; it is also significant at each level of intensity.

In addition, the second order interaction between the response and the factors intensity and cultural meaning is significant; it is also significant at each level of anatomical meaning.
5. Remarks

It would be noted that the test criteria for the corresponding hypotheses in the first three situations are identical while those for the last situation are, in general, quite different. Moreover, even when the test criteria are identical the meaningful interpretations of the corresponding hypotheses are quite different for the first three probability models depending on the adopted experimental scheme. This point has already been stressed in our earlier paper [1965] to which reference may be made for a more detailed discussion. In particular, it may be pointed out that in many experimental situations, it is more or less obvious whether a particular dimension is a "response" or a "factor", so that the problem belongs to one of the four situations discussed in a unique manner; on the other hand, in some instances a particular dimension may be viewed either as a "response" or a "factor" in which case the problem can be tackled from different points of view.

Note also that the hypotheses for the first three situations are formulated in terms of specific measures of association between two or three responses as the case may be. The criteria are offered only for those measures which have appeared in the literature so far and which seem to be satisfactory. An alternative measure of association between two responses has been offered in our earlier paper [1965] and is mentioned briefly in 2.3 (see (2.26)). Formulations corresponding to hypotheses (2.5) or (2.17) can be offered in terms of this measure and test criteria can be constructed for these, but this has not been pursued further here.

The methods of this paper can be now immediately generalized to multi-dimensional (multi-response and/or multi-factor) contingency tables. Note
that the four-dimensional tables, rather than the three-dimensional ones, point out the different problems that may arise in the higher-dimensional tables; this is mainly because a multi-response, multi-factor situation cannot arise in a three-dimensional table.

Bhapkar, V. P. [1965], "A note on the equivalence of some test criteria", Institute of Statistics, University of North Carolina, Mimeo Series No. 421.

Bhapkar, V. P. and Koch, Gary G., [1965], "On the hypothesis of 'no interaction' in three-dimensional contingency tables", Institute of Statistics, University of North Carolina, Mimeo Series No. 440.


Wald, A. [1943], "Tests of statistical hypotheses concerning several parameters when the number of observations is large", Transactions American Mathematical Society, 54, 426-82.