A SECOND RENEWAL THEOREM FOR NONIDENTICALLY DISTRIBUTED
VARIABLES AND ITS APPLICATION TO THE THEORY OF QUEUES

by

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(1.3) should read

\[ \lim_{t \to \infty} q^{SR}(t) = \frac{1}{\mu_1} \int_0^\infty q(u) \, du. \]

Secondly, Lemma 1 (bottom of page 4) omitted the hypotheses \( A(0) = 0 \) and \( B(0) = 0 \). It should read

**Lemma 1.** Suppose that \( R(t) \) is the renewal function given by (1.2) and that \( q(t) \) is a non-negative function for which (1.3) is true. If \( \mu_1 = \int_0^\infty x dB(x) \), \( \mu_2 = \int_0^\infty x^2 dB(x) \) are both finite, and \( A(0) = 0, B(0) = 0 \), then the inequality

\[ \boxed{q^{SR}(t) \leq \frac{2t}{2\mu_1 t - \mu_2} \int_0^{2t} q(u) \, du} \]  \hspace{1cm} (1.4)

holds for \( t > \mu_2 / 2\mu_1 \), uniformly in \( \theta \).

**Proof:** Since \( R(0) = 0 \), integration by parts shows that \( q^{SR}(t) = q^{SB}(t) \).

From (1.2)

\[ q^{SR} \{ U(t) - B(t) \} = q^A(t). \]

Integrating this expression from \( 0 \) to \( t \) and then integrating the resulting left-hand side by parts shows that

\[ \mu_1^{-1} \left\{ \frac{1}{2} (U(t) - B(t)) \right\} = \mu_1^{-1} \int_0^t q^A(u) \, du \leq \mu_1^{-1} \int_0^t q(u) \, du. \]

The rest of the proof is as before.
A Second Renewal Theorem for nonidentically
distributed variables and its application to
the Theory of Queues.

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Summary.

Let \( \{X_j\} \) be a sequence of independent, positive, non-lattice
variables with renewal function
\[
H(t) = \sum_{j=1}^{\infty} \Pr \left\{ X_1 + X_2 + \ldots + X_j \leq t \right\}.
\]

If the \( X_j \) are identically distributed with finite first two moments \( \mu_1 \) and \( \mu_2 \)
the second renewal theorem states that as \( t \to \infty \)
\[
H(t) \to \frac{t}{\mu_1} + \frac{\mu_2}{2\mu_1^2} - 1.
\]

Provided certain conditions are satisfied an analogous result (Theorem 2)
is derived for nonidentically distributed variables. An upper bound for
the quantity \( H(t) - \frac{t}{\mu_1} \) and a limiting result concerning the distribution
of the forward delay are established in \( \S 3 \). \( \S 4 \) contains two illustrative
examples. The remainder of the paper treats the renewal sequence formed
by the first passage times from queue length \( j \) to queue length \( j+1 \), \( j = 0, 1, \ldots \),
of the single server queue with renewal input and negative exponential
service time. It is shown that, provided the queue is transient, the expected
value of maximum queue length in \( (0,t) \) is a renewal function satisfying
the second renewal theorem derived earlier.
1. Introduction

Let $X_1, X_2, \ldots$ be independent, positive, non-lattice random variables with distribution functions $F_1(x), F_2(x), \ldots$ and finite expectations $m_1, m_2, \ldots$ respectively. We assume that the sequence $\{X_i\}$ does not degenerate. If

$$G_j(t) = \Pr \{X_1 + X_2 + \ldots + X_j \leq t\} \quad , \quad j = 1, 2, \ldots ,$$

the renewal function associated with the sequence $\{X_i\}$ is

$$H(t) = \sum_{j=1}^{\infty} G_j(t) . \quad (1.1)$$

That is, for each $t > 0$, $H(t)$ is the expectation of the maximum $n$ such that the inequality

$$X_1 + X_2 + \ldots + X_n \leq t$$

is satisfied.

If the $X_i$ are identically distributed the sequence $\{X_i\}$ is called a renewal process, and for renewal processes one has available the following three principal results:

$$\lim_{t \to \infty} \frac{H(t)}{t} = \frac{1}{\mu_1} \quad , \quad \mu_1 \leq \infty ,$$

$$\lim_{t \to \infty} \int_0^t q(t-u) \, dH(u) = \frac{1}{\mu_1} \int_0^\infty q(u) \, du \quad , \quad \mu_1 \leq \infty ,$$

$$\lim_{t \to \infty} \left[ H(t) - \frac{t}{\mu_1} \right] = \frac{\mu_2}{2\mu_1^2} - 1 , \quad \mu_1 < \infty , \mu_2 < \infty .$$
Here, $\mu_1$ and $\mu_2$ are the first two moments of $X_i$. The function $q(x)$ has been variously defined but we will take it to be of bounded variation, zero for negative argument, approaches zero as $x \to \infty$, and integrable over $(0, \infty)$. For the widest definition of the class $K$ of which $q(x)$ is a typical member, see Smith (1961, page 469). An equivalent statement to the second result is Blackwell's Theorem,

$$\lim_{t \to \infty} \frac{H(t + x) - H(t)}{\mu_1} = \frac{\alpha}{\mu_1}.$$

The above three results are known as the elementary renewal theorem, the key renewal theorem, and the second renewal theorem, respectively. One may ask whether similar theorems hold for sequences $\{X_i\}$. Smith (1963, 1961) has obtained an elementary renewal theorem and a key renewal theorem for general aperiodic (i.e., not all $X_i$ are lattice) sequences under mild restrictions on the random variables of the sequence. One such restriction is a weak convergence property, namely that

$$\lim_{j \to \infty} j^{-1} \left\{ \frac{EX_{n+1}}{EX_{n+1}} + \frac{EX_{n+2}}{EX_{n+2}} + \ldots + \frac{EX_{n+j}}{EX_{n+j}} \right\} = \mu_1 < \infty$$

uniformly in $n$. This defines the constant $\mu_1$ appearing in the two theorems. To date there is no analogue for sequences to the second renewal theorem, and the main purpose of the first part of this paper is to derive such a result, (Theorem 2).

The second renewal theorem for renewal processes is a simple consequence of the key renewal theorem (by a suitable choice of $q(x)$) and the fact that the renewal function satisfies the integral equation

$$H(t) = F(t) + \int_0^t F(t-u) \, dH(u),$$
where $F(t)$ is the common distribution function of the $X_i$, (Smith, 1954 page 29). However this approach fails in the case of sequences since the renewal function does not, in general, satisfy an integral equation of the above type. The approach adopted here is described in the next section where our main result is also derived. In § 5 this theorem is applied to a problem in queueing theory in which the rather stringent conditions of the theorem are satisfied.

Convolutions will be written as follows:
If $F_1(x), F_2(x), \ldots$ are distribution functions of independent positive random variables then

$$G_j(t) = \int_0^t F_1 * F_2 * \ldots * F_{j-1}(t-u) \, dF_j(u)$$

$$= F_1 * F_2 * \ldots * F_j(t) \quad , \quad j = 1, 2, \ldots$$

The $j$ fold iterated convolution of a distribution function $F(x)$ with itself will be written

$$F^{*(j)}(t) = \int_0^t F^{*(j-1)}(t-u) \, dF(u) \quad , \quad j = 2, 3, \ldots$$

$$F^{*(1)}(t) = F(t) \text{ and } F^{*(0)}(t) = U(t),$$

where $U(t)$ is the Heaviside function

$$U(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

On occasion we will use $t * B(t) = \int_0^t B(u) \, du$.

To close this section we establish a preliminary result which will be needed later. Let $\{Z_i, \ i = 1, 2, \ldots\}$, be a sequence of
independent, positive, non-lattice, random variables with distribution functions

\[ \Pr \{ Z_1 \leq x \} = A(x) \]
\[ \Pr \{ Z_1 \leq x \} = B(x) \quad , \quad i = 2, 3, \ldots \]

The \( \{Z_i\} \) constitute a so-called delayed renewal process and the associated renewal function

\[ R(t) = \sum_{j=0}^{\infty} A^{*B^{\ast}}(j)(t) \]

satisfies the integral equation

\[ R(t) = A(t) + B^{\ast}R(t) . \]

If we wish to draw attention to the fact that \( A(t) = A(t; \theta) \) depends on a parameter \( \theta \), the integral equation will be written as

\[ R(t) = A(t; \theta) + B^{\ast}R(t) \quad (1.2) \]

Suppose \( q(x) \) is a member of the class of functions for which the key renewal theorem holds;

\[ \lim_{t \to \infty} R^{\ast}q(t) = \frac{1}{\mu_1} \int_{0}^{\infty} q(u) \, du . \quad (1.3) \]

The result we require relates to distribution functions \( B(x) \) with finite coefficient of variation and is as follows:

**Lemma 1.** Suppose that \( R(t) \) is the renewal function given by (1.2) and that \( q(t) \) is a non-negative function for which (1.3) is true. If
\[ \mu_1 = \int_0^\infty x \, dB(x) \text{ and } \mu_2 = \int_0^\infty x^2 \, dB(x) \]

are both finite, then the inequality

\[ R^*q(t) \leq \frac{2t}{(\frac{2}{\mu_1} + \mu_2)} \int_0^t q(u) \, du \quad (1.4) \]

holds for \( t > \frac{\mu_2}{\mu_1} \) uniformly in \( \theta \).

**Proof:** From the integral equation satisfied by \( R(t) \),

\[ R^*q(t) = A^*q(t) + B^*R^*Q(t), \]

and hence, for all \( t \),

\[ \mu_1^{-1} t^* (U-B) * R^*q(t) = \mu_1^{-1} t^* A^*q(t) \]

\[ \leq \mu_1^{-1} t^* q(t) = \mu_1^{-1} \int_0^t q(u) \, du. \]

The function

\[ B(1)(t) = \mu_1^{-1} t^* (U-B) = \mu_1^{-1} \int_0^t [1-B(u)] \, du \]

is an honest distribution function with finite expectation \( \mu_2/\mu_1 \). Thus

\[ \mu_1^{-1} t^* (U-B)*R^*q(2t) = \int_0^t B(1)(2t-u) \, d[R^*q(u)] \]

\[ + \int_t^{2t} B(1)(2t-u) \, d[R^*q(u)]. \]

\[ \geq B(1)(t) R^*q(t). \]
Combining the above two inequalities we have

\[ R^*q(t) \leq \frac{1}{\mu_1 B_1(t)} \int_0^{2t} q(u) \, du . \quad (1.5) \]

But

\[ \frac{\mu_2}{2 \mu_1} \geq \int_t^\infty y \, dB_1(y) \geq t \, [1 - B_1(t)] \]

so that

\[ \frac{1}{B_1(t)} \leq \frac{2\mu_1 t}{\mu_1 t + \mu_2} \quad \text{for} \quad t > \frac{\mu_2}{\mu_1} . \]

and (1.4) follows from (1.5)
§2. A Second Renewal Theorem

Let \( Y \) be a positive random variable with distribution function \( F(x) \). Let

\[
K(t) = \sum_{j=1}^{\infty} F^{*}(j)(t)
\]

be the ordinary renewal function associated with \( F(t) \). Suppose the sequence of independent random variables \( \{X'_i\} \) is formed from the \( \{X_i\} \) of §1 by the truncation procedure

\[
X'_i = \begin{cases} X_i & , i = 1, 2, \ldots, n, \\
Y & , i = n+1, n+2, \ldots .
\end{cases}
\]

For a fixed integer \( n \), the renewal function of \( \{X'_i\} \) is

\[
H_n(t) = \sum_{j=1}^{n} G_j(t) + \sum_{j=n+1}^{\infty} G_{n}^{*}(j-n)(t)
\]

\[
= \sum_{j=1}^{n} G_j(t) + G_{n}^{*}K(t) . \tag{2.1}
\]

It is evident that for each fixed \( n \), renewal theorems for \( H_n(t) \) can be deduced without difficulty from the corresponding theorems for \( K(t) \). In fact \( \{X'_i\} \) is a simple extension of a so-called delayed renewal process. Furthermore, since the \( X_i \) do not degenerate, we have for all finite \( t \),

\[
G_{n}^{*}K(t) \leq G_{n}(t) K(t) \to 0 \text{ as } n \to \infty
\]

so that

\[
\lim_{n \to \infty} H_n(t) = H(t) .
\]
To obtain asymptotic formulae for $H(t)$ from those valid for each member of the sequence $(H_n(t))$, one requires

$$\lim_{t \to \infty} \lim_{n \to \infty} H_n(t) = \lim_{n \to \infty} \lim_{t \to \infty} H_n(t). \tag{2.2}$$

Clearly this imposes restrictions on the choice of the random variable $Y$ used to form the truncated sequence $\{X'_i\}$. Our procedure is to derive renewal theorems for $H_n(t)$, valid for general $Y$, and subsequently choose $Y$ so that the interchange of limits (2.2) holds for the formulae of interest.

**Lemma 2.** If $Y$ is non-lattice with expectation $\mu_1 \leq \infty$, then for each finite $n$, as $t \to \infty$,

$$\begin{align*}
(i) \quad & \frac{H_n(t)}{t} \to \frac{1}{\mu_1}, \\
(ii) \quad & q^*H_n(t) \to \frac{1}{\mu_1} \int_0^\infty q(u) \, du, \\
(iii) \quad & \text{If } \mu_1 < \infty, \mu_2 = \mathbb{E}Y^2 < \infty, \text{ then } \\
& H_n(t) - t \to \frac{\mu_2}{2\mu_1^2} - 1 + \frac{1}{\mu_1} \sum_{i=1}^n (\mu_1 - m_i). 
\end{align*}$$

The class $Q$ of which $q(x)$ of (ii) is a member has been defined in §. Parts (i) and (ii) of the Lemma state that the elementary renewal theorem and key renewal theorem valid for $H_n(t)$ are those valid for $H(t)$, and this is immediately apparent from the definition (2.1).
To establish (iii), note that

\[ H_n(t) - K(t) = \sum_{j=1}^{n} A_j(t) - (U - A_n) * K(t) \]

\[ + \frac{1}{\mu_1} \int_0^\infty [1-A_n(u)] \, du = \frac{1}{\mu_1} \sum_{i=1}^{n} (\mu_1 - m_i) \]

by the key renewal theorem for K(t). Thus, by the second renewal theorem for K(t),

\[ H_n(t) - \frac{t}{\mu_1} = [ H_n(t) - K(t) ] + [ K(t) - \frac{t}{\mu_1} ] \]

\[ + \frac{\mu_2}{2\mu_1^2} - 1 + \frac{1}{\mu_1} \sum_{i=1}^{n} (\mu_1 - m_i). \]

A deduction from (iii) of Lemma 2 is that if in fact the interchange of limits (2.2) is permissible, then the form of a second renewal theorem for H(t) is

\[ H(t) - \frac{t}{\mu_1} + \frac{\mu_2}{2\mu_1^2} - 1 + \frac{1}{\mu_1} \sum_{i=1}^{\infty} (\mu_1 - m_i). \]

Apparently we require the series \( \sum_{i=1}^{\infty} (\mu_1 - m_i) \) to converge, implying

\[ \lim_{n \to \infty} m_n = \mu_1, \]

so that \( F(x) \) is related to the sequence \( \{ F_n(x) \} \) by
\[
\lim_{n \to \infty} (\mu_n - m_n) = \lim_{n \to \infty} \int_0^\infty [F_n(x) - F(x)] \, dx = 0.
\]

The conditions C stated below, which are sufficient for \(2.2\) to hold, impose a stronger mode of convergence on \(\{F_n(x)\}\) and uniquely determine the constant \(\mu_2\).

We will say that the sequence of random variables \(\{X_n\}\) satisfies condition C if it is true that there exists a positive non-lattice random variable \(Y\) with distribution function \(F(x)\) and finite first two moments \(\mu_1\) and \(\mu_2\) such that

\[
C(1) \quad F_n(x) \neq F(x) \text{ or } F_n(x) \neq F(x) \quad \text{for all } n \text{ and } x
\]

\[
C(ii) \quad \sum_{n=1}^\infty (\mu_n - m_n) \text{ converges.}
\]

An immediate consequence is

**Theorem 1.** Suppose \(\{X_n\}\) satisfies condition C, with the exception that \(\mu_2\) need not be finite. Then

\[
\lim_{n \to \infty} F_n(x) = F(x) \text{ almost everywhere.}
\]

**Proof:** Without loss of generality suppose \(F_n(x) \neq F(x)\) for all \(n\) and \(x\). Then both

\[
F_n(x) - F(x) \text{ and } \mu_n - m_n = \int_0^\infty [F_n(x) - F(x)] \, dx
\]

are non-negative. Thus \(\{F_n(x) - F(x)\}\) is a sequence of non-negative integrable functions for which, by C(ii),
\[ \sum_{n=1}^{\infty} (\mu_{1m} - m_n) = \sum_{n=1}^{\infty} \int_{0}^{\infty} [F_n(x) - F(x)] \, dx < \infty. \]

By the Lebesgue monotone convergence theorem
\[ \int_{0}^{\infty} \sum_{n=1}^{\infty} [F_n(x) - F(x)] \, dx = \sum_{n=1}^{\infty} \int_{0}^{\infty} [F_n(x) - F(x)] \, dx, \]

Thus \( \sum_{n=1}^{\infty} [F_n(x) - F(x)] \) converges almost everywhere (Titchmarsh, 1960, page 347), implying
\[ F_n(x) - F(x) \to 0 \quad \text{almost everywhere.} \]

The random variable \( Y \) is thus specified as the almost everywhere limit in distribution of the sequence \( \{X_n\} \). As its second moment the constant \( \mu_2 \) is uniquely determined. Note that \( \mu_2 \) need not bear any relation to the sequence of second moments \( \{EX_1^2\} \), and in fact it is not necessary that the latter exist (although this will be the case if \( F_n(x) \equiv F(x) \)). The finiteness of \( \mu_2 \) is not required by Theorem 1 but is essential for the main result (2.3).

**Theorem 2.** If \( \{X_n\} \) satisfies condition C then (2.2) is true and
\[
\lim_{t \to \infty} \left[ H(t) - \frac{t}{\mu_1} \right] = \frac{\mu_2}{2\mu_1^2} - 1 + \frac{1}{\mu_1} \sum_{i=1}^{\infty} (\mu_i - m_i) \tag{2.3}
\]
Proof: Suppose again that $F_n(x) \geq F(x)$ for all $n$ and $x$. Then $H_j(t)$ is a non-decreasing function of both $j$ and $t$.

$$H_j(t) - H_{j-1}(t) = G_j(t) + G_j K(t) - G_{j-1} K(t)$$

$$= G_{j-1} F_j(t) + G_{j-1} F_j K(t) - G_{j-1} F(t) - G_{j-1} F K(t)$$

$$= (G_{j-1} + G_{j-1} K) \ast (F_j(t) - F(t)).$$

The function

$$R(t) = G_{j-1}(t) + G_{j-1} K(t) = G_{j-1}(t) + F R(t)$$

is a delayed renewal function. Also, $F_j(t) - F(t)$ is non-negative, of bounded variation, $L_1(0, \infty)$, and approaches zero as $t \to \infty$. Thus Lemma 1 applies and

$$H_j(t) - H_{j-1}(t) \leq \frac{2t}{2_{12}} \int_0^{2t+\alpha} [F_j(u) - F(u)] \, du \quad \text{for} \quad t > \frac{\mu_2}{\mu_1}.$$

If $t \geq \ell > \frac{\mu_2}{\mu_1}$,

$$\frac{2t}{2_{12}} < \left(1 - \frac{\mu_2}{\mu_1 \ell} \right)^{-1} \leq \frac{c}{\mu_1} < \infty.$$

Thus for $t \geq \ell$

$$H_j(t) - H_{j-1}(t) \leq \frac{c}{\mu_1} \int_0^{\infty} [F_j(u) - F(u)] \, du = \frac{c(\mu_1 - m_j)}{\mu_1}.$$
and
\[ H_{n+k}(t) - H_n(t) = \sum_{j=n+1}^{n+k} [H_j(t) - H_{j-1}(t)] \]
\[ \leq c \frac{\sum_{j=n+1}^{n+k} (\mu_1 - m_j)}{\mu_1} \cdot \tag{2.4} \]

Since \( \sum_{j} (\mu_1 - m_j) \) is convergent by C(ii), the sequence \( (H_n(t) - t/\mu_1) \)
is uniformly convergent to \( (H(t) - t/\mu_1) \) for \( t \geq t > \frac{\mu_2}{\mu_1} \), whichproves (2.2). (2.3) follows on applying this result to (iii) of Lemma 2.
The proof when \( F_n(x) \leq F(x) \) is similar.

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It was mentioned earlier than an elementary renewal theorem
and a key renewal theorem have been established by Smith for generalaperiodic sequences \( \{X_i\} \). If \( \{X_i\} \) satisfies condition C, simple proofs
of these theorems can be obtained and for completeness we state without proof

**Theorem 3:** If \( \{X_i\} \) satisfies condition C then

\[ \lim_{t \to \infty} \frac{H(t)}{t} = \frac{1}{\mu_1} , \]

\[ \lim_{t \to \infty} q^* H(t) = \frac{1}{\mu_1} \int_{0}^{\infty} q(u) \, du , \]

where \( q(x) \) is a non-negative function satisfying the key renewal theorem
for \( K(t) \).
§3. Some further properties of the renewal sequence

The first result is an upper bound for the renewal function $R(t)$, valid for large but finite values of the argument.

**Lemma 3:** If $R(t)$ is the delayed renewal function defined in (1.2) and if $v_1 = \int_0^\infty xdA(x)$ and $v_2 = \int_0^\infty x^2dA(x)$ and both finite, then for $t > \frac{\mu_2}{\mu_1}$

$$
R(t) - \frac{t}{\mu_1} \leq \left[1 - \frac{\mu_2}{\mu_1 t}\right]^{-1} \frac{1}{\mu_1} \int_0^{2t} \left[1 - B_1(u)\right] du + \frac{v_2}{\mu_1 t} - \frac{v_1}{\mu_1} \leq \left[1 - \frac{\mu_2}{\mu_1 t}\right]^{-1} \frac{\mu_2}{\mu_1^2} + \frac{v_2}{\mu_1 t} - \frac{v_1}{\mu_1}
$$

(3.1)

**Proof:** Take $q(t) = 1 - B_1(t)$ in Lemma 1. Then

$$
q*R(t) = R(t) - \frac{t}{\mu_1} + \frac{1}{\mu_1} \int_0^t [1 - A(u)] du
$$

and, for $t > \frac{\mu_2}{\mu_1}$,

$$
R(t) - \frac{t}{\mu_1} \leq \left[1 - \frac{\mu_2}{\mu_1 t}\right]^{-1} \frac{1}{\mu_1} \int_0^{2t} \left[1 - B_1(u)\right] du - \frac{1}{\mu_1} \int_0^t [1 - A(u)] du.
$$

But $\frac{1}{\nu_1} \int_0^t [1 - A(u)] du$ is a distribution function with finite expectation $\frac{v_2}{2\nu_1}$. Hence by the Chebyshev type inequality used in the proof of Lemma 1

$$
-\frac{1}{\mu_1} \int_0^t [1 - A(u)] du \leq \frac{v_1}{\mu_1} \left\{ \frac{v_2}{2\nu_1 t} - 1 \right\}
$$
The Lemma follows. Note that the right hand side of (3.1) is a decreasing function of $t$ and has the same limit as the left hand side as $t \to \infty$.

Sharper upper bounds for $R(t) - t/\mu_1$ can be obtained if the finiteness of higher moments is assumed, and one can obtain lower bounds by similar arguments. We do not pursue this question further at the moment but state for sequences $\{X_i\}$ that are bounded below

**Theorem 4:** If $F_n(x) \preceq F(x)$ for all $n$ and $x$ and if $\{X_i\}$ satisfies condition $C$ then for $t > \mu_2/\mu_1$, 

\[ H(t) - \frac{t}{\mu_1} \leq \left[ 1 - \frac{\mu_2}{\mu_1} \right]^{-1} \left\{ \frac{\mu_2}{\mu_1^2} - 1 + \frac{1}{\mu_1} \sum_{i=1}^{\infty} \left( \mu_1 - m_i \right) \right\} + \frac{\mu_2}{\mu_1} e(t), \tag{3.2} \]

where

\[ e(t) = \frac{m_2^{(1)}}{\mu_2} + \frac{m_1}{\mu_1} \left[ 1 - \frac{\mu_2}{\mu_1} \right]^{-1}, \]

\[ m_2^{(1)} = \int_0^\infty x^2 dF_1(x). \]

**Proof:** Since $F_n(x) \preceq F(x)$ and $\mu_2$ is assumed finite, each $X_i$ has a finite second moment, and in particular $m_2^{(1)}$ is finite. In the proof of Theorem 2 it was shown that for $t > \mu_2/\mu_1$

\[ H_j(t) - H_{j-1}(t) \leq \left[ 1 - \frac{\mu_2}{\mu_1} \right]^{-1} \frac{(\mu_1 - m_j)}{\mu_1}. \]
It follows that for these values of $t$

$$H_n(t) - \frac{t}{\mu_1} \leq \left[ 1 - \frac{\mu_2}{\lambda_1 t} \right]^{-1} \frac{1}{\mu_1} \sum_{j=2}^{n} (\mu_1 - m_j) + H_1(t) - \frac{t}{\mu_1},$$

and hence

$$H(t) - \frac{t}{\mu_1} \leq \left[ 1 - \frac{\mu_2}{\lambda_1 t} \right]^{-1} \frac{1}{\mu_1} \sum_{j=2}^{\infty} (\mu_1 - m_j) + H_1(t) - \frac{t}{\mu_1}. \quad (3.3)$$

But $H_1(t) = F_1(t) + F^*H_1(t)$ is a delayed renewal function to which Lemma 3 applies. (3.2) follows from (3.1) and (3.3) on rearranging terms.

Consider now the forward and backward delay of the sequence $\{X_i\}$.

If $N(t)$ denotes the maximum $n$ for which

$$S_n = X_1 + X_2 + \ldots + X_n \leq t$$

is true, the forward delay $\xi(t)$ is defined as

$$\xi(t) = S_{N(t)+1} - t.$$ 

That is, $\xi(t)$ is the time from $t$ until the next renewal occurs. The backward delay $\beta(t)$ is the 'age' of the process at time $t$,

$$\beta(t) = t - S_N(t).$$

If $\{X_i\}$ is a renewal process with renewal function $K(t)$ and forward delay $\eta(t)$ it is known (Smith, 1958, page 246 and also Takac's discussion to the paper) that

$$t + E\eta(t) = \mu_1 \left[ K(t) + 1 \right].$$
Furthermore,

\[
\lim_{t \to \infty} E\eta(t) = \frac{\mu_2}{\mu_1} \quad \text{if} \ \mu_2 < \infty
\]

and

\[
\lim_{t \to \infty} \Pr[\eta(t) \leq x] = \frac{1}{\mu_1} \int_0^x [1 - F(u)] \, du.
\]

The backward delay of a renewal process has the same limiting
distribution function as \(\eta(t)\), although its expectation does not have a
simple relation to the renewal function \(E(t)\).

In the case of a sequence \(\{X_i\}\) we have (with the same notation
as before)

**Theorem 5:** If \(\mu_1 < \infty\) then

\[
t + E\xi(t) = m_1 + \sum_{j=1}^{\infty} m_{j+1} G_j(t).
\] (3.4)

If \(\{X_i\}\) satisfies condition C then

\[
\lim_{t \to \infty} \Pr[\xi(t) \leq x] = \frac{1}{\mu_1} \int_0^x [1 - F(u)] \, du
\] (3.5)

and

\[
\lim_{t \to \infty} E\xi(t) = \frac{\mu_2}{\mu_1}.
\] (3.6)

**Proof:** (3.4) is a case of Wald's equation for independent but not
identically distributed random variables;

\[
E\xi(t) = \sum_{j=0}^{\infty} E \left[ \xi(t) \mid S_j < t \leq S_{j+1} \right] \Pr \left[ S_j < t \leq S_{j+1} \right]
\]

\[
= \sum_{j=0}^{\infty} \left( m_1 + m_2 + \ldots + m_{j+1} - t \right) \left[ G_j(t) - G_{j+1}(t) \right],
\]
where $G_0(t)$ is the Heaviside function $U(t)$. A simple manipulation yields

$$
E_{\xi}(t) = \mu_1 [H(t) + 1] - t - \sum_{j=0}^{\infty} (\mu_1 - m_{j+1}) G_j(t)
$$

which is the same as (3.4). An appeal to Theorem 2 shows that when condition C is satisfied $\lim_{t \to \infty} E_{\xi}(t)$ has the required value.

To prove (3.5) proceed as follows. For fixed $n$ let $\xi_n(t)$ be the forward delay of the sequence $\{X_j\}$, whose renewal function is $H_n(t)$. Then

$$
\Pr[\xi_n(t) \leq x] = \sum_{j=1}^{\infty} \Pr[t < S_j \leq t+x < S_{j+1}]
$$

where $S_j = X_1 + X_2 + \ldots + X_j$. It follows that

$$
\Pr[\xi_n(t) \leq x] = \sum_{j=1}^{n-1} \int_{t}^{t+x} [1 - F_{j+1}(t+u)] dG_j(u) + \sum_{j=n}^{\infty} \int_{t}^{t+x} [1 - F(t+u)] d[G_n * \Phi_j(u)]
$$

$$
= \sum_{j=1}^{n-1} \int_{t-x}^{t} [F(t+u) - F_{j+1}(t+u)] dG_j(u) + \int_{t}^{t+x} [1 - F(t+u)] dH_n(u) + \int_{t}^{t+x} [1 - F(t+u)] dH_n(u)
$$

(3.7)

For fixed $n$ the sum on the right hand side has the limit zero as $t \to \infty$. The function

$$
q(y) = 1 - F(y) , \quad t \leq y \leq t+x ,
$$

$$
= 0 \quad \text{otherwise}
$$
satisfies the conditions of the key renewal theorem for $H_n(t)$, and hence

$$
\lim_{t \to \infty} \Pr [\xi_n(t) \leq x] = \lim_{t \to \infty} \int_t^{t+x} [1 - F(t+x-u)] \, dH_n(u) = \frac{1}{\mu_1} \int_0^x [1 - F(u)] \, du.
$$

We have then that

$$
\lim_{n \to \infty} \lim_{t \to \infty} \Pr [\xi_n(t) \leq x] = \frac{1}{\mu_1} \int_0^x [1 - F(u)] \, du, \tag{3.8}
$$

and (3.5) is established if we can justify the interchange of limits.

Note that the uniform convergence of $H_n(t)$ to $H(t)$ for sufficiently large $t$ does not necessarily imply uniform convergence of (3.7).

Suppose first that $F_n(x) \equiv F(x)$ for all $n$ and $x$. Then

$$
v_n(t,x) = \Pr[\xi_n(t) \leq x] - \Pr [\xi_{n-1}(t) \leq x]
$$

$$
= \int_t^{t+x} [1 - F(t+x-u)] \, d[H_n(u) - H_{n-1}(u)] - \int_t^{t+x} [F_n(t+x-u) - F(t+x-u)] \, dC_{n-1}(u)
$$

The first member on the right hand side is dominated by

$$
|H_n(t+x) - H_{n-1}(t+x) - H_n(t) + H_{n-1}(t)| \leq 2c(\mu_1 - m_n)
$$

for $t > \mu_2/\Delta_1$. Since $F_n(y) - F(y)$ is non-negative, the absolute value of the second term on the right is dominated by
\[
\int_{t}^{t+x} \left[ F_n(t+x-u) - F(t+x-u) \right] \, du \leq \mu_1 - m_n
\]

for large enough \(t\), uniformly in \(x\). Recalling that \(\sum_{n} (\mu_1 - m_n)\) is convergent we can repeat the argument used in Theorem 2 to justify interchanging limits in (3.8). The proof when \(F_n(x) \leq F(x)\) is similar.

The corresponding result for the backward delay is

\[
\lim_{t \to \infty} \Pr[\beta(t) \leq x] = \frac{1}{\mu_1} \int_{0}^{x} \left[ 1 - F(u) \right] \, du. \tag{3.9}
\]

The proof is based on the same principle as the proof of (3.5), but is slightly longer because of the fact that \(\beta(t)\) is bounded above by \(t\), the distribution function \(\Pr[\beta(t) \leq x]\) having a positive jump into the value unity when \(x = t\). We do not give the proof of (3.9).

Another quantity of interest is the variance of the number of renewals in \((0,t]\). To date we have had no success in studying this aspect of the sequence \(\{X_i\}\).
The following example illustrates the point made in §2 that the second moments $E X^2_1$ are not required to converge for Theorem 2 to hold.

For any $\delta > 0$ and $n = 2, 3, \ldots$ suppose the sequence $\{X_n\}$ is defined by the distribution functions

$$f_n(x) = \begin{cases} 0, & 0 < x < 1, \\ \frac{1}{2} - \frac{1}{n \sqrt{\delta}} & , 1 \leq x < \sqrt{2} , \\ 1 - \frac{2}{n \sqrt{\delta}} & , \sqrt{2} \leq x < n^2 \\ 1 & , n^2 \leq x \end{cases}$$

The sequence has the following properties:

$$E X_n = m_n = \frac{(1 + \sqrt{2})}{2} - \frac{(1 + \sqrt{2})}{n \sqrt{\delta}} + \frac{2}{n \delta}$$

$$E X^2_n = \frac{3}{2} - \frac{3}{n \sqrt{\delta}} + \frac{2}{n \delta - 1}$$

$$\lim_{n \to \infty} m_n = \frac{1}{2}(1 + \sqrt{2}) = \mu_1$$

$$\sum_{n=2}^{\infty} (m_n - \mu_1) \text{ converges}$$

$$\lim_{n \to \infty} F_n(x) = F(x)$$

$F(x)$ is the two point distribution function with equal jumps at 1 and $\sqrt{2}$, and whose mean is $\mu_1$ and second moment $\mu_2 = \frac{3}{2}$. 
Since $F_n(x) \leq F(x)$ for all $n$ and $x$, Theorem 2 holds and in fact
\[ H(t) = \frac{2t}{1+\sqrt{2}} + \frac{-2\sqrt{2}}{(1+\sqrt{2})^2} + 2 \sum_{n=2}^{\infty} \frac{1}{n^{3+\delta}} - \frac{4}{(1+\sqrt{2})} \sum_{n=2}^{\infty} \frac{1}{n^{1+\delta}}. \]

However,
\[ \lim_{n \to \infty} E^2_n = \begin{cases} \infty & \text{if } 0 < \delta < 1, \\ \frac{3}{2} + 2 & \text{if } \delta = 1, \\ \frac{3}{2} & \text{if } \delta > 1, \end{cases} \]

so that only for $\delta > 1$ does $E^2_n + \mu^2$.

It may appear from the result of Theorem 2 that the convergence of $\sum (\mu_1 - m_1)$ is essential for a second renewal theorem to hold. This seems a plausible induction from the fact that if only a finite number of the $X_i$ differ (the rest being identically distributed) then indeed a second renewal theorem takes the form given by Lemma 2(iii). On the other hand the boundedness condition C(i) does appear too restrictive and one suspects that it can be weakened considerably. The next example shows that a second renewal theorem exists for a sequence $\{X_i\}$ for which condition C is not fulfilled in any respect. In fact, this example is such that it is difficult to find any distribution function $F(x)$ to utilise the truncation method adopted in §2.
Suppose \( \{Y_i\} \) and \( \{Z_i\} \) are two independent renewal processes with

\[
\Pr \left[ Y_i \leq x \right] = A(x), \quad \Pr \left[ Z_i \leq x \right] = B(x),
\]

and first two moments (assumed finite)

\[
\begin{align*}
\mathbb{E}Y_1 &= a_1, & \mathbb{E}Y_1^2 &= a_2, \\
\mathbb{E}Z_1 &= b_1, & \mathbb{E}Z_1^2 &= b_2.
\end{align*}
\]

Form a renewal sequence \( \{X_i\} \) by taking

\[
X_i = Y_i \quad \text{if } i \text{ is odd},
\]

\[
X_i = Z_i \quad \text{if } i \text{ is even}.
\]

If

\[
\Pr \left[ Y_i + Z_i \leq x \right] = A*B(x) = C(x)
\]

and

\[
G_j(x) = \Pr \left[ X_1 + X_2 + \ldots + X_j \leq x \right]
\]

then for \( j = 1, 2, \ldots \),

\[
G_{2j-1}(x) = A*C^{*(j-1)}(x),
\]

\[
G_{2j}(x) = C^{*(j)}(x).
\]

The renewal function \( H(t) = \sum_{j=1}^{\infty} G_j(t) \) is absolutely convergent so we can rearrange terms to write

\[
H(t) = H_0(t) + H_e(t)
\]
\( H_0(t) \) is a delayed renewal function consisting of the odd terms in the sum \( \Sigma G_j(t) \) and satisfies the integral equation
\[
H_0(t) = A(t) + C^*H_0(t)
\]

\( H_e(t) \) consists of the even terms of \( \Sigma G_j(t) \) and satisfies
\[
H_e(t) = C(t) + C^*H_e(t).
\]

If
\[
v_1 = \int_0^\infty x dC(x) = a_1 + b_1,
\]
and
\[
v_2 = \int_0^\infty x^2 dC(x) = a_2 + 2a_1b_1 + b_2,
\]
then
\[
H_0(t) - \frac{t}{v_1} = \frac{v_2}{2v_1^2} - \frac{a_1}{v_1},
\]

\[
H_e(t) - \frac{t}{v_1} = \frac{v_2}{2v_1^2} - 1.
\]

The second renewal theorem for \( H(t) \) is then
\[
H(t) - \frac{2t}{v_1} = \left[ H_0(t) - \frac{t}{v_1} \right] + \left[ H_e(t) - \frac{t}{v_1} \right] + \frac{v_2}{2v_1^2} - 1 - \frac{a_1}{v_1},
\]

that is,
\[
H(t) = \frac{2t}{a_1 + b_1} + \frac{(a_2 + b_2)}{(a_1 + b_1)^2} + \frac{a_1(b_1 - a_1)}{(a_1 + b_1)^2} - 1 \quad (4.1)
\]
If one takes $F(x) = \left[ A(x) + B(x) \right] 2^{-1}$, so that

$$\mu_1 = \int_0^\infty x dF(x) = (a_1 + b_1) 2^{-1}$$

$$\mu_2 = \int_0^\infty x^2 dF(x) = (a_2 + b_2) 2^{-1},$$

and assumes that the method of §2 is valid, the result is

$$H(t) = \frac{2t}{a_1 + b_1} + \frac{(a_2 + b_2)}{(a_1 + b_1)^2} - 1.$$  \hspace{1cm} (4.2)

(4.2) agrees with (4.1) only if $b_1 = a_1$. It is in fact not hard to show for general $a_1$ and $b_1$ and the above choice of $F(x)$ that the interchange of limits (2.2) is not valid. Furthermore there seems to be no obvious choice of a distribution function $F(x)$ with the required first two moments that will yield (4.1) by the truncation method. If $F(x)$ is chosen so that its first moment is $\mu_1 = \int_0^\infty \frac{a_1 + b_1}{2} = 2^{-1} \mu$, the truncated renewal functions ($H_n(t)$) have the following property:

$$H_{2n-1}(t) = \frac{2t}{\nu_1} + \frac{\mu_2}{\nu_1^2} - 1 + \frac{(b_1 - a_1)}{2}.$$  \hspace{1cm} (4.3)

$$H_{2n}(t) = \frac{2t}{\nu_1} + \frac{\mu_2}{\nu_1^2} - 1.$$  \hspace{1cm} (4.4)

It is apparent that not even Cesaro limits of the right hand sides of (4.3) and (4.4) will yield the required result.

The second example is a special case of a semi-Markov renewal sequence. Suppose there is given a matrix of transition probabilities
\( (p_{ij}), i, j = 1, 2, \ldots, 0 \leq p_{ij} \leq 1, \quad \sum_j p_{ij} = 1 \) for every \( i \); an initial probability distribution \( (a_j) \); and a renewal sequence \( (Y_i) \). We form a new renewal sequence \( (X_i) \) by the following rule: if at any instant the current 'lifetime' is distributed as \( Y_i \) then the next 'lifetime' will be distributed as \( Y_j \) with probability \( p_{ij} \). Thus \( (X_n) \) is described by

\[
\Pr \left[ X_1 = Y_1 \right] = a_1
\]

\[
\Pr \left[ X_{n+2} = Y_j \mid X_{n+1} = Y_i \right] = p_{ij}, \quad n = 2, 3, \ldots
\]

The example above results if \( a_1 = 1, p_{12} = p_{21} = 1, p_{1i} = 0 \) otherwise.

Conditions ensuring a second renewal theorem for these sequences are not yet known. However, the following is not without interest. Let \( P_i(t) \) be the probability that at time \( t \) a lifetime distributed as \( Y_i \) is in progress and suppose \( \lim_{t \to \infty} P_i(t) = p_i \). In the example, it is not hard to show that

\[
p_1 = \frac{a_1}{a_1 + b_1}, \quad p_2 = \frac{b_1}{a_1 + b_1}
\]

Thus (4.1) can be rewritten as

\[
H(t) = \frac{2_t}{a_1 + b_2} + \frac{(a_2 + b_2)}{(a_1 + b_1)^2} - 1 + p_1 (p_2 - p_1)
\]

Any given renewal sequence is, of course, a semi-Markov sequence with \( P_{ii+1} = 1, i = 1, 2, \ldots \). Some results for the general case are available and will be discussed elsewhere.
§ 5. The maximum of the queue GI/M/1.

The renewal theorems developed in §§1-3 will now be applied to a problem in the theory of queues, namely, given that the system is not ergodic how fast does the maximum size of the queue grow? We consider the single server queueing process commonly characterised by the symbols GI/M/1. The intervals between the arrival of successive customers are of 'General Independent' type, that is, constitute a renewal process, the common distribution function being, say, $A(x)$ with

$$A(0+) = 0 \text{ and } a_1 = \int_0^\infty x dA(x) < \infty.$$ 

In most of what follows we will also assume that the second moment

$$a_2 = \int_0^\infty x^2 dA(x) \text{ is finite.} \text{ The symbol 'M' indicates that the service times of customers are independently and identically distributed with common negative exponential distribution, say}$$

$$C(x) = 1 - e^{-\mu x}.$$ 

Also, the sequence of interarrival times is assumed to be independent of the service times. In general the queue discipline must also be specified, but this is not relevant to the present work. One further point is that we are here dealing with the 'reflecting barrier' situation, the server remaining idle until the next customer arrives so that zero queue length constitutes a reflecting barrier.

Let $Q(t) = 0, 1, 2, \ldots$ denote the length of the queue at time $t$ (including the customer in service, if any), and let

$$M(t) = \max_{0 < r < t} Q(r)$$
be the largest length attained by the queue in \((0,t]\). Suppose that

\[ Q(0) = 0 \]

so that successive maxima are attained when the queue first reaches the lengths \(1,2,3,\ldots\). Let \(T_{ij}\) denote the first passage time from queue size \(i\) to queue size \(j\) (measured from the instant queue size first attained the value \(i\)), \(j = i+1, i+2, \ldots, i = 0, 1, \ldots\), and write

\[ \Pr [T_{ij} \leq x] = F_{ij}(x). \]

Furthermore, since \(A(0+) = 0\),

\[ T_{ij} = T_{i,j+1} + T_{i+1,i+2} + \cdots + T_{j-1,j}. \]

Then, given \(Q(0) = 0\),

\[ \Pr [M(t) \geq j] = F_{Oj}(t) = \Pr[T_{Oj} \leq t], \]

and

\[ \Pr [M(t) = j] = F_{Oj}(t) - F_{Oj+1}(t), \]

\[ EM(t) = \sum_{j=1}^{\infty} F_{Oj}(t). \]

The last expression shows that \(EM(t)\) is the renewal function of the sequence \(\{T_{i,i+1}\}, i = 0, 1, 2,\ldots\), so that the asymptotic behaviour of \(EM(t)\) is given by the results of \(\S\S2-3\) if it can be shown that \(\{T_{i,i+1}\}\) satisfies condition \(C\).

Firstly note that the assumption of negative exponential service times ensures the mutual independence of the \(T_{i,i+1}\). For other
distributions of service time this independence would not hold and it
is for this reason that the present remarks apply only to the queue
GI/M/1. The restriction to one server is for convenience in calculation
only. If explicit expressions for the \( F_{oj}(t) \) are available then \( M(t) \)
can be studied directly. However, this seems to be the case only
when the queueing process is also a birth and death process and even then
direct calculations can be very lengthy. The first result we require is

**Lemma 4:** The sequence \( \{F_{i-1i}(x)\} \) is monotone non-increasing for all \( x \).

**Proof:** For fixed \( i \), first passage from queue length \( i \) to queue length \( i+1 \)
can be achieved in the following mutually exclusive ways. If \( k = 0,1,2,...,i \),
service completions are achieved before the next arrival occurs then,
consequent on this arrival, we must have first passage from \( i-k+1 \) to \( i+1 \).
Suppose this next arrival occurs at time \( u \), the probability of which is
\( dA(u) \). The probability of \( k \) service completions in \( (0,u] \) is

\[
e^{-\mu u} \left( \frac{\mu u}{r!} \right)^k \quad \text{if } k = 0,1,...,i-1,
\]

\[
1 - \sum_{r=0}^{i-1} e^{-\mu u} \left( \frac{\mu u}{r!} \right)^r \quad \text{if } k = i.
\]

Thus

\[
F_{i+1i}(x) = \int_0^x e^{-\mu u} dA(u) + \sum_{r=1}^{i-1} \int_0^x e^{-\mu u} \frac{\left( \frac{\mu u}{r!} \right)^r}{r!} F_{i+r} \ (x-u) \ dA(u)
\]

\[
+ \int_0^x \left[ 1 - \sum_{r=0}^{i-1} e^{-\mu u} \frac{\left( \frac{\mu u}{r!} \right)^r}{r!} \right] F_{i+1i}(x-u) dA(u) \tag{5.1}
\]
The Lemma is proved if we can show

\[ F_{i-1j}(x) = F_{j+1i}(x) , \quad i = 1,2,3, \ldots, \]

for all \( x \), and we do this by induction on \( i \). We will assume that the time from the origin until the first arrival occurs has the distribution function \( F_{01}(x) = A(x) \). Now

\[ F_{12}(x) = \int_0^x e^{-\mu u} \, dA(u) + \int_0^x [1 - e^{-\mu u}] \, F_{12}(x-u) \, dA(u) \]

\[ \leq \int_0^x [e^{-\mu u} + 1 - e^{-\mu u}] \, dA(u) = A(x). \]

Assume that

\[ F_{jj+1}(x) \leq F_{j-1j}(x) , \quad j = 2,3, \ldots, i-1. \]

Writing \( 1 - \sum_{r=0}^{i-2} e^{-\mu u} \frac{(\mu u)^r}{r!} \) as \( \sum_{r=i}^{\infty} e^{-\mu u} \frac{(\mu u)^r}{r!} \), we find from (5.1) on rearranging terms that

\[ F_{i-11}(x) - F_{ii+1}(x) = \sum_{r=1}^{i-1} \int_0^x e^{-\mu u} \frac{(\mu u)^r}{r!} \left[ F_{i-r1}(x-u) - F_{i+1-r i+1}(x-u) \right] \, dA(u) \]

\[ + \int_0^x \left[ 1 - \sum_{r=0}^{i-1} e^{-\mu u} \frac{(\mu u)^r}{r!} \right] \left[ F_{11}(x-u) - F_{li+1}(x-u) \right] \, dA(u). \]

Since \( F_{i-r1}(x) = F_{i-r i-r+1} * F_{i-r+1 i-r+2} * \ldots * F_{i-1 i}(x) \), by the independence of the \( \{T_{i-11}\} \), we have
\[ F_{i-r}(x) - F_{i+1-r+1}(x) = \left[ F_{i-r+1} - F_{i+1} \right] * F_{i-r+1}(x), \quad r=1,2,\ldots,i-1, \]

\[ \geq \left[ F_{i-1} - F_{i+1} \right] * F_{i}(x), \]

the inequality following from the induction hypothesis and the fact that \( F_{i}(x) \leq F_{i-r+1}(x) \) for \( r = 1,2,\ldots,i-1 \).

Hence for all \( x \),

\[ F_{i-1}(x) - F_{i+1}(x) \geq \sum_{r=1}^{i-1} \int_{0}^{X} e^{-u \left( \frac{\mu u}{r} \right)^{r}} \left[ F_{i-1} - F_{i+1} \right] * F_{i}(x-u) \, dA(u) \]

\[ + \int_{0}^{X} \left[ 1 - \sum_{r=0}^{i-1} e^{-u \left( \frac{\mu u}{r} \right)^{r}} \right] \left[ F_{i-1} - F_{i+1} \right] * F_{i}(x-u) \, dA(u) \]

\[ = \int_{0}^{X} (1-e^{-u}) \left[ F_{i-1} - F_{i+1} \right] * F_{i}(x-u) \, dA(u) \quad (5.2) \]

But the right hand side of (5.2) is less in absolute value than

\[ |F_{i-1}(x) - F_{i+1}(x)| \]

so that (5.2) can hold for all \( x \geq 0 \) only when

\[ F_{i-1}(x) - F_{i+1}(x) \geq 0. \]

A consequence of Lemma 4 is that the expectations \( E T_{i+1} = m_{i+1} \) form a monotone non-decreasing sequence. The next result concerns these quantities. Note that the traffic intensity of the queue is \((\mu a_1)^{-1}\).
If \( \mu a_1 < 1 \) the queue is not ergodic and an equilibrium distribution of queue length does not exist.

**Lemma 5:** If \( \mu a_1 < 1 \), then

\[
\lim_{i \to \infty} m_{i+1} = \frac{a_1}{1 - \mu a_1} < \infty. \tag{5.3}
\]

If in addition, \( a_2 = \int_0^\infty x^2 dA(x) < \infty \), then

\[
\sum_{i=0}^{\infty} \left( \frac{a_1}{1 - \mu a_1} - m_{i+1} \right)
\]

converges.

**Proof:** Since \( \{m_{i+1}\} \) is a non-decreasing sequence, \( \lim_{i \to \infty} m_{i+1} \) certainly exists, although it may be infinite. (5.3) is established by the following argument. For a fixed queue length \( i \), the probability of \( k \) service completions in an interarrival interval is, for \( k = 0,1,\ldots,i-1 \),

\[
\alpha_k = \int_0^\infty e^{-\mu u} \frac{(\mu u)^k}{k!} \, dA(u),
\]

and when \( k = i \), is

\[
\int_0^\infty \left[ 1 - \sum_{k=0}^{i-1} e^{-\mu u} \frac{(\mu u)^k}{k!} \right] \, dA(u) = 1 - \sum_{k=0}^{i-1} \alpha_k.
\]

If in fact \( k \) service completions eventuate before the first arrival after the queue reaches length \( i \), then

\[
E\left[ T_{i+1}^k \right] = a_1 + m_{i-k+1} + \ldots + m_{i+1}.
\]
Unconditionally,

\[
E_{\ell+1} = m_{\ell+1} = a_{\ell} + \sum_{k=1}^{\ell-1} \left( a_{\ell} + m_{\ell-k+1} \right) \alpha_k + m_{\ell+1} \sum_{k=1}^{\infty} \alpha_k
\]

\[
= a_{\ell} + \sum_{k=1}^{\ell-1} m_{\ell-k+1} \alpha_k + m_{\ell+1} \sum_{k=1}^{\infty} \alpha_k , \quad i = 2, 3, \ldots ,
\]

(5.4)

\[
m_{12} = a_{\ell} + m_{12} \sum_{k=1}^{\infty} \alpha_k ,
\]

(5.5)

\[
m_{01} = a_{\ell} .
\]

(5.6)

But

\[
m_{\ell-k+1} i+1 = m_{\ell-k+1} i-\ell+2 + m_{\ell-i+2} i-\ell+3 + \ldots + m_{\ell+1}
\]

\[
\leq k m_{\ell+1} ,
\]

and

\[
m_{\ell+1} \leq i m_{\ell+1} .
\]

(5.4) thus yields the inequality

\[
m_{\ell+1} \leq a_{\ell} + m_{\ell+1} \sum_{k=1}^{\ell} \alpha_k + m_{\ell+1} \sum_{k=1}^{\infty} \alpha_k .
\]

Noting that

\[
\sum_{k=1}^{\infty} \alpha_k = \sum_{k=1}^{\infty} k \int_{0}^{1} e^{-\mu u} \frac{(\mu u)^k}{k!} \, dA(u) = \int_{0}^{\infty} \mu udA(u) = \mu a_{\ell} ,
\]

we find

\[
m_{\ell+1} \leq \frac{a_{\ell}}{1-\mu a_{\ell}} ,
\]
so that, if \( \mu a_1 < 1 \),

\[
\lim_{i \to \infty} m_{ii+1} = \lim_{i \to \infty} \sup m_{ii+1} \leq \frac{a_1}{1-\mu a_1} < \infty.
\]

Suppose \( m_{ii+1} \to l \). Then the generating function

\[
\sum_{i=0}^{\infty} z^i (l - m_{ii+1}) = G(z)
\]

converges for at least \( |a| < 1 \), since

\[
|G(z)| \leq \left| \sum_{i=0}^{\infty} z^i (l - a_1) \right| = (l - a_1) \left| (1 - z)^{-1} \right|.
\]

From (5.4) - (5.6), calculations show that

\[
G(z) = \frac{l}{1-z} - \frac{a_1 A^*(\mu - \mu z)}{A^*(\mu - \mu z) - z},
\]

(5.7)

where

\[
A^*(\mu - \mu z) = \int_{0}^{\infty} e^{-u(\mu - \mu z)} dA(u).
\]

By virtue of the convergence of \( l - m_{ii+1} \) to zero, a standard Abelian argument yields

\[
0 = \lim_{i \to \infty} (l - m_{ii+1}) = \lim_{z \to 1-} (1-z) G(z)
\]

\[
= l - \frac{a_1}{1-\mu a_1},
\]

which proves (5.3). To prove the second part of the lemma we recognise that for \( \mu a_1 < 1 \),
\[ Q(z) = \sum_{r=0}^{\infty} z^r q_r = \frac{(1-z)A*(\mu-\mu z)(1-\mu a_1)}{A*(\mu-\mu z) - z} \]  

(5.8)

is the generating function of an honest probability distribution \( \{q_r\} \) (see below). If \( a_2 \) is finite then

\[ \bar{q} = \sum_{r=1}^{\infty} r q_r = \sum_{r=1}^{\infty} \sum_{k=r}^{\infty} q_k < \infty, \]

implying that the generating function

\[ \frac{1 - Q(z)}{1 - z} = \sum_{r=0}^{\infty} z^r \left( \sum_{k=r+1}^{\infty} q_k \right) \]

converges at \( z = 1 \). However,

\[ G(z) = \frac{a_1}{1-\mu a_1} \left[ \frac{1 - Q(z)}{1 - z} \right] \]

(5.9)

Thus

\[ \sum_{i=0}^{\infty} \left( \frac{a_1}{1-\mu a_1} - m_{i+1} \right) = G(1) \]

is finite and has the value

\[ \frac{a_1 \bar{q}}{1-\mu a_1} = \mu a_1 + \frac{a_2 \mu^2}{2(1-\mu a_1)} \]  

(5.10)

A further deduction from (5.9) is the formula

\[ m_{i+1} = \frac{a_1}{1-\mu a_1} \sum_{r=0}^{i-l} q_r, \]

(5.11)

valid when \( \mu a_1 < 1 \).
The probability distribution \( q_r \) defined by (5.8) will be recognized as the equilibrium distribution of queue length of the queue M/G/1 (Kendall, 1951, equation (15)) in which the service distribution function is \( A(x) \) and arrival process is Poisson with parameter \( \mu \). This queue is the dual of the system GI/M/1 with which we are concerned, the roles of interarrival and service distributions being interchanged. That the results of the lemma are not surprising can be seen by the following considerations. In random walk terminology the \( T_{i+1} \) are first passage times from \( i \) to \( i+1 \) with paths that are step functions on the integers \( 0, 1, \ldots, i+1 \), the barrier at 0 being reflecting and the barrier at \( i+1 \) absorbing. 'Upward' jumps occur according to a renewal process with distribution function \( A(x) \) and 'downward' jumps according to a Poisson process with parameter \( \mu \). If we reverse the roles of the distribution functions \( A(x) \) and \( 1-e^{-\mu x} \), place an absorbing barrier at 0 and a reflecting one at \( i+1 \), and commence the walk in state 1, then \( T_{i+1} \) becomes the first passage from state 1 to the origin of the dual process. This latter formulation shows that \( T_{i+1} \) is also the length of a busy period of the finite capacity M/G/1 queue in which the number of waiting customers is limited to \( i \). The limit \( a_1 \left(1-\mu a_1\right)^{-1} \) is well known as the expected length of a busy period of the unlimited M/G/1 queue.
§6. The queue GI/M/1 (continued)

Let \( H(t) = EM(t) \) be the expectation of maximum queue length in \((0,t]\).

**Theorem 6:** If \( \mu a_1 < 1 \), \( A(x) \) is not a lattice distribution function, and \( a_2 \) is finite, then

\[
\lim_{t \to \infty} \left\{ H(t) - t \frac{(1-\mu a_1)}{a_1} \right\} = \frac{a_2(x + 1)}{2a_2^2(1 - \mu a_1)} - (1 - \mu a_1). \tag{6.1}
\]

**Proof:** Lemma 4 implies that \( F_{i+1}(x) \downarrow F(x) \) as \( i + \infty \). Provided it can be shown that \( F(x) \) is non-lattice with mean \( \mu_1 = a_1(1-\mu a_1)^{-1} \) and finite second moment, Lemma 5 ensures the satisfaction of condition C. First note that for \( |\gamma(s)| \leq 1 \) and \( Re s \geq 0 \)

\[
A^*(s+\mu-\mu_1(s)) = \int_0^\infty e^{-x(s+\mu-\mu_1(s))} dA(x)
= \sum_{j=0}^\infty \gamma_j(s) \int_0^\infty e^{-x(s+\mu)} \frac{\mu x_j^j}{j!} dA(x)
= \sum_{j=0}^\infty \gamma_j(s) \alpha_j(s). \tag{6.2}
\]

Taking Laplace transforms of (5.1) we have

\[
F_{i+1}(s) = \alpha(s) + \sum_{r=1}^{i-1} \alpha_r(s) F_{i-1-r+1}(s) + F_{i+1}(s) \sum_{r=1}^\infty \alpha_r(s). \tag{6.3}
\]

As a trial solution of (6.3) take

\[
F_{i+1-r+1}(s) = [\gamma_i(s)]^r.
\]
Then

\[ \gamma_i(s) = \alpha_0(s) + \sum_{r=1}^{i-1} \alpha_r(s) \gamma_i(s) + \gamma_i(s) \sum_{r=1}^{\infty} \alpha_r(s). \]  \hspace{1cm} (6.4)  

Since \( F_{i+1}(x) \) has a limit, \( \lim_{i \to \infty} \gamma_i(s) = \gamma(s) \) exists for \( \text{Re } s \geq 0 \) and is the Laplace-Stieltjes transform of \( F(x) \). Furthermore, the series \( \sum_{r=0}^{\infty} \alpha_r(s) \) is convergent. Taking the limit as \( i \to \infty \) in (6.4) shows that, from (6.2),

\[ \gamma(s) = \alpha_0(s) + \sum_{r=1}^{\infty} \alpha_r(s) \gamma(s) \]

\[ = A^s(s+\mu-\mu \gamma(s)). \] \hspace{1cm} (6.5)

The functional equation (6.5) has been extensively studied (see, for example, Takács, 1961, Lemma 1) and it is known to have a unique root \( \gamma = F^*(s) \) within the unit circle. The inverse transform, \( F(x) \), of \( F^*(s) \) is the distribution function of the duration of a busy period of the queue \( M/G/1 \) which is dual to the \( GI/M/1 \) system under consideration. If \( \mu a_1 < 1 \) then \( F(\infty) = 1 \) and

\[ \mu_1 = \int_0^{\infty} x dF(x) = \frac{a_1}{1-\mu a_1}. \] \hspace{1cm} (6.6)

If in addition, \( a_2 \) is finite, then

\[ \mu_2 = \int_0^{\infty} x^2 dF(x) = \frac{a_2}{(1-\mu a_1)^3}. \] \hspace{1cm} (6.7)

We have only to show that \( F(x) \) is non-lattice under the hypothesis that \( A(x) \) is non-lattice. But this is easily proved if we rewrite (6.5) in terms of
Fourier transforms and appeal to the criterion for lattice distributions (Lukacs, 1960, page 25). The limit (6.1) follows on substituting the above values of \( \mu_1 \) and \( \mu_2 \) in (2.3) and using (5.10).

Further deductions are, from (3.2),

\[
H(t) - \frac{t(1-\mu a_1)}{a_1} \leq \frac{a_2}{2a_1} \left\{ \frac{2t(1+\mu_1^2)}{2a_1 t(1-\mu a_1)^2 - a_2} + \frac{1}{t} \right\} \lambda (1-\mu a_1)
\]  \hspace{1cm} (6.8)

for \( t > \frac{a_2}{2a_1 (1-\mu a_1)^2} \), and from (3.5),

\[
\psi(x) = \lim_{t \to \infty} \Pr \left[ \xi(t) \leq x \right] = \frac{(1-\mu a_1)}{a_1} \int_0^x \left[ 1 - F(u) \right] du.
\]  \hspace{1cm} (6.9)

\( \xi(t) \) is the time one must wait, at \( t \), before the queue attains its next maximum.

\[
\psi^*(s) = \int_0^\infty e^{-sx} \psi(x) dx
\]

\[
\psi^*(s) = \frac{(1-\mu a_1)[1-\gamma(s)]}{a_1 s}
\]  \hspace{1cm} (6.10)

where \( \gamma(s) \) is the smallest root of (6.5).

\[
\int_0^\infty x \psi(x) \ dx = \frac{a_2}{sa_1(1-\mu a_1)^2},
\]
\[ \int_{0}^{\infty} x^2 \, d\Phi(x) = \frac{3\mu_2^2 + a_3(1-\mu_1)}{3a_1(1-\mu_1)^4}, \]

provided \( a_3 = \int_{0}^{\infty} x^3 dA(x) \) is finite.

The distribution function \( A(x) \) has so far been assumed to be non-lattice. Queueing systems of interest are often ones in which \( A(x) \) is in fact lattice, a case of particular importance being that of deterministic input:

\[
A(x) = \begin{cases} 
0 & , x < a_1, \\
1 & , a_1 \leq x.
\end{cases}
\]

Suppose now that \( A(x) \) is lattice and without loss of generality suppose its possible points of increase occur only at the positive integers. Successive queue maxima can then only be attained at times \( 1, 2, \ldots \), and each \( F_{i+1}(x) \) is lattice. The renewal sequence \( \{ T_{i+1} \} \) is discrete and it is clear that theorems corresponding to the above can be developed in this case. The explicit formulae will differ in one important respect due to the fact that if \( K(t) \) is now the renewal function of a discrete process \( \{ Y_i \} \), its second renewal theorem takes the form (Feller, 1949, page 103)

\[
\lim_{t \to \infty} \left\{ K(t) - \frac{t}{\mu_1} \right\} = \frac{\mu_2}{a_1^2} - 1 - \frac{1}{a_1}.
\]

(6.11)

The time parameter \( t \) now takes only the values \( 0, 1, 2, \ldots \). The final result (6.1) must be modified by the subtraction of \((a_1)^{-1}\) from the right hand side, otherwise the argument is essentially unchanged. We have
Theorem 7. If \( \mu a_1 < 1 \), \( A(x) \) is a lattice distribution function with unit period, and \( a_2 \) is finite, then

\[
\lim_{t \to \infty} \left\{ H(t) - t(1-\mu a_1) \right\} = \frac{a_2(1+\mu^2 a_1^2)}{2a_1^2(1-\mu a_1)} \left[ 1 + (2a_1)^{-1} \right].
\] (6.12)

In the special case \( D/M/1 \), where

\[
A(x) = \begin{cases} 
0 & , x < 1, \\
1 & , x \geq 1,
\end{cases}
\] (6.13)

(6.12) reduces to

\[
H(t) - t(1-\mu) + \frac{\mu}{1-\mu} - (1-\mu). 
\] (6.14)

By way of comparison, for the queue \( M/M/1 \) in which

\[
A(x) = 1 - e^{-x},
\] (6.15)

(6.1) yields

\[
H(t) - t(1-\mu) + \frac{\mu}{1-\mu}.
\] (6.16)

Intermediate between these two cases is the \( E_k/M/1 \) queue, \( k=1,2,\ldots \), with

\[
A(x) = \int_0^x e^{-ku} \frac{k^k u^{k-1}}{(k-1)!} \, du,
\] (6.17)

\[
a_1 = 1, \quad a_2 = 1 + k^{-1}.
\]

(6.17) reduces to (6.15) when \( k = 1 \) and approaches (6.13) as \( k \to \infty \).

For this choice of \( A(x) \) (6.1) is
\[ H(t) = t(1-\mu) + \frac{\mu}{1-\mu} - \left( \frac{1+\mu}{1-\mu} \right) \left( \frac{1-k^{-1}}{2} \right) \]

\[ = \frac{\mu}{1-\mu} - \frac{(1-\mu)}{2} + \frac{(1+\mu^2)}{k^2(1-\mu)} \]

(6.18)

The limit on the right side of (6.18) is equal to the limit in (6.16) when \( k = 1 \) and decreases as \( k \) increases. However the former does not approach the corresponding limit in (6.14) as \( k \to \infty \). The reason for this is, of course, that the sequence \( \{X_i\} \) to which (6.14) relates is lattice whereas that corresponding to (6.18) is non lattice for each finite \( k \).

An important problem still unsolved concerns the asymptotic behaviour of \( H(t) \) when \( \mu a_1 = 1 \). In this case both the original GI/M/1 queue and its M/G/1 dual are null recurrent and a stationary distribution of queue length does not exist for either system. The first passage distribution functions \( F_{i+1}(x) \) still have the limit \( F(x) \) defined in (6.5), and although \( F(x) \) is honest it has infinite expectation. The only known result (Karlin and McGregor, 1961) when \( \mu a_1 = 1 \) is for the Poisson queue M/M/1,

\[ H(t) \sim t^{\frac{1}{2}} . \]
References


