A PRIORI ERROR ESTIMATES FOR NUMERICAL METHODS FOR SCALAR CONSERVATION LAWS. PART III: MULTIDIMENSIONAL FLUX-SPLITTING MONOTONE SCHEMES ON NON-CARTESIAN GRIDS

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This paper is dedicated to the memory of Professor Ami Harten.

Abstract. This paper is the third of a series in which a general theory of *a priori* error estimates for scalar conservation laws is constructed. In this paper, we consider multidimensional flux-splitting monotone schemes defined on non-Cartesian grids, we identify those schemes which are *consistent* and prove that the $L^\infty(0,T;L^1(\mathbb{R}^d))$-norm of the error goes to zero as $(\Delta x)^{1/2}$ when the discretization parameter Δx goes to zero. Moreover, we show that non-*consistent* schemes can converge at optimal rate of $(\Delta x)^{1/2}$ because (i) the *conservation form* of the schemes and (ii) the so-called *consistency of the numerical fluxes* allow the regularity properties of the approximate solution to compensate for their lack of *consistency*.

Key words. A priori error estimates, irregular grids, monotone schemes, conservation laws, supraconvergence

AMS(MOS) subject classifications. 65M60, 65N30, 35L65

1. Introduction.

This is the third of a series of papers in which we develop a theory of *a priori* error estimates, that is, estimates given solely in terms of the exact solution, for numerical methods for the scalar conservation law [16]

$$v_t + \nabla\cdot f(v) = 0, \quad \text{in } (0,T)\times\mathbb{R}^d, \tag{1.1a}$$

$$v(0) = v_0, \qquad \text{on } \mathbb{R}^d. \tag{1.1b}$$

In the first paper of this series [6], a new, general approach for obtaining *a priori* error estimates for numerical methods for scalar conservation laws was introduced. The approach was then used to obtain optimal *a priori* error estimates for the Engquist-Osher scheme [9] on one-dimensional uniform grids; in contrast with previous work, [4], [5], [18-21], [23], [24], [25], [27], [29], [31], (see also [28]), no regularity properties of the approximate solution were *explicitly* used. In the second paper of this series [7], an extension of the above result to the case of flux-splitting monotone schemes defined on irregular, Cartesian grids was undertaken. As it is well-known, see, for example, [13], [26], and [30], if the numerical fluxes do not properly take into account the irregularity of the grids, a *loss of consistency* is generated and, as a consequence, a new term in the truncation error appears. In [7] it was shown that this new term can be controlled without using regularity properties of the approximate solution *only* for a subset of the schemes which were called *consistent* schemes. For this *new* class

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of schemes, optimal *a priori* error estimates were proven. In this paper, we consider the case of flux-splitting monotone schemes of the form

$$u_K^{n+1} = u_K^n - \frac{\Delta t}{|\,K\,|} \sum_{e \in \partial K} |\,e\,|\, f_{e,K}(u_K^n, u_{K_e}^n), \tag{1.2}$$

defined on non-Cartesian grids (see §2.a), we identify those schemes which are *consistent* and we prove an optimal *a priori* error estimate for them; more precisely, we show that the $L^\infty(0,T;L^1(\mathbb{R}^d))$-norm of the error goes to zero as $(\Delta x)^{1/2}$ where Δx is the maximum diameter of the finite volumes K. Moreover, we uncover the mechanism that allows optimal *a priori* error estimates to hold even for non-*consistent* schemes. Extending a similar result obtained in the one-dimensional case [7], we show that the regularity properties of the approximate solution can compensate for the loss of *consistency* of the scheme thanks to (i) the *conservation form* of the schemes and to (ii) the so-called *consistency of the numerical fluxes*; the nonlinear nature of the equations does not play any role in this mechanism.

The error estimate presented in this paper is the first *a priori* error estimate and the first optimal error estimate for numerical schemes for scalar conservation laws defined on non-Cartesian grids. As pointed out above, see also the discussion in [5], all previous work in error estimation for nonlinear conservation laws has been devoted to *a posteriori* error estimates. As a consequence, upper bounds of the total variation of the approximate solution had to be obtained. Since it is not a trivial matter to obtain such bounds even for simple schemes, this requirement has significantly slowed down the progress in this area. Indeed, there are only a few upper bounds of the total variation available in the literature. In [17] and [8], for monotone schemes in uniform Cartesian grids, and in [27], for monotone schemes in non-uniform Cartesian grids, it was proven that the total variation of the approximate solution does not increase in time. Unfortunately, the techniques used in the above mentioned papers are of no use in the non-Cartesian grids case since in this case the total variation of the approximate solutions *does* increase with time, even when the grids are uniform. For example, consider the simple case of the linear conservation law in two dimensions with $f(v) = (0,b)\,v$ with initial condition v_0 equal to one in the shaded area shown in Figure 1 and equal to zero elsewhere.

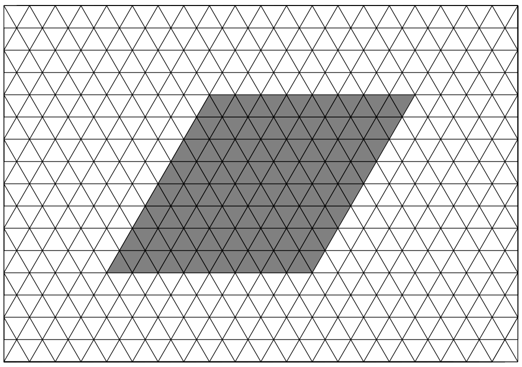

FIGURE 1. *Initial condition v_0 and triangulation with $\Delta x = 1/8$.*

Then for the simple Lax-Friedrichs-like scheme (1.2) with

$$f_{e,K}(v,w) = \frac{1}{2}(f(v)+f(w))\cdot n_{e,K} - \frac{1}{2\,\lambda}(w-v), \qquad (\lambda = \frac{\Delta t}{\Delta x}),$$

and grids of equilateral triangles as shown in Figure 1, a computation shows that the ratio R of the total variation of the approximate solution $u_h(\Delta t)$ after a single time-step to the total variation of the initial data $u_h(0) = v_0$ as a function of $\Delta x = 1/N$ is bigger than one, as is shown in Figure 2; therein, we have taken $\lambda\, b = .9$ and $cfl = 2\sqrt{3}\frac{\Delta t}{\lambda\,\Delta x} = .9$.

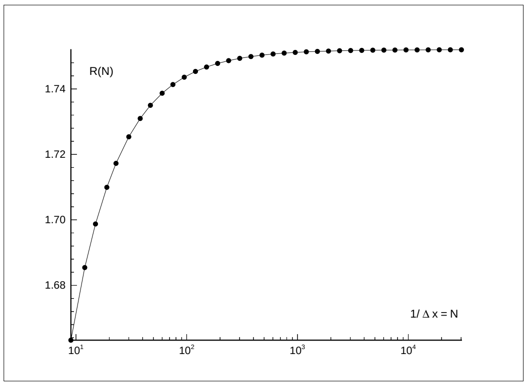

FIGURE 2. *The ratio* $R(N) = |\, u_h(\Delta t)\,|_{TV(R^2)} / |\, u_h(0)\,|_{TV(R^2)}$.

The other technique to obtain upper bounds for the total variation of approximate solutions of numerical schemes for nonlinear conservation laws is an extension, obtained in [4], see also [15], of an argument introduced in [2] to obtain the convergence of finite difference methods [3] in the framework of measure-valued solutions. This technique, which is essentially an *a priori* L^2-estimate, was used in [4], [31], [25], and [5] to obtain that the total variation of the approximate solution blows up like $(\Delta x)^{-1/2}$ as Δx goes to zero. This upper bound results in *a posteriori* error estimates with a rate of convergence of $(\Delta x)^{1/4}$ for finite volume schemes, [4], [31],[25]; of $(\Delta x)^{1/4}$ for the Discontinuous Galerkin method, [5]; and of $(\Delta x)^{1/8}$ for the Streamline Diffusion method, [5]. Unfortunately, this technique does not lead to optimal error estimates since the total variation is bounded only in terms of the L^2-norm of the initial data instead of in terms of its total variation. The advantage of our approach is that no estimate of the total variation is required to obtain optimal error estimates for the so-called *consistent* schemes.

We have to point out, however, that *consistent* schemes of the form (1.2), (i) can only be defined on fairly uniform grids and (ii) have to have fairly restrictive numerical fluxes. For example, upwinding-like schemes are not *consistent*; see a related result in [12]. This unfortunate situation is, of course, related to the fact that the schemes of the form (1.2) update the value at the volume K by *only* using the information stored in those volumes K_e sharing a common face e with K. We believe that the consistency requirements for schemes of a wider stencil, like, for example, the Lax-Friedrichs-like scheme considered in [1], should be weaker. However, we do not explore this issue in this paper since this would only further complicate an already very technical analysis.

For schemes that are not *consistent*, we show that optimal error estimates can still be obtained in spite of the loss of consistency. This phenomenon, sometimes called *supraconvergence*, has been studied by several authors in different contexts; see, for example, [22], [14], [11] and [10]. *Supraconvergence* for schemes for the linear conservation laws has been studied in [33, 34], for the one-dimensional case, and in [32], for non-uniform, Cartesian grids. Therein, the degeneracy from second-order to first-order accuracy, due to the irregularities of the grids, "predicted" by a naive consideration of the L^∞-like truncation error is studied. By a careful consideration of the *structure* of that truncation error, it is shown that this degeneracy does not actually take place provided that the scheme is stable and the exact solution is smooth enough. Our approach is similar in that we show that the *structure* of the truncation error allows the stability properties of the scheme to compensate for its lack of consistency. However, we study the general nonlinear conservation laws, work with an L^1-like truncation error and half-order accurate schemes, require minimal regularity of the exact solution, and consider general, non-Cartesian grids.

Finally, let us emphasize that the results presented in this paper are not a straightforward extension of those obtained in our second paper of this series [7] for irregular, Cartesian grids. Although the technique to obtain *a priori* error estimates introduced in [6] is independent of the space dimension, the irregularity of the grids plays a greater role in the multidimensional case and renders the manipulation of the truncation error a much more delicate undertaking.

The paper is organized as follows. In §2, the numerical schemes under consideration and related technical assumptions are presented, and the main results are stated and discussed. In §3, we give a proof of our main *a priori* error estimate, Theorem 2.1. In §4, we give an explanation of the *supraconvergence* phenomenon and the proof of the corresponding *a priori* error estimate, Theorem 2.3. We end with some concluding remarks in §5.

2. The numerical schemes and the main results.

a. The numerical schemes. We consider, for the sake of simplicity, uniform partition of $\mathbb{R}^+$ of the form $\{t^n = n\Delta t\}_{n\in\mathbb{N}}$ and triangulations of $\mathbb{R}^d$, $\mathcal{T}_{\Delta x} = \{K\}$, made of non-overlapping polyhedra. We require that for any two elements K and K', $\overline{K}\cap\overline{K}'$ is either a face e of both K and K' with nonzero $(d-1)$-Lebesgue measure $|e|$, or has Hausdorff dimension less than $d-1$. The maximum diameter of the sets K of the triangulation $\mathcal{T}_{\Delta x}$ is denoted by Δx.

Given these partitions, we define an approximation u to the entropy solution v of (1.1) as the piecewise-constant function

$$u(t,x) = u_K^n, \qquad \text{for } (t,x) \in [t^n, t^{n+1}) \times K, \tag{2.1}$$

constructed as follows. At $t=0$, the degrees of freedom of u are given by

$$u_K^0 = \frac{1}{|K|}\int_K u_0(s)\,ds, \tag{2.2a}$$

where $|K|$ is the measure of the finite volume K. The remaining degrees of freedom are defined by the following flux-splitting scheme:

$$u_K^{n+1} = u_K^n - \frac{\Delta t}{|K|}\sum_{e\in\partial K} |e|\, f_{e,K}(u_K^n, u_{K_e}^n), \tag{2.2b}$$

where K_e denotes the finite volume sharing the face e with the finite volume K, and the numerical flux $f_{e,K}(v,w)$ has the form

$$f_{e,K}(v,w) = f_{cent;e,K}(v,w) - f_{visc;e,K}(v,w), \tag{2.3a}$$

with

$$f_{cent;e,K}(v,w) = a_{e,K}\, f(v)\cdot n_{e,K} + b_{e,K}\, f(w)\cdot n_{e,K}, \tag{2.3b}$$

where $n_{e,K}$ is the outward unit normal at the face e of the finite volume K and

$$f_{visc;e,K}(v,w) = \alpha_e\,(N_e(w) - N_e(v)). \tag{2.3c}$$

We assume that the flux $f_{e,K}$ is *consistent* with the nonlinearity $f\cdot n_{e,K}$ and *conservative*, i.e., that $f_{e,K}(u,u) = f(u)\cdot n_{e,K}$ and that $f_{e,K}(u_K, u_{K_e}) + f_{e,K_e}(u_{K_e}, u_K) = 0$, respectively. For the schemes under consideration, these properties are equivalent to the following conditions:

$$a_{e,K} + b_{e,K} = 1, \qquad\qquad \text{(consistency)}, \tag{2.3d}$$

$$a_{e,K_e} = b_{e,K}, \qquad b_{e,K_e} = a_{e,K}, \qquad \text{(conservativity)}. \tag{2.3e}$$

We ask that

$$0 \le a_{e,K}, b_{e,K} \le 1. \tag{2.3f}$$

Finally, we require that for every finite volume K and each of its faces e we have

$$\alpha_e\, N_e'(\cdot) \ge b_{e,K}\, |\, f'(\cdot)\cdot n_{e,K}\,|, \tag{2.3g}$$

and normalize α_e and N_e' by asking that

$$\alpha_e \le 1, \quad N_e'(\cdot) \le \|\, f'(\cdot)\,\|_{\ell^\infty}. \tag{2.3h}$$

As it is well-known, this condition ensures the *monotonicity* of the scheme under the following condition on the size of the time-step Δt:

$$\frac{\Delta t}{|\,K\,|}\sum_{e\in\partial K} |\,e\,|\,\big(\alpha_e\, N_e'(u_K^n) - b_{e,K}\, f'(u_K^n)\cdot n_{e,K}\big) \le 1, \qquad \forall K\in\mathcal{T}_{\Delta x}, n\ge 0. \tag{2.4}$$

Our results also involve two grid related quantities which we introduce next. Given a face $e\in\mathcal{E}_{\Delta x}\equiv\bigcup\{e : e\in\partial K \text{ for all } K\in\mathcal{T}_{\Delta x}\}$, we define C_e to be the convex hull of $K_e^+\cup K_e^-$, d_e to be the diameter of C_e, and we set

$$\mathbb{D}_{\Delta x} = \sup_{e\in\mathcal{E}_{\Delta x}} \frac{|\,e\,|\, d_e}{\max\{\,|\,K_e^+\,|, |\,K_e^-\,|\,\}}. \tag{2.5}$$

Also, given the finite volume K, we denote by ι_K the number of sets C_e for which $|\,K\cap C_e\,| \ne 0$ for some $e\in\mathcal{E}_{\Delta x}$ and we set

$$\iota_{\Delta x} = \sup_{K\in\mathcal{T}_{\Delta x}} \iota_K. \tag{2.6}$$

The following table gives the values of those parameters in some elementary cases

mesh type	*aspect ratio*	d	$\mathbb{D}_{\Delta x}$	$\iota_{\Delta x}$
equilateral	N/A	2	4	3
cartesian triangular	λ	2	$2\sqrt{4+\lambda^2}$	3
cartesian rectangular	λ	2	$\sqrt{4+\lambda^2}$	4
cartesian rectangular	λ, μ	3	$\sqrt{4+\lambda^2+\mu^2}$	6

TABLE 1. *Values of the mesh related parameters on simple grids*

b. Consistency and *a priori* error estimates.

The generalization of the notion of *consistency* introduced in [7] for schemes on irregular, Cartesian grids will be based on the following two quantities:

$$\Delta_K = \max_{1\le j\le d} \sum_{i=1}^{d} \left| \sum_{e\in\partial K} |\, e\,|\,(\delta_e)_j\,(n_{e,K})_i \right|, \tag{2.7a}$$

$$\mathbb{A}_K = \sup_{c\in\mathbb{R}} \frac{1}{\|\, f'(c)\,\|} \sum_{i=1}^{d} \left| \sum_{e\in\partial K} |\, e\,|\,\alpha_e\, N_e'(c)\,(x_{K_e} - x_K)_i \right|, \tag{2.7b}$$

where

$$\delta_e = (x_e - x_K) - b_{e,K_e}\,(x_{K_e} - x_K), \tag{2.7c}$$

where x_K denotes the barycenter of the finite volume K and x_e the barycenter of the face e. We say that the scheme (2.2b) with fluxes satisfying the conditions (2.3) is *consistent* with respect to the family of triangulations $\{\, \mathcal{T}_{\Delta x}\,\}_{\Delta x>0}$ if $\Delta_K = A_K \equiv 0$. If we consider for instance a grid of equilateral triangles, or any affine transformation of it, and choose

$$a_{e,K} = b_{e,K} = 1/2, \quad N_e'(c) = \|\, f'(c)\,\|, \quad \alpha_e\, N_e'(c) = \frac{\beta}{|\, e\,|},$$

where β does not depend on e, then the corresponding scheme is *consistent.* Note that to get monotonicity, β has to satisfy (still on the equilateral grid)

$$\frac{\sqrt{3}}{4}\Delta x\|\, f'(c)\,\| \le \beta \le \frac{\sqrt{3}}{2}\Delta x\|\, f'(c)\,\|.$$

The conditions for *consistency* for the schemes under consideration impose heavy restrictions on the grids and on the numerical fluxes. Indeed, $\Delta_K \equiv 0$ if $\delta_e \equiv 0$, that is, if

$$(x_e - x_K) = b_{e,K_e}\,(x_{K_e} - x_K),$$

or, equivalently, if

$$x_e = b_{e,K_e}\, x_{K_e} + b_{e,K}\, x_K.$$

This means not only that the value of the scalar b_{e,K_e} is determined, but that the barycenters x_{K_e} and x_{K_e} and the barycenter of the face e, x_e, must lie on the same line! This condition is trivially satisfied for Cartesian or affine transformations of Cartesian grids, but that is not the case for general non-Cartesian grids. Moreover,

$\mathbb{A}_K \equiv 0$ if and only if

$$x_K = \frac{\sum_{e\in\partial K} |\, e \,|\, \alpha_e\, N'_e(c)\, x_{K_e}}{\sum_{e\in\partial K} |\, e \,|\, \alpha_e\, N'_e(c)}.$$

This means, not only that the quantities

$$\frac{|\, e \,|\, \alpha_e\, N'_e(c)}{\sum_{e\in\partial K} |\, e \,|\, \alpha_e\, N'_e(c)},$$

should be independent of c, but that the barycenter x_K has to be equal to a very precise convex combination of the barycenters x_{K_e}. This condition is *impossible* to satisfy for upwinding-like fluxes unless the grids are Cartesian or affine transformations of Cartesian grids.

For example, consider the linear case $f(v) = (0, b)\, v$ in two-space dimensions and triangulations made of acute triangles. For the upwinding scheme $N'_e(c) = |\,(0, b) \cdot n_{e,K}\,|$. If for some face e of the finite volume K the velocity $(0, b)$ is perpendicular to $n_{e,K}$, we have $N'_e(c) = 0$ and the condition above is impossible to satisfy. This lack of consistency of upwinding-like schemes has been pointed out in [12] in the framework of a formal L^∞-like truncation error analysis.

Fortunately, the consistency condition can be relaxed. We say that a scheme is *$p-$consistent* if there are two nonnegative constants C_δ and C_α such that

$$|\, \delta \,|_{var,1/2} \le C_\delta\, (\Delta x)^p, \quad |\, \alpha \,|_{var,1/2} \le C_\alpha\, (\Delta x)^p, \tag{2.8}$$

where

$$|\, \delta \,|_{var,1/2} = \sup_{x\in\mathbb{R}^d} \frac{\sum_{x:|\, x - x_K \,| \le (\Delta x)^{1/2}} \Delta_K}{(\Delta x)^{d/2}}, \tag{2.9}$$

$$|\, \alpha \,|_{var,1/2} = \sup_{x\in\mathbb{R}^d} \frac{\sum_{x:|\, x - x_K \,| \le (\Delta x)^{1/2}} \mathbb{A}_K}{(\Delta x)^{d/2}}. \tag{2.10}$$

For instance if we consider the two-dimensional case and define

$$b_{e,K} = \frac{(x_e - x_{K_e}) \cdot (x_K - x_{K_e})}{\|\, x_K - x_{K_e} \|^2}, \qquad a_{e,K} = 1 - b_{e,K},$$

$$N'_e(c) = \|\, f'(c) \,\|, \qquad \alpha_e N'_e(c) = \frac{\beta}{|\, e \,|} \qquad (\beta \text{ independent of } e),$$

then the conservativity condition (2.3e) is satisfied. Moreover, (2.3f) is verified provided that

$$\|\, x_e - x_{K_e} \| \le \|\, x_K - x_{K_e} \| \quad \text{and} \quad |\, \theta_e \,| \le \frac{\pi}{2},$$

where $\theta_e = \measuredangle\,(x_e - x_{K_e}, x_K - x_{K_e})$. In particular, those conditions are satisfied if the triangulation is acute. Furthermore, to get monotonicity, β has to satisfy

$$b_{e,K} \|\, f'(c) \,\| \,|\, e \,| \le \beta \le \|\, f'(c) \,\| \,|\, e \,|.$$

We observe that

$$\Delta_K \le 2\iota_{\Delta x} \mathbb{D}_{\Delta x} \sup_{e\in\mathcal{E}_{\Delta x}} \left(\frac{\|\, \delta_e \,\|_{\ell^\infty}}{d_e} \max\{K_e^+, K_e^-\} \right),$$

$$\mathbb{A}_K \le \beta \sum_{i=1}^{2} |\sum_{e\in\partial K} (x_{K_e} - x_K)_i|.$$

Note that by definition of $b_{e,K}$, δ_e is here the orthogonal projection of $x_e - x_K$ on the direction perpendicular to $x_{K_e} - x_K$. Assuming now the triangulation to be uniformly regular, i.e., $\Delta x \le \kappa\, h_K$ for any $K \in \mathcal{T}_{\Delta x}$ (h_K = diameter of K), gives

$$|\,\delta\,|_{var,1/2} \le 2\,\kappa^2\,\iota_{\Delta x}\,\mathbb{D}_{\Delta x} \sup_{e\in\mathcal{E}_{\Delta x}} \frac{\|\,\delta_e\,\|_{\ell^\infty}}{d_e},$$
$$|\,\alpha\,|_{var,1/2} \le 2\,\frac{\kappa^2}{\Delta x^2}\,\beta \sup_{e\in\mathcal{E}_{\Delta x}} \|\sum_{e\in\partial K}(x_{K_e} - x_K)\,\|_{\ell^\infty}.$$

Without further assumptions on the grid, we have $|\,\delta\,|_{var,1/2} = \mathcal{O}(1)$ and $|\,\alpha\,|_{var,1/2} = \mathcal{O}(1)$ and the above scheme is 0-consistent, i.e., inconsistent (case treated in Theorem 2.3). However, if we assume for instance

$$\|\,\delta_e\,\|_{\ell^\infty} = \mathcal{O}(\Delta x^{1+p}),$$
$$\|\sum_{e\in\partial K}(x_{K_e} - x_K)\,\|_{\ell^\infty} = \mathcal{O}(\Delta x^{1+p}),$$

then the scheme is p-consistent (case treated in Corollary 2.2). The first of the above assumptions amounts to having, for $e \in \mathcal{E}_{\Delta x}$, the three nodes x_K, x_e, x_{K_e} lining up as Δx goes to zero. The second assumption means for instance that the grid becomes asymptotically equilateral, if simplicial elements are used.

We are now ready to state our error estimate which, following [6], is expressed in terms of the numerical viscosity associated to the scheme under consideration and in terms of the measure of consistency introduced above.

THEOREM 2.1. *Let the Courant-Friedrichs-Levy condition* (2.4) *be satisfied. Let* u *be the piecewise-constant solution given by the scheme* (2.2) *with coefficients satisfying* (2.3)*, let* v *be the entropy solution. Then*

$$\begin{aligned}\|\,u(t^N) - v(t^N)\,\|_{L^1(\mathbb{R}^d)} \le{}& 2\,\|\,u_0 - v_0\,\|_{L^1(\mathbb{R}^d)} + 8\,|\,v_0\,|_{TV(\mathbb{R}^d)}\sqrt{2t^N\,\|\,\nu_v\,\|}\,(\Delta x)^{1/2}\\ &+ C\,|\,v_0\,|_{TV(\mathbb{R}^d)}\big(\,|\,\delta\,|_{var,1/2} + |\,\alpha\,|_{var,1/2}\big)\\ &+ |\,v_0\,|_{TV(\mathbb{R}^d)}\,(\,b_1(\Delta x)^{3/4} + b_2\,\Delta x\,),\end{aligned}$$

where $\|\,\nu_v\,\| = \sup_K \sup_{\substack{t\in(0,t^N)\\ x\in\mathbb{R}^d}} \max_{1\le i,j\le d} |\,\nu_K^{ij}(v(t,x))/\Delta x\,|$ *and the local viscosity coefficient* ν_K^{ij} *is given by*

$$\nu_K^{ij}(c) = \frac{\Delta t}{2}\,f_i'(c)\,f_j'(c) - \frac{1}{|\,K\,|}\sum_{e\in\partial K}|\,e\,|\,\big[b_{e,K_e}\,f'(c)\cdot n_{e,K} + \alpha_e\,N_e'(c)\big]\,\xi_{e,K}^{ij},$$
$$\xi_{e,K}^{ij} = \frac{1}{2}\big\{(x_{K_e} - x_K)_i\,(x_{K_e} - x_K)_j + \mathbb{I}_{K_e}^{ij} - \mathbb{I}_K^{ij}\big\},$$
$$\mathbb{I}_\Omega^{ij} = \frac{1}{|\,\Omega\,|}\int_\Omega (x' - x_\Omega)_i\,(x' - x_\Omega)_j\,dx'.$$

The constant C *is given by*

$$C = 4\,\|\,f'(v)\,\|\,(t^N + \sqrt{2t^N/\|\,\nu_v\,\|}\,)\,(1 + b_0\,(\Delta x)^{1/4}),$$

and $\| f'(v) \|$ *is given by*

$$\| f'(v) \| = \sup_{\substack{t \in (0,t^N) \\ x \in \mathbb{R}^d}} \| f'(v(t,x)) \|_{\ell^\infty} .$$

The constants b_0, b_1, *and* b_2 *are locally bounded functions that depend solely on the quantities* $\| f'(v) \| \Delta t / \Delta x$, $\| f'(v) \| / \| \nu_v \|$, $\{t^N \| \nu_v \|\}^{1/2}$, $\mathbb{D}_{\Delta x}$ *and* ι_Δ.

An immediate consequence of this result is the following.

COROLLARY 2.2 ($p-$CONSISTENT SCHEMES). *With the notation and under the assumptions of Theorem* 2.1, *if the scheme is* $p-$*consistent, we have*

$$\| u(t^N) - v(t^N) \|_{L^1(\mathbb{R}^d)} \le 2 \| u_0 - v_0 \|_{L^1(\mathbb{R}^d)} + 8 \, | v_0 |_{TV(\mathbb{R}^d)} \sqrt{2 t^N \| \nu_v \|} \, (\Delta x)^{1/2} + \mathcal{O}((\Delta x)^{\min\{p, 3/4\}}).$$

c.*Supraconvergence* **and** *a posteriori* **error estimates.** For non-*consistent* schemes in arbitrary triangulations, and this includes notably the upwinding-like schemes mentioned in the previous section, it is possible to obtain an error estimate in terms of the following quantities

$$\| \delta \|_{\ell^\infty(\mathcal{E}_{\Delta x})/\mathbb{R}^d} = \inf_{\hat\delta \in \mathbb{R}^d} \sup_{e \in \mathcal{E}_{\Delta x}} \max_{1 \le j \le d} | (\delta_e - \hat\delta)_j | \tag{2.11a}$$

$$\| \alpha \|_{\ell^\infty(\mathcal{E}_{\Delta x})} = \sup_{e \in \mathcal{E}_{\Delta x}} | \alpha_e | \max_{1 \le j \le d} | (x_{K_e} - x_K)_j |, \tag{2.11b}$$

which are always of order Δx, by (2.7c), (2.3f) and (2.3h), instead of in terms of the quantities $| \delta |_{var,1/2}$ and $| \alpha |_{var,1/2}$. The price to pay is that the error estimate now must depend on the smoothness properties of the approximate solution; see Lemma 4.1.

THEOREM 2.3. *Let the Courant-Friedrichs-Levy condition* (2.4) *be satisfied. Let* u *be the piecewise-constant solution given by the scheme* (2.2) *with coefficients satisfying* (2.3), *let* v *be the entropy solution. Then*

$$\begin{aligned} \| u(t^N) - v(t^N) \|_{L^1(\mathbb{R}^d)} \le {} & 2 \| u_0 - v_0 \|_{L^1(\mathbb{R}^d)} + 8 \, | v_0 |_{TV(\mathbb{R}^d)} \sqrt{2 t^N \| \nu_v \|} \, (\Delta x)^{1/2} \\ & + \frac{2\sqrt{2}\, d}{\sqrt{t^N \| \nu_v \| \Delta x}} \{ \| \delta \|_{\ell^\infty(\mathcal{E}_{\Delta x})/\mathbb{R}^d} + \| \alpha \|_{\ell^\infty(\mathcal{E}_{\Delta x})} \} \\ & \qquad \Big\{ \mathbb{D}_{\Delta x} \, \iota_{\Delta x} \| f'(v) \| \, t^N |v_0|_{TV(\mathbb{R}^d)} + \frac{1}{\iota_{inf}} \| f'(u) \| \, \| u \|_{L^1(0,t^N; TV(\mathbb{R}^d))} \Big\} \\ & + | v_0 |_{TV(\mathbb{R}^d)} \, (b_1 (\Delta x)^{3/4} + b_2 \, \Delta x), \end{aligned}$$

where $\iota_{inf} = \inf_{K \in \mathcal{T}_{\Delta x}} \iota_K$. *The constants* b_1 *and* b_2 *are locally bounded functions that depend solely on the quantities* $\| f'(v) \| \Delta t / \Delta x$, $\| f'(v) \| / \| \nu_v \|$, $\{t^N \| \nu_v \|\}^{1/2}$, $\mathbb{D}_{\Delta x}$ *and* $\iota_{\Delta x}$, *where* $\| \nu_v \|$ *is defined in Theorem 2.1.*

Beside the estimate itself, which is new, an explanation of why it is at all possible is worth considering. As will be shown in §4, it is a combination of both *consistency* and *conservativity* of the numerical fluxes which makes it possible by forcing the quantity $D_{e,K} = (x_e - x_K) - b_{e,K_e} (x_{K_e} - x_K)$ to be such that $D_{e,K} = D_{e,K_e} \equiv \delta_e$. Indeed, we have

$$\begin{aligned}
D_{e,K} &= (x_e - x_K) - b_{e,K_e}\,(x_{K_e} - x_K) \\
&= (x_e - x_K) - a_{e,K}\,(x_{K_e} - x_K), && \text{by conservativity (2.3e),} \\
&= (x_e - x_K) - (1 - b_{e,K})\,(x_{K_e} - x_K), && \text{by consistency (2.3d),} \\
&= (x_e - x_{K_e}) - b_{e,K}\,(x_K - x_{K_e}) \\
&= D_{e,K_e}.
\end{aligned}$$

3. Proof of Theorem 2.1. This section is entirely devoted to the proof of Theorem 2.1. It has exactly the same structure as §3 of [7]. In all the proofs, we assume that the entropy solution v is smooth since the general case can be obtained by a standard density argument; see the proof of Proposition 5.5 in [6]. The case in which v is piecewise smooth is treated in [35].

a. The approximation inequality. Along the lines of [6], we consider the form

$$E(u,v;t^N) = \int_0^{t^N}\!\!\int_{\mathbb{R}^d} \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_{\Delta x}} \Psi_K^n(v(t,x))\,\phi(t,x,t^{n+1},x_K)\,|\,K\,|\,\Delta t\; dx\; dt,$$

where $U(\cdot)$ is the absolute value function $|\cdot|$, and Ψ_K^n is given by

$$\Psi_K^n(c) = U'(u_K^n - c)\left\{\frac{u_K^{n+1}-u_K^n}{\Delta t} + \frac{1}{|\,K\,|}\sum_{e\in\partial K} |\,e\,|\, f_{e,K}(u_K^n, u_{K_e}^n)\right\}.$$

The term $\phi(t,x,t^{n+1},x_K)$ is an averaged test function defined by

$$\phi(t,x,t^{n+1},x_K) = \frac{1}{|\,K\,|}\int_K \varphi(t,x,t^{n+1},x')\,dx', \tag{3.1}$$

where x_K denotes the barycenter of the finite volume K, and where the function $\varphi(t,x,t',x')$ is defined as follows:

$$\varphi(t,x,t',x') = w_{\epsilon_t}(t-t')\prod_{i=1}^{d}\eta_{\epsilon_x}(x_i - x_i'), \tag{3.2a}$$

$x = (x_1,\cdots,x_d)$, $x' = (x_1',\cdots,x_d')$ and t, t' being respectively points in $\mathbb{R}^d$ and $\mathbb{R}^+$. Finally, the functions w_{ϵ_t} and η_{ϵ_x} are constructed as follows

$$w_{\epsilon_t}(s) = \frac{1}{\epsilon_t}\,w(\frac{s}{\epsilon_t}), \quad \eta_{\epsilon_x}(s) = \frac{1}{\epsilon_x}\,\eta(\frac{s}{\epsilon_x}), \quad \forall s\in\mathbb{R}, \tag{3.2b}$$

where ϵ_t and ϵ_x are two arbitrary positive numbers. Both w and η satisfy the following conditions: (i) $\eta(t)\geq 0$, for $t>0$, (ii) $\eta(t)=\eta(-t)$, for $t>0$, (iii) the support of η is $[-1,1]$, and (iv) $\int_0^1 \eta(r)dr = 1/2$. For future reference, we set

$$W(t) = \int_0^t w_{\epsilon_t}(s)\,ds. \tag{3.2c}$$

Furthermore, the functions w and η are taken to be smooth approximations to $\chi\equiv\chi_0$

and χ_ϵ, where

$$\chi_\epsilon(x) = \begin{cases} (1+\epsilon)/2, & \text{for } |x| \le (1-\epsilon)/(1+\epsilon), \\ (1+\epsilon)^2\,(1-|x|)/4\,\epsilon, & \text{for } |x| \in [(1-\epsilon)/(1+\epsilon), 1], \\ 0, & \text{elsewhere .} \end{cases}$$

It is easy to verify that we can find a sequence of functions η such that

$$\lim_{\eta\to\chi_\epsilon} |\,\eta\,|_{TV(\mathbb{R})} = |\,\chi_\epsilon\,|_{TV(\mathbb{R})} = 1+\epsilon, \tag{3.3a}$$

$$\lim_{\eta\to\chi_\epsilon} |\,\eta'\,|_{TV(\mathbb{R})} = |\,\chi_\epsilon'\,|_{TV(\mathbb{R})} = 2+\epsilon+1/\epsilon. \tag{3.3b}$$

$$\lim_{\eta\to\chi_\epsilon} \|\,\eta\,\|_{L^\infty(\mathbb{R})} = \|\,\chi_\epsilon\,\|_{L^\infty(\mathbb{R})} = (1+\epsilon)/2. \tag{3.3c}$$

We now describe the main ideas on which the derivation of our estimates is based. From the identity

$$0 = E(u,v;t^N) = -E^\star_{div}(u,v;t^N) + E_{diss}(u,v;t^N) + T_{err}(u,v;t^N),$$

and the structure of the term $T_{err}(u,v;t^N)$, the following approximation inequality can be derived, [6, Proposition 7.6]:

$$\begin{aligned} e(t^N) &\le 2\,e(0) + 8\,\left(\epsilon_x + \epsilon_t \|f'(v)\|\right) |\,v_0\,|_{TV(\mathbb{R}^d)} + 2\,\|\,f'(v)\,\|\,|\,v_0\,|_{TV(\mathbb{R}^d)}\,\Delta t \\ &\quad + 2 \lim_{w\to\chi}\ \sup_{1\le n\le N} \left\{E^\star_{div}(u,v;t^n)/W(t^n) - E_{diss}(u,v;t^n)/W(t^n)\right\}, \end{aligned}$$

where $e(t)$ denotes the error $\|\,u(t)-v(t)\,\|_{L^1(\mathbb{R}^d)}$. Now, to obtain the error estimate, it is enough to estimate the forms $E^\star_{div}(u,v;t^n)$ and $E_{diss}(u,v;t^n)$. The remaining of this section is devoted to identifying these forms and estimating them.

We start by rewriting $\Psi^n_K(c)$

$$\begin{aligned} \Psi^n_K(c) &= U'(u^n_K - c)\,\frac{1}{\Delta t}\left\{u^{n+1}_K - u^n_K + \frac{\Delta t}{|\,K\,|}\sum_{e\in\partial K} |\,e\,|\,f_{e,K}(u^n_K, u^n_{K_e})\right\} \\ &= U'(u^n_K - c)\,\frac{1}{\Delta t}\left\{u^{n+1}_K - u^n_K + \frac{\Delta t}{|\,K\,|}\sum_{e\in\partial K} (p_{e,K}(u^n_K) - p_{e,K}(u^n_{K_e}))\right\}, \end{aligned}$$

where

$$p_{e,K}(s) = -|\,e\,|\,\left[b_{e,K}\,f(s)\cdot n_{e,K} - \alpha_e\,N_e(s)\right],$$

and proceed as in [6]. Here, the form $E_{diss}(u,v;t^N)$ is

$$E_{diss}(u,v;t^N) = \int_0^{t^N}\!\!\int_{\mathbb{R}^d} \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_{\Delta x}} LRED^n_K(v(t,x))\,\phi(t,x,t^{n+1},x_K)\,|\,K\,|\,\Delta t\;dx\;dt,$$

where the local rate of entropy dissipation $LRED^n_K(c)$ is given by

$$\begin{aligned} LRED^n_K(c) &= \frac{1}{\Delta t}\int_{u^{n+1}_K}^{u^n_K} (p_K(u^n_K) - p_K(s))\,U''(s-c)\,ds \\ &\quad + \frac{1}{|\,K\,|}\sum_{e\in\partial K}\int_{u^{n+1}_K}^{u^n_{K_e}} (p_{e,K}(u^n_{K_e}) - p_{e,K}(s))\,U''(s-c)\,ds, \end{aligned}$$

and $p_K(s) = s - \frac{\Delta t}{|K|}\sum_{e\in\partial K} p_{e,K}(s)$.

The dual form $E^{\star}_{div}(u,v;t^N)$ reads

$$\begin{aligned}
E^{\star}_{div}(u,v;t^N) = &- \int_0^{t^N}\int_{\mathbb{R}^d}\int_0^{t^N}\int_{\mathbb{R}^d} U(u(t',x')-v(t,x))\,\varphi_t(t,x,t',x')\,dx\,dt\,dx'\,dt' \\
&+ \int_0^{t^N}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} U(u(t',x')-v(t^N,x))\,\varphi(t^N,x,t',x')\,dx\,dx'\,dt' \\
&- \int_0^{t^N}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} U(u(t',x')-v_0(x))\,\varphi(0,x,t',x')\,dx\,dx'\,dt' \\
&- \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_{\Delta x}}\int_0^{t^N}\int_{\mathbb{R}^d}\sum_{e\in\partial K} |\,e\,|\, b_{e,K}\,\big[\phi(t,x,t^{n+1},x_K) \\
&\qquad\qquad - \phi(t,x,t^{n+1},x_{K_e})\big]\,F(u^n_K,v)\cdot n_{e,K}\,\Delta t\,dx\;dt \\
&+ \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_{\Delta x}}\int_0^{t^N}\int_{\mathbb{R}^d}\sum_{e\in\partial K} |\,e\,|\,\alpha_e\,\big[\phi(t,x,t^{n+1},x_{K_e}) \\
&\qquad\qquad - \phi(t,x,t^{n+1},x_K)\big]\,\mathcal{N}_e(u^n_K,v)\,\Delta t\;dx\;dt,
\end{aligned}$$

where the functions $F(u,c)$ and $\mathcal{N}_e(u,c)$ is defined as follows:

$$F(u,c) = \int_c^u f'(s)\,U'(s-c)\,ds, \qquad \mathcal{N}_e(u,c) = \int_c^u N'_e(s)\,U'(s-c)\,ds. \tag{3.4}$$

b. Estimate of $E_{diss}(u,v;t^n)$.

PROPOSITION 3.1. *Under condition* (2.3g) *on the viscosity term N, and if the Courant-Friedrichs-Levy* (CFL) *condition* (2.4) *is satisfied, the local rate of entropy dissipation $LRED^n_j(c)$ is nonnegative. Hence*

$$-E_{diss}(u,v;t^n) \le 0.$$

Sketch of the proof. The above conditions ensure that the functions $p_{e,K}(s)$ and $p_K(s)$ are nondecreasing in s. The result follows from this fact and the definition of $E_{diss}(u_h,v;t^n)$. □

c. Estimate of $E^{\star}_{div}(u,v;t^N)$.

PROPOSITION 3.2. *We have*

$$\lim_{w\to\chi}\sup_{1\le n\le N}\left\{E^{\star}_{div}(u,v;t^n)/W(t^n)\right\} \le TEW_{visc} + TEW_{cons} + TEW_{h.o.t.},$$

where

$$TEW_{visc}(u,v;t^n) \le\; C_0\,t^N\,d\left\{2\,\frac{|\,\eta\,|_{TV(\mathbb{R})}}{\epsilon_x}\,(1+\frac{\Delta t}{\epsilon_t})\right\}\|\nu_v\|\,\Delta x,$$

$$TEW_{cons}(u,v;t^n) \le\; 2\,C_1\,t^N\;(1+\frac{\Delta t}{\epsilon_t})\,(1+\frac{(\Delta x)^{d/2}}{\epsilon_x^d})\,\|\,\eta\,\|^d_{L^\infty(\mathbb{R})}\big(|\,\delta\,|_{var,1/2} + |\,\alpha\,|_{var,1/2}\big),$$

$$\begin{aligned}
TEW_{h.o.t.}(u,v;t^n) \le &C_1\left\{\frac{1}{2}\,\|\,f'(v)\,\|\frac{(\Delta t)^2\,d\,|\,\eta\,|_{TV(\mathbb{R})}}{\epsilon_t\epsilon_x}\,t^N + 4\,(1+\frac{\Delta t}{\epsilon_t})\Delta t\right. \\
&\left. +\; 32\,\mathbb{D}_{\Delta x}\,\iota_{\Delta x}\,(\Delta x)^2\left\{\frac{d^2|\,\eta\,|^2_{TV(\mathbb{R})}}{\epsilon_x^2} + \frac{d\,|\,\eta'\,|_{TV(\mathbb{R})}}{\epsilon_x^2}\right\}(1+\frac{\Delta t}{\epsilon_t})\,t^N\right\}.
\end{aligned}$$

where $C_0 = |v_0|_{TV(\mathbb{R}^d)}$ and $C_1 = C_0\,\|f'(v)\|$.

To prove this result, we proceed in several steps.

First step: Relating the dual form $E^{\star}_{div}(u,v;t)$ to the truncation error. We have the following upper bound for $E^{\star}_{div}(u_h,v;t^N)$.

LEMMA 3.3. *We have*

$$\begin{aligned}
E^{\star}_{div}(u_h,v;t^N) \le & \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_{\Delta x}}\int_0^{t^N}\int_{\mathbb{R}^d} F(u_K^n, v(t,x))\cdot\nabla_x\overline{\phi}(t,x,t^{n+1},x_K)\,|\,K\,|\,\Delta t\;dx\,dt \\
& -\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_{\Delta x}}\int_0^{t^N}\int_{\mathbb{R}^d}\sum_{e\in\partial K}|\,e\,|\,b_{e,K_e}\big[\phi(t,x,t^{n+1},x_K) \\
&\qquad - \phi(t,x,t^{n+1},x_{K_e})\big]\,F(u_K^n,v(t,x))\cdot n_{e,K}\,\Delta t\,dx\;dt \\
& +\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_{\Delta x}}\int_0^{t^N}\int_{\mathbb{R}^d}\sum_{e\in\partial K}|\,e\,|\,\alpha_e\big[\phi(t,x,t^{n+1},x_{K_e}) \\
&\qquad - \phi(t,x,t^{n+1},x_K)\big]\,\mathcal{N}_e(u_K^n,v(t,x))\,\Delta t\,dx\;dt.
\end{aligned}$$

To prove this result, we use the fact the v is the entropy solution and make some algebraic manipulations; see the proof of the similar result [6, Lemma 7.9].

Next, a relation between the functions $\phi(x,x_{K_e})$ and $\phi(x,x_K)$, defined by (3.1), is needed. That relation is displayed in the following lemma where, for the sake of clarity, we drop the dependence on t and t'. This simple result is a keystone of the proof of Theorem 2.1.

LEMMA 3.4. *We have*

$$\phi(x,x_{K_e})-\phi(x,x_K) = -(x_{K_e}-x_K)\cdot\nabla_x\phi(x,x_K)+\overline{H}_{e,K}(x,x_K)+\overline{\overline{H}}_{e,K}(x),$$

where

$$\begin{aligned}
\overline{H}_{e,K}(x,x_K) &= \sum_{i,j=1}^{d}\xi_{e,K}^{ij}\,\partial^2_{x_i\,x_j}\,\phi(x,x_K),\\
\xi_{e,K}^{ij} &= \frac{1}{2}\big\{(x_{K_e}-x_K)_i\,(x_{K_e}-x_K)_j+\mathbb{I}_{K_e}^{ij}-\mathbb{I}_K^{ij}\big\},\\
\mathbb{I}_{\Omega}^{ij} &= \frac{1}{|\,\Omega\,|}\int_{\Omega}(x'-x_\Omega)_i\,(x'-x_\Omega)_j\,dx',
\end{aligned}$$

and

$$\begin{aligned}
\overline{\overline{H}}_{e,K}(x) &= \frac{1}{|\,K_e\,|}\int_{K_e}\frac{1}{|\,K\,|^3}\int_{K^3}\int_0^1\!\!\int_0^1\!\!\int_0^1\Psi(x)\;d\lambda\,d\mu\,d\nu\,dz'\,dy'\,dw'\,dx',\\
\Psi(x) &= \sum_{i,j,k=1}^{d}\zeta^{ijk}\,\partial^3_{x_i\,x_j\,x_k}\varphi(x,\lambda(\mu(\nu x'+(1-\nu)w')+(1-\mu)y')+(1-\lambda)z'),\\
\zeta^{ijk} &= (x'-w')_i\,(\nu x'+(1-\nu)w'-y')_j\,(\mu(\nu x'+(1-\nu)w')+(1-\mu)y'-z')_k.
\end{aligned}$$

Proof of Lemma 3.4. Since

$$\varphi(x,x') = \varphi(x,w') - \int_0^1 (x'-w')\cdot\nabla_x\varphi(x,\nu x'+(1-\nu)w')\,d\nu,$$

by the definition of φ (3.2), averaging the above equality over K with respect to w'

yields

$$\varphi(x,x') = \phi(x,x_K) - \frac{1}{|K|}\int_K\int_0^1 (x'-w')\cdot\nabla_x\varphi(x,\nu x'+(1-\nu)w')\,d\nu\,dw',$$

by the definition of ϕ, (3.1). Using the above relation recursively twice, we easily obtain

$$\varphi(x,x') = \phi(x,x_K) + \sum_{i=1}^{d} A^i(x')\,\partial_{x_i}\phi(x,x_K) + \sum_{i,j=1}^{d} B^{ij}(x')\,\partial^2_{x_i\,x_j}\phi(x,x_K) + Res(x,x'),$$

where

$$\begin{aligned}
A^i(x') &= -\frac{1}{|K|}\int_K (x'-w')_i\,dw',\\
B^{ij}(x') &= \frac{1}{|K|}\int_K\int_0^1 (x'-w')_i\left\{\frac{1}{|K|}\int_K(\nu x'+(1-\nu)w'-y')_j\,dy'\right\}d\nu\,dw',\\
Res(x,x') &= \frac{1}{|K|^3}\int_{K^3}\int_0^1\int_0^1\int_0^1 \Psi(x)\,d\lambda\,d\mu\,d\nu\,dz'\,dy'\,dw'.
\end{aligned}$$

Averaging the above equation K_e with respect to x' and using the definition of ϕ, (3.1), yields

$$\phi(x,x_{K_e}) = \phi(x,x_K) + \sum_{i=1}^{d} \mathbb{A}^i\,\partial_{x_i}\phi(x,x_K) + \sum_{i,j=1}^{d} \mathbb{B}^{ij}\,\partial^2_{x_i\,x_j}\phi(x,x_K) + \overline{\overline{\overline{H}}}_{e,K}(x),$$

where

$$\mathbb{A}^i = \frac{1}{|K_e|}\int_{K_e} A^i(x')\,dx', \qquad \mathbb{B}^{ij} = \frac{1}{|K_e|}\int_{K_e} B^{ij}(x')\,dx'.$$

The result follows if we show that $\mathbb{A}^i = -(x_{K_e}-x_K)_i$ and $\mathbb{B}^{ij} = \xi^{ij}_{e,K}$. But, since $A^i(x') = -(x'-x_K)_i$, we do have that $\mathbb{A}^i = -(x_{K_e}-x_K)_i$, and since

$$\begin{aligned}
B^{ij}(x') &= \frac{1}{|K|}\int_K\int_0^1 (x'-w')_i\,(\nu x'+(1-\nu)w'-x_K)_j\,d\nu\,dw'\\
&= \frac{1}{|K|}\int_K (x'-w')_i\,((x'+w')/2-x_K)_j\,dw'\\
&= \frac{1}{2\,|K|}\int_K ((x'-x_K)-(w'-x_K))_i\,((x'-x_K)+(w'-x_K))_j\,dw'\\
&= \frac{1}{2}\{\,(x'-x_K)_i\,(x'-x_K)_j - \mathbb{I}^{ij}_K\},
\end{aligned}$$

we have

$$\begin{aligned}
\mathbb{B}^{ij} &= \frac{1}{2}\Big\{\,\frac{1}{|K_e|}\int_{K_e}(x'-x_K)_i\,(x'-x_K)_j\,dx' - \mathbb{I}^{ij}_K\Big\}\\
&= \frac{1}{2}\Big\{(x_{K_e}-x_K)_i\,(x_{K_e}-x_K)_j + \frac{1}{|K_e|}\int_{K_e}(x'-x_{K_e})_i\,(x'-x_{K_e})_j\,dx' - \mathbb{I}^{ij}_K\Big\}\\
&= \frac{1}{2}\Big\{(x_{K_e}-x_K)_i\,(x_{K_e}-x_K)_j + \mathbb{I}^{ij}_{K_e} - \mathbb{I}^{ij}_K\Big\} = \xi^{ij}_{e,K}.
\end{aligned}$$

This completes the proof. ∎

With the above lemma, we can now rewrite the upper bound of $E^{\star}_{div}(u_h, v; t^N)$ as follows.

LEMMA 3.5. *We have*

$$E^{\star}_{div}(u_h, v; t^N) \leq TE_{visc}(u, v; t^N) + TE_{cons}(u, v; t^N) + TE_{h.o.t.}(u, v; t^N),$$

where

$$\begin{aligned} TE_{visc}(u, v; t^N) \; &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_{\Delta x}} \int_0^{t^N} \int_{\mathbb{R}^d} VISC^n_K(v(t,x); t, x)) \; dx \, dt \, \Delta t \\ &- \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_{\Delta x}} \int_{\mathbb{R}^d} F(u^n_K, v(t^N, x)) \cdot \frac{\Delta t}{2} \nabla_x \phi(t^N, x, t^{n+1}, x_K) \, dx \, | \, K \, | \, \Delta t \\ &+ \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_{\Delta x}} \int_{\mathbb{R}^d} F(u^n_K, v(t^0, x)) \cdot \frac{\Delta t}{2} \nabla_x \, \phi(0, x, t^{n+1}, x_K) \, dx \, | \, K \, | \, \Delta t, \\ TE_{cons}(u, v; t^N) \; &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_{\Delta x}} \int_0^{t^N} \int_{\mathbb{R}^d} CONS^n_K(v(t,x); t, x)) \; dx \, dt \, \Delta t, \\ TE_{h.o.t.}(u, v; t^N) &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_{\Delta x}} \int_0^{t^N} \int_{\mathbb{R}^d} HOT^n_K(v(t,x); t, x)) \; dx \, dt \, \Delta t \\ &+ \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_{\Delta x}} \int_{\mathbb{R}^d} F(u^n_K, v(t^N, x)) \cdot \frac{\Delta t}{2} \nabla_x \, \phi(t^N, x, t^{n+1}, x_K) \, dx \, | \, K \, | \, \Delta t \\ &- \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}_{\Delta x}} \int_{\mathbb{R}^d} F(u^n_K, v(t^0, x)) \cdot \frac{\Delta t}{2} \nabla_x \, \phi(0, x, t^{n+1}, x_K) \, dx \, | \, K \, | \, \Delta t. \end{aligned}$$

The 'viscosity' term $VISC^n_K(c; t, x)$ is given by

$$VISC^n_K(c; t, x) = \overset{time}{VISC}{}^n_K(c; t, x) + \overset{cent}{VISC}{}^n_K(c; t, x) + \overset{visc}{VISC}{}^n_K(c; t, x),$$

where

$$\begin{aligned} \overset{time}{VISC}{}^n_K(c; t, x) &= F(u^n_K, c) \cdot \frac{\Delta t}{2} | \, K \, | \, \nabla_x \phi_t(t, x, t^{n+1}, x_K), \\ \overset{cent}{VISC}{}^n_K(c; t, x) &= F(u^n_K, c) \cdot \sum_{e \in \partial K} | \, e \, | \, b_{e, K_e} \, \overline{H}_{e,K}(t, x, t^{n+1}, x_K) \, n_{e,K}, \\ \overset{visc}{VISC}{}^n_K(c; t, x) &= \sum_{e \in \partial K} | \, e \, | \, \alpha_e \, \overline{H}_{e,K}(t, x, t^{n+1}, x_K) \, \mathcal{N}_e(u^n_K, c). \end{aligned}$$

The 'consistency' term $CONS^n_K(c; t, x)$ is given by

$$CONS^n_K(c; t, x) = \overset{cent}{CONS}{}^n_K(c; t, x) + \overset{visc}{CONS}{}^n_K(c; t, x),$$

where

$$\overset{cent}{CONS}{}^n_K(c; t, x) = F(u^n_K, c) \cdot \sum_{e \in \partial K} | \, e \, | \, \nabla_x \phi(t, x, t^{n+1}, x_K) \cdot \delta_e \, n_{e,K},$$

$$\overset{visc}{CONS}{}_K^n(c;t,x) = -\sum_{e\in\partial K} |\,e\,|\,\alpha_e\,(x_{K_e} - x_K)\cdot\nabla_x\phi(t,x,t^{n+1},x_K)\,\mathcal{N}_e(u_K^n,c).$$

Finally, the 'high-order' term $HOT_K^n(c;t,x)$ *is given by*

$$HOT_K^n(c;t,x) = \overset{time}{HOT}{}_K^n(c;t,x) + \overset{cent}{HOT}{}_K^n(c;t,x) + \overset{visc}{HOT}{}_K^n(c;t,x),$$

where

$$\begin{aligned}\overset{time}{HOT}{}_K^n(c;t,x) =& F(u_K^n,c)\cdot\big[|\,K\,|\nabla_x\overline{\phi}(t,x,t^{n+1},x_K) - |\,K\,|\,\nabla_x\phi(t,x,t^{n+1},x_K)\\ &- \frac{\Delta t}{2}\,|\,K\,|\,\nabla_x\phi_t(t,x,t^{n+1},x_K)\big],\\ \overset{cent}{HOT}{}_K^n(c;t,x) =& F(u_K^n,c)\cdot\sum_{e\in\partial K}|\,e\,|\,b_{e,K_e}\,\overline{\overline{H}}_{e,K}(t,x,t^{n+1})\,n_{e,K},\\ \overset{visc}{HOT}{}_K^n(c;t,x) =& \sum_{e\in\partial K}|\,e\,|\,\alpha_e\,\overline{\overline{H}}_{e,K}(t,x,t^{n+1})\,\mathcal{N}_e(u_K^n,c).\end{aligned}$$

Proof of Lemma 3.5. The result can be easily obtained by inserting the expression for $\phi(x,x_{K_e}) - \phi(x,x_K)$ of Lemma 3.4 into the upper bound for the dual form $E^\star(u,v;t^N)$ in Lemma 3.3 and rearranging terms.

The only manipulation that deserves mentioning is the following. After the above mentioned rearrangement, we get for $\overset{cent}{CONS}{}_K^n(c;t,x)$ the expression

$$F(u_K^n,c)\cdot\big[|\,K\,|\,\nabla_x\phi(t,x,t^{n+1},x_K) - \sum_{e\in\partial K}|\,e\,|\,b_{e,K_e}\,(x_{K_e}-x_K)\cdot\nabla_x\phi(t,x,t^{n+1},x_K)\,n_{e,K}\big].$$

This expression can be rewritten as

$$F(u_K^n,c)\cdot\sum_{e\in\partial K}|\,e\,|\,\nabla_x\phi(t,x,t^{n+1},x_K)\cdot[(x_e - x_K) - b_{e,K_e}(x_{K_e} - x_K)]\,n_{e,K},$$

or, equivalently, see the definition of δ_e, (2.7c), as

$$F(u_K^n,c)\cdot\sum_{e\in\partial K}|\,e\,|\,\nabla_x\phi(t,x,t^{n+1},x_K)\cdot\delta_e\,n_{e,K},$$

by using the identity

$$|K|\,m = \sum_{e\in\partial K}|\,e\,|\,\big[m\cdot(x_e - x_K)\big]\,n_{e,K},$$

with $m = \nabla_x\phi(t,x,t^{n+1},x_K)$. The above identity can be proved as follows:

$$\begin{aligned}|K|\,m &= \int_K m\,dx' = \int_K \nabla_x\big[m\cdot(x'-x_K)\big]\,dx'\\ &= \sum_{e\in\partial K}\int_e \big[m\cdot(x'-x_K)\big]\,n_{e,K}\,d\Gamma(x')\\ &= \sum_{e\in\partial K}|\,e\,|\,\big[m\cdot(x_e - x_K)\big]\,n_{e,K}.\end{aligned}$$

This completes the proof. □

Second step: Estimating $TE_{visc}(u,v;t^N)$. In this section, we prove the following result.

LEMMA 3.6. *We have*

$$TE_{visc}(u,v;t^N) \;\leq\; 2\,C_0\,\frac{d\,|\,\eta\,|_{TV(\mathbb{R})}}{\epsilon_x}\,(1+\frac{\Delta t}{\epsilon_t})\,\|\nu_v\|\,\Delta x,$$

where $C_0 = t^N|v_0|_{TV(\mathbb{R}^d)}\,W(t^N)$.

Proof of Lemma 3.6. We have, by the definition of $TE_{visc}(u,v;t^N)$ in Lemma 3.5,

$$TE_{visc}(u,v;t^N) = \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_{\Delta x}}\Xi(t^{n+1},x_K)\,|\,K\,|\,\Delta t$$

where

$$\begin{aligned}\Xi(t^{n+1},x_K) = \int_0^{t^N}\!\!\int_{\mathbb{R}^d}\Big[&F(u_K^n,v(t,x))\cdot\frac{\Delta t}{2}\,\nabla_x\phi_t(t,x,t^{n+1},x_K)\\ &+F(u_K^n,v(t,x))\cdot\frac{1}{|\,K\,|}\sum_{e\in\partial K}|\,e\,|\,b_{e,K_e}\,\overline{H}_{e,K}(t,x,t^{n+1},x_K)\,n_{e,K}\\ &+\frac{1}{|\,K\,|}\sum_{e\in\partial K}|\,e\,|\,\alpha_e\,\overline{H}_{e,K}(t,x,t^{n+1},x_K)\,\mathcal{N}_e(u_K^n,v(t,x))\Big]\,dx\,dt\\ &-\int_{\mathbb{R}^d}F(u_K^n,v(t^N,x))\cdot\frac{\Delta t}{2}\nabla_x\phi(t^N,x,t^{n+1},x_K)\,dx\\ &+\int_{\mathbb{R}^d}F(u_K^n,v(t^0,x))\cdot\frac{\Delta t}{2}\nabla_x\phi(0,x,t^{n+1},x_K)\,dx.\end{aligned}$$

A couple of integrations by parts yield

$$\begin{aligned}\Xi(t^{n+1},x_K) = -\int_0^{t^N}\!\!\int_{\mathbb{R}^d}\Big\{&F_t(u_K^n,v(t,x))\cdot\frac{\Delta t}{2}\,\nabla_x\phi(t,x,t^{n+1},x_K)\\ +\frac{1}{|\,K\,|}\sum_{e\in\partial K}&|\,e\,|\,b_{e,K_e}\sum_{i,j=1}^{d}\partial_{x_i}\big[F(u_K^n,v(t,x))\cdot n_{e,K}\big]\,\xi_{e,K}^{ij}\,\partial_{x_j}\phi(t,x,t^{n+1},x_K)\\ +\frac{1}{|\,K\,|}\sum_{e\in\partial K}&|\,e\,|\,\alpha_e\sum_{i,j=1}^{d}\partial_{x_i}\mathcal{N}_e(u_K^n,v(t,x))\,\xi_{e,K}^{ij}\,\partial_{x_j}\phi(t,x,t^{n+1},x_K)\Big\}\,dx\,dt\,,\end{aligned}$$

Then, by using the definition of F and $\mathcal{N}$, (3.4), we get

$$\Xi(t^{n+1},x_K) = -\sum_{i,j=1}^{d}\int_0^{t^N}\!\!\int_{\mathbb{R}^d}U'(u_K^n-v)\,\partial_{x_i}v\;\nu_K^{ij}(v(t,x))\,\partial_{x_j}\phi(t,x,t^{n+1},x_K)\,dx\,dt\,,$$

where the entries of the matrix ν are

$$\nu_K^{ij}(c) = \frac{\Delta t}{2}\,f_i'(c)\,f_j'(c) - \frac{1}{|\,K\,|}\sum_{e\in\partial K}|\,e\,|\,\big[b_{e,K_e}\,f'(c)\cdot n_{e,K}+\alpha_e\,N_e'(c)\big]\,\xi_{e,K}^{ij}$$

We have now

$$TE_{visc}(u,v;t^N) \leq T_{aux}\,|v_0|_{TV(\mathbb{R}^d)}\,t^N$$

where

$$T_{aux} = \sup_{\substack{t\in(0,t^n)\\ x\in\mathbb{R}^d}} \Big\{ \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_{\Delta x}} \max_{1\le i\le d} \sum_{j=1}^{d} \nu_K^{ij}(v(t,x))\, \partial_{x_j} \phi(t,x,t^{n+1},x_K)\,|\,K\,|\,\Delta t \Big\}$$
$$\le \frac{d\,|\eta|_{TV(\mathbb{R})}}{\epsilon_x}\, 2\,(1+\frac{\Delta t}{\epsilon_t})\, W(t^N)\,\|\nu_v\|\,\Delta x,$$

where we have made use of Lemma 3.7 of [7], the definition of ϕ, (3.1), and the definition of $\|\nu_v\|$. This completes the proof. ∎

Third step: Estimating $TE_{cons}(u,v;t^N)$.
LEMMA 3.7. *We have*

$$TE_{cons}(u,v;t^N) \le 2\,C_1\,\big(\,1+\frac{\Delta t}{\epsilon_t}\,\big)\,\big(\,1+\frac{(\Delta x)^{d/2}}{\epsilon_x^d}\,\big)\,\|\,\eta\,\|^d_{L^\infty(\mathbb{R})}\,\big(|\,\delta\,|_{var,1/2}+|\,\alpha\,|_{var,1/2}\big),$$

where $C_1 = t^N |v_0|_{TV(\mathbb{R}^d)}\,\|f'(v)\|\,W(t^N)$.

Proof of Lemma 3.7. By definition of the term $TE_{cons}(u,v;t^N)$ in Lemma 3.5, we can write

$$TE_{cons}(u,v;t^N) = \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_{\Delta x}} \int_0^{t^N}\!\!\int_{\mathbb{R}^d} \nabla_x \phi(t,x,t^{n+1},x_K)\cdot A_K(u_K^n, v(t,x))\,dx\,dt\,\Delta t,$$

where

$$A_K(u,c) = \sum_{e\in\partial K} |\,e\,|\big(F(u,c)\cdot n_{e,K}\,\delta_e - \alpha_e\,\mathcal{N}_e(u,c)\,(x_{K_e}-x_K)\,\big).$$

After a simple integration by parts, we obtain

$$TE_{cons}(u,v;t^N) = -\sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_{\Delta x}} \int_0^{t^N}\!\!\int_{\mathbb{R}^d} \phi(t,x,t^{n+1},x_K)\,\nabla_x v(t,x)\cdot\partial_v A_K(u_K^n, v(t,x))\,dx\,dt\,\Delta t$$
$$\le \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_{\Delta x}} \int_0^{t^N}\!\!\int_{\mathbb{R}^d} \phi(t,x,t^{n+1},x_K)\,\sum_{i=1}^{d} |\,\partial_{x_i} v(t,x)\,|\,B_K(v(t,x))\,dx\,dt\,\Delta t,$$

where,

$$B_K(c) = \max_{1\le i\le d} \Big|\sum_{e\in\partial K} |\,e\,|\,\big(f'(c)\cdot n_{e,K}\,\delta_e - \alpha_e\,N_e'(c)\,(x_{K_e}-x_K)\,\big)\,\Big|.$$

Hence,

$$TE_{cons}(u,v;t^N) \le T_{aux}\,t^N\,|v_0|_{TV(\mathbb{R}^d)},$$

where

$$
\begin{aligned}
T_{aux} &= \sup_{\substack{t\in(0,t^n)\\ x\in\mathbb{R}^d}} \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_{\Delta x}} \phi(t,x,t^{n+1},x_K)\, B_K(v(t,x))\,\Delta t \\
&= \sup_{\substack{t\in(0,t^n)\\ x\in\mathbb{R}^d}} \sum_{n=0}^{N-1} \sum_{K:|\,x-x_K\,|\le\epsilon_x} \phi(t,x,t^{n+1},x_K)\, B_K(v(t,x))\,\Delta t \\
&\le \sup_{\substack{t\in(0,t^n)\\ x\in\mathbb{R}^d}} \Big\{ \sum_{n=0}^{N-1} \sup_{x\in\mathbb{R}^d} \phi(t,x,t^{n+1},x_K)\,\Delta t \Big\}\, \Lambda(\epsilon_x),
\end{aligned}
$$

with

$$
\Lambda(\epsilon_x) = \sup_{\substack{t\in(0,t^n)\\ x\in\mathbb{R}^d}} \Big\{ \sum_{K:|\,x-x_K\,|\le\epsilon_x} B_K(v(t,x)) \Big\}.
$$

Now, by Lemma 3.7 of [7], we have

$$
\sup_{\substack{t\in(0,t^n)\\ x\in\mathbb{R}^d}} \Big\{ \sum_{n=0}^{N-1} \sup_{x\in\mathbb{R}^d} \phi(t,x,t^{n+1},x_K)\,\Delta t \Big\} \le \frac{\|\,\eta\,\|^d_{L^\infty(\mathbb{R})}}{\epsilon_x^d}\, 2\,\big(\,1+\frac{\Delta t}{\epsilon_t}\,\big)\, W(t^N);
$$

observing also

$$
\begin{aligned}
\Lambda(\epsilon_x) &\le \epsilon_x^d\,\big(\,1+\frac{(\Delta x)^{d/2}}{\epsilon_x^d}\,\big)\,\Lambda((\Delta x)^{1/2}) \\
&\le \epsilon_x^d\,\big(\,1+\frac{(\Delta x)^{d/2}}{\epsilon_x^d}\,\big)\,\|\,f'(v)\,\|\,\big(|\,\delta\,|_{var,1/2}+|\,\alpha\,|_{var,1/2}\big),
\end{aligned}
$$

and using the definition of $|\,\delta\,|_{var,1/2}$ and $|\,\alpha\,|_{var,1/2}$, (2.9) and (2.10), we get

$$
T_{aux} \le 2\,\big(\,1+\frac{\Delta t}{\epsilon_t}\,\big)\,W(t^N)\big(\,1+\frac{(\Delta x)^{d/2}}{\epsilon_x^d}\,\big)\,\|\,\eta\,\|^d_{L^\infty(\mathbb{R})}\,\big(|\,\delta\,|_{var,1/2}+|\,\alpha\,|_{var,1/2}\big),
$$

and the results follows. ∎

Fourth step: Estimating $TE_{h.o.t.}(u,v;t^N)$.

LEMMA 3.8. *Suppose that the conditions* (2.3) *are satisfied. Then,*

$$
\begin{aligned}
TE_{h.o.t.}(u,v;t^N) \le C_0\Big\{ &\frac{1}{2}\,\|\,f'(v)\,\|\,\frac{(\Delta t)^2\,d\,|\,\eta\,|_{TV(\mathbb{R})}}{\epsilon_t\epsilon_x}\,t^N + 4\,(1+\frac{\Delta t}{\epsilon_t})\Delta t \\
&+ 32\,\mathbb{D}_{\Delta x}\,\iota_{\Delta x}\,(\Delta x)^2\,\Big\{ \frac{d^2|\,\eta\,|^2_{TV(\mathbb{R})}}{\epsilon_x^2} + \frac{d\,|\,\eta'\,|_{TV(\mathbb{R})}}{\epsilon_x^2} \Big\}(1+\frac{\Delta t}{\epsilon_t})\,t^N \Big\},
\end{aligned}
$$

where $C_0 = |v_0|_{TV(\mathbb{R}^d)}\,\|f'(v)\|\,W(t^N)$.

To prove the above result, we rewrite $TE_{h.o.t.}(u,v;t^N)$ as the sum $\overset{time}{TE}_{h.o.t.}(u,v;t^N)+\overset{cent}{TE}_{h.o.t.}(u,v;t^N)+\overset{visc}{TE}_{h.o.t.}(u,v;t^N)$, with the obvious notation, see Lemma 3.5, and estimate each of the above three terms.

LEMMA 3.9. *We have*

$$\overset{time}{TE}_{h.o.t.}(u,v;t^N) \leq C_0 \left\{ \frac{1}{2} \| f'(v) \| \frac{(\Delta t)^2\, d\, |\, \eta\, |_{TV(\mathbb{R})}}{\epsilon_t \epsilon_x} \, t^N + 4\,(1+\frac{\Delta t}{\epsilon_t})\Delta t \right\},$$

where $C_0 = |v_0|_{TV(\mathbb{R}^d)} \, \| f'(v) \| \, W(t^N)$.

Proof. Integrating by parts in time and inserting the definition of ϕ (3.1) leads to

$$\overset{time}{TE}_{h.o.t.}(u,v;t^N) = \sum_{n=0}^{N-1} \int_{\mathbb{R}^d} \Theta(t^{n+1},x')\, dx'\, \Delta t,$$

where

$$\Theta(t^{n+1},x') = \int_0^{t^N}\!\!\int_{\mathbb{R}^d} F(u(t^n,x'),v(t,x)) \cdot \nabla_x \big(\bar{\varphi}(t,x,t^{n+1},x') - \varphi(t,x,t^{n+1},x')\big)\, dx\, dt$$
$$+ \frac{\Delta t}{2} \int_0^{t^N}\!\!\int_{\mathbb{R}^d} F_t(u(t^n,x'),v(t,x)) \cdot \nabla_x \varphi(t,x,t^{n+1},x')\, dx\, dt.$$

Noticing that $\bar{\varphi}(t,t^{n+1}) - \varphi(t,t^{n+1}) = \dfrac{1}{\Delta t} \int_{t^n}^{t^{n+1}} (s-t^n)\, \varphi_t(t,s)\, ds$, one gets after integrating by parts and some elementary manipulations,

$$\Theta(t^{n+1},x') = \int_0^{t^N}\!\!\int_{\mathbb{R}^d} F_t(u(t^n,x'),v(t,x)) \cdot \nabla_x \int_{t^n}^{t^{n+1}} \frac{(s-t^n)^2}{2}\, \varphi_{t'}(t,x,s,x')\, ds\, dx\, dt$$
$$- \int_{\mathbb{R}^d} \nabla_x \cdot F(u(t^n,x'),v(t^N,x)) \int_{t^n}^{t^{n+1}} (s-t^n)\, \varphi(t^N,x,s,x')\, ds\, dx$$
$$+ \int_{\mathbb{R}^d} \nabla_x \cdot F(u(t^n,x'),v_0(x)) \int_{t^n}^{t^{n+1}} (s-t^n)\, \varphi(0,x,s,x')\, ds\, dx.$$

This allows us to write $\overset{time}{TE}_{h.o.t.}(u,v;t^N) = I - II + III$, with the obvious notation. Now we only need to estimate terms I, II and III. First we have

$$|\, I\, | \leq T_{aux} \, \| \, f'(v) \, \|^2 \, |\, v_0\, |_{TV(\mathbb{R}^d)} t^N ,$$

where

$$T_{aux} = \frac{(\Delta t)^2}{2} \sup_{\substack{t\in(0,t^n)\\ x\in\mathbb{R}^d}} \int_0^{t^N}\!\!\int_{\mathbb{R}^d} |\nabla_x \varphi_{t'}(t,x,t',x')|\, dx'\, dt'$$
$$\leq \frac{(\Delta t)^2}{2} \frac{d\, |\, \eta\, |_{TV(\mathbb{R})}}{\epsilon_t \epsilon_x} W(t^N)$$

where we follow the lines of [6, Lemma 7.12] and [6, Lemma 7.13].

Next we estimate term II:

$$|\, II\, | \leq T_{aux} \, \| \, f'(v) \, \| \, |\, v_0\, |_{TV(\mathbb{R}^d)} \, \Delta t \, ,$$

where

$$T_{aux} = \sup_{x\in\mathbb{R}^d} \int_0^{t^N}\!\!\int_{\mathbb{R}^d} |\phi(t^N,x,t',x')|\, dx'\, dt' \, ,$$
$$\leq 2\,(1+\frac{\Delta t}{\epsilon_t})\, W(t^N),$$

by Lemma 3.7 of [7]. The term III can be estimated in exactly the same way. Combining the estimates of terms I, II, III completes the proof. □

Before we estimate $\overset{cent}{TE}_{h.o.t.}(u,v;t^N)$ and $\overset{visc}{TE}_{h.o.t.}(u,v;t^N)$, we need to prove a couple of simple auxiliary results.

LEMMA 3.10. *We have*

$$\sum_{K\in\mathcal{T}_{\Delta x}}\sum_{e\in\partial K} |\,e\,|\,d_e\,\mathcal{I}_{e,K}(\Phi) \leq 2\,\mathbb{D}_{\Delta x}\,\iota_{\Delta x}\|\,\Phi\,\|_{L^1(\mathbb{R}^d)},$$

where

$$\mathcal{I}_{e,K}(\Phi) = \frac{1}{|\,K_e\,|}\int_{K_e}\frac{1}{|\,K\,|^3}\int_{K^3}\int_0^1\int_0^1\int_0^1 \Phi\; d\lambda\, d\mu\, d\nu\, dz'\, dy'\, dw'\, dx',$$
$$\Phi \equiv \Phi(\lambda(\mu(\nu x' + (1-\nu)w') + (1-\mu)y') + (1-\lambda)z').$$

Proof. Since the point $\lambda(\mu(\nu x' + (1-\nu)w') + (1-\mu)y') + (1-\lambda)z'$ belongs to the convex hull of K and K_e, C_e, we have

$$\big|\,\mathcal{I}_{e,K}(\Phi)\,\big| \leq \frac{1}{\max\{|\,K\,|, |\,K_e\,|\}}\,\|\,\Phi\,\|_{L^1(C_e)}.$$

Hence

$$\begin{aligned}
\sum_{K\in\mathcal{T}_{\Delta x}}\sum_{e\in\partial K} |\,e\,|\,d_e\,\mathcal{I}_{e,K}(\Phi) &\leq \sum_{K\in\mathcal{T}_{\Delta x}}\sum_{e\in\partial K}\frac{|\,e\,|\,d_e}{\max\{|\,K\,|, |\,K_e\,|\}}\|\,\Phi\,\|_{L^1(C_e)}\\
&\leq \mathbb{D}_{\Delta x}\sum_{K\in\mathcal{T}_{\Delta x}}\sum_{e\in\partial K}\|\,\Phi\,\|_{L^1(C_e)}, \quad \text{by (2.5)},\\
&\leq 2\,\mathbb{D}_{\Delta x}\sum_{e\in\mathcal{E}_{\Delta x}}\|\,\Phi\,\|_{L^1(C_e)}\\
&\leq 2\,\mathbb{D}_{\Delta x}\sum_{K\in\mathcal{T}_{\Delta x}}\iota_{\Delta x}\|\,\Phi\,\|_{L^1(K)}\\
&\leq 2\,\mathbb{D}_{\Delta x}\,\iota_{\Delta x}\|\,\Phi\,\|_{L^1(\mathbb{R}^d)}, \quad \text{by (2.6)}.
\end{aligned}$$

This completes the proof. □

LEMMA 3.11. *We have*

$$\Theta_{aux} \leq 4\,t^N\,(\Delta x)^2\,\mathbb{D}_{\Delta x}\,\iota_{\Delta x}\,(1+\frac{\Delta t}{\epsilon_t})\,W(t^N)\left\{\frac{d^2|\,\eta\,|^2_{TV(\mathbb{R})}}{\epsilon_x^2} + \frac{d|\,\eta'\,|_{TV(\mathbb{R})}}{\epsilon_x^2}\right\},$$

where

$$\Theta_{aux} = \sum_{1\leq j,k\leq d}\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_{\Delta x}}\sum_{e\in\partial K}|\,e\,|b_{e,K_e}\,d_e\mathcal{I}_{e,K}(\big|\,\partial_{x_j\,x_k}\varphi(t,x)\,\big|)\,\Delta t.$$

Proof. By the definition of the operator $\mathcal{I}_{e,K}$ in Lemma 3.10 and the definition of φ, (3.2), we can write $\Theta_{aux} = \Theta_{aux,t}\cdot\Theta_{aux,x}$, where

$$\Theta_{aux,t} = \sum_{n=0}^{N-1}\omega_{\epsilon_t}(t-t^{n+1})\,\Delta t \leq 2\,(1+\frac{\Delta t}{\epsilon_t})\,W(t^N),$$

by [7, Lemma 3.7] and

$$\Theta_{aux,x} = \sum_{1\leq j,k\leq d}\sum_{K\in\mathcal{T}_{\Delta x}}\sum_{e\in\partial K}|\,e\,|b_{e,K_e}\,d_e\mathcal{I}_{e,K}(\big|\,\partial_{x_j\,x_k}\mathbb{R}^{jk}(x,\cdot)\big|),$$

where $\mathbb{R}^{jk}(x,x') = \prod_{i=1}^{d} \eta_{\epsilon_x}(x_i - x'_i)$. Since, by Lemma 3.10,

$$\Theta_{aux,x} \leq 2\,\mathbb{D}_{\Delta x}\,\iota_{\Delta x} \sum_{1\leq j,k\leq d} \|\,\mathbb{R}^{jk}(x,\cdot)\,\|_{L^1(\mathbb{R}^d)},$$

and since

$$\begin{aligned}\sum_{1\leq j,k\leq d} \left| \partial_{x_j x_k} \prod_{i=1}^{d} \eta_{\epsilon_x}(x_i - x'_i) \right| = \sum_{1\leq j,k\leq d} \Bigg| &\prod_{\substack{i=1\\ i\neq j\neq k}}^{d} \eta_{\epsilon_x}(x_i - x'_i)\,\eta'_{\epsilon_x}(x_k - x'_k)\,\eta'_{\epsilon_x}(x_j - x'_j) \\ &+ \prod_{\substack{i=1\\ i\neq j}}^{d} \eta_{\epsilon_x}(x_i - x'_i)\,\eta''_{\epsilon_x}(x_j - x'_j)\delta_{jk} \Bigg|,\end{aligned}$$

we have that

$$\Theta_{aux,x} \leq 2\,\mathbb{D}_{\Delta x}\,\iota_{\Delta x} \left\{ \frac{d^2|\,\eta\,|^2_{TV(\mathbb{R})}}{\epsilon_x^2} + \frac{d|\,\eta'\,|_{TV(\mathbb{R})}}{\epsilon_x^2} \right\}.$$

This completes the proof. ∎

We are now ready to estimate the remaining term in $TE_{h.o.t.}$

LEMMA 3.12. *Suppose that the conditions* (2.3) *are satisfied. Then,*

$$\overset{cent}{TE}_{h.o.t.}(u,v;t^N) \leq C_2\,(\Delta x)^2 \left\{ \frac{d^2|\,\eta\,|^2_{TV(\mathbb{R})}}{\epsilon_x^2} + \frac{d|\,\eta'\,|_{TV(\mathbb{R})}}{\epsilon_x^2} \right\}(1+\frac{\Delta t}{\epsilon_t}),$$

$$\overset{visc}{TE}_{h.o.t.}(u,v;t^N) \leq C_2\,(\Delta x)^2 \left\{ \frac{d^2|\,\eta\,|^2_{TV(\mathbb{R})}}{\epsilon_x^2} + \frac{d|\,\eta'\,|_{TV(\mathbb{R})}}{\epsilon_x^2} \right\}(1+\frac{\Delta t}{\epsilon_t})\,\|\,\alpha\,\|_{\ell^\infty(\mathcal{E}_{\Delta x})},$$

where $C_2 = 16\,t^N|v_0|_{TV(\mathbb{R}^d)}\,\|f'(v)\|\,\mathbb{D}_{\Delta x}\,\iota_{\Delta x}\,W(t^N)$.

Proof. We only have to prove the first estimate since the second is similar. We have

$$\overset{cent}{TE}_{h.o.t.}(u,v;t^N) = \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_{\Delta x}} \sum_{e\in\partial K} |\,e\,|\,b_{e,K_e}\,\Theta^n_{e,K}\,\Delta t,$$

where

$$\Theta^n_{e,K} = \sum_{1\leq i,j,k\leq d} \int_0^{t^N}\!\!\int_{\mathbb{R}^d} F(u^n_K, v(t,x))\cdot n_{e,K}\;\partial_{x_i}\mathcal{I}_{e,K}(\Xi^{ijk}(t,x))\,dx\,dt.$$

Moreover, $\Xi^{ijk}(t,x)) = \zeta^{ijk}\,\partial_{x_j\,x_k}\phi$, where ζ^{ijk} and ϕ are as in Lemma 3.4.

Performing integrations by parts with respect to x_i, we get

$$\Theta^n_{e,K} = -\sum_{1\leq i,j,k\leq d} \int_0^{t^N}\!\!\int_{\mathbb{R}^d} \big(\partial_v F(u^n_K, v(t,x))\cdot n_{e,K}\,\big)\;\partial_{x_i}v(t,x)\;\mathcal{I}_{e,K}(\Xi^{ijk}(t,x))\,dx\,dt,$$

and hence,

$$\begin{aligned}\overset{cent}{TE}_{h.o.t.}(u,v;t^N) &\leq T_{aux}\,\|f'(v)\| \int_0^{t^N}\!\!\int_{\mathbb{R}^d} \sum_{1\leq i\leq d} |\,\partial_{x_i}v(t,x)\,|\,dx\,dt \\ &\leq T_{aux}\,\|\,f'(v)\,\|\,t^N\,|\,v_0\,|_{TV(\mathbb{R}^d)},\end{aligned}$$

where

$$T_{aux} = \max_{1\le i\le d} \sup_{\substack{t\in(0,t^N)\\ x\in\mathbb{R}^d}} \sum_{1\le j,k\le d} \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_{\Delta x}} \sum_{e\in\partial K} |\,e\,| b_{e,K_e} \mathcal{I}_{e,K}(\Xi^{ijk}(t,x))\,\Delta t.$$

Since

$$|\,\zeta^{ijk}\,| \le 4\,(\Delta x)^2\, d_e,$$

we have that

$$\big|\,\mathcal{I}_{e,K}(\Xi^{ijk}(t,x))\,\big| \le\; 4\,(\Delta x)^2\, d_e\, \mathcal{I}_{e,K}(\big|\,\partial_{x_j\,x_k}\varphi(t,x)\,\big|),$$

and so,

$$\begin{aligned} T_{aux} &\le\; 4\,(\Delta x)^2\,\Theta_{aux} \\ &\le\; 16\,t^N\,(\Delta x)^2\,\mathbb{D}_{\Delta x}\,\iota_{\Delta x}\,(1+\frac{\Delta t}{\epsilon_t})\,W(t^N)\left\{\frac{d^2|\,\eta\,|^2_{TV(\mathbb{R})}}{\epsilon_x^2} + \frac{d|\,\eta'\,|_{TV(\mathbb{R})}}{\epsilon_x^2}\right\}, \end{aligned}$$

by Lemma 3.11. This completes the proof. □

d. Proof of the error estimate. To obtain the error estimate, we proceed exactly as in [6]. Inserting the estimates obtained in §3.b and §3.c into the approximation inequality in §3.a, and taking the auxiliary function η as in (3.3), we obtain

$$\begin{aligned} e(t^N) \le\;& 2\,e(0) + 8\,\big(\epsilon_x + \epsilon_t\|\,f'(v)\,\|\big)\,|\,v_0\,|_{TV(\mathbb{R}^d)} + 2\,\|\,f'(v)\,\|\,|\,v_0\,|_{TV(\mathbb{R}^d)}\,\Delta t \\ &+ 2\,C_0\,t^N\left\{2\,\frac{d\,|\,\eta\,|_{TV(\mathbb{R})}}{\epsilon_x}\,\big(1+\frac{\Delta t}{\epsilon_t}\big)\right\}\|\nu_v\|\,\Delta x \\ &+ 4\,C_1\,t^N\,\big(1+\frac{\Delta t}{\epsilon_t}\big)\,\big(1+\frac{(\Delta x)^{d/2}}{\epsilon_x^d}\big)\,\|\,\eta\,\|^d_{L^\infty(\mathbb{R})}\,\big(|\,\delta\,|_{var,1/2} + |\,\alpha\,|_{var,1/2}\big) \\ &+ 2\,C_1\left\{\frac{1}{2}\,\|\,f'(v)\,\|\,\frac{(\Delta t)^2\,d\,|\,\eta\,|_{TV(\mathbb{R})}}{\epsilon_t\epsilon_x}\,t^N + 4\,(1+\frac{\Delta t}{\epsilon_t})\Delta t\right. \\ &\left.+\;32\,\mathbb{D}_{\Delta x}\,\iota_{\Delta x}\,(\Delta x)^2\left\{\frac{d^2|\,\eta\,|^2_{TV(\mathbb{R})}}{\epsilon_x^2} + \frac{d\,|\,\eta'\,|_{TV(\mathbb{R})}}{\epsilon_x^2}\right\}(1+\frac{\Delta t}{\epsilon_t})\,t^N\right\}, \end{aligned}$$

where $C_0 = |v_0|_{TV(\mathbb{R}^d)}$ and $C_1 = C_0\,\|\,f'(v)\,\|$. The estimate of Theorem 2.1 is then obtained by eliminating the parameter Δt through the use of the CFL condition (2.4) and then taking the very same optimal values as those in [6], namely,

$$\epsilon_x^\star = \sqrt{t^N\,\|\,\nu_v\,\|\,\Delta x/2}, \quad \epsilon_t = A_t\,(\Delta x)^{3/4}, \quad \epsilon = A\,(\Delta x)^{1/4}.$$

This concludes the proof of Theorem 2.1.

4. Proof of Theorem 2.3. In this section, we prove Theorem 2.3. The only difference between the proof of Theorem 2.3 and that of Theorem 2.1 is the following estimate of the consistency term of the truncation error, $TE_{cons}(u,v;t^N)$:

LEMMA 4.1. *We have*

$$\begin{aligned} TE_{cons}(u,v;t^N) \le\; C_1\,\big\{\,& \mathbb{D}_{\Delta x}\,\iota_{\Delta x}\,\|\,f'(v)\,\|\,t^N\,|v_0|_{TV(\mathbb{R}^d)} \\ &+ \|\,f'(u)\,\| \sum_{n=0}^{N-1} \sum_{e\in\mathcal{E}_{\Delta x}} |\,e\,|\,|\,u^n_{K_e^+} - u^n_{K_e^-}\,|\,\Delta t\,\big\}, \end{aligned}$$

where $C_1 = 2\frac{d\,|\,\eta\,|_{TV}}{\epsilon_x}\left(1+\frac{\Delta t}{\epsilon_t}\right)W(t^N)\left\{\,\|\,\delta\,\|_{\ell^\infty(\mathcal{E}_{\Delta x})/\mathbb{R}^d}+\|\,\alpha\,\|_{\ell^\infty(\mathcal{E}_{\Delta x})}\,\right\}$ *and* $\mathbb{D}_{\Delta x}$ *and* $\iota_{\Delta x}$ *are defined in* (2.5) *and* (2.6), *respectively.*

Theorem 2.3 follows easily from this result, §3, as well as the following observation

$$\sum_{n=0}^{N-1}\sum_{e\in\mathcal{E}_{\Delta x}}|\,e\,|\,|\,u^n_{K_e^+}-u^n_{K_e^-}\,|\,\Delta t\le\frac{1}{\iota_{inf}}\,\|\,u\,\|_{L^1(0,t^N;TV(\mathbb{R}^d))},$$

where $\iota_{inf}=\inf_{K\in\mathcal{T}_{\Delta x}}\iota_K$. We note again that beyond merely proving the above estimate, we aim at understanding the mechanism that allows it to be possible; see the fourth step of the proof below. This provides an explanation of the *supraconvergence* phenomenon for the schemes under consideration.

To prove Lemma 4.1, we will need the following simple auxiliary result.

LEMMA 4.2. *We have*

$$\sum_{e\in\mathcal{E}_{\Delta x}}|\,e\,|\,|\,\psi_{K_e^+}-\psi_{K_e^-}\,|\le\mathbb{D}_{\Delta x}\,\iota_{\Delta x}\,|\,\psi\,|_{TV(\mathbb{R}^d)},$$

where $\psi_K=\dfrac{1}{|\,K\,|}\int_K\psi(x')\,dx'$.

Proof. Let us call Θ the left hand side of the inequality we want to prove. Then by the proof of Lemma 3.4,

$$\Theta=\sum_{e\in\mathcal{E}_{\Delta x}}|\,e\,|\left|\frac{1}{|K_e^+|\,|K_e^-|}\int_{K_e^+}\int_{K_e^-}\int_0^1(x'-w')\cdot\nabla_{x'}\psi(\nu x'+(1-\nu)w')\,d\nu\,dw'\,dx'\right|.$$

Assume that $|K_e^-|=\max\{\,|\,K_e^+\,|,|\,K_e^-\,|\,\}$, then, since $|\,x'-w'\,|\le d_e$ and since the point $\nu x'+(1-\nu)w'$ belongs to the convex hull of K and K_e, C_e, we have

$$\begin{aligned}
\Theta&\le\sum_{e\in\mathcal{E}_{\Delta x}}|\,e\,|\left|\frac{d_e}{|K_e^+|\,|K_e^-|}\int_{K_e^+}\int_0^1\int_{C_e}\sum_{i=1}^d|\,\partial_{x_i}\psi(w')\,|\,dw'\,d\nu\,dx'\right|\\
&\le\sum_{e\in\mathcal{E}_{\Delta x}}\frac{|\,e\,|\,d_e}{|K_e^-|}\,|\,\psi\,|_{TV(C_e)}\\
&\le\mathbb{D}_{\Delta x}\sum_{e\in\mathcal{E}_{\Delta x}}|\,\psi\,|_{TV(C_e)},\quad\text{by (2.5)},\\
&\le\mathbb{D}_{\Delta x}\sum_{K\in\mathcal{T}_{\Delta x}}\iota_K\,|\,\psi\,|_{TV(K)}\\
&\le\mathbb{D}_{\Delta x}\,\iota_{\Delta x}\sum_{K\in\mathcal{T}_{\Delta x}}|\,\psi\,|_{TV(K)},\quad\text{by (2.6)},\\
&\le\mathbb{D}_{\Delta x}\,\iota_{\Delta x}\,|\,\psi\,|_{TV(\mathbb{R}^d)}.
\end{aligned}$$

This completes the proof. ∎

We are now ready to prove Lemma 4.1.

Proof of Lemma 4.1. We proceed in several steps.

First step. First, we rewrite the term $TE_{cons}(u,v;t^N)$ as follows:

$$TE_{cons}(u,v;t^N)=\sum_{n=0}^{N-1}\int_0^{t^N}\int_{\mathbb{R}^d}\Theta(t,x,t^{n+1};v(t,x))\,dx\,dt\,\Delta t,$$

where

$$\Theta(t,x,t^{n+1};c) = \sum_{K\in\mathcal{T}_{\Delta x}} \sum_{e\in\partial K} \nabla_x\phi(t,x,t^{n+1},x_K)\cdot \mathcal{A}_{e,K}(u_K^n,c),$$

and

$$\mathcal{A}_{e,K}(u,c) = \,|\,e\,|\big(F(u,c)\cdot n_{e,K}\,\delta_e - \alpha_e\,\mathcal{N}_e(u,c)\,(x_{K_e}-x_K)\,\big).$$

Note that, since $\sum_{e\in\partial K}\,|\,e\,|\,n_{e,K}=0$, we can modify $\mathcal{A}_{e,K}(u,c)$ as follows:

$$\mathcal{A}_{e,K}(u,c) = \,|\,e\,|\,\big(F(u,c)\cdot n_{e,K}\,(\delta_e-\hat{\delta}) - \alpha_e\,\mathcal{N}_e(u,c)\,(x_{K_e}-x_K)\,\big),$$

where $\hat{\delta}$ is an arbitrary constant vector.

Second step. Next, we rewrite $\Theta(t,x,t^{n+1};c)$ as a sum on all the faces e of finite volumes K of the triangulation, $\mathcal{E}_{\Delta x}$. To perform this discrete integration by parts, we denote by K_e^+ and K_e^- the two finite volumes sharing the face e, drop the superscript n in u_K^n, and write ϕ_K instead of $\phi(t,x,t^{n+1},x_K)$; we get

$$\begin{aligned}
\Theta(t,x,t^{n+1};c) &= \sum_{K\in\mathcal{T}_{\Delta x}} \sum_{e\in\partial K} \nabla_x\phi_K\cdot\mathcal{A}_{e,K}(u_K,c)\\
&= \sum_{e\in\mathcal{E}_{\Delta x}} \big\{\,\nabla_x\phi_{K_e^+}\cdot\mathcal{A}_{e,K_e^+}(u_{K_e^+},c) + \nabla_x\phi_{K_e^-}\cdot\mathcal{A}_{e,K_e^-}(u_{K_e^-},c)\big\}\\
&= \Theta_1(t,x,t^{n+1};c) + \Theta_2(t,x,t^{n+1};c),
\end{aligned}$$

where

$$\begin{aligned}
\Theta_1(t,x,t^{n+1};c) &= \sum_{e\in\mathcal{E}_{\Delta x}} \nabla_x\big(\phi_{K_e^+}-\phi_{K_e^-}\big)\cdot\mathcal{A}_{e,K_e^+}(u_{K_e^+},c),\\
\Theta_2(t,x,t^{n+1};c) &= \sum_{e\in\mathcal{E}_{\Delta x}} \nabla_x\phi_{K_e^-}\cdot\big(\mathcal{A}_{e,K_e^+}(u_{K_e^+},c)+\mathcal{A}_{e,K_e^-}(u_{K_e^-},c)\big)\\
&\equiv \sum_{e\in\mathcal{E}_{\Delta x}} \nabla_x\phi_{K_e^-}\cdot\mathcal{B}_e(u_{K_e^-},u_{K_e^+},c),
\end{aligned}$$

with the obvious notation for $\mathcal{B}_e$. The above results allow us to write $TE_{cons}(u,v;t^N) = TE_{1,cons}+TE_{2,cons}$.

Third step. To estimate $TE_{1,cons}$, we proceed as follows. After a simple integration by parts, we obtain

$$TE_{1,cons} = -\sum_{n=0}^{N-1}\sum_{e\in\mathcal{E}_{\Delta x}} \int_0^{t^N}\!\!\int_{\mathbb{R}^d} (\phi_{K_e^+}-\phi_{K_e^-})\nabla_x v(t,x)\cdot\partial_v\mathcal{A}_{e,K_e^+}(u_{K_e^+},v(t,x))\,dx\,dt\,\Delta t,$$

and after taking absolute values,

$$TE_{1,cons} \le \sum_{n=0}^{N-1}\sum_{e\in\mathcal{E}_{\Delta x}} \int_0^{t^N}\!\!\int_{\mathbb{R}^d} |\,\phi_{K_e^+}-\phi_{K_e^-}\,| \sum_{i=1}^{d} |\,\partial_{x_i}v(t,x)\,|\mathbb{A}_e(v(t,x))\,dx\,dt\,\Delta t,$$

where

$$\mathbb{A}_e(c) = |\,e\,|\max_{1\le i\le d}|\,f'(c)\cdot n_{e,K_e^+}\,(\delta_e-\hat{\delta})_i - \alpha_e\,N_e'(c)\,(x_{K_e^-}-x_{K_e^+})_i\,|.$$

Hence

$$TE_{1,cons} \le T_{aux}\,\mathbb{C}\,\|\,f'(v)\,\|\,t^N\,|v_0|_{TV(\mathbb{R}^d)},$$

where

$$\mathbb{C} = \| \delta_e - \hat{\delta} \|_{\ell^\infty(\mathcal{E}_{\Delta x})} + \| \alpha \|_{\ell^\infty(\mathcal{E}_{\Delta x})},$$

and

$$\begin{aligned} T_{aux} &= \sup_{\substack{t\in(0,t^n)\\ x\in\mathbb{R}^d}} \sum_{n=0}^{N-1} \sum_{e\in\mathcal{E}_{\Delta x}} |\, e \,|\, |\, \phi_{K_e^+} - \phi_{K_e^-} \,|\, \Delta t \\ &\leq 2\,\big(1 + \frac{\Delta t}{\epsilon_t}\big)\, W(t^N)\, \mathbb{D}_{\Delta x}\, \iota_{\Delta x}\, \frac{d\,|\,\eta\,|_{TV(\mathbb{R})}}{\epsilon_x}, \end{aligned}$$

by Lemma 3.7 of [7] and Lemma 4.2. This implies

$$TE_{1,cons} \leq 2\, \frac{d\,|\,\eta\,|_{TV(\mathbb{R})}}{\epsilon_x}\, \big(1 + \frac{\Delta t}{\epsilon_t}\big)\, W(t^N)\, \mathbb{C}\, \mathbb{D}_{\Delta x}\, \iota_{\Delta x}\, \|\, f'(v) \,\|\, t^N\, |v_0|_{TV(\mathbb{R}^d)}.$$

Note that the term $TE_{1,cons}$ has been estimated without using regularity properties of the approximate solution. This is not the case for the term $TE_{2,cons}$, as we see next.

Fourth step. To estimate $TE_{2,cons}$, we simply take absolute values to obtain

$$TE_{2,cons} \leq \sum_{n=0}^{N-1} \sum_{e\in\mathcal{E}_{\Delta x}} \int_0^{t^N}\!\! \int_{\mathbb{R}^d} \sum_{i=1}^{d} |\partial_{x_i} \phi_{K_e^-}\,|\, \mathbb{B}_e(u_{K_e^-}, u_{K_e^+})\, dx\, dt\, \Delta t,$$

where

$$\begin{aligned} \mathbb{B}_e(u_{K_e^-}, u_{K_e^+}) &= \sup_{\substack{t\in(0,t^n)\\ x\in\mathbb{R}^d}} |\, \mathcal{B}_e(u_{K_e^-}, u_{K_e^+}, v(t,x)) \,| \\ &\leq |\, e \,|\, \big\{ |\, \big(f(u_{K_e^+}) - f(u_{K_e^-})\big) \cdot n_{e,k} \,|\, \|\, \delta_e - \hat{\delta} \,\|_{\ell^\infty(\mathcal{E}_{\Delta x})} \\ &\qquad + |\, \big(N_e(u_{K_e^+}) - N_e(u_{K_e^-})\big) \cdot n_{e,k} \,|\, \|\, \alpha \,\|_{\ell^\infty(\mathcal{E}_{\Delta x})} \big\} \\ &\leq |\, e \,|\, \|\, f'(u) \,\|\, \mathbb{C}\, |u_{K_e^+} - u_{K_e^-}|\, . \end{aligned}$$

This is the crucial estimate that allows the *supraconvergence* phenomenon to take place: The estimate of $\mathbb{B}_e(u_{K_e^-}, u_{K_e^+})$ depends *only* on ℓ^∞-norms of δ_e and $\alpha_e\,(x_{K_e^+} - x_{K_e^-})$. This is possible because the above quantities depend *only* on the face e and this fact is a direct consequence of the conservativity and consistency properties of the fluxes, see §2.c.

However, the price to pay, is that now the estimate of $TE_{2,cons}$ must depend on the smoothness properties of the approximate solution since

$$TE_{2,cons} \leq \frac{d\,|\,\eta\,|_{TV(\mathbb{R})}}{\epsilon_x}\, W(t^N)\, \mathbb{C}\, \|\, f'(u) \,\| \sum_{n=0}^{N-1} \sum_{e\in\mathcal{E}_{\Delta x}} |\, e \,|\, |\, u^n_{K_e^+} - u^n_{K_e^-} \,|\, \Delta t.$$

Fifth step. The result now follows from the estimates of $TE_{1,cons}$ and $TE_{2,cons}$, from the fact that $TE_{cons}(u, v; t^N) = TE_{2,cons} + TE_{2,cons}$, and from the fact that, since the vector $\hat{\delta}$ is arbitrary, we can take $\mathbb{C} = \|\, \delta \,\|_{\ell^\infty(\mathcal{E}_{\Delta x})/\mathbb{R}^d} + \|\, \alpha \,\|_{\ell^\infty(\mathcal{E}_{\Delta x})}$. ☐

5. Concluding remarks.

In [6], a general theory of *a priori* estimates for scalar conservation laws was proposed. The approach is based on a modification of the original Kuznetsov approximation theory [17]. The key feature and advantage of this new and modified method is that *no* regularity properties of the approximate solution are needed. This point is significant since establishing such properties is a well known bottleneck in this area. This approach was applied to general multidimensional flux-splitting monotone schemes in [7]. For grids consisting of Cartesian products of possibly nonuniform one dimensional meshes, optimal order of convergence were obtained.

In the present paper, we generalize this result to general grids. It should be emphasized that, to the authors' knowledge, this is the first derivation of optimal orders of convergence for truly multidimensional numerical methods for conservation laws. By truly multidimensional, we mean that, in the spirit of the finite volume methods considered, no "dimension by dimension" argument is used. This fact is not a mere detail since, as shown in the introduction, standard properties like TVD do not hold in general on non-Cartesian meshes, which accounts for the lack similar results so far.

The relationship between the consistency of the method and the properties of the grid is analyzed. In particular, it is shown that first, requiring consistency imposes extremely strong restrictions on both the mesh and the numerical fluxes, second, those restrictions are not needed, neither theoretically nor practically. A much larger class of "nearly consistent" (but *not* consistent) schemes is studied. In both cases, optimal *a priori* estimates are obtained. It is shown how the combination of the *conservation form* of the scheme together with the *consistency of the numerical fluxes* allows this *supraconvergence* phenomenon to take place. For schemes that are non-consistent all together, optimal error estimates are obtained as well. However, the nature of the term describing the loss of consistency does require a bound on the total variation $L^1(0,T;TV(\mathbb{R}^d))$ of the approximate solution (which is used there and only there).

In conclusion, we have shown that the approach taken in [6, 7] keeps its promises, by being able to tackle problems previously unsolved. The application to non-splitting numerical fluxes, and to higher-order methods will be the object of future publications. The generalization to *systems* of conservation laws constitutes an exciting challenge.

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