HERMITIAN VARIETIES IN A FINITE PROJECTIVE SPACE $\text{PG}(N, q^2)$

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SUMMARY

The elements $x$ and $x^q$ of $\text{GF}(q^2)$ are defined to be conjugate to one another. The square matrix $H = ((h_{ij}))$ is called Hermitian if $h_{ij}$ is conjugate to $h_{ji}$ for all $i,j$. If $x^T = (x_0, x_1, \ldots, x_N)$, then $x^{(q)}$ is defined to be the column vector whose elements are $x_0^q, x_1^q, \ldots, x_N^q$. A Hermitian variety $V_{N-1}$ in the finite projective space $\text{PG}(N,q^2)$ has the equation $x^T H x^{(q)} = 0$, where $H$ is a Hermitian matrix of order $N+1$. The present paper studies the geometrical properties of Hermitian varieties.

The theory of pole and polars has been developed, and the sections of these varieties by hyperplanes have been studied. The number of flat spaces of a given dimension contained in a Hermitian variety has been obtained.

Finally, the geometry of the surface $x_0^{q+1} + x_1^{q+1} + x_2^{q+1} + x_3^{q+1} = 0$, has been studied in some detail leading to a geometric interpretation of some designs.
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1. Introduction.

The geometry of quadric varieties (hypersurfaces) in finite projective spaces of N dimensions has been studied by Primrose (12) and Ray-Chaudhuri (13). In this paper we study the geometry of another class of varieties, which we call Hermitian varieties which have many properties analogous to quadrics. Hermitian varieties are defined only for finite projective spaces for which the ground (Galois field) GF(q^2), has order q^2, where q is the power of a prime. If h is any element of GF(q^2), then \( \overline{h} = h^q \) is defined to be conjugate to h. Since \( h^{q^2} = h \), h is conjugate to \( \overline{h} \). A square matrix \( H = (h_{ij}) \), \( i, j = 0, 1, \ldots, N \) with elements from GF(q^2) is called Hermitian if \( h_{ij} = \overline{h_{ji}} \) for all \( i, j \). The set of all points in PG(N, q^2) whose row vectors \( x^T = (x_0, x_1, \ldots, x_N) \) satisfy the equation \( x^T H x(q) = 0 \) are said to form a Hermitian variety V_{N-1}, if H is Hermitian and \( x(q) \) is the column vector whose transpose is

\[
(x_0^q, x_1^q, \ldots, x_N^q).
\]

The properties of the curve \( x_0^{q+1} + x_1^{q+1} + x_2^{q+1} = 0 \) in PG(2, q^2), which is a Hermitian variety, were studied in some detail by one of the authors in (3). The present paper generalizes these results to N dimensions. The theory of tangent and polar hyperplanes of Hermitian varieties has been developed, and the sections of these varieties by hyperplanes have been studied. The number of points on a Hermitian variety has been obtained.

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It has been shown that if \( N = 2t+1 \) or \( 2t+2 \), a non-degenerate Hermitian variety \( V_{N-1} \) contains flat spaces of \( t \) dimensions and no higher. The number of such subspaces contained in \( V_{N-1} \) has been derived. Finally the geometry of the surface \( x_0^{q+1} + x_1^{q+1} + x_2^{q+1} + x_3^{q+1} = 0 \), has been studied in some detail, leading to a geometric interpretation of some designs. For example if \( q = 2 \), the surface contains 45 points and is ruled by 27 lines three of which pass through each point. Corresponding to any point \( P \) on the surface we get a set of 12 points which are joined to \( P \) by a line on the surface. The 45 sets so obtained form the blocks of a balanced incomplete block design with parameters \( v = b = 45, r = k = 12, \lambda = 3 \). There are many other interesting designs and configurations connected with Hermitian varieties. These will be discussed in a separate communication.

2. Correspondence between the elements of \( GF(q) \) and \( GF(q^2) \).

Let \( q = p^m \), where \( p \) is a prime number and \( m \) is a positive integer. Let \( GF(q) \) be a Galois field with \( q \) elements, and \( GF(q^2) \) an extension of \( GF(q) \). If \( \theta \) is a primitive element of \( GF(q^2) \), then the elements of \( GF(q^2) \) are

\[
(2.1) \quad 0, \, \theta, \, \theta^2, \ldots, \theta^{q^2-1} = 1.
\]

Any non-zero element \( x \) of \( GF(q^2) \) satisfies the fundamental equation

\[
(2.2) \quad x^{q^2-1} = 1.
\]

Writing

\[
(2.3) \quad \phi = \theta^{q+1},
\]

it follows that

\[
(2.4) \quad 0, \, \phi, \, \phi^2, \ldots, \phi^{q-1} = 1,
\]
are all different and are elements of $\text{GF}(q)$. Hence $\phi$ is a primitive element of $\text{GF}(q)$, and all the elements of $\text{GF}(q)$ are given by (2.4).

Corresponding to a given element $x$ of $\text{GF}(q^2)$, there is a unique element $y$ belonging to $\text{GF}(q)$, given by

$$y = x^{q+1}.$$  \hspace{1cm} (2.5)

But for a given non-zero $y$ belonging to $\text{GF}(q)$, there are precisely $q + 1$ distinct elements $x$ of $\text{GF}(q^2)$ which satisfy (2.5).

Thus if

$$y = \phi^i = \phi^{i(q+1)}, \quad 1 \leq i \leq q-1$$  \hspace{1cm} (2.6)

then

$$x = \phi^{i+j(q-1)}, \quad j = 1, 2, \ldots, q-1.$$  \hspace{1cm} (2.7)

If however $y$ is zero, then the corresponding element $x$ of $\text{GF}(q^2)$ is zero.

3. Conjugate elements of $\text{GF}(q^2)$.

The primitive element $\theta$ of $\text{GF}(q^2)$ satisfies a quadratic equation

$$x^2 - sx + t = 0,$$  \hspace{1cm} (3.1)

where $s$ and $t$ belong to $\text{GF}(q)$, and the left hand side of (3.1) is irreducible over $\text{GF}(q)$.

Using the relation $\theta^2 - s\theta + t = 0$, every element $x$ of $\text{GF}(q^2)$ can be expressed as

$$x = a + b\theta,$$  \hspace{1cm} (3.2)

Where $a$ and $b$ belong to $\text{GF}(q)$. We then define

$$\overline{x} = x^q,$$  \hspace{1cm} (3.3)
as the conjugate of \( x \). Since

\[(3.4) \quad x^q = x, \]

the conjugate of \( x \) is \( \bar{x} \). If \( x \) is given by (3.3), then

\[(3.5) \quad \bar{x} = (a+b\theta)^q = a+b\theta^q = a+b\bar{\theta}. \]

Since \( x \to x^q \) is an automorphism of \( GF(q^2) \), the second root of (3.1) is \( \theta^q \) or \( \bar{\theta} \). Hence

\[(3.6) \quad \theta + \bar{\theta} = s, \quad \theta \bar{\theta} = t, \]

\[(3.7) \quad x + \bar{x} = 2a + bs, \]

\[(3.8) \quad x \bar{x} = a^2 + abs + b^2t. \]

Hence the sum as well as the product of two conjugate elements of \( GF(q^2) \) belongs to \( GF(q) \).

It should be noted that the necessary and sufficient condition for any element of \( GF(q^2) \) to be self-conjugate is that it belongs to \( GF(q) \).

The elements \( s \) and \( t \) of \( GF(q) \) appearing in the equation (3.1) are non-zero. From (3.6), \( t = \theta^{q+1} \neq 0 \) since \( \theta \) is a primitive element of \( GF(q^2) \). Again if \( s = 0 \), it would follow from (3.6) that \( \theta + \theta^q = 0 \), i.e., either \( \theta = 0 \) or \( \theta^{q-1} = -1 \). Obviously \( \theta \neq 0 \). Also \( \theta^{q-1} \neq -1 \), otherwise \( \theta^{2q-2} = 1 \), which is contradicted by the fact that \( \theta \) is a primitive element of \( GF(q^2) \).

Lemma (3.1). If \( h \) is a non-zero element of \( GF(q^2) \), we can find a non-zero element \( \lambda \) of \( GF(q^2) \) such that \( h\lambda + \bar{h}\lambda \neq 0 \).

Let \( h = a+b\theta \) and \( \lambda = u+v\theta \), where \( a, b, u, v \) belong to \( GF(q) \). Then using (3.6)

\[ h\lambda + \bar{h}\lambda = (2a + bs)u + (2bt + as)v. \]
Case I. If $2a + bs \neq 0$, we can choose $u = 1$, $v = 0$, i.e. $\lambda = 1$.

Case II. If $2a + bs = 0$, then $a \neq 0$, since $a = 0$ would make $b = 0$, contradicting $h \neq 0$. Now $(2bt + as) = a(s^2 - 4t)/s \neq 0$, since $s^2 - 4t = 0$ is the condition for the roots of $(3,1)$ to coincide i.e. for $\theta$ to be equal to $0^q$, which is obviously false since $\theta$ is a primitive element of $\text{GF}(q^2)$. Hence in this case we can choose $u = 0$, $v = 1$, i.e. $\lambda = 0$.

4. Hermitian matrices and Hermitian forms.

A square matrix

(4.1) \[ H = \begin{pmatrix} h_{ij} \end{pmatrix}, \quad i, j = 0, 1, \ldots, N \]

with elements from $\text{GF}(q^2)$, will be defined to be Hermitian if

(4.2) \[ h_{ij} = \overline{h_{ji}} \]

for all $i, j$. Hence the diagonal elements of a Hermitian matrix belong to $\text{GF}(q)$, and symmetrically situated off diagonal elements are conjugate to each other.

Given a matrix $A = \begin{pmatrix} a_{ij} \end{pmatrix}$ with elements from $\text{GF}(q^2)$ we define the conjugate of $A$ by

(4.3) \[ A^q = \begin{pmatrix} a_{ij}^q \end{pmatrix} = \begin{pmatrix} \overline{a_{ij}} \end{pmatrix}. \]

Clearly, the conjugate of $A^q$ is $A$ itself. Thus the relation of conjugacy is symmetric. So far as this definition is concerned $A$ may or may not be a square matrix. In particular $A$ may be a row vector or a column vector. Clearly, the necessary and sufficient condition for $A$ to be self conjugate is that all its elements belong to $\text{GF}(q)$.

The transpose of $A$ will be denoted by $A^T$. Clearly the transpose of the conjugate is the conjugate of the transpose, i.e.
\[(4.4)\quad A^T(q) = A(q)^T.\]

Lemma \((4.1)\). A square matrix \(G = ((g_{ij}))\) with elements from \(GF(q^2)\) is Hermitian if and only if
\[(4.5)\quad g(q) = g^T.\]

The proof is obvious.

Lemma \((4.2)\). Suppose \(A\) and \(B\) are two matrices of order \(m \times n\) and \(n \times h\) with elements from \(GF(q^2)\), and \(C = AB\), then
\[(4.6)\quad c(q) = A(q)B(q).\]

Proof: Now \(C = ((c_{ik}))\), where
\[
c_{ik} = \sum_j a_{ij}b_{jk}.
\]
\[
c_{ik}^q = c_{ik}^q = \left(\sum_{j=1}^{n} a_{ij}^q b_{jk}^q\right)^q = \sum_{j=1}^{n} a_{ij}^q b_{jk}^q = \sum_{j=1}^{n} \bar{a}_{ij} \bar{b}_{jk}.
\]
Hence by definition \(c(q) = A(q)B(q)\).

Lemma \((4.3)\). If \(H\) is a Hermitian matrix of order \(N+1\), and \(A\) is any matrix of order \((N+1) \times m\) with elements from \(GF(q^2)\), then
\[G = A^T H A(q),\]
is a Hermitian matrix of order \(m\).

From Lemmas \((4.1)\) and \((4.2)\), and the equation \((4.4)\),
\[G^T = A(q)^T H^T A = A(q)^T H(q) A = G(q),\]
The required result follows from Lemma \((4.1)\).

Corollary. If \(x\) is a \((N+1) \times 1\) column vector, and \(H\) is a Hermitian matrix of order \(N+1\) then \(x^T H x(q)\) is an element of \(GF(q)\).

Proof. \(x^T H x(q)\) is a \(1 \times 1\) Hermitian matrix. Hence it is a self conjugate element of \(GF(q^2)\).
The elements of all the vectors and matrices which we shall consider belong to $\mathbb{GF}(q^2)$. When we speak of the dependence or independence of a set of vectors we shall mean dependence and independence over $\mathbb{GF}(q^2)$. The rank of a vector space or the rank of a matrix shall mean rank over $\mathbb{GF}(q^2)$.

Two Hermitian matrices $H$ and $G$ of the same order $N+1$ with elements from $\mathbb{GF}(q^2)$, will be called equivalent if we can find a non-singular square matrix $A$, with elements from $\mathbb{GF}(q^2)$ such that

$$A^T H A^{(q)} = G.$$ 

If $H$ and $G$ are equivalent we may write $H \sim G$. It is readily seen that this relation satisfies the three axioms of equivalence i.e. (i) $H \sim H$, (ii) if $H \sim G$, then $G \sim H$, (iii) if $H \sim G$ and $G \sim K$, then $H \sim K$.

The above follows by noting that (i) $I^T = I^{(q)} = I$ where $I$ is the unit matrix of order $N+1$ (ii) $(A^T)^{-1} = (A^{-1})^T$, $(A^{(q)})^{-1} = (A^{-1})^{(q)}$ (iii) $B^T A^T = (AB)^T$, $A^{(q)} B^{(q)} = (AB)^{(q)}$ from Lemma (4.2).

Theorem (4.1). A Hermitian matrix $H$ of order $N+1$ and rank $r > 0$, with elements from $\mathbb{GF}(q^2)$, is equivalent to a diagonal matrix of order $N + 1$, the first $r$ diagonal elements of which are unity and the rest zero.

(a) We can permute the columns of $H$ in any desired manner and permute its rows in the corresponding manner by post multiplying $H$ with a suitable permutation matrix $P = P^{(q)}$, and premultiplying $H$ with $P^T$. Hence by such operations we can rearrange the rows and columns of $H$ so that all null rows and columns are at the end. The transformed matrix is equivalent to $H$.

(b) We shall denote by $E_{uv}(\lambda)$, a matrix of order $N+1$, for which each diagonal element is unity, the element in the $u$-th row and $v$-th column is $\lambda$, and all other elements are zero. Such a matrix will be called an elementary
matrix of order $N+1$. Clearly

$$E^T_{uv}(\lambda) = E_{vu}(\lambda).$$

The effect of premultiplying $H$ with $E^T_{uv}(\lambda)$ is to replace the $v$-th row of $H$, by the sum of the $v$-th row and the $u$-th row multiplied by $\lambda$. The effect of post multiplying the matrix so obtained with $E^{(q)}_{uv}(\lambda)$ is to replace its $v$-th column by the sum of the $v$-th column and the $u$-th column multiplied with $\overline{\lambda}$. Thus if $H$ is given by $(4,1)$,

$$E^T_{uv}(\lambda) H E^{(q)}_{uv}(\lambda) = G = ((g_{ij}))$$

where

$$g_{v-1,v-1} = h_{v-1,v-1} + \lambda h_{u-1,v-1} + \overline{\lambda} h_{v-1,u-1},$$

$$g_{v-1,j} = h_{v-1,j} + h_{u-1,j}, \quad g_{j,v-1} = h_{j,v-1} + h_{j,u-1}; \quad j \neq v-1,$$

$$g_{ij} = h_{ij}, \quad (i \neq v-1, j \neq v-1).$$

If the $v$-th row and column of $H$ are non-null but the diagonal element $h_{v-1,v-1} = 0$, then we can find non-zero conjugate elements $h_{u-1,v-1}$ and $h_{v-1,u-1}$ belonging to the $(u-1)$-th row and column respectively. By Lemma (3.1) there exists an element $\lambda$ of $GF(q^2)$ such that

$$\lambda h_{u-1,v-1} + \overline{\lambda} h_{v-1,u-1} \neq 0.$$ Then the matrix $E^T_{uv}(\lambda) H E^{(q)}_{uv}(\lambda)$ is equivalent to $H_1$ and the element $g_{v-1,v-1}$ in the $v$-th row and column is non-zero.

(c) By using (a) and (b) suppose $H$ has already been transformed to an equivalent form such that the first row and column are non-null and $h_{00} \neq 0$. We now reduce the non-diagonal elements of the first row and column to zero, in $N$ steps, the $(v-1)$-th step consisting of premultiplying the matrix obtained in the previous step by $E^T_{1v}(-h_{v-1,0}/h_{00})$ and post multiplying it by $E^{(q)}_{1v}(-h_{v-1,0}/h_{00})$, $v = 2, 3, \ldots, N+1$.  

8
If any null rows and columns appear they are transferred to the end by using (a). If now the diagonal element in the second row is zero, (and the second row is non-null) we can make it non-zero by using (b). Then as in (c), we can reduce the non-diagonal elements of the first row and column to zero. Proceeding in this manner we reduce $H$ to an equivalent diagonal matrix $D$, in which the first $r_0$ diagonal elements are non-null, and the remaining diagonal elements are null. Since all our transformations have been rank-preserving $r_0 = r$.

(d) Since $D$ is Hermitian the diagonal elements belong to $GF(q)$. Let the $i$-th diagonal element be $d_i$. From the correspondence described in section 2, we can find an element $\alpha_i$ of $GF(q^2)$ such that $d_i = \alpha_i^{q+1} = \alpha_i \bar{\alpha}_i$ ($i = 0, 1, \ldots, r-1$). We denote by $\Delta(\alpha_i)$ the diagonal matrix, whose $i$-th diagonal element is $\alpha_i$ and the other diagonal elements are zero. We can finally reduce $D$ to the form desired in the theorem in $r$ steps, the $(i+1)$-th step consisting of premultiplying the matrix obtained in the previous step by $D^T(\alpha_i)^{-1}$ and post multiplying it by $D(q) (\alpha_i)^{-1}$, ($i = 0, 1, \ldots, r-1$). This completes the proof of the theorem.

Let $\mathbf{x}^T$ be the row vector $(x_0, x_1, \ldots, x_N)$, and $\mathbf{x}$ the corresponding column vector, where $x_0, x_1, \ldots, x_N$ are indefinites. Then the form $\mathbf{x}^T H \mathbf{x}(q)$ is called a Hermitian form, if $H$ is a Hermitian matrix. $H$ is called the matrix of the form. The order and rank of the form are defined to be the order and rank of $H$. Note that $\mathbf{x}^T H \mathbf{x}(q)$ is a homogeneous polynomial of the $(q+1)$-th degree in the indefinites $x_0, x_1, \ldots, x_N$.

The Hermitian form $\mathbf{x}^T H \mathbf{x}(q)$ is transformed into $\mathbf{y}^T A^T H A(q) \mathbf{y}(q)$ by the linear transformation $\mathbf{x} = A \mathbf{y}$. Two Hermitian forms are defined to be equivalent if one can be transformed to the other by a non-singular linear
transformation. Clearly the necessary and sufficient condition for two Hermitian forms to be equivalent is that their matrices are equivalent.

**Corollary** The Hermitian form $\mathbf{x}^T \mathbf{H} \mathbf{x}^{(q)}$ of order $N+1$ and rank $r$ can be reduced to the canonical form $y_1 \bar{y}_1 + y_2 \bar{y}_2 + \ldots + y_r \bar{y}_r$ by a suitable non-singular linear transformation $\mathbf{x} = \mathbf{A} \mathbf{y}$.

5. **Hermitian varieties in $\text{PG}(N, q^2)$**

We denote by $\text{PG}(N, s)$ the finite projective space of $N$ dimensions over the Galois field $\text{GF}(s)$ where $s$ is a prime power. The points of the space can be made to correspond to ordered $(N+1)$-plets

$$(5.1) \quad (x_0, x_1, \ldots, x_N),$$

where the $x_i$'s belong to $\text{GF}(s)$, and are not all zero. The ordered $n$-plets $(x_0, x_1, \ldots, x_N)$ and $(x_0^*, x_1^*, \ldots, x_N^*)$ correspond to the same point if and only if there exists a non-zero element $\rho$ of $\text{GF}(s)$ such that

$$\rho x_i^* = x_i^* \quad i = 0, 1, \ldots, N.$$

If $P$ is the point corresponding to (5.1), then the row vector $\mathbf{x} = (x_0, x_1, \ldots, x_N)$ is called the row vector of $P$, and its transpose $\mathbf{x}^T$ is called the column vector of $P$. The elements $x_0, x_1, \ldots, x_N$ are called the coordinates of $P$.

If $C$ is a matrix with $N+1$ rows and of rank $N-m$ with elements from $\text{GF}(s)$, then the set of points whose row vectors satisfy

$$(5.2) \quad \mathbf{x}^T \mathbf{C} = 0,$$

is called an $m$-**flat** or a linear subspace of $m$ dimensions, and (5.2) is called the equation of the $m$-flat. Points are linear subspaces of zero dimensions. Linear subspaces of 1, 2 and $N-1$ dimensions are respectively
called lines, planes, and hyperplanes.

A set of points will be said to be dependent or independent according as the corresponding row (column) vectors are dependent or independent. Any \( m + 1 \) independent points determine a unique \( m \)-flat containing them.

Let \( E_i \) be the point for which the \((i+1)\)-th coordinate is unity, and other coordinates are zero \((i = 0, 1, \ldots, N)\). Also let \( E \) be the point all of whose coordinates are unity. Then \( E_0, E_1, \ldots, E_N \) are called the fundamental points and \( E \) is called the unit point. Clearly any \( N \) of the \( N+1 \) points \( E_0, E_1, \ldots, E_N \), \( E \) are independent. Together they are said to constitute the reference system.

Let \( A \) be an \((N+1) \times (N+1)\) non-singular matrix with elements from \( GF(s) \). Then the homogeneous linear transformation

\[
\begin{align*}
\mathbf{y} &= A \mathbf{x} \\
(5.3)
\end{align*}
\]

defines a transformation of coordinates. If \( \mathbf{x} \) is the original column vector of \( P \), the transformed column vector is \( \mathbf{y} \). This transformation defines new fundamental points \( F_0, F_1, \ldots, F_N \) and a new unit point \( F \). Their transformed coordinates are \((1,0,\ldots,0)\), \((0,1,\ldots,0)\), \((0,0,\ldots,1)\) and \((1,1,\ldots,1)\) and the original coordinates can be calculated from (5.3). An important theorem states that given any \( N+2 \) points \( P_0, P_1, \ldots, P_N \) and \( P \), no \( N+1 \) of which are dependent, there exists a unique linear transformation, which would make \( P_0, P_1, \ldots, P_N \) the fundamental points and \( P \) the unit point. Thus in \( PG(N,s) \) any \( N+2 \) points no \( N+1 \) of which are dependent may be chosen as the fundamental points and the unit point. This choice uniquely determines the coordinates of all other points (up to a non-zero multiple of \( GF(s) \)). Projective geometry studies those properties which are invariant under linear homogeneous transformations and are thus independent of the choice of a
reference system. An excellent account of finite projective spaces will be found in (1,10,15).

In particular let us choose \( s = q^2 \), where \( q \) is a prime power and consider the finite projective space \( \text{PG}(N,q^2) \). If \( H \) is a Hermitian matrix of order \( N+1 \) and rank \( r \) with elements from \( \text{GF}(q^2) \) then the set of points whose coordinates satisfy the \((q+1)\)-th degree equation

\[
X^T H X(q) = 0,
\]

are said to be the points of a Hermitian variety \( V_{N-1} \) of \( N-1 \) dimensions and rank \( r \). The equation (5.4) is said to be the equation of \( V_{N-1} \). If we apply the linear transformation (5.3) the new equation of \( V_{N-1} \) becomes

\[
Y^T A^T H A(q) Y = 0.
\]

Now \( A^T H A(q) \) is a Hermitian matrix of rank \( r \) equivalent to \( H \). Hence the rank of a Hermitian variety is invariant under a non-singular linear transformation. It follows from Theorem (4.1) and its corollary that by a suitable choice of the frame of reference, the equation of a Hermitian variety of \( N-1 \) dimensions and rank \( r \) can be reduced to the canonical form

\[
x_0\bar{x}_0 + x_1\bar{x}_1 + \cdots + x_{r-1}\bar{x}_{r-1} = 0.
\]

A Hermitian variety \( V_{N-1} \) of \( N-1 \) dimensions is said to be non-degenerate if its rank is \( N+1 \). Now \( \text{PG}(N,q^2) \) contains linear subspaces of dimensions \( r<N \). Let \( \Sigma_r \) be such a subspace. Then each point of \( \Sigma_r \) can be characterized by a set of \( r+1 \) coordinates \((y_0, y_1, \ldots, y_r)\). For example if we choose the frame of reference so that the equations of \( \Sigma_r \) are \( y_{r+1} = y_{r+2} = \cdots = y_N = 0 \); then if the point \( P \), when regarded as a point of \( \text{PG}(N,q^2) \), has the row vector

\[
X^T = (y_0, y_1, \ldots, y_r, 0, 0, \ldots 0);
\]

then regarded as a point of \( \Sigma_r \) it has row vector \( \bar{X}^T = (y_0, y_1, \ldots, y_r) \). Then if \( H^* \) is a Hermitian matrix of order \( r+1 \), the points of \( \Sigma_r \) which satisfy the equation \( \bar{X}^T H^* X(q) = 0 \) will be said to form the Hermitian variety \( V_{r-1} \) of dimensions \( r-1 \) and rank equal to the rank of \( H^* \).

We shall in what follows always denote
a Hermitian variety with the letter \( V \) and choose our notation so that the subscript of \( V \) denotes the number of dimensions of \( V \).

Let us consider the special case \( N = 1 \). Our space is now the projective line \( \text{PG}(1,q^2) \). Let \( V_0 \) be a non-degenerate Hermitian variety in this space. Then the equation of \( V_0 \) can be taken as

\[
(5.1') \quad x_0 x_1 + x_1 x_2 = 0 \quad \text{or} \quad x_0^{q+1} + x_1^{q+1} = 0.
\]

The point \((0,1)\) obviously does not lie on \( V_1 \). Hence for points satisfying \((5.7)\), \( x_0 \neq 0 \). Now \((5.7)\) gives \( x_1 / x_0 \)^{(q+1)} = -1 \). The correspondence described in section 2, shows that there are precisely \( q + 1 \) values of \( x_1 / x_0 \) which satisfy \((5.7)\). Hence a non-degenerate Hermitian variety \( V_0 \) on a projective line (over a field of order \( q^2 \)) contains exactly \( q+1 \) distinct points. Again suppose the rank of \( V_0 \) is one. Then by a suitable choice of the frame of reference its equation can be reduced to \( x_0^{q+1} = 0 \). The only point satisfying this equation is \((0,1)\). Hence in this case \( V_0 \) consists of a single point.

The properties of the curve

\[
(5.8) \quad x_0^{q+1} + x_1^{q+1} + x_2^{q+1} = 0,
\]

were studied in some detail in \((5)\). In particular it was shown that a non-degenerate Hermitian variety \( V_1 \) in \( \text{PG}(2,q^2) \) has exactly \( q^3 + 1 \) points.

6. Conjugate points, polar spaces and tangent spaces.

Consider a Hermitian variety \( V_{N-1} \) with equation \((5.4)\). A point \( C \) with row vector \( C^T = (c_0, c_1, \ldots, c_N) \) will be called a singular point of \( V_{N-1} \) if \( C^T H = 0 \) or equivalently \( H C^{(q)} = 0 \).

Of course a singular point must lie on \( V_{N-1} \). A point of \( V_{N-1} \) which is not singular is called a regular point of \( V_{N-1} \). A point \( C \) will be called a
non-singular point if it is not a singular point of $V_{N-1}$. Thus a non-
singular point may be a regular point of $V_{N-1}$ or a point not lying on $V_{N-1}$.

A non-degenerate Hermitian variety cannot possess a singular point,
since in this case there cannot exist a non-null $c^T$ satisfying $c^T H = 0$ as
$H$ is non-singular. If $V_{N-1}$ is degenerate let $r < N+1$ be the rank of $H$.
Then (6.1) has $N+1-r$ independent solutions

\[(6.1) \quad c^T, \; c^T, \; \ldots, \; c^T_{N+1-r}.\]

Thus the singular points of $V_{N-1}$ are the points of the $(N-r)$-flat
determined by the points with row vectors (6.1). This will be said to
constitute the singular space of $V_{N-1}$.

All points whose row vectors satisfy the equation

\[(6.2) \quad x^T H c(q) = 0 ,\]

constitute the polar space of the point $C$ with row vector $c^T$. When $C$ is
a singular point of $V_{N-1}$ the polar space of $C$ is identical with the whole
space PG$(N, q^2)$. When however $C$ is non-singular the rank of $H c(q)$ is one
and (6.2) is the equation of a hyperplane, which may be called the polar
hyperplane of $C$. If $d^T$ is the row vector of a point $D$ then the necessary
and sufficient condition for the polar space of $C$ to pass through $D$ is
$d^T H c(q) = 0$, which is equivalent to $c^T H d(q) = 0$. This shows that
if the polar space of $C$ passes through $D$, then the polar space of $D$ passes
through $C$. Two such points whose polar spaces mutually pass through each
other are said to be conjugate to each other with respect to $V_{N-1}$. In
case $V_{N-1}$ is degenerate the polar space of $C$ passes through every singular
point of $V_{N-1}$ and thus contains the singular space of $V_{N-1}$. Hence any two
points at least one of which is singular, are always conjugate to one
another.
The condition for the point \( C \) to be self conjugate i.e. to lie on its own polar space is that \( c^T H c(q) = 0 \). Hence a point is conjugate to itself with respect to \( V_{N-1} \), if and only if \( C \) lies on \( V_{N-1} \).

The polar hyperplane of a regular point \( C \) of \( V_{N-1} \) is defined to be the tangent hyperplane to \( V_{N-1} \) at \( C \). The tangent hyperplane is defined only for regular points of \( V_{N-1} \) and when \( V_{N-1} \) is degenerate contains the singular space of \( V_{N-1} \).

When \( V_{N-1} \) is non-degenerate, there is no singular point. To every point there corresponds a unique polar hyperplane, and at every point of \( V_{N-1} \) there is a unique tangent hyperplane.

7. Sections of Hermitian varieties with flat spaces.

Let \( V_{N-1} \) be a Hermitian variety of rank \( r \) in \( \mathbb{P} G(N, q^2) \) with equation (5.4). The set of points common to \( V_{N-1} \) and the \( m \)-flat \( \Sigma_m \) with equation (5.2), is defined to be the section of \( V_{N-1} \) by \( \Sigma_m \). If \( m = N \), then \( \Sigma_m \) is the whole space and the section is \( V_{N-1} \) itself. Let \( m < N \). Let \( \Sigma_m \) be defined by \( m+1 \) independent points \( F_0, F_1, \ldots, F_m \). We can find a non-singular linear transformation \( \mathcal{X} = A \mathcal{X} \) such that \( F_0, F_1, \ldots, F_m \) become the fundamental points of the reference system. Then the points of \( \Sigma_m \) will satisfy the equations

\[
(7.1) \quad y_{m+1} = y_{m+2} = \cdots = y_{N-1} = 0,
\]

while the equation of \( V_N \) will become

\[
(7.2) \quad x^T G y(q) = 0.
\]

where \( G \) is a Hermitian matrix equivalent to \( H \). Writing (7.2) in full we have
\[(7.3) \quad \sum_{j=0}^{N} \sum_{i=0}^{N} s_{ij} y_{i}^{(q)} y_{j} = 0.\]

Hence the points common to \(\Sigma_{m}\) and \(V_{N-1}\) satisfy \((7.1)\) and

\[(7.4) \quad \sum_{j=0}^{m} \sum_{i=0}^{m} s_{ij} y_{i}^{(q)} y_{j} = 0.\]

Let \(G^{*}\) be the matrix obtained from \(G\) by retaining only the first \(m+1\) rows and columns of \(G\). Evidently \(G^{*}\) is Hermitian and the points on the section of \(V_{N-1}\) by \(\Sigma_{m}\) satisfy \((7.1)\) and

\[(7.5) \quad \nabla^{*} G^{*} \nabla^{*} = 0,\]

where \(\nabla^{*} = (y_{0}, y_{1}, \ldots, y_{m})\). Regarding \(\Sigma_{m}\) as a projective space of \(m\) dimensions, it is clear that the section of a Hermitian variety \(V_{N-1}\) in \(PG(N,q^{2})\) by a flat space \(\Sigma_{m}\) of \(m\) dimensions, is a Hermitian variety \(V_{m-1}\) contained in \(\Sigma_{m}\). Clearly the rank of \(V_{m-1}\) cannot exceed \(m+1\). However this rank could be less and in particular it may happen that \(G^{*}\) is null so that every point of \(\Sigma_{m}\) belongs to the section, which therefore coincides with the section. In this case the flat space \(\Sigma_{m}\) is contained in \(V_{N-1}\). We will therefore adopt the convention that a flat-space \(\Sigma_{m}\) of dimensions \(m\), can be regarded as a Hermitian variety \(V_{m-1}\) of dimensions \(m-1\) and rank zero.

As a particular case let \(m = 1\). Then \(\Sigma_{m}\) is a line. Since the intersection of a line with \(V_{N-1}\) must be a Hermitian variety \(V_{0}\) of rank 2, 1 or 0 we see that a line intersects \(V_{N-1}\) in \(i\) \(q+1\) points \((i)\) a single point or \((i i)\) lies completely in \(V_{N-1}\). We shall now prove the following theorem:

Theorem \((7.1)\). If the Hermitian variety \(V_{N-1}\) with equation \(X^{*} H X^{(q)} = 0\) is degenerate with rank \(r < N+1\) and \(\Sigma_{r-1}\) is a flat space of dimensions \(r-1\) disjoint with the singular space \(\Sigma_{N-r}\) of \(V_{N-1}\), then \(V_{N-1}\)
and $\Sigma_{r-1}$ intersect in a non-degenerate Hermitian variety $V_{r-2}$ contained in $\Sigma_{r-1}$.

Let $F_0, F_1, \ldots, F_{r-1}$ be any $r$ independent points in $\Sigma_{r-1}$. Also let $F_r, F_{r+1}, \ldots, F_N$ be any $N-r+1$ independent points in $\Sigma_{N-r}$. Now make a non-singular linear transformation $X = AY$ such that $F_0, F_1, \ldots, F_{r-1}, F_r, \ldots, F_N$ become the fundamental points of the reference system. The equation of $V_{N-1}$ now becomes $Y^T G Y^{(q)} = 0$, where $G = (g_{ij})$, $(i, j = 0, 1, \ldots, N)$, is equivalent to $H$ and is therefore of rank $r$. Using $y$-coordinates the row vector of $F_i$ is $e_i^T$ for which the $(i+1)$-th coordinate is equal to unity and all other coordinates are equal to zero. The condition for $F_i$ and $F_j$ to be conjugate to each other is $e_i^T G e_j^{(q)} = 0$ or $g_{ij} = 0$.

Since $F_r, F_{r+1}, \ldots, F_N$ are singular points of $V_{N-1}$, $F_i$ and $F_j$ are conjugate if (i) $0 \leq i \leq r-1, r \leq j \leq N$ (ii) $r \leq i \leq N, r \leq j \leq N$. This shows that we may write

$$G = \begin{bmatrix} G^* & 0 \\ 0 & 0 \end{bmatrix},$$

where $G^*$ is a Hermitian matrix of order $r$. Since the rank of $G$ is $r$, the rank of $G^*$ must also be equal to $r$. Now the points of $V_{r-2}$ satisfy

$$Y^*T G^* Y^{*(q)} = 0; \quad Y_r = Y_{r+1} = \cdots = Y_N = 0,$$

where $Y^* = (y_0, y_1, \ldots, y_{r-1})$. Hence $V_{r-2}$ is a Hermitian variety of rank $r$ contained in $\Sigma_{r-1}$, and is therefore non-degenerate.

**Corollary** If $V_{N-1}$, $\Sigma_{N-r}$, $\Sigma_{r-1}$ and $V_{r-2}$ have the same meanings as in the theorem, $C^*$ is a point on $V_{r-2}$ and $\Sigma_{r-2}$ is the tangent space to $V_{r-2}$ at $C^*$, then the tangent space to $V_{N-1}$ at $C^*$, is the flat space $\Sigma_{N-1}$ of $N-1$ dimensions containing $\Sigma_{r-2}$ and $\Sigma_{N-r}$.
Theorem (7.2). If \( V_{N-1} \) is a degenerate Hermitian variety of rank \( r < N+1 \), in \( \text{PG}(N, q^2) \), and if \( C \) is any point belonging to the singular space of \( V_{N-1} \) and \( D \) is an arbitrary point of \( V_{N-1} \), then any point on the line \( CD \) belongs to \( V_{N-1} \).

Let the equation of \( V_{N-1} \) be \( X^T H X(q) = 0 \), and let \( c^T \) and \( d^T \) be the row vectors of \( C \) and \( D \) respectively. Now \( C \) and \( D \) are self conjugate, and also \( C \) and \( D \) are conjugate to each other. Hence

\[
(7.1) \quad c^T H c(q) = 0, \quad d^T H d(q) = 0, \quad c^T H d(q) = 0, \quad d^T H c(q) = 0.
\]

If \( B \) is any point on the line \( CD \), then its row vector \( b^T \) must be of the form \( l_1 c^T + l_2 d^T \) or \( (l_1 c + l_2 d)^T \). But

\[
(l_1 c + l_2 d)^T H (l_1 c + l_2 d)(q) = 0,
\]

which proves the theorem.

Corollary. If \( V_{N-1} \) is as in the theorem and \( V_{r-2} \) is the section of \( V_{N-1} \) by an \((r-1)\)-flat \( \Sigma_{r-1} \) disjoint with the singular space \( \Sigma_{N-r} \) of \( V_{N-1} \), then every point of \( V_{N-1} \) lies on some line joining a point of \( \Sigma_{N-r} \) with a point of \( V_{r-2} \).

From the theorem if \( C \) is a point of \( \Sigma_{N-r} \) and \( D \) is a point of \( V_{r-2} \), then any point on the line joining \( CD \) belongs to \( V_{N-1} \).

Conversely, let \( D \) be any point on \( V_{N-1} \). We have to show that it lies on some line joining a point of \( \Sigma_{N-r} \) with a point of \( V_{r-2} \). This is obviously true if \( D \) belongs to \( \Sigma_{N-r} \) or \( \Sigma_{r-1} \). We may therefore suppose that \( D \) does not lie on either of these flat spaces. Let \( \Sigma_r \) be the \( r \)-flat containing \( D \) and \( \Sigma_{r-1} \). Then \( \Sigma_r \) intersects \( \Sigma_{N-r} \) in a point \( C_0 \) and \( C_0 D \) intersects \( \Sigma_{r-1} \) in a point \( P_0 \). From the theorem \( P_0 \) must be on \( V_{N-1} \) and therefore on \( V_{r-2} \). This proves the corollary.
We shall next study the nature of the section of a Hermitian variety with a tangent space. We shall first prove the following:

Theorem (7.3). The tangent spaces at two distinct regular points \( A \) and \( B \) of a Hermitian variety \( V_{N-1} \) are identical if and only if the line joining \( A \) and \( B \) meets the singular space of \( V_{N-1} \) in a point. In particular if \( V_{N-1} \) is non-degenerate then the tangent spaces at \( A \) and \( B \) must be distinct.

Let the equation of \( V_{N-1} \) be \( x^T H x(q) = 0 \), and let the row vectors of \( A \) and \( B \) be \( a^T \) and \( b^T \). Then the tangent spaces at \( A \) and \( B \) have the equations \( x^T H a(q) = 0 \) and \( x^T H b(q) = 0 \). Hence the two tangent spaces are identical if and only if there exists a non-zero element \( \ell \) of \( GF(q^2) \) such that

\[
(7.2) \quad H a(q) = \ell H b(q) \quad \text{or} \quad (a - \ell(q)b) H = 0.
\]

First suppose \( V_N \) is non-degenerate. In this case \( H \) is non-singular. Hence the homogeneous linear equations \( c^T H = 0 \) can only be satisfied by \( c^T = 0 \). Hence \( a^T = \ell(q)b^T \). The vectors of \( A \) and \( B \) differ only by a non-zero multiple of an element of \( GF(q^2) \). Hence the points \( A \) and \( B \) must be identical.

Now suppose that \( V_{N-1} \) is degenerate and of rank \( r < N + 1 \). The singular space \( \Sigma_{N-r} \) of \( V_{N-1} \) consists of all points with row vector \( c^T \) satisfying \( c^T H = 0 \). Hence \( (7.2) \) implies that \( a^T - \ell(q)b^T = c^T \), where \( c^T \) is the row vector of some point \( C \) belonging to \( \Sigma_{N-r} \). Hence the line \( AB \) meets \( \Sigma_{N-r} \) at \( C \).

**Corollary** Let \( \Sigma_{N-r+1} \) be the flat space of \( N-r+1 \) dimensions containing a regular point \( A \) and the singular space \( \Sigma_{N-r} \) of a degenerate Hermitian variety \( V_{N-1} \) of rank \( r < N + 1 \). Then any point \( B \) on \( \Sigma_{N-r+1} \) (which is not on
\( \Sigma_{N-2} \) has the same tangent space as \( A \).

Theorem (7.4). Given a non-degenerate Hermitian variety \( V_{N-1} \) the tangent space at a point \( C \) of \( V_{N-1} \) intersects \( V_{N-1} \) in a degenerate Hermitian variety \( V_{N-2} \) of rank \( N-1 \) contained in \( \Sigma_{N-1} \). The singular space of \( V_{N-2} \) consists of the single point \( C \).

Let the equation of \( V_{N-1} \) be \( x^T H x(q) = 0 \). Let \( \Sigma_{N-1} \) be the tangent space to \( V_{N-1} \) at \( C \). Let \( F_0 = C, F_1, \ldots, F_{N-1} \) be \( N \) independent points in \( \Sigma_{N-1} \). We can find a non-singular linear transformation \( y = A x \), such that \( F_0, F_1, \ldots, F_{N-1} \) become the fundamental points of the reference system. Then the equation of \( \Sigma_{N-1} \) becomes \( x_N = 0 \), and the equation of \( V_{N-1} \) becomes \( y^T G y(q) = 0 \), where \( G = ( (g_{ij}) ), (i,j = 0,1, \ldots, N) \), is Hermitian and of rank \( N+1 \). Since \( F_0 \) is self conjugate and is conjugate to \( F_1, F_2, \ldots, F_{N-1} \), we have \( g_{0j} = 0, j = 0,1, \ldots, N-1 \), and \( g_{i0} = 0, i = 0,1, \ldots N-1 \). We can therefore write

\[
(7.3) \quad G = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & g_{ON} \\
0 & \ddots & & & & \\
0 & & \ddots & & & \\
\vdots & & & \ddots & & \\
0 & & & & \ddots & \ddots \\
g_{NO} & g_{NL} & g_{N2} & \cdots & g_{N-1,N} & g_{NN}
\end{bmatrix},
\]

where \( G^{**} = ( (g_{ij}) ), i,j = 1,2, \ldots, N-1 \). Now \( g_{ON} \) and \( g_{NO} \) are non-null since \( G \) is non-singular. Also

\[
(7.4) \quad \det G = -g_{ON} g_{NO} \det G^{**}.
\]

It follows that \( \det G^{**} \) is non-null so that the rank of \( G^{**} \) is \( N-1 \). Regarding \( \Sigma_{N-1} \) as a projective space of \( N-1 \) dimensions, we have seen that
the equation of the section \( V_{N-2} \) is \( \mathbf{y}^* G^* \mathbf{y}(q) = 0 \), where \( \mathbf{y}^* = (y_0, y_1, \ldots, y_{N-1}) \) and \( G^* \) is the Hermitian matrix obtained by retaining only the first \( N \) rows and columns of \( G \). Since \( G^* \) is the same as \( G^{**} \) except that it has an additional null row and column, \( \text{rank } G^* = \text{rank } G^{**} = N-1 \).

The row vector of \( C = F_0 \) when regarded as a point of projective space.
\( \Sigma_{N-1} \) is \( \Sigma_0^* = (1,0, \ldots, 0) \). Since \( \Sigma_0^* G^* = 0 \), \( C \) is a singular point of \( V_{N-1} \). Since the rank of \( V_{N-2} \) is \( N-1 \), the singular space has dimension 0, and must therefore consist of the single point \( C \).

Corollary. Let \( V_{N-1} \) be a degenerate Hermitian variety of rank \( r < N+1 \), with singular space \( \Sigma_{N-r} \). Let \( \Sigma_{N-1} \) be the tangent space to \( V_{N-1} \) at a regular point \( C \). Then \( \Sigma_{N-1} \) intersects \( V_{N-1} \) in a Hermitian variety \( V_{N-2} \) of \( N-2 \) dimensions and rank \( r-1 \), whose singular space is the \((N-r+1)\)-flat \( \Sigma_{N-r+1} \) containing \( C \) and \( \Sigma_{N-r} \).

8. Number of points on a Hermitian variety. Let \( S_{N+1}(q^2) \) denote the vector space of row vectors of order \( N+1 \), with elements from \( GF(q^2) \), and let \( S_{N+1}(q) \) have a similar meaning with relation to \( GF(q) \). To any vector \( \mathbf{x}^T = (x_0, x_1, \ldots, x_N) \) belonging to \( S_{N+1}(q^2) \) let there correspond a vector \( \mathbf{y}^T = (y_0, y_1, \ldots, y_N) \) belonging to \( S_{N+1}(q) \) where \( y_1 = x_1^{q+1}, (i = 0, 1, \ldots, N) \).

It follows from the correspondence between the elements of \( GF(q^2) \) and \( GF(q) \) discussed in section \( \rho \), that to each \( \mathbf{x}^T \) there corresponds a unique \( \mathbf{y}^T \), but to each \( \mathbf{y}^T \) with \( r \) non-zero coordinates there correspond \((q+1)^r \) vectors \( \mathbf{x}^T \) belonging to \( S_{N+1}(q) \), each with \( r \) non-zero coordinates.

Now let \( X \) be any point of \( PG(N, q^2) \) with row vector \( \mathbf{x}^T \) having \( r \) non-zero coordinates. Then any one of the \( q^2-1 \) row vectors \( \rho \mathbf{x}^T \) of \( S_{N+1}(q^2) \) will represent \( X \), where \( \rho \) is any arbitrary non-zero element of \( GF(q^2) \). Let \( \mathbf{y}^T \) be the row vector of \( S_{N+1}(q) \) which corresponds to \( \mathbf{x}^T \), and let \( Y \) be
the point of $PG(N,q)$ with row vector $\boldsymbol{y}^T$. We then say that $Y$ corresponds to $X$. The point $Y$ of $PG(N,q)$ is given uniquely by the point $X$ of $PG(N,q)$ for if we take $\rho \boldsymbol{x}^T$ as the vector representing $X$, then the corresponding vector of $S_{N+1}(q)$ is $a \boldsymbol{y}^T$ where $a = \rho^{q+1}$, and represents the same point of $PG(N,q)$ as $\boldsymbol{y}^T$. Conversely let $\boldsymbol{y}^T$ be a vector of $S_{N+1}(q)$ representing a point $Y$ of $PG(N,q)$. If $\boldsymbol{y}^T$ has $r$ non-zero coordinates then we get $(q+1)^r$ distinct vectors of $S_{N+1}(q^2)$ corresponding to $\boldsymbol{y}^T$. Now $Y$ can be represented by any one of the $q-1$ row vectors $a \boldsymbol{y}^T$ of $S_{N+1}(q)$, where $'a'$ is any arbitrary non-zero element of $GF(q)$. To each of these vectors there correspond $(q+1)^r$ vectors of $S_{N+1}(q^2)$. Thus to the $q-1$ vectors $a \boldsymbol{y}^T$ (where $'a'$ ranges over all the non-zero elements of $GF(q)$) there correspond $(q-1)(q+1)^r$ vectors of $S_{N+1}(q^2)$. But any $q^2-1$ of these vectors which differ merely by a multiple of some non-zero element $\rho$ of $GF(q^2)$ represent the same point of $PG(N,q^2)$. Hence to each point of $PG(N,q)$ with $r$ non-zero coordinates there correspond $(q-1)(q+1)^r/(q^2-1)$ or $(q+1)^{r-1}$ points of $PG(N,q^2)$.

Now let $V_{N-1}$ be a non-degenerate Hermitian variety in $PG(N,q^2)$. By a suitable choice of the frame of reference we take its equation in the canonical form

$$\sum_{i=0}^{N} x_i \bar{x}_i = 0$$

(8.1)

$$x_0^{q+1} + x_1^{q+1} + \cdots + x_N^{q+1} = 0$$

Let $\Sigma$ be the hyperplane of $PG(N,q)$ with equation

(8.2)

$$y_0 + y_1 + \cdots + y_N = 0$$

In the correspondence between the points of $PG(N,q)$ and $PG(N,q^2)$ just described, if $X$ lies on $V_{N-1}$ then $Y$ lies on $\Sigma$ and conversely. Let $n_r = (q^{r+1}-1)/(q-1)$ denote the number of points on an $r$-flat in $PG(N,q)$. 

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The number of points on $\Sigma$ which have exactly $r$ non-zero coordinates is

\[(\mathfrak{c}3) \quad \binom{n+1}{r} \binom{n-r}{r-3} + \binom{r}{2} \binom{n-r}{4} + \ldots + (-1)^{r-2} \binom{r}{r-2} n_0\]

\[= \binom{N+1}{r} ((-1)^{r-1} ((q-1)(-1)^{r-1})]/q .\]

Hence the total number of points on $V_{N-1}$ is

\[\phi(N,q^2) = \sum_{r=1}^{N+1} \binom{N+1}{r} ((-1)^{r-1} ((q-1)(-1)^{r-1})]/q\]

\[= (q^{N+1}-(-1)^{N+1})[q^N-(-1)^N]/(q^2-1) .\]

We have thus proved:

Theorem ($\mathfrak{c}4$). The number of points on a non-degenerate Hermitian variety $V_{N-1}$ in $P(G(N,q^2))$ is

\[(\mathfrak{c}4) \quad \phi(N,q^2) = [q^{N+1}(-1)^{N+1}][q^N-(-1)^N]/(q^2-1) .\]

Corollary. The number of points on a degenerate Hermitian variety $V_{N-1}$ of rank $r < N+1$ in $P(G(N,q^2))$ is

\[(\mathfrak{c}5) \quad (q^2-1)f(N-r,q^2) \phi(r-1,q^2) + f(N-r,q^2) + \phi(r-1,q^2) ,\]

where $\phi(N,q^2)$ is given by ($\mathfrak{c}4$), $f(k,q^2) = [q^{2(k+1)}-1]/(q^2-1) .$

Let $\Sigma_{N-r}$ be the singular space of $V_{N-1}$ then the number of points in $\Sigma_{N-r}$ is $f(N-r,q^2)$. Also let $\Sigma_{r-1}$ be an $(r-1)$-flat disjoint from $\Sigma_{N-r}$.

Then from theorem (7.1), $\Sigma_{r-1}$ intersects $V_{N-1}$ in a non-degenerate Hermitian variety $V_{r-2}$ contained in $\Sigma_{r-1}$. The number of points on $V_{r-2}$ is $\phi(r-1,q^2)$. Now from the corollary to theorem (7.2) every point of $V_{N-1}$ belongs to some line joining a point of $\Sigma_{N-r}$ with a point of $V_{r-2}$. Two such lines cannot have a point in common outside of $V_{r-2}$ or $\Sigma_{N-r}$. Suppose
if possible the points $A_1$ and $A_2$ be in $\Sigma_{N-r}$ and the points $B_1$ and $B_2$ in $V_{r-2}$. If possible let the lines $A_1B_1$ and $A_2B_2$ intersect in $P$, a point not in $\Sigma_{N-r}$ or $V_{r-2}$. Then $A_1$ and $A_2$ are distinct. If not they would coincide with the point of intersection of the two lines which would mean that $P$ lies in $\Sigma_{N-r}$. Similarly $B_1$ and $B_2$ are distinct. However both $A_1B_1$ and $A_2B_2$ lie in the plane $PA_1A_2$. Hence the lines $A_1A_2$ and $B_1B_2$ intersect in a point $Q$, which therefore must be common to $\Sigma_{N-r}$ and $\Sigma_{r-1}$. This contradicts the fact that $\Sigma_{r-1}$ and $\Sigma_{N-r}$ are disjoint. Each line joining a point of $\Sigma_{N-r}$ and $V_{r-2}$ contains $q^2-1$ points not contained in either $\Sigma_{N-r}$ and $V_{r-2}$. Hence $V_{N-1}$ contains $f(N-r,q^2)\cdot (r-1,q^2)(q^2-1)$ points not on $\Sigma_{N-r}$ or $V_{r-2}$. This proves the corollary.

9. Flat spaces contained in Hermitian varieties.

We shall first prove the following lemma.

Lemma (9.1). The line joining two points $C$ and $D$ on a Hermitian variety $V_{N-1}$ is completely contained in $V_{N-1}$ if and only if $C$ and $D$ are conjugate with respect to $V_{N-1}$.

Let the equation of $V_{N-1}$ be $x^T H x(q) = 0$, where $H$ is Hermitian or order $N+1$. Let $c^T$ and $d^T$ be the row vectors of $C$ and $D$. Then

\begin{equation}
(9.1) \quad c^T H c(q) = 0, \quad d^T H d(q) = 0.
\end{equation}

The row vector of any point $A$ lying on the line $CD$ can be written as $a^T = \ell_1 c^T + \ell_2 d^T = (\ell_1 c + \ell_2 d)^T$. If $CD$ is completely contained in $V_{N-1}$ we must have

\begin{equation}
(9.2) \quad (\ell_1 c + \ell_2 d)^T H (\ell_1 c + \ell_2 d)(q) = 0 \text{ for any } (\ell_1, \ell_2) \neq (0,0).
\end{equation}
Hence from (9.1)

(9.3) \[ \lambda_{1} \lambda_{2} \mathbf{c}^T \mathbf{d}(q) + \lambda_{2} \lambda_{1} \mathbf{d}^T \mathbf{c}(q) = 0 \quad \text{if} \quad (\lambda_{1}, \lambda_{2}) \neq (0,0). \]

This implies that \( \mathbf{c}^T \mathbf{d}(q) = 0 \), i.e., \( C \) and \( D \) are conjugate. If not suppose \( \mathbf{c}^T \mathbf{d}(q) = h \neq 0 \). Then from lemma (3.1) we can find a non-zero element \( \lambda \) of \( GF(q^2) \) such that \( h \lambda + \lambda h \neq 0 \). Now let us choose \( \lambda_{1} = 1, \lambda_{2} = \lambda \), then \( \lambda_{1} \lambda_{2} = \lambda, \lambda_{2} \lambda_{1} = \lambda \), \( \mathbf{d}^T \mathbf{H} \mathbf{c}(q) = h \) so that from (9.3) we have \( h \lambda + \lambda h = 0 \). This is a contradiction.

Conversely suppose \( C \) and \( D \) are conjugate. Then \( \mathbf{c}^T \mathbf{d}(q) = 0 \) and \( \mathbf{d}^T \mathbf{H} \mathbf{c}(q) = 0 \), so that (9.2) is satisfied. Hence every point of the line \( CD \) is on \( V_{N-1} \).

Corollary. The necessary and sufficient condition for any \( t \)-flat \( \Sigma_t \) to be completely contained in \( V_{N-1} \) is that any two points of \( \Sigma_t \) are conjugate with respect to \( V_{N-1} \). If \( \Sigma_t \) is contained in \( V_{N-1} \), and a point \( C \) of \( \Sigma_t \) is a regular point of \( V_{N-1} \) then \( \Sigma_t \) is contained in the tangent space to \( V_{N-1} \) at \( C \).

The first part is obvious. For the second part we observe that if \( D \) is any point of \( \Sigma_t \), then \( D \) is conjugate to \( C \), and is therefore contained in the polar hyperplane of \( C \), which in this case is the tangent space to \( V_{N-1} \) at \( C \).

Theorem (9.1). If \( N = 2t+1 \) or \( 2t+2 \), then a non-degenerate Hermitian variety \( V_{N-1} \) contains flat spaces of dimensions \( t \) and no higher.

We can without loss of generality take the equation of \( V_N \) in the canonical form \( X^T X(q) = 0 \), i.e., we take \( H = I_{N+1} \), the unit matrix of order \( N+1 \). Suppose \( V_N \) contains a \( t \)-flat determined by the \( t+1 \) independent points \( U_0, U_1, \ldots, U_t \) with row vectors \( u_0, u_1, \ldots, u_t \). Any two of these points are conjugate to each other with respect to \( V_{N-1} \). Hence
(9.4) \[ u_i^T u_j(q) = 0 \quad i, j = 0, 1, \ldots, t. \]

Let

\[
U^T = \begin{bmatrix}
  u_{00} & u_{01} & \cdots & u_{0N} \\
  u_{10} & u_{11} & \cdots & u_{1N} \\
  \vdots & \vdots & \ddots & \vdots \\
  u_{t0} & u_{t1} & \cdots & u_{tN}
\end{bmatrix},
\]

be the \((t+1) \times (N+1)\) matrix whose row vectors are \(u_0^T, u_1^T, \ldots, u_t^T\).

Since the rows of \(U^T\) are independent its rank is \(t+1\). Hence we can find at least \(t+1\) independent columns. We can suppose the first \(t+1\) columns of \(U^T\) to be dependent, for if this is not true we can achieve it merely by a permutation of coordinates. Now the equation (9.4) may be re-written as

(9.6) \[ U^T u(q) = 0. \]

Let \(U_1^T\) be the matrix consisting of the first \(t+1\) columns of \(U^T\), and \(U_2^T\) the matrix consisting of the last \(N-t\) columns. Then \(\text{rank}(U_1^T) = t+1\), \(\text{rank}(U_2^T) = N-t\). Now from (9.6)

(9.7) \[ [U_1^T, U_2^T] \begin{bmatrix} u_1(q) \\ u_2(q) \end{bmatrix} = 0. \]

(9.8) \[ : U_1^T u_1^q + U_2^T u_2(q) = 0. \]

Since \(x \rightarrow x(q)\) is an automorphism of \(GF(q^2)\),

\(\text{rank } U_1^T = \text{rank } U_1^q = t+1\). Hence
(9.9) \[ t+1 = \text{rank} \left( U_{1}^{T} U_{1}^{(q)} \right) \]
\[ = \text{rank}(-U_{1}^{T} U_{2}^{(q)}) \]
\[ \leq N - t , \]

which shows that \( N \geq 2t+1 \).

Changing \( t \) to \( t+1 \) we find that if \( V_{N-1} \) contains a flat space of dimensions \( t+1 \), then \( N \geq 2t+3 \). Hence if \( N = 2t+1 \) or \( 2t+2 \) then \( V_{N-1} \) cannot contain a flat space of dimensions higher than \( t \).

We shall next show that if \( N = 2t+1 \) or \( 2t+2 \), we can always find \( t+1 \) mutually conjugate points on \( V_{N-1} \). The flat space of \( t \) dimensions determined by these points must lie in \( V_{N-1} \). Choose any point \( U_{0} \) on \( V_{N-1} \). Let \( \Sigma_{N-1} \) be the polar space of \( U_{0} \). Then \( \Sigma_{N-1} \) intersects \( V_{N-1} \) in a degenerate Hermitian variety \( V_{N-2} \) contained in \( \Sigma_{N-1} \) of which \( U_{0} \) is the singular space (cf. Theorem (7.4)). Now we can find an \((N-2)\)-flat \( \Sigma_{N-2} \) lying in \( \Sigma_{N-1} \), disjoint from \( U_{0} \) and intersecting \( V_{N-2} \) in a non-degenerate Hermitian variety \( V_{N-3} \) contained in \( \Sigma_{N-2} \) (cf. Theorem (7.1)). Let \( U_{1} \) be any point on \( V_{N-3} \). Since \( U_{1} \) lies in \( \Sigma_{N-1} \) it is conjugate to \( U_{0} \). Now let \( \Sigma_{N-3} \) be the tangent space to \( V_{N-3} \) when considered as a variety of the space \( \Sigma_{N-2} \), and let it intersect \( V_{N-3} \) in the Hermitian variety \( V_{N-4} \) of which \( U_{1} \) is the singular space. Again in \( \Sigma_{N-3} \) we can find a flat space \( \Sigma_{N-4} \) of dimension \( N-4 \) disjoint from \( U_{1} \) and intersecting \( V_{N-4} \) in a non-degenerate Hermitian variety \( V_{N-5} \) contained in \( \Sigma_{N-4} \). Let \( U_{2} \) be in \( V_{N-5} \). Then \( U_{2} \) is conjugate to both \( U_{1} \) and \( U_{0} \). Continuing in this way we get points \( U_{0}, U_{1}, \ldots, U_{t} \) mutually conjugate to one another, \( U_{t} \) lying on \( V_{N-2t-1} \). If \( N = 2t+1 \) or \( 2t+2 \), we will not be able to carry this process further. The flat space \( \Sigma_{t} \) determined by \( U_{0}, U_{1}, \ldots, U_{t} \) lies completely in \( V_{N-1} \).
Let $\psi(N, t, q^2)$ denote the number of $t$-flats contained in a non-degenerate Hermitian variety $V_{N-1}$ in $\text{PG}(N, q^2)$. Then we know that

\[(9.10) \quad \psi(N, 0, q^2) = \Phi(N, q^2); \quad \psi(2t+1, k, q^2) = \psi(2t+2, k, q^2) = 0 \quad \text{for} \ k > t,
\]

where $\Phi(N, q^2)$ is given by (8.4).

We shall next calculate the value of $\psi(N, t, q^2)$ when $N = 2t+1$ or $2t+2$.

First suppose $N = 2t+1$. Let $C$ be any point on $V_{2t}$. Then the tangent space $\Sigma_{2t}$ at $C$ to $V_{2t}$ cuts it in a Hermitian variety $V_{2t-1}$ contained in $\Sigma_{2t}$, for which $C$ is the singular space. We can find a $(2t-1)$-flat $\Sigma_{2t-1}$ contained in $\Sigma_{2t}$ and disjoint from $C$ intersecting $V_{2t-1}$ in a non-degenerate Hermitian variety $V_{2t-2}$. Now $V_{2t-2}$ contains $\psi(2t-1, t-1, q^2)$ $(t-1)$-flats. Any of these $(t-1)$-flats together with $C$ determines a $t$-flat contained in $V_{N-1}$.

Conversely if $\Sigma_t$ is a $t$-flat contained in $V_{N-1}$ and passing through $C$, then it is contained in $\Sigma_{2t}$ and intersects $\Sigma_{2t-1}$ in a $(t-1)$-flat contained in $V_{2t-2}$. Hence the number of $t$-flats contained in $V_{N-1}$ and passing through a fixed point $C$ is $\psi(2t-1, t-1, q^2)$. But the number of points on $V_{2t}$ is $\Phi(2t+1, q^2)$, where $\Phi(N, q^2)$ is given by (8.4). We thus get $\psi(2t-1, t-1, q^2) = \Phi(2t+1, q^2)$ $t$-flats by considering all points of $V_{2t}$. Here each $t$-flat has been counted $f(t, q^2) = (q^{2(t+1)} - 1)/(q^2 - 1)$ times since this is the number of points on a $t$-flat. Hence

\[
\psi(2t+1, t, q^2) = \frac{\psi(2t-1, t-1, q^2) \Phi(2t+1, q^2)}{f(t, q^2)}
\]

\[
= (q^{2t+1} + 1) \psi(2t-1, t-1, q^2).
\]

By successive reduction

\[
\psi(2t, t, q^2) = (q^{2t+1} + 1)(q^{2t-1} + 1) \cdots (q+1),
\]

since $\psi(1, 0, q^2) = q+1$.
In the same manner we can prove that

\[ \psi(2t+2, t, q^2) = (q^{2t+2}+1)(q^{2t+1}) \ldots (q^3+1). \]

Theorem (9.2). If \( \psi(N, t, q^2) \) denotes the number of \( t \)-flats on a non-degenerate Hermitian variety \( V_{N-1} \) in \( PG(N, q^2) \), then

\[
(9.11) \quad \psi(2t+1, t, q^2) = (q^{2t+1}+1)(q^{2t-1}+1) \ldots (q+1),
\]

\[
(9.12) \quad \psi(2t+2, t, q^2) = (q^{2t+3}+1)(q^{2t+1}+1) \ldots (q^3+1).
\]

10. Some designs associated with a non-degenerate Hermitian variety of 2 dimensions in \( PG(3, q^2) \).

Let \( V_2 \) be a non-degenerate Hermitian variety in \( PG(3, q^2) \).

It follows from theorem (8.1), that \( V_2 \) contains \( (q^3+1)(q^2+1) \) points.

The case \( q = 2 \) is of special interest. In this case \( V_2 \) is a cubic surface with 45 points. Again from theorems (9.1) and (9.2), \( V_2 \) does not contain any plane but is ruled by lines, \( (q^3+1)(q+1) \) in number. In the special case \( q = 2 \) the number of lines is 27. The lines lying on \( V_2 \) will be called generators of \( V_2 \).

From theorem (7.4), the tangent plane to \( V_2 \) at any point \( C \), intersects \( V_2 \) in a degenerate Hermitian variety \( V_1 \) of rank 2, with \( C \) as a singular point. It follows from theorem (7.1), that if we take a line \( \ell \) in the tangent plane at \( C \), disjoint from \( C \), then \( \ell \) would intersect \( V_1 \) in a non-degenerate Hermitian variety \( V_0 \) of dimension 0, contained in \( \ell \). It was shown in section 5 that \( V_0 \) consists of a set of \( q+1 \) distinct points \( P_0, P_1, \ldots, P_q \). It now follows from the corollary to theorem (7.2) that \( V_1 \) consists of the set of \( q+1 \) concurrent lines \( CP_0, CP_1, \ldots, CP_q \). Thus the tangent plane to \( V_2 \) at any point \( C \), meets \( V_2 \) in a set of \( q+1 \) generators passing through \( C \). Conversely, from the corollary to Lemma (9.4), any
generator through \( C \) is contained in the tangent plane at \( C \). We have thus shown: Through any point \( C \) of \( V_2 \), there pass exactly \( q+1 \) generators which constitute the intersection with \( V_2 \) of the tangent plane at \( C \). Now through \( C \) there pass \( q^2+1 \) lines lying in the tangent plane at \( C \), out of which \( q+1 \) are generators. The remaining \( q^2-q \) lines through \( C \), which lie in the tangent plane meet \( V_2 \) only in the single point. Lines meeting \( V_2 \) in a single point \( C \) will be called tangents to \( V_2 \) at the point where they meet \( V_2 \). Through \( C \) there will pass \( q^4 \) lines not lying in the tangent plane at \( C \). Now the \((q+1)\) generators through \( C \) contain \( q^3+q^2+1 \) points of \( V_2 \). Hence there are \( q^5 \) points of \( V_2 \) not lying on the tangent plane at \( C \). On the other hand any line through \( C \) not lying in the tangent plane must meet \( V_2 \) in a Hermitian variety \( V_0^* \), which must consist of either \( q+1 \) points or a single point, according as the rank is 2 or 1. Since each of the \( q^5 \) points of \( V_2 \), not contained in the tangent plane must lie on some line through \( C \), each of the \( q^4 \) lines passing through \( C \) and not contained in the tangent plane at \( C \) must intersect \( V_2 \) in exactly \( q+1 \) points, one of which is \( C \). This shows that \( V_0^* \) must be of rank 2. Lines intersecting \( V_2 \) in \( q+1 \) points may be called secants. Any arbitrary line not a generator of \( V_2 \) must either be a tangent or a secant.

Let \( l \) be any generator of \( V_2 \). From theorem (7.3), the tangents to \( V_2 \) at two distinct points of \( l \) must be distinct. There are exactly \( q^2 + 1 \) points on \( l \), and exactly \( q^2+1 \) planes pass through \( l \). Hence any plane through a generator is tangent to \( V_2 \) at some point, and intersects \( V_2 \) in a set of \( q+1 \) generators through the point of contact. Let \( P \) be a point on \( V_2 \) disjoint from a given generator \( l \). Then the plane \( \pi \) containing \( P \) and \( l \) must be tangent to \( V_2 \) at some point \( C \) on \( l \). Since \( P \) is on
the intersection of $V_2$ and $\pi$, CP must be a generator of $V_2$. Since $\pi$ can
be tangent to $V_2$ at only one point on CP so $\pi$ is not the tangent plane at
P. Let $\pi^*$ be the tangent plane to $V_2$ at P. Then the $q+1$ generators of $V_2$
through P lie on $\pi^*$. Thus $\pi$ and $\pi^*$ intersect in a single generator CP.
We have thus shown that given a generator $l$ of $V_2$ and a point $P$ of $V_2$ not on
$l_1$ there passes through $P$ exactly one generator which meets $l$ in a point.
The concept of a partial geometry $(r,k,t)$ was introduced by one of
the authors in (4). It is a system of two kinds of undefined elements called
'points' and 'lines' and an undefined relation of 'incidence' satisfying the
following axioms.

A1. Any two points are incident with not more than one line.
A2. Each point is incident with $r$ lines.
A3. Each line is incident with $k$ points.
A4. If the point $P$ is not incident with the line $l$, there are
exactly $t$ lines ($t \geq 1$) through $P$ intersecting $l$.

Theorem (10.1). The configuration of points and generators of a
Hermitian variety $V_2$ in PG(3, $q^2$) form a partial geometry $(q+1, q^{2+1}, 1)$.

All the axioms A1-A4 are satisfied in view of the results already
proved.

From the connection established between partial geometries and
partially balanced incomplete block (PBIB) designs in (4), it follows that
by identifying the points of $V_2$ with treatments, and the generators of $V_2$
with blocks we get the PBIB design with parameters

\[(10,1) \quad v = (q^2+1)(q^3+1), \quad b = (q+1)(q^3+1), \quad r = q+1, \quad k = q^{2+1},
\]
\[n_1 = q^2(q+1), \quad n_2 = q^5, \quad p_{11}^1 = q^2-1, \quad p_{11}^2 = q+1, \quad \lambda_1 = 1, \lambda_2 = 0.\]
This design was otherwise obtained by Ray Chaudhuri (14). The case
q = 2 was obtained earlier by Clatworthy and one of the authors (6). For the
definition and other properties of PBIB designs the reader is referred to
(5, 7, 8, 9).

Let \( C_0 \) and \( C_1 \) be two distinct points of \( V_2 \) not on the same gener-
or. Denote the line joining \( C_0 \) and \( C_1 \) by \( l_1 \). Then \( l_1 \) must be a
secant to \( V_2 \) and intersects \( V_3 \) in \( q-1 \) other points \( C_2, \ldots, C_q \). Let \( \pi_0 \) and
\( \pi_1 \) be the tangent planes to \( V_2 \) at \( C_0 \) and \( C_1 \) respectively. Now \( \pi_0 \)
cannot pass through \( C_1 \). Otherwise \( C_1 \) would be on the section of \( V_2 \) by \( \pi_0 \) and this
would make \( C_0 C_1 \) a generator contrary to the hypothesis. Similarly \( \pi_1 \) can-
not pass through \( C_0 \). Let \( l_2 \) be the line of intersection of \( \pi_0 \) and \( \pi_1 \).
Then \( l_2 \) must be skew to \( l_1 \). Since \( l_2 \) is a line on \( \pi_0 \) disjoint with \( C_0 \),
\( l_2 \) meets \( V_2 \) in \( q+1 \) distinct points \( D_0, D_1, \ldots, D_q \). Now \( D_i \) (\( i=0,1,\ldots,q \))
is conjugate to both \( C_0 \) and \( C_1 \). Hence the tangent plane \( \Sigma_i \) at \( D_i \) passes
through \( C_0 \) and \( C_1 \) and so through the line \( l_1 \). Thus \( C_1 \) and \( D_j \) are conjugate
(\( i,j = 0,1,\ldots,q-1 \)) and the lines \( C_1 D_j \) are generators of \( V_2 \). We have
incidentally shown that if two points \( C_0 \) and \( C_1 \) on \( V_2 \) do not lie on a
generator then there are exactly \( q+1 \) points \( D_i \) on \( V_3 \) such that both \( D_i C_0 \)
and \( D_1 C_1 \) are generators of \( V_2 \).

Now consider the special case \( q=2 \). Then \( V_2 \) is the cubic surface
\[ x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0 \] in \( PG(3,2^2) \). It has 45 points and 27 generators.
Through each point there pass 3 generators and on each generator lie 5 points.
To each point \( P \) of \( V_2 \) we may associate a set of 12 points viz the points
(other than \( P \)) lying on the three generators through \( P \). This set of points
will be called the block corresponding to \( P \). There are exactly 45 blocks,
and each point \( P \) of \( V_2 \) belongs to 12 blocks viz the blocks corresponding to
the 12 points (other than P) lying on the three generators through P.

Given two distinct points P and Q on \( V_2 \) we shall show that there are exactly three blocks which contain both P and Q. We have to consider two separate cases. Firstly let P and Q lie on a generator \( f^* \). Now \( f^* \) contains three other points besides P and Q, and both P and Q belong to the blocks corresponding to each of these points. Again suppose P and Q lie on a secant. Then from what has been shown above the line of intersection of the tangent planes at P and Q meets \( V_2 \) in \( q+1 = 3 \) points \( D_0, D_1, D_2 \) such that \( D_i P \) and \( D_i Q \) (\( i = 0, 1, 2 \)) are both generators. Hence both P and Q belong to the blocks corresponding to \( D_0, D_1 \) and \( D_2 \).

Now a balanced incomplete block (BIB) design is a set of \( v \) objects or treatments, arranged into \( b \) sets or blocks such that (i) Each block contains \( k \) distinct treatments (ii) Each treatment appears in exactly \( r \) blocks (iii) Any pair of objects occurs in exactly \( \lambda \) blocks. The numbers \( v, b, r, k, \lambda \) are called the parameters of the BIB design \((2,11)\). We may thus state:

Theorem (10.2). If \( V_2 \) is a non-degenerate Hermitian variety in \( PG(3,q^2) \), and if the points of \( V_2 \) are identified with treatments, and if corresponding to each point P on \( V_2 \) we define a block consisting of all points (other than P) on the generators through P, then the treatments and the blocks form a BIB design with parameters

\[ v = b = 45, \quad r = k = 12, \quad \lambda = 3. \]

This design has been otherwise obtained by Shrikhande and Singh \((16)\) and by Takeuchi \((17)\).

There are many other interesting balanced and partially balanced
incomplete block designs associated with Hermitian varieties. These shall be discussed in a separate communication.

References


13. D. K. Ray-Chaudhuri. Some results on quadrics in finite projective

14. Application of the geometry of quadrics for

15. B. Segre. Lectures on modern geometry (Rome: Edizioni Cremonese, 1960),
chapter 17.

16. S. S. Shrikhande and N. K. Singh. On a method of constructing symmetri-
cal balanced incomplete block designs. Sankhya, series A, 24 (1962),
25-32.