ON CERTAIN TYPES OF ASYMPTOTIC EQUIVALENCE

OF REAL PROBABILITY DISTRIBUTIONS

I

DEFINITIONS AND SOME OF THEIR PROPERTIES

by

Sadao Ikeda
University of North Carolina

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DEPARTMENT OF STATISTICS

UNIVERSITY OF NORTH CAROLINA

Chapel Hill, N. C.

SUMMARY AND INTRODUCTION

With the aim of application to some problems of asymptotic approximation a notion of asymptotic equivalence of probability distributions, called type I asymptotic equivalence, has been introduced by the present author [1], basing upon a distance defined over the class of all probability distributions which are absolutely continuous with respect to a σ-finite measure over the basic space. This type of asymptotic equivalence is regarded as an asymptotic equivalence version of the so-called mean convergence in the usual convergence theory, and it appears to be too severe for some situations of practical application: For example, let us consider the following situation. Let $\{(X_1^s, X_2^s, \ldots, X_{n_s}^s)\}$ $(s = 1,2,\ldots)$ be a sequence of n_s-dimensional real random variables where n_s is assumed to tend to infinity as s tends to infinity, and suppose that, for any fixed n independently of s, the sequence $\{(X_1^s,\ldots,X_n^s)\}$ $(s = 1,2,\ldots)$ converges in law to the n-dimensaionl normal distribution with mean vector $(0,0,\ldots,0)$ and variance matrix I_n, the unit matrix of order n. Then, what additional conditions will be needed in order that the random variable $X_1^{s^2} + \ldots + X_{n^2}^{s^2}$ can be approximated, in a certain non-trivial sense, by a chi-square distribution with n_s degrees of freedom, asymptotically as s tends to infinity, and what conditions assure us that the random variable $(X_1^{s^2} + \ldots + X_{k_s}^{s^2})/(X_1^{s^2} + \ldots + X_{k_s}^{s^2} + \ldots + X_{n_s}^{s^2})$ is approximated by a Beta-distribution with parameters $(k_s, n_s - k_s)$ asymptotically as s tends to infinity? Let us consider another example: Let $X_{(1)} < X_{(2)} < \ldots < X_{(N)}$ be an order statistic of size N from a one-dimensional distribution of the continuous type, and let $(X_{(1)}, \ldots, X_{(n)})$ and $(X_{(N-m+1)},\ldots,X_{(N)})$ be the joint distributions of n lower extremes and of m upper extremes respectively. It has been shown that if $n+m = o(\sqrt{N})$, then these two random variables are asymptotically independent in the sense of type I as N tends to infinity [1]. However, the condition on the order of magnitude of m+n would be improved, if the requirement of type I asymptotic independence is weakened to, for instance, a notion of asymptotic independence parallel to that of 'in law' convergence in the usual convergence theory. To treat these types of problems we need certain weaker types of asymptotic equivalence or of asymptotic independence than that of type I.

The purpose of the present article is to introduce some weaker types of asymptotic equivalence and of asymptotic independence which are useful for practical applications, and to discuss some of their fundamental properties. Many unsolved questions are left open, which will be stated in the final section.

From the view point of practical applications, we confine ourselves, throughout this paper, to real random variables, i.e., the 'basic spaces', over which the random variables under consideration are defined, are euclidean. In most of the problems of asymptotic approximation, we are given a sequence of subsets of the basic spaces, whose probabilities we want to evaluate approximately by a certain other sequence of probability distributions asymptotically, and the types of these subsets are usually the same for a given problem of asymptotic approximation. A sequence of classes to which the above stated subsets belong, will be called the basic classes for a given problem of asymptotic approximations. Our definitions of the types of asymptotic equivalence is essentially based upon the types of the basic classes.

In section 1, four types of asymptotic equivalence are introduced based upon a sort of distance between probability distributions associated with each of four types of basic classes. In section 2, some of the fundamental properties of four types of asymptotic equivalence given in section 1, are stated where the strongest two notions are shown to be mutually equivalent, and to be equivalent to that of the type I asymptotic equivalence given in [1] in some special case. Section 3 is devoted to introduce two types of asymptotic independence of a set of random variables, which is done by specializing the weaker two notions of asymptotic equivalence given in section 1. Some of the fundamental properties of these notions of asymptotic independence are also discussed.

In section 4, some classes of transformations of random variables which preserve the properties of asymptotic equivalence in the two weaker types, are discussed. In the last section, we list some open questions which are important for practical applications.

1. Definition of four types of asymptotic equivalence

Let $\mathfrak{R}_{(n)}$ be the n-dimensional euclidean space and $\mathfrak{B}_{(n)}$ be the Borel field of subsets of $\mathfrak{R}_{(n)}$. Denote the family of all probability distributions defined over the measurable space $(\mathfrak{R}_{(n)}, \mathfrak{B}_{(n)})$ by $\mathfrak{F}(\mathfrak{R}_{(n)},\mathfrak{B}_{(n)})$, a member of which is designated by a random variable $X_{(n)}$ or equivalently by a probability measure $P^{X_{(n)}}$, according to which $X_{(n)}$ is distributed.

Let $\mathfrak{C}_{(n)}$ be any subclass of $\mathfrak{B}_{(n)}$, and let us define, for $X_{(n)}$ and $Y_{(n)}$ belonging to $\mathfrak{F}(\mathfrak{R}_{(n)},\mathfrak{B}_{(n)})$, a quantity associated with the subclass $\mathfrak{C}_{(n)}$

$$(1.1) \qquad \delta_{\mathfrak{C}_{(n)}}(X_{(n)},Y_{(n)}) = \sup_{E_{(n)}\in \mathfrak{C}_{(n)}} |P^{X_{(n)}}(E_{(n)}) - P^{Y_{(n)}}(E_{(n)})| \quad .$$

Clearly, this is a metric over $\mathfrak{F}(\mathfrak{R}_{(n)}, \mathfrak{B}_{(n)})$ if we identify those random variables which have the same probability measure over $C_{(n)}$, i.e., $P^{X_{(n)}}(E_{(n)}) = P^{Y_{(n)}}(E_{(n)})$ for all $E_{(n)}$ belonging to $C_{(n)}$.

Let us define three subclass of $\mathfrak{B}_{(n)}$ as follows: $\mathcal{M}_{(n)}$ is the subclass of $\mathfrak{B}_{(n)}$ consisting of all subsets of $\mathfrak{R}_{(n)}$ which have the following form

$$E_{(n)} = \{Z_{(n)} = (Z_1, \ldots, Z_n) \mid -\infty \leq Z_i < a_i,\ a_i \text{ real adm.} \pm \infty;\quad i = 1,2,\ldots,n\},$$

$S_{(n)}$ is the subclass of $\mathfrak{B}_{(n)}$ consisting of all subsets of $R_{(n)}$ which have the following form

$$E_{(n)} = \{Z_{(n)} = (Z_1,\ldots,Z_n) \mid -b_i \leq Z_i < a_i,\ a_i \leq b_i \text{ real adm.} \pm \infty;\quad i = 1,2,\ldots,n\},$$

and finally, $G_{(n)}$ is the finitely additive field generated by $S_{(n)}$ or equivalently by $\mathcal{M}_{(n)}$. Then, it is clear that the implication relations

$$\mathcal{M}_{(n)} \subset S_{(n)} \subset G_{(n)} \subset \mathfrak{B}_{(n)} \tag{1.2}$$

hold and $G_{(n)}$ is constructed from $S_{(n)}$ in the following manner:

$$G_{(n)} = \{ \bigcup_{i=1}^{k} E^i_{(n)} \mid E^i_{(n)} \in S_{(n)},\ i = 1,\ldots,k,\ k \text{ is any finite positive integer} \}$$

Now, let us consider two sequences of random variables $\{X^s_{(n_s)}\}(s = 1,2,\ldots)$ and $\{Y^s_{(n_s)}\}$ $(s=1,2,\ldots,)$ with $X^s_{(n_s)}$ and $Y^s_{(n_s)}$ belonging to $\mathfrak{F}(\mathfrak{R}_{(n_s)}, \mathfrak{B}_{(n_s)})$ for each s, where n_s may or may not depend on s. The case where n_s is fixed independently of s will be called the case of equal basic spaces. Another important case is that where n_s tends to infinity as s tends to infinity.

Corresponding to these sequences of random variables, we give four types of sequence of basic classes: $\{\mathcal{M}_{(n_s)}\}(s=1,2,\ldots)$, $\{\mathbb{S}_{(n_s)}\}(s=1,2,\ldots)$, $\{\mathbb{G}_{(n_s)}\}(s=1,2,\ldots)$ and $\{\mathbb{B}_{(n_s)}\}$ $(s=1,2,\ldots)$. Let $\{\mathbb{C}_{(n_s)}\}$ $(s=1,2,\ldots)$ designate any of these four sequences:

DEFINITION 1.1 Two sequences of random variables $\{X^s_{(n_s)}\}(s=1,2,\ldots)$ and $\{Y^s_{(n_s)}\}(s=1,2\ldots)$ are said to be asymptotically equivalent in the sense of type $(\mathbb{C})$ as $s\to\infty$ and are denoted by

$$(\mathbb{C}) \qquad X^s_{(n_s)} \sim Y^s_{(n_s)}(\mathbb{C}_{(n_s)}), \quad (s\to\infty)$$

if

$$(1.3) \qquad \delta_{\mathbb{C}_{(n_s)}}(X^s_{(n_s)}, Y^s_{(n_s)}) \to 0, \quad (s\to\infty) \ .$$

We shall call this type of asymptotic equivalence the type $(\mathbb{C})$ asymptotic equivalence, and say that $X^s_{(n_s)}$ and $Y^s_{(n_s)}$ are asymptotically equivalent $(\mathbb{C})$, briefly.

Identifying $\{\mathbb{C}_{(n_s)}\}$ $(s=1,2\ldots)$ to each one of the four sequences $\{\mathcal{M}_{(n_s)}\}$ $(s-1,2,\ldots)$, $\{\mathbb{S}_{(n_s)}\}(s=1,2\ldots)$, $\{\mathbb{G}_{(n_s)}\}(s=1,2\ldots)$ and $\{\mathbb{B}_{(n_s)}\}$ $(s=1,2\ldots)$, we get four types of asymptotic equivalence, type $(\mathcal{M})$, $(\mathbb{S})$, $(\mathbb{G})$ and $(\mathbb{B})$.

For these four types of asymptotic equivalence, it is clear from (1.2) that the implication relations

$$(1.4) \qquad (\mathbb{B}) \Longrightarrow (\mathbb{G}) \Longrightarrow (\mathbb{S}) \Longrightarrow (\mathcal{M})$$

hold over $\mathfrak{F}$, i.e., for any two sequences $\{X^s_{(n_s)}\}$ $(s=1,2\ldots)$ and $\{Y^s_{(n_s)}\}(s=1,2\ldots)$ with $X^s_{(n_s)}$ and $Y^s_{(n_s)}$ belonging to $\mathfrak{F}(\mathfrak{R}_{(n_s)}, \mathbb{B}_{(n_s)})$ for each s .

It should be noted that the type $(\mathfrak{B})$ asymptotic equivalence is an equivalent notion to that of the type I asymptotic equivalence given in [1] under the situation in which $X^s_{(n_s)}$ and $Y^s_{(n_s)}$ under consideration belong to the family $P(\mathfrak{R}_{(n_s)}, \mathfrak{B}_{(n_s)}, \nu_{(n_s)})$, the family of all probability distributions which are absolutely continuous with respect to a preassigned σ-finite measure $\nu_{(n_s)}$ over $(\mathfrak{R}_{(n_s)}, \mathfrak{B}_{(n_s)})$. In fact, under such a situation we have

$$(1.5) \qquad 2\,\delta_{\mathfrak{B}_{(n_s)}}(X^s_{(n_s)}, Y^s_{(n_s)}) = \int_{\mathfrak{R}_{(n_s)}} |\, f_{(n_s)} - g_{(n_s)} | \, d\,\mu_{(n_s)}$$

for each s, where $f_{(n_s)}$ and $g_{(n_s)}$ stand for the generalized probability density function of $X^s_{(n_s)}$ and $Y^s_{(n_s)}$ with respect to $\nu_{(n_s)}$ respectively, and our previous notion of the type I asymptotic equivalence was based upon the distance which is the right-hand member of the above equality.

2. Properties of four types of asymptotic equivalence.

In the present section, we shall discuss some of the fundamental properties of four types of asymptotic equivalence given in the preceding section.

First, it is clear that

LEMMA 2.1. In general, the type $(\mathfrak{C})$ asymptotic equivalence is transitive in the sense that, if $X^s_{(n_s)} \sim Y^s_{(n_s)}(\mathfrak{C}_{(n_s)})$ and $Y^s_{(n_s)} \sim Z^s_{(n_s)}(\mathfrak{C}_{(n_s)})$, then $X^s_{(n_s)} \sim Z^s_{(n_s)}(\mathfrak{C}_{(n_s)})$. Thus, each one of four types of asymptotic equivalence given in the preceding section has this property.

LEMMA 2.2. Let $\{\mathfrak{C}_{(n_s)}\}$ (s=1,2,...) be any one of the four sequences $\{\mathcal{M}_{(n_s)}\}$(s=1,2,..), $\{\mathfrak{S}_{(n_s)}\}$ (s=1,2,...), $\{\mathfrak{G}_{(n_s)}\}$ (s=1,2,...) and $\{\mathfrak{B}_{(n_s)}\}$(s=1,2,...). Then, $X^s_{(n_s)} \sim Y^s_{(n_s)}$ $(\mathfrak{C}_{(n_s)})$ implies that

$\overline{X}^s_{(n_s)} \sim \overline{Y}^s_{(n_s)}(\mathcal{C}_{(n_s)})$, where $\overline{X}^s_{(n_s)}$ and $\overline{Y}^s_{(n_s)}$ are marginals of $X^s_{(n_s)}$ and $Y^s_{(n_s)}$ such that

$$\overline{X}^s_{(n_s)} = (X^s_{i_1}, \ldots, X^s_{i_{m_s}}) \text{ and } \overline{Y}^s_{(n_s)} = (Y^s_{i_1}, \ldots, Y^s_{i_{m_s}}) ,$$

for which the choice of $\{i_1, \ldots, i_{m_s}\}$ out of $\{1,2,\ldots,n_s\}$ may or may not depend on s .

The following result would be helpful for practical applications of our notions of asymptotic equivalence.

LEMMA 2.3. In order that $X^s_{(n_s)} \sim Y^s_{(n_s)}(\mathcal{C}_{(n_s)})$ as $s \to \infty$, it is necessary and sufficient that

$$(2.1) \qquad | P^{X^s_{(n_s)}}(E^s_{(n_s)}) - P^{Y^s_{(n_s)}}(E^s_{(n_s)})| \to 0 , \ (s \to \infty)$$

for every sequence $\{E^s_{(n_s)}\}$ $(s=1,2\ldots)$ with $E_{(n_s)} \in \mathcal{C}_{(n_s)}$ for each s . Thus, this is true for each of the four types of asymptotic equivalence given in the preceding section.

The proof of this lemma is quite similar to that of Lemma 1.3.2 of [1] and is omitted.

In the next place, we shall discuss the implication relations of our four types of asymptotic equivalence.

First, we can show the following

THEOREM 2.1. It holds that

$$(2.2) \qquad (\mathcal{A}) \Longleftrightarrow (\mathcal{B})$$

over $\mathfrak{F}$.

PROOF. This is proved by making use of the so-called extension theorem [2].

To prove the theorem, it is sufficient to show that $(\mathfrak{G}) \Longrightarrow (\mathfrak{B})$.

Let $\{\epsilon_s\}(s=1,2,\dots)$ be a sequence of positive numbers such that $\epsilon_s \to 0$ as $s \to \infty$. Then, for each s, there exists a member of $\mathfrak{B}_{(n_s)}$, $E^s_{(n_s)}$ say, such that

$$(2.3) \qquad 0 \leq \delta_{\mathfrak{B}_{(n_s)}}(X^s_{(n_s)}, Y^s_{(n_s)}) - |P^{X^s_{(n_s)}}(E^s_{(n_s)}) - P^{Y^s_{(n_s)}}(E^s_{(n_s)})| < \epsilon_s$$

By the extension theorem, one can find two coverings of $E^s_{(n_s)}$, $\Gamma^s_{(n_s)1} = \{F^s_{(n_s)1i}\}(i=1,2,\dots)$ and $\Gamma^s_{(n_s)2} = \{F^s_{(n_s)2j}\}$ $(j=1,2,\dots)$ with $F^s_{(n_s)1i}$ and $F^s_{(n_s)2j}$ belonging to $\mathfrak{G}_{(n_s)}$ for all i and j, such that

$$(2.4) \qquad \begin{aligned} P^{X^s_{(n_s)}}(E^s_{(n_s)}) &\leq \sum_{i=1}^{\infty} P^{X^s_{(n_s)}}(F^s_{(n_s)1i}) < P^{X^s_{(n_s)}}(E^s_{(n_s)}) + \epsilon_s \quad \text{and} \\ P^{Y^s_{(n_s)}}(E^s_{(n_s)}) &\leq \sum_{j=1}^{\infty} P^{Y^s_{(n_s)}}(F^s_{(n_s)2j}) < P^{Y^s_{(n_s)}}(E^s_{(n_s)}) + \epsilon_s \quad , \end{aligned}$$

where, we can assume, without loss of generality, $F^s_{(n_s)1i} \cap F^s_{(n_s)1i'} = \emptyset (i \neq i')$ and $F^s_{(n_s)2j} \cap F^s_{(n_s)2j'} = \emptyset$ $(j \neq j')$. Putting

$$\Gamma^s_{(n_s)} = \{F^s_{(n_s)1i} \cap F^s_{(n_s)2j} \mid F^s_{(n_s)1i} \in \Gamma^s_{(n_s)1},\ F^s_{(n_s)2j} \in \Gamma^s_{(n_s)2}, \text{ for all i and j}\},$$

and rearranging them, let $\Gamma^s_{(n_s)} = \{F^s_{(n_s)k}\}$ $(k=1,2,\dots)$. Then, it is clear that $F^s_{(n_s)k} \cap F^s_{(n_s)k'} = \emptyset$ $(k \neq k')$ and $\Gamma^s_{(n_s)}$ is a covering of $E^s_{(n_s)}$. Since $F^s_{(n_s)} \equiv \sum_{k=1}^{\infty} F^s_{(n_s)k}$, $\sum_{i=1}^{\infty} F^s_{(n_s)1i}$ and $\sum_{j=1}^{\infty} F^s_{(n_s)2j}$ are members of $\mathfrak{B}_{(n_s)}$ and

$$F^s_{(n_s)} \subseteq (\Sigma_{i=1}^{\infty} F^s_{(n_s)1i}) \cap (\Sigma_{j=1}^{\infty} F^s_{(n_s)2j})$$

we have

$$(2.5)\quad \begin{aligned} &P^{X^s_{(n_s)}}(F^s_{(n_s)}) \leq \Sigma_{i=1}^{\infty} P^{X^s_{(n_s)}}(F^s_{(n_s)1i}) \quad \text{and} \\ &P^{Y^s_{(n_s)}}(F^s_{(n_s)}) \leq \Sigma_{j=1}^{\infty} P^{Y^s_{(n_s)}}(F^s_{(n_s)2j}) , \end{aligned}$$

and, of course

$$(2.6)\quad \begin{aligned} &P^{X^s_{(n_s)}}(E^s_{(n_s)}) \leq P^{X^s_{(n_s)}}(F^s_{(n_s)}) \quad \text{and} \\ &P^{Y^s_{(n_s)}}(E^s_{(n_s)}) \leq P^{Y^s_{(n_s)}}(F^s_{(n_s)}) . \end{aligned}$$

Hence, it follows from (2.4), (2.5) and (2.6) that

$$\begin{aligned} &P^{X^s_{(n_s)}}(E^s_{(n_s)}) \leq P^{X^s_{(n_s)}}(F^s_{(n_s)}) < P^{X^s_{(n_s)}}(E^s_{(n_s)}) + \epsilon_s \quad \text{and} \\ &P^{Y^s_{(n_s)}}(E^s_{(n_s)}) \leq P^{Y^s_{(n_s)}}(F^s_{(n_s)}) < P^{Y^s_{(n_s)}}(E^s_{(n_s)}) + \epsilon_s \end{aligned}$$

Now, since $F^s_{(n_s)} = \sum_{k=1}^{\infty} F^s_{(n_s)k}$, and $\Sigma_{k=1}^{N} F^s_{(n_s)k}$ is a member of $G'_{(n_s)}$ for any fixed N, we can find a positive integer $N = N_{(s)}$ such that

$$(2.8)\quad \begin{aligned} &P^{X^s_{(n_s)}}(F^s_{(n_s)}) - \epsilon_s < P^{X^s_{(n_s)}}\Big(\sum_{k=1}^{N} F^s_{(n_s)k}\Big) \leq P^{X^s_{(n_s)}}(F^s_{(n_s)}) \quad \text{and} \\ &P^{Y^s_{(n_s)}}(F^s_{(n_s)}) - \epsilon_s < P^{Y^s_{(n_s)}}\Big(\sum_{k=1}^{N} F^s_{(n_s)k}\Big) \leq P^{Y^s_{(n_s)}}(F^s_{(n_s)}) \end{aligned}$$

simultaneously. Hence, it follows from (2.7) that

$$|P^{X^s_{(n_s)}}(E^s_{(n_s)}) - P^{Y^s_{(n_s)}}(E^s_{(n_s)})| \leq |P^{X^s_{(n_s)}}\Big(\sum_{k=1}^{N} F^s_{(n_s)k}\Big)| + 4\epsilon_s$$

from which we get

(2.9) $$\delta_{\mathfrak{B}(n_s)}(X^s_{(n_s)}, Y^s_{(n_s)}) \leq \delta_{\mathfrak{a}(n_s)}(X^s_{(n_s)}, Y^s_{(n_s)}) + 4\epsilon_s \quad .$$

Therefore, $\delta_{\mathfrak{a}(n_s)}(X^s_{(n_s)}, Y^s_{(n_s)}) \to 0$ $(s \to \infty)$ implies that $\delta_{\mathfrak{B}(n_s)}(X^s_{(n_s)}, Y^s_{(n_s)}) \to 0$ $(s \to \infty)$, which completes the proof of the theorem.

Equivalence relation between the type ($\mathfrak{B}$) asymptotic equivalence and the type (S) asymptotic equivalence is not valid for many practical cases: For example, let $\{X^s_{(n_s)}\}$ (s=1,2,...) be a sequence of random variables of the discrete type, while $\{Y^s_{(n_s)}\}$(s=1,2...) be those of the continuous type. Then, for these two sequences, the type ($\mathfrak{B}$) asymptotic equivalence never holds. It would be useful to give conditions, under which the two types of asymptotic equivalence, type ($\mathfrak{B}$) and type (S), are mutually equivalent.

We do not enter into further discussions of the properties of the type ($\mathfrak{B}$) asymptotic equivalence, because we have done this already in the previous paper [1] for some practically important cases.

Now, we shall consider the implication relation between two notions of asymptotic equivalence, the type ($\mathcal{M}$) and the type (S). First we show the following

<u>THEOREM 2.2</u> If n_s is bounded above uniformly for all s, then it holds that

(2.10) $$(\mathcal{M}) \iff (S)$$

over $\mathfrak{F}$.

<u>PROOF</u>. It suffices to show that $(\mathcal{M}) \Longrightarrow (S)$. Let, for each s, $E_{(n_s)}$ be any fixed member of $S_{(n_s)}$. Then, there can be found a positive integer $N_s(\leq 2^{n_s})$ and a sequence $\{F_{(n_s)i}\}$ $(i=1,2,\ldots,N_s)$ such that

$$P^{X^s_{(n_s)}}(E_{(n_s)}) = \sum_{i=1}^{N_s} c_i^s \, P^{X^s_{(n_s)}}(F_{(n_s)i})$$

and

$$P^{Y^s_{(n_s)}}(E_{(n_s)}) = \sum_{i=1}^{N_s} c_i^s \, P^{Y^s_{(n_s)}}(F_{(n_s)i}) \; ,$$

where $c_i^s = +1$ or -1, $i = 1,2,\ldots,N_s$. Hence we have

$$\delta_{\mathcal{S}_{(n_s)}}(X^s_{(n_s)}, Y^s_{(n_s)}) \le 2^{n_s} \, \delta_{\mathcal{M}_{(n_s)}}(X^s_{(n_s)}, Y^s_{(n_s)}) \; . \tag{2.11}$$

from which the theorem follows.

By this theorem it is known that the equivalence relation (2.10) holds true for the cases of equal basic spaces, and hence, for convergence cases.

The following example shows that the equivalence relation (2.10) can not necessarily hold in general case.

<u>EXAMPLE 2.1</u>. Let $X_1, \ldots, X_n$ be a random sample of size n drawn from the uniform distribution on $[0,1)$, while $Y_1^n, \ldots, Y_n^n$ be that from a distribution with probability density function given by

$$g_n(z) = \begin{cases} 1/n, & \text{if } -1 \le z < 0 \; , \\ 1-1/n, & \text{if } 0 \le z < 1, \quad (n=1,2,\ldots\ldots) \\ 0 \; , & \text{otherwise} \; . \end{cases}$$

Then, for the set $E_{(n)}$ in $\mathcal{S}_{(n)}$, defined by

$$E_{(n)} = [0,1) \times [0,1) \times \ldots \times [0,1) \; ,$$

we obtain, for $X_{(n)} = (X_1, \ldots, X_n)$ and $Y^n_{(n)} = (Y_1^n , \ldots, y_n^n)$,

$$|P^{X_{(n)}}(E_{(n)}) - P^{Y^n_{(n)}}(E_{(n)})| = 1 - (1 - \tfrac{1}{n})^n \to 1 - e^{-1}, \; (n \to \infty)$$

which means that $X_{(n)} \sim Y^n_{(n)}$ $(\mathcal{S}_{(n)})$, $(n \to \infty)$ can not hold true.

On the other hand, we can easily see that

$$\delta_{\mathcal{M}_{(n)}}(X_{(n)}, Y^n_{(n)}) \leq 1/n, \qquad n=1,2,\dots,$$

which implies that $X_{(n)} \sim Y^n_{(n)}(\mathcal{M}_{(n)})$, $(n \to \infty)$.

3. Asymptotic independence of a set of random variables.

In the present section, we introduce two types of asymptotic independence of a set of random variables, by specializing the notions of asymptotic equivalence of the type $(\mathcal{M})$ and the type $(\mathcal{S})$. The method is quite similar to that of the previous paper [1].

Under the same situation as in section 1, let $X^s_{(n_s)} = (X^s_1, X^s_2, \dots, X^s_{n_s})$ be a member of $\mathfrak{F}(\mathfrak{R}_{(n_s)}, \mathfrak{B}_{(n_s)})$, s=1,2,... . Corresponding to a decomposition of $\mathfrak{R}_{(n_s)}$ in the product form:

$$\mathfrak{R}_{(n_s)} = \mathfrak{R}_{(m_1)} \times \mathfrak{R}_{(m_2)} \times \dots \times \mathfrak{R}_{(m_{k_s})} \quad ,$$

let us write $X^s_{(n_s)}$ in the form

$$X^s_{(n_s)} = (X^s_{(m_1)}, X^s_{(m_2)}, \dots, X^s_{(m_{k_s})}) \tag{3.1}$$

where $X^s_{(m_j)}$ belongs to $\mathfrak{F}(\mathfrak{R}_{(m_j)}, \mathfrak{B}_{(m_j)})$ for each j and s, and $m_1, m_2, \dots, m_{k_s}$ and k_s may or may not depend on s under the restriction that $m_j \geq 1$ and $m_1 + m_2 + \dots + m_{k_s} = n_s$. For each s, we consider the set of marginal variables of $X^s_{(n_s)}$:

$$\{X^s_{(m_1)}, X^s_{(m_2)}, \dots, X^s_{(m_{k_s})}\} \quad . \tag{3.2}$$

Let us consider, for each s, a member of $\mathfrak{F}(\mathfrak{R}_{(n_s)}, \mathfrak{B}_{(n_s)})$

$$Y^s_{(n_s)} = (Y^s_{(m_1)}, Y^s_{(m_2)}, \ldots, Y^s_{(m_{k_s})}) , \tag{3.3}$$

whose marginals

$$\{Y^s_{(m_1)}, Y^s_{(m_2)}, \ldots, Y^s_{(m_{k_s})}\} \tag{3.4}$$

constitute an independent set of random variables, i.e., satisfying the condition

$$P^{Y^s_{(n_s)}}(E_{(n_s)}) = \prod_{i=1}^{k_s} P^{Y^s_{(m_i)}}(E_{(m_i)}) \tag{3.5}$$

for any $E_{(n_s)} = E_{(m_1)} \times E_{(m_2)} \times \ldots \times E_{(m_{k_s})}$ with $E_{(m_i)} \in \mathfrak{B}_{(m_i)}$, $i=1,\ldots,k_s$.

It is known that for any given $X^s_{(n_s)}$ there exists a $Y^s_{(n_s)}$, having the property stated above, such that

$$P^{X^s_{(m_i)}}(E_{(m_i)}) = P^{Y^s_{(m_i)}}(E_{(m_i)}) , \quad i = 1,2,\ldots,k_s \tag{3.6}$$

for every $E_{(m_i)}$ belonging to $\mathfrak{B}_{(m_i)}$, $i=1,2,\ldots,k_s$.

Under this situation, we give the following

DEFINITION 3.1. A set of random variable (3.2) is said to be asymptotically independent in the sense of type $(\mathcal{M})$ as $s \to \infty$, if two sequences $\{X^s_{(n_s)}\}$ $(s=1,2,\ldots)$ and $\{Y^s_{(n_s)}\}$ $(s=1,2,\ldots)$ are asymptotically equivalent in the sense of type $(\mathcal{M})$ as $s \to \infty$.

Likewise

DEFINITION 3.2. A set of random variable (3.2) is said to be asymptotically independent in the sense of type $(\mathbf{S})$ as $s \to \infty$, if $\{X^s_{(n_s)}\}$ $(s=1,2,\ldots)$ and $\{Y^s_{(n_s)}\}$ $(s=1,2,\ldots)$ are asymptotically equivalent in the sense of type $(\mathbf{S})$ as $s \to \infty$.

In these two cases, we shall call the set (3,2) an <u>asymptotically independent ($\mathcal{M}$) set of random variables</u> and an <u>asymptotically independent (S) set of random variables</u>, respectively.

Since (S) implies ($\mathcal{M}$) in general, it is straightforward that

<u>LEMMA 3.1</u>. If the set of random variables (3.2) is asymptotically independent in the sense of type (S) as $s\to\infty$, then it is asymptotically independent in the sense of type ($\mathcal{M}$) as $s\to\infty$.

We list some fundamental properties of the notions of asymptotic independence given above in the following lemmas, whose proofs are easy and will be omitted.

<u>LEMMA 3.2</u>. Let $\{X^s_{(m_{i1})}, \ldots, X^s_{(m_{i\ell_s})}\}$ be any subset of (3.2), where the choice of $\{i1, \ldots, i\ell_s\}$ out of^s $\{1, s, \ldots, k_s\}$ may or may not depend on s . Then, the asymptotic independence ($\mathcal{M}$) ((S)) of $\{X^s_{(m1)}, \ldots, X^s_{(m_{ks})}\}$ implies the asymptotic independence ($\mathcal{M}$) ((S)) of $\{X^s_{(m_{i1})}, \ldots, X^s_{(m_{i\ell_s})}\}$.

<u>LEMMA 3.3</u>. If $\{X^s_{(n_s)} = (X^s_{(m_1)}, \ldots, X^s_{(m_{k_s})}\}$ (s=1,2,...) and $\{Z^s_{(n_s)} = (Z^s_{(m_1)}, \ldots, Z^s_{(m_{k_s})})\}$ (s=1,2...) are asymptotically equivalent in the sense of type ($\mathcal{M}$) ((S)) as $s\to\infty$, and if $\{X^s_{(m1)}, \ldots, X^s_{(m_{ks})}\}$ is asymptotically independent in the sense of type ($\mathcal{M}$) ((S)) as $s\to\infty$, then $\{Z^s_{(m1)}, \ldots, Z^s_{(m_{ks})}\}$ (s=1,2...) are asymptotically independent in the sense of type ($\mathcal{M}$) ((S)) as $s\to\infty$.

<u>LEMMA 3.4</u>. If n_s is bounded above uniformly for all s, then both types of asymptotic independence, type ($\mathcal{M}$) and type (S), of the set of random variables (2.3) are mutually equivalent.

The following result is also immediate.

LEMMA 3.5. Let $f_i^s(x)$'s be measurable transformations from $\Re_{(1)}$ into $\Re_{(1)}$, and

(3.2) $$U_{(n_s)}^s = (U_1^s, \ldots, U_{n_s}^s), \quad U_i^s = f_i^s(X_i^s) \quad i=1,\ldots,n_s$$

and

(3.3) $$V_{(n_s)}^s = (V_1^s, \ldots, V_{n_s}^s), \quad V_i^s = f_i^s(Y_i^s), \quad i=1,\ldots,n_s,$$

for every s. Suppose that $X_{(n_s)}^s \sim Y_{(n_s)}^s \ (\mathcal{M}_{(n_s)})$ $(s \to \infty)$ implies that $U_{(n_s)}^s \sim V_{(n_s)}^s (\mathcal{M}_{(n_s)})$ $(s \to \infty)$, that is, the function

(3.4) $$f_{(n_s,n_s)}^s(z_{(n_s)}) = (f_1^s(z_1), \ldots, f_{n_s}^s(z_{n_s}))$$

preserves type $(\mathcal{M})$ asymptotic equivalence. Then, the asymptotic independence of $\{X_1^s, \ldots, X_{n_s}^s\}$ in the sense of type $(\mathcal{M})$ implies that of $\{U_1^s, \ldots, U_{n_s}^s\}$ in the same sense.

The same is true for type (S) asymptotic independence.

4. Some classes of transformations of random variables preserving the asymptotic equivalence in the sense of type $(\mathcal{M})$ and (S)

From the viewpoint of practical application, it is important to decide the classes of transformations of random variables preserving the type $(\mathcal{M})$ and type (S) asymptotic equivalence. In this section, we discuss some simple classes of such transformations.

The problem is formulated as follows: Given two sequences of random variables, $\{X_{(n_s)}^s\}$ (s=1,2...) and $\{Y_{(n_s)}^s\}$ (s=1,2...) with $X_{(n_s)}^s$ and $Y_{(n_s)}^s$ belonging to $\mathfrak{F}(\Re_{(n_s)}, \mathcal{B}_{(n_s)})$ for each s, let

(4.1) $$U^s_{(m_s)} = f^s_{(n_s,m_s)}(X^s_{(n_s)}) \text{ and } V^s_{(m_s)} = f^s_{(n_s,m)}(Y^s_{(n_s)}),\ s=1,2\ldots.$$

where $f^s_{(n_s,m_s)}(z_{(n_s)})$ is a measurable transformation from $\Re_{(n_s)}$ into $\Re_{(m_s)}$, $(n_s \geq m_s)$, for each s. Then, what conditions should be imposed on the function $f^s_{(n_s,m_s)}$ and on the sequences $\{X^s_{(n_s)}\}$ (s=1,2...) and $\{Y^s_{(n_s)}\}$ (s=1,2...), in order that $X^s_{(n_s)} \sim Y^s_{(n_s)} (\mathcal{M}_{(n_s)})$ $(s \to \infty)$ implies $U^s_{(m_s)} \sim V^s_{(m_s)}$ $(\mathcal{M}_{(n_s)})$ $(s \to \infty)$, or that $X^s_{(n_s)} \sim Y^s_{(n_s)} (S_{(n_s)}) (s \to \infty)$ implies $U^s_{(m_s)} \sim V^s_{(m_s)} (\mathcal{M}_{(n_s)})$ $(s \to \infty)$ and/or $U^s_{(m_s)} \sim V^s_{(m_s)} (S_{(n_s)}) (s \to \infty)$?

This problem would be very difficult to solve in the general case. In the present section, we consider some special cases and discuss this problem. Before entering into our discussion, we shall state the following well-known result.

__LEMMA 4.1.__ Suppose $n_s = n$ and $m_s = m$ $(m \leq n)$ for all s, and $Y^s_{(n)}$'s are all identical to some fixed $Y_{(n)}$. Then the type $(\mathcal{M})$ asymptotic equivalence of $\{X^s_{(n)}\}$(s=1,2...) and $Y_{(n)}$ (i.e., __the type $(\mathcal{M})$ convergence__ in this case) implies that of $U^s_{(m)} = f_{(n,m)}(X^s_{(n)})$ and $V_{(m)} = f_{(n,m)}(Y_{(n)})$ in the same sense, if $f_{(n,m)}$ is independent of s and is continuous.

Now, in the first place, we consider the case when

(4.2) $$f^s_{(n_s,n_s)}(z_{(n_s)}) = (c^s_1 z_1 + d^s_1, c^s_2 z_2 + d^s_2, \ldots, c^s_{n_s} z_{n_s} + d^s_{n_s})$$

where $z_{(n_s)} = (z_1, \ldots, z_{n_s})$ and c^s_i's and d^s_i's are constants such that $c^s_i > 0$ for all i and s. Then, we can show the following

__THEOREM 4.1.__ For the function given by (4.2), $X^s_{(n_s)} \sim Y^s_{(n_s)} (\mathcal{M}_{(n_s)}) (s \to \infty)$ implies that $U^s_{(n_s)} \sim V^s_{(n_s)} (\mathcal{M}_{(n_s)}) (s \to \infty)$, and similarly $X^s_{(n_s)} \sim Y^s_{(n_s)} (S_{(n_s)}) (s \to \infty)$

implies that $U^s_{(n_s)} \sim V^s_{(n_s)} (\mathbf{S}_{(n_s)}) (s \to \infty)$. The converses are also true.

<u>PROOF.</u> This theorem follows easily from the facts that

$$f^{s\,-1}_{(n_s,n_s)} (\mathcal{M}_{(n_s)}) \equiv \{E_{(n_s)} | f^s_{(n_s,n_s)}(E_{(n_s)}) \in \mathcal{M}_{(n_s)}\} = \mathcal{M}_{(n_s)}$$

and

$$f^{s\,-1}_{(n_s,n_s)} (\mathbf{S}_{(n_s)}) \equiv \{E_{(n_s)} | f^s_{(n_s,n_s)}(E_{(n_s)}) \in \mathbf{S}_{(n_s)}\} = \mathbf{S}_{(n_s)}$$

for each s. In fact, these imply that

$$\delta_{\mathcal{M}_{(n_s)}} (X^s_{(n_s)}, Y^s_{(n_s)}) = \delta_{\mathcal{M}_{(n_s)}} (U^s_{(n_s)}, V^s_{(n_s)})$$

and

$$\delta_{\mathbf{S}_{(n_s)}} (X^s_{(n_s)}, Y^s_{(n_s)}) = \delta_{\mathbf{S}_{(n_s)}} (U^s_{(n_s)}, V^s_{(n_s)}) \quad ,$$

respectively.

As a special case, let us consider a case of equal basic spaces for which $n_s = n$ $(s=1,2,\ldots)$. Then, it is easy to see that the conditions $c^s_i > 0$ $(i=1,\ldots,n;\ s=1,2\ldots)$ can be replaced by the conditions $c^s_i \neq 0$ $(i=1,2,\ldots,n;\ s=1,2,\ldots)$. This gives a theoretical foundation to the so-called "asymptotic distribution": For example, let U_s and Y be random variables distributed according to a binomial distribution $B(s,p)$ and the standard normal distribution $N(0,1)$. Then, under the limiting $s \to \infty$, $p \to \lambda (\geq 0)$, and $sp \to \infty$, $X_s = (U_s - sp)/\sqrt{sp(1-p)}$ tends to Y in the sense of type $(\mathbf{S})$, and therefore, by the above theorem (for $n_s = n$), $\{U_s\}$ $(s=1,2\ldots)$ and $\{V_s = \sqrt{sp(1-p)}\, Y + sp\}$ $(s=1,2\ldots)$ are asymptotically equivalent in the sense of type $(\mathbf{S})$ under the same limiting. For sufficiently large s, V_s is called an <u>asymptotic distribution</u> of U_s.

Next, let us consider the case of equal basic spaces, and let

$$f_{(n,n)}(z_{(n)}) = \left(\frac{1}{z_1}, \frac{1}{z_2}, \ldots, \frac{1}{z_n} \right) . \tag{4.3}$$

for all $z_{(n)} = (z_1, \ldots, z_n)$ belonging to $\mathfrak{R}_{(n)}$. Then, we can prove the following.

<u>THEOREM 4.2.</u> Suppose that $Y^s_{(n)}$'s are absolutely continuous with respect to the Lebesgue measure $\mu_{(n)}$ over $\mathfrak{R}_{(n)}$ uniformly for all s, i.e., for any given $\epsilon > 0$, there exists a positive constant δ not depending on s such that, if $\mu_{(n)}(N_{(n)}) < \delta$ then $P^{Y^s_{(n)}}(N_{(n)}) < \epsilon$, for all s. Then under the situation stated above, if $X^s_{(n)} \sim Y^s_{(n)} (\mathcal{M}_{(n)}) (s \to \infty)$, then $U^s_{(n)} \sim V^s_{(n)} (\mathcal{M}_{(n)}) (s \to \infty)$.

<u>PROOF.</u> Let $\mathfrak{S}^*_{(n)}$ be the class of all subsets of $\mathfrak{R}_{(n)}$ of the forms

$$E^*_{(n)} = \{ z_{(n)} = (z_1, \ldots, z_n) \mid -\infty \leq z_i < a_i \text{ or } -\infty \leq z_i \leq a_i;\ a_i \text{ real adm.} \pm \infty \quad i = 1, \ldots, n \}$$

and

$$E^{**}_{(n)} = \{ z_{(n)} = (z_1, \ldots, z_n) \mid b_i \leq z_i < a_i \text{ or } b_i \leq z_i \leq a_i \text{ or } b_i < z_i < a_i \text{ or } b_i < z_i \leq a_i;\ a_i, b_i \text{ real adm.} \pm \infty;\ i = 1, \ldots, n \} .$$

Then, under the assumption of this theorem, it is easy to see that the two conditions $\delta_{\mathfrak{S}^{**}_{(n)}}(X^s_{(n)}, Y^s_{(n)}) \to 0\ (s \to \infty)$ and $\delta_{\mathfrak{S}_{(n)}}(X^s_{(n)}, Y^s_{(n)}) \to 0\ (s \to \infty)$ are mutually equivalent.

Now, since

$$\{ z_i \mid \frac{1}{z_i} < a_i \} = \begin{cases} \{ z_i \mid \frac{1}{a_i} < z_i \text{ or } z_i \leq 0 \}, & \text{if } a_i > 0 \\ \{ z_i \mid z_i < 0 \}, & \text{if } a_i = 0 \\ \{ z_i \mid \frac{1}{a_i} < z_i \leq 0 \}, & \text{if } a_i < 0 \end{cases}$$

for each i, i=1,...,n, the set

$$\{z_{(n)} = (z_1 \dots z_n) \mid \frac{1}{z_i} < a_i,\ i=1,\dots,n\}$$

is expressed in the form

$$F^*_{(n)} = \sum_{p=1}^{N} E^*_{(n)j}$$

where $E^*_{(n)j} \in \mathbf{S}^*_{(n)}$ (j=1,...,N) and $N = N(a_1,\dots,a_n)$ is a certain positive integer not greater than 2^n uniformly for all $(a_1,\dots,a_n)$. Hence, it follows that

$$\delta_{\mathcal{M}(n)}(U^s_{(n)}, V^s_{(n)}) \leq 2^n \cdot \delta_{\mathbf{S}^*_{(n)}}(X^s_{(n)} Y^s_{(n)}) \; .$$

whence the theorem follows.

Let us consider another situation where

(4.4) $$f^s_{(n_s,n_s)}(z_{(n_s)}) = (f^s_1(z_1),\dots,f^s_{n_s}(z_{n_s})) \quad s=1,2\dots$$

are such that each $f^s_i(z_i)$ is a monotone increasing function of z_i, $i=1,2,\dots,n_s$; s=1,2... Then, clearly

$$f^{s\,-1}_{(n_s,n_s)}(\mathcal{M}_{(n_s)}) \subseteq \mathcal{M}_{(n_s)} \text{ and } f^{s\,-1}_{(n_s,n_s)}(\mathbf{S}_{(n_s)}) \subseteq \mathbf{S}_{(n_s)},\ s=1,2\dots,$$

from which we obtain the following

<u>THEOREM 4.3</u>. Under the above situation, $X^s_{(n_s)} \sim Y^s_{(n_s)}(\mathcal{M}_{(n_s)})(s \to \infty)$ implies that $U^s_{(n_s)} \sim V^s_{(n_s)}(\mathcal{M}_{(n_s)})$ $(s \to \infty)$. The same is true for type (**S**) asymptotic equivalence.

In the last place, let us take a transformation from $\mathfrak{R}_{(2)}$ to $\mathfrak{R}_{(1)}$ defined by

(4.5) $$f_{(2,1)}(z_{(2)}) = z_1 + c\, z_2$$

$c(\neq 0)$ being any fixed constant, and suppose that two sequences of random variables $\{X^s_{(2)} = (X^s_1, X^s_2)\}$ $(s=1,2\ldots)$ and $\{Y^s_{(2)} = (Y^s_1, Y^s_2)\}$ $(s=1,2\ldots)$ satisfy the following conditions:

(i) One of the marginals of $Y^s_{(2)}$, Y^s_1 say tends to a certain distribution Z of the continuous type in the sense of type $(\mathcal{M})$ as $s \to \infty$.

(ii) $Y^s_{(2)}$'s are absolutely continuous with respect to the two-dimensional Lebesgue measure over $(\mathfrak{R}_{(2)}, \mathfrak{B}_{(2)})$ uniformly for all s.

Then, we can show the following

THEOREM 4.4. Under the above situation, if $X^s_{(2)} \sim Y^s_{(2)} (\mathcal{M}_{(2)})(s \to \infty)$,, then $U^s \sim V^s$ $(\mathcal{M}_{(1)})(s \to \infty)$, where

$$U^s = f_{(2.1)}(X^s_{(2)}) \text{ and } V^s = f_{(2,1)}(Y^s_{(2)}) . \tag{4.6}$$

PROOF. By the condition (i), for any given $\epsilon > 0$, there exist a set $E^\epsilon_1 \in S_{(1)}$ and a positive integer s_ϵ such that

$$|1 - P^{Y^s_1}(E^\epsilon_1)| < \epsilon \text{ and } |1 - P^{X^s_1}(E^\epsilon_1)| < \epsilon \tag{4.7}$$

for all $s \geq s_\epsilon$. Put

$$E^\epsilon_{(2)} = E^\epsilon_1 \times \mathfrak{R}_{(1)} .$$

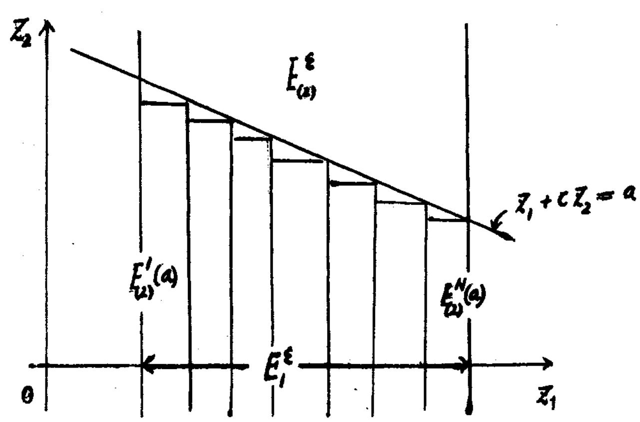

Fig. 1

Dividing E_1^ϵ into N subintervals, and giving a straight line $z_1 + cz_2 = a$, consider the N members of $\mathbf{S}_{(2)}$, $E^1_{(2)}(a), E^2_{(2)}(a), \ldots, E^N_{(2)}(a)$ as in Figure 1. Then, we can choose N such that

$$\mu_{(2)}(\{z_{(2)} | z_1 + cz_2 < a\} \cap E^\epsilon_{(2)} - \sum_{i=1}^{N} E^i_{(2)}(a)) < \epsilon$$

independently of the value of a, and hence, there exists $N = N(\epsilon)$ such that

$$| P^{X^s_{(2)}}(f^{-1}_{(2,1)}(E(a)) - P^{X^s_{(2)}}(\sum_{i=1}^{N} E^i_{(2)}(a)) | < 2\ \epsilon$$

and

$$|P^{Y^s_{(2)}}(f^{-1}_{(2,1)}(E(a))) - P^{Y^s_{(2)}}(\sum_{i=1}^{N} E^i_{(2)}(a))| < 2\ \epsilon$$

uniformly for a and for sufficiently large s, where $E(a) = \{z | z < a\} \in \mathcal{M}_{(1)}$.

$$(4.8) \qquad | P^{U^s}(E(a)) - P^{V^s}(E(a)) | \leq | P^{X^s_{(2)}}(\sum_{i=1}^{N} E^i_{(2)}(a)) - P^{Y^s_{(2)}}(\sum_{i=1}^{N} E^i_{(2)}(a)) | + 4\epsilon$$

uniformly for a, and for sufficiently large s, from which we have

$$(4.9) \qquad \delta_{\mathcal{M}_{(1)}}(U^s, V^s) \leq N\ \delta_{S_{(2)}}(X^s_{(2)}, Y^s_{(2)}) + 4\epsilon$$

for sufficiently large values of s. This results the theorem.

<u>COROLLARY 4.1.</u> Under the situation of the above theorem, suppose X^s_i and Y^s_i are positive random variables i=1,2., and let

$$(4.10) \qquad U^s = \frac{X^s_2}{X^s_1} \quad \text{and} \quad V^s = \frac{Y^s_2}{Y^s_1} .$$

Then, $X^s_{(2)} \sim Y^s_{(2)}(\mathcal{M}_{(2)})(s \to \infty)$ implies that $U^s \sim V^s(\mathcal{M}_{(2)})(s \to \infty)$.

<u>PROOF.</u> Applying Theorems 4.3 and 4.4, we have

(4.11) $$\log U^s \sim \log V^s \quad (\mathcal{M}_{(1)}) \ (s \to \infty) .$$

Hence, using Theorem 4.3 again, we have

(4.12) $$U^s = \exp(\log U^s) \sim V^s = \exp(\log V^s) \ (\mathcal{M}_{(1)}) \ (s \to \infty)$$

which proves the corollary.

We conclude this section by the following example.

<u>EXAMPLE 4.1.</u> Suppose that

(i) $\{X^s_{(2)} = (X^s_1, X^s_2)\}(s=1,2\ldots)$ and $\{Y^s_{(2)} = (Y^s_1, Y^s_2)\}$ $(s=1,2\ldots)$ are asymptotically equivalent in the sense of type $(\mathcal{M})$ as $s \to \infty$,

(ii) X^s_i and Y^s_i are positive random variables, $i=1,2$; $s=1,2,\ldots$,

(iii) Y^s_1 tends to a certain distribution Z of the continuous type in the sense of type $(\mathcal{M})$, i.e., in the sense of 'in law' convergence, as $s \to \infty$, and

(iv) $Y^s_{(2)}$'s and Y^s_2/Y^s_1's are absolutely continuous with respect to the Lebesgue measure over $(\mathfrak{R}_{(2)}, \mathfrak{B}_{(2)})$ uniformly for all s.

Put

(4.13) $$U^s = \frac{X^s_1}{X^s_1 + X^s_2} \quad \text{and} \quad V^s = \frac{Y^s_1}{Y^s_1 + Y^s_2} \quad , \ s=1,2,\ldots,$$

then it holds that

(4.14) $$U^s \sim V^s \ (\mathcal{M}_{(2)}), \ (s \to \infty).$$

In fact, from Corollary 4.1 and Theorem 4.1 it follows that

(4.15) $$\frac{1}{U^s} = 1 + \frac{X^s_2}{X^s_1} \sim \frac{1}{V^s} = 1 + \frac{Y^s_2}{Y^s_1} \ (\mathcal{M}_{(1)}), \ (s \to \infty) .$$

Using Theorem 4.3, we have (4.14).

5. Unsolved Problems

To establish the theory of asymptotic equivalence in the sense of type ($\mathcal{M}$) and (S) which is enough for practical applications, many questions are left open. We list some of them below.

(a) First of all, it is desirable to give certain criteria for type ($\mathcal{M}$) and type (S) asymptotic equivalence in both of the cases of equal and unequal basic spaces.

(b) More general results are desirable for the problem stated in the beginning of Section 4, i.e., the problem of determining the class of measurable transformations which preserve type ($\mathcal{M}$) and type (S) asymptotic equivalance property.

In practical applications, we sometimes meet the following types of problem: (c) For $\{X^s_{(n_s)}\}$ (s=1,2...) and $\{Y^s_{(n_s)}\}$ (s=1,2...), the two sequences of k_s marginals of them, $\{\overline{X}^s_{(k_s)} = (X^s_{i_1}, \ldots, X^s_{i_{k_s}})\}$ (s=1,2,...) and $\{\overline{Y}^s_{(k_s)} = (Y^s_{i_1}, \ldots, Y^s_{i_{k_s}}\}$(s=1,2,...) are asymptotically equivalent in the sense of type ($\mathcal{M}$) or (S) as $s \to \infty$, where $k_s \leq n_s$ and the choice of $\{i_1, \ldots, i_{k_s}\}$ out of $\{1,2,\ldots,n_s\}$ and k_s may or may not depend on s. Then, what conditions will be needed for the validity of $U^s_{(m_s)} \sim V^s_{(m_s)} (\mathcal{M}_{(m_s)}) (s \to \infty)$; for $U^s_{(m_s)}$ and $V^s_{(m_s)}$ given by (4.1), or (d) for $\{X^s_{(n_s)}\}$ (s=1,2...) and $\{Y^s_{(n_s)}\}$ (s=1,2,...), for any given finite integer k the two sequences of any k marginals of them, $\{\overline{X}^s_{(k)} = X^s_{i_1}, \ldots, X^s_{i_k}\}$ (s=1,2,...) and $\{\overline{Y}^s_{(k)} = (Y^s_{i_1}, \ldots, Y^s_{i_k})\}$ (s=1,2,...), are asymptotically equivalent in the sense of type ($\mathcal{M}$) as $s \to \infty$, where $\{i_1, \ldots, i_k\}$ is dependent or not dependent on s. Then, what conditions will be needed for the validity of $U^s_{(m_s)} \sim V^s_{(m_s)} (\mathcal{M}_{(m_s)})$ $(s \to \infty)$, for $U^s_{(m_s)}$ and $V^s_{(m_s)}$ given by (4.1).

(e) Suppose we are given a sequence of random variables $\{(X^s_{(n_s)}, Y^s_{(m_s)})\}$ $(s=1,2,\ldots)$ such that the first marginal $X^s_{(n_s)}$ is of the continuous type, whose conditional probability density function given $Y^s_{(m_s)} = y_{(m_s)}$ being $p_s(x_{(n_s)}|y_{(m_s)})$, and for the second marginal $Y^s_{(m_s)}$ there can be found another probability distribution $Z^s_{(m_s)}$ which is asymptotically equivalent to $Y^s_{(m_s)}$ in the sense of type (S) as $s \to \infty$. Then, what conditions are sufficient in order that two probability distributions $X^s_{(n_s)}$ and $\tilde{X}^s_{(n_s)}$ are asymptotically equivalent in the sense of type ($\mathcal{M}$) or (S) as $s \to \infty$, where $\tilde{X}^s_{(n_s)}$ is a random variable whose probability density function being given by taking the expectation of $p_s(x_{(n_s)}|Z^s_{(n_s)})$ with respect to $Z^s_{(n_s)}$. Here, n_s and m_s may or may not depend on s.

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