ON SOME TESTS OF GROWTH CURVE MODEL
UNDER BEHRENS–FISHER SITUATION

By

S. R. Chakravorti

Department of Biostatistics
University of North Carolina

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S. R. CHAKRAVORTI

Department of Biostatistics, University of North Carolina, Chapel Hill, N. C. 27514

Abstract

In this article, we have considered the problem of testing equality of several growth curves under Behrens–Fisher situation. In this context the robustness of the existing test criteria have been studied. Also some exact test procedures have been considered and the exact and asymptotic non-central distribution problems have been studied.

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1. INTRODUCTION

The analysis of growth curve model, which is a special case of generalised MANOVA model, has been studied by several authors (Potthoff and Roy [13], Rao [14, 15], Khatri [12], Grizzle and Allen [11] among others). Following Rao [14], this model can be treated as a multivariate analysis of covariance model with stochastic predictors. So far, inferences on problems of this model have been studied under the usual assumptions of normality and homoscedasticity.

In this article, an attempt is made on testing the hypothesis of equality of several growth curves when the dispersion matrices are different and unknown. At the outset, we have studied the robustness of the existing test criteria, under the violation of the assumption of homoscedasticity. Also some exact test criteria have been considered in the line of Anderson [2] and Bhargava [4] and their distribution problems have been studied.

2. THE MODEL AND THE HYPOTHESIS

Let $X_t^{(j)} (1 \times p)$ be the $t$-th observation vector in $j$-th sample ($t=1,2,\ldots,n_j; j=1,2,\ldots,m$). Then the multivariate analysis of covariance model, under Behrens–Fisher situation, can be written as

$$X_t^{(j)} = \xi_t^{(j)} + Z_t^{(j)} B + \varepsilon_t^{(j)}$$  \hspace{1cm} (2.1)

where, $\xi_t^{(j)} (1 \times p)$ is the vector of unknown constants, which is a $p$-th degree polynomial in growth curve model, $Z_t^{(j)} (1 \times q-p)$, the vector of concomitant variables in $j$-th group, ($p \leq q$)

$B(q-p \times p)$, the common matrix of regression coefficients,

$\varepsilon_t^{(j)} (1 \times p)$, the error component in $j$-th group, distributed as $N_p(0, \Sigma_j)$ and independently for all $j(j=1,2,\ldots,m)$, where $\Sigma_j (p \times p)$ is the conditional dispersion matrix of $X_t^{(j)}$ given $Z_t^{(j)}$. Under this
set up, we want to test the hypothesis

\[ H_0[\xi^{(1)} = \ldots = \xi^{(m)}] \] (2.2)

against the alternative of at least one inequality of \(\xi^{(j)}\)'s.

3. ROBUSTNESS OF EXISTING TEST CRITERIA

When dispersion matrices, \(\Sigma_j\)'s in the model (2.1) are equal, the maximum likelihood estimates of \(\xi^{(j)}\) and \(B\) are given by

\[ \hat{\xi}^{(j)} = \bar{x}^{(j)} - \bar{X}^{(j)} \hat{B} \] (3.1)

\[ \hat{B} = S_{Z}^{-1} \sum_{j} \sum_{t} (Z^{(j)} - \bar{Z}^{(j)})' (X^{(j)} - \bar{X}) \] (3.2)

where

\[ S_{Z} = \sum_{j} \sum_{t} (Z^{(j)} - \bar{Z}^{(j)})' (Z^{(j)} - \bar{Z}^{(j)}). \]

Then for testing (2.2), by analysis of variance technique, we have the unconditional residual S.P. matrix

\[ Q_w = \sum_{j} \sum_{t} (X^{(j)} - \bar{X}^{(j)} - (Z^{(j)} - \bar{Z}^{(j)}) \hat{B})' (X^{(j)} - \bar{X}^{(j)} - (Z^{(j)} - \bar{Z}^{(j)}) \hat{B}), \] (3.3)

and S.P. matrix due to hypothesis

\[ Q_B = [\Sigma (X^{(j)} - \bar{X}^{(j)} - (Z^{(j)} - \bar{Z}^{(j)}) \hat{B})' (X^{(j)} - \bar{X}^{(j)} - (Z^{(j)} - \bar{Z}^{(j)}) \hat{B}) \Sigma ]^{-1} \]

\[ - \Sigma (X^{(j)} - \bar{X}^{(j)} - (Z^{(j)} - \bar{Z}^{(j)}) \hat{B})' (X^{(j)} - \bar{X}^{(j)} - (Z^{(j)} - \bar{Z}^{(j)}) \hat{B}), \] (3.4)

where \(\hat{B}\) is the same as (3.2) and \(\hat{B}_0\), the estimate of \(B\) when \(H_0\) is true, is given by

\[ \hat{B}_0 = S_{Z}^{-1} \sum (Z^{(j)} - \bar{Z}^{(j)})' (X^{(j)} - \bar{X}); \quad S_{Z} = \Sigma (Z^{(j)} - \bar{Z})' (Z^{(j)} - \bar{Z}). \] (3.5)

Unconditional distribution of \(Q_w\) is \(W_p(n-m-q+p, \Sigma)\) whether \(H_0\) is true or not. When \(H_0\) is true, \(Q_B\) is distributed as central Wishart distribution,
$W_p (m-1, E)$ and conditional and unconditional distributions remain the same. However, under alternative hypothesis, the distribution of $Q_B$ depends on $Z_t^{(j)}$'s, through the non-centrality parameter.

Thus, when $E_j$'s are equal, the hypothesis (2.2) can be tested by any one of the criteria, viz., the largest root criterion, the likelihood ratio criterion and Lawley-Hotelling's $T^2_o$-statistic. Following Potthoff-Roy's approach Khatri [12] discussed the test procedure using largest root criterion, while in the line of Rao [14], Grizzle and Allen [11] discussed the test procedure using likelihood ratio criterion.

To study the robustness of the existing test criteria, under Behrens-Fisher situation, let us consider Lawley-Hotelling's $T^2_o$-test given by

$$T^2_o = (n-m-q+p)\text{tr} Q_B Q_W^{-1}. \quad (3.6)$$

To study the robustness of the $T^2_o$-test in ordinary MANOVA model under Behrens-Fisher situation, Ito and Schull [9] made the following

Assumption (i) - The sample sizes $n_j$'s and $n = \sum_{j=1}^{m} n_j$ are so large as to keep $r_j = n_j/n$ finite such that the sample residual dispersion matrices can be replaced by the corresponding population dispersion matrices.

Assumption (ii) - That the statistic obtained under assumption (i) is distributed like a constant multiple of a central chi-square.

In order to study the robustness of (3.6) under Behrens-Fisher situation utilizing the assumption (i) and (ii), we observe that $E(Q_W) = \sum_j W_j E_j$, where $Q_W$ is defined by (3.3) and

$$W_j = n_j^{-1} - \sum_t (Z_t^{(j)} - \bar{Z}^{(j)}) S_Z^{-1} (Z_t^{(j)} - \bar{Z}^{(j)}),$$

$$= n_j^{-1} \text{tr} \left( \frac{S_Z}{n-m} \right) \left( \frac{1}{n-m} \right) \frac{1}{n_j^{-1} \sum_t (Z_t^{(j)} - \bar{Z}^{(j)}), (Z_t^{(j)} - \bar{Z}^{(j)})}. \quad (3.7)$$
Hence by the Stutsky theorem (Cramér [7]) \( \sum (Z_t^{(j)} - \bar{Z}(j)) S^{-1}_Z (Z_t^{(j)} - \bar{Z}(j)) \), \( p \), \( c_j \), where \( c_j \) is some finite constant. Hence under assumption (i), at least asymptotically, we can replace \( Q_w/(n-m-q+p) \) by \( \Sigma \Sigma_j \). Thus the asymptotic distribution of \( T_o^2 \) defined by (3.6) is the same as that of

\[ \hat{\chi}_o^2 = \text{tr} Q_B \Sigma^{-1}. \]  

(3.7)

Let us first consider the distribution problem of \( \hat{\chi}_o^2 \) for fixed \( \bar{Z}_t^{(j)} \)'s. When \( \bar{Z}_t^{(j)} \)'s are equal, \( \hat{\chi}_o^2 \) is distributed as a central chi-square under \( H_o \) and a non-central chi-square under alternative hypothesis. However, when \( \bar{Z}_t^{(j)} \)'s are different \( \hat{\chi}_o^2 \) is distributed as a linear combination of chi-square variates. Hence utilizing the assumption (ii) of Ito and Schull, the asymptotic distribution of \( \hat{\chi}_o^2 \) is considered as \( c\chi^2(f) \), where \( \chi^2(f) \) is a central chi-square with \( f \) d.f. and \( c \) and \( f \) are adjusted to give the correct first two moments.

To calculate the mean and variance of (3.7), we write \( \hat{\chi}_o^2 \), using (2.1), (3.2), (3.4) and (3.5) as follows:

\[ \hat{\chi}_o^2 = \{ \Sigma \Sigma [Z_t^{(j)} - \bar{Z}(j)] (\theta + (Z_t^{(j)} - \bar{Z}(j)) \Sigma^{-1}_Z (Z_t^{(j)} - \bar{Z}(j)) \Sigma^{-1}_Z (Z_t^{(j)} - \bar{Z}(j))) \}, \]

(3.8)

where

\[ \theta = \Sigma^{-1}_Z \Sigma_j n_j (Z_t^{(j)} - \bar{Z}(j)), \]

\[ \bar{Z}(j) = \Sigma_j \bar{Z}(j)/n, \]

(3.9)

Then consider the following non-singular transformations:
\[ F_{t}^{-1} F' = I, \quad \xi_{t}^{(j)} F = \gamma_{t}^{(j)}, \quad \xi_{t} F = \bar{\gamma}, \quad \xi F = \bar{\gamma} \]  

(3.10)

where \( F \) is a non-singular matrix of order \((p \times p)\) and \( F^{-1} = F' F \), elements of vectors \( \gamma_{t}^{(j)} \), \( \bar{\gamma} \) and \( \bar{\gamma} \) follow jointly the distributions \( N_{p}(0, F'_{j} F) \), \( N_{p}(0, n_{j}^{-1} F'_{j} F) \) and \( N_{p}(0, n^{-1} I) \) respectively, where \( I \) is the identity matrix of order \((p \times p)\).

Applying this set of transformations in (3.8), wherever necessary and after some simplifications, we obtain the mean and variance of \( \chi_{o}^{2} \) as follows:

\[
\begin{align*}
E(\chi_{o}^{2}) &= \text{tr}[\Sigma(1-r_{j})E_{j}E_{j}^{-1} \\
&\quad + (\Sigma_{j}^{-1}E_{j}^{(j)} - \bar{\gamma})'(\Sigma_{j}^{-1}E_{j}^{(j)}) \otimes \Sigma_{j}^{-1}E_{j}^{-1} - \Sigma_{j}^{-1}E_{j}^{(j)}'(\Sigma_{j}^{-1}E_{j}^{(j)}) \otimes \Sigma_{j}^{-1}E_{j}^{-1}) \\
&\quad + \Sigma_{j}^{-1}E_{j}^{(j)}'(\Sigma_{j}^{-1}E_{j}^{(j)}) \otimes \Sigma_{j}^{-1}E_{j}^{-1}] \\
\end{align*}
\]  

(3.11)

\[
\begin{align*}
V(\chi_{o}^{2}) &= 2\text{tr}[\Sigma(1-2r_{j})(E_{j}E_{j}^{-1})^{2}] + I \\
&\quad + (\Sigma_{j}^{-1}E_{j}^{(j)} - \bar{\gamma})'(\Sigma_{j}^{-1}E_{j}^{(j)}) \otimes \Sigma_{j}^{-1}E_{j}^{-1} - \Sigma_{j}^{-1}E_{j}^{(j)}'(\Sigma_{j}^{-1}E_{j}^{(j)}) \otimes \Sigma_{j}^{-1}E_{j}^{-1})^{2} \\
&\quad + 2(\Sigma_{j}^{-1}E_{j}^{(j)} - \bar{\gamma})'(\Sigma_{j}^{-1}E_{j}^{(j)}) \otimes (\Sigma_{j}^{-1}E_{j}^{-1} - \Sigma_{j}^{-1}E_{j}^{(j)}'(\Sigma_{j}^{-1}E_{j}^{(j)}) \otimes \Sigma_{j}^{-1}E_{j}^{-1}) \\
&\quad + 2(\Theta E_{j}^{(j)} - \bar{\gamma})'(\Sigma_{j}^{-1}E_{j}^{(j)}) \otimes \Sigma_{j}^{-1}E_{j}^{-1} + \Sigma_{j}^{-1}E_{j}^{(j)}'(\Sigma_{j}^{-1}E_{j}^{(j)}) \otimes \Sigma_{j}^{-1}E_{j}^{-1} \\
&\quad - 2\Sigma_{j}^{-1}E_{j}^{(j)}'(\Sigma_{j}^{-1}E_{j}^{(j)}) \otimes \Sigma_{j}^{-1}E_{j}^{-1}] \\
\end{align*}
\]  

(3.12)

where \( \Theta \) is defined in (3.9) and \( \otimes \) stands for the Kronecker product. In particular, when \( E_{j} \)'s are equal,

\[
E(\chi_{o}^{2}) = (m-1)p + \text{tr}[\Sigma_{j}^{(j)}(\Sigma_{j}^{-1}E_{j}^{(j)} - \bar{\gamma})'(\Sigma_{j}^{-1}E_{j}^{(j)}) \Theta \otimes \Sigma_{j}^{-1}] \\
\]

and

\[
V(\chi_{o}^{2}) = 2(m-1)p + 4\text{tr}[\Sigma_{j}^{(j)}(\Sigma_{j}^{-1}E_{j}^{(j)} - \bar{\gamma})'(\Sigma_{j}^{-1}E_{j}^{(j)}) \Theta \otimes \Sigma_{j}^{-1}] - 2\Sigma_{j}^{(j)}(\Sigma_{j}^{-1}E_{j}^{(j)}) \Theta \otimes \Sigma_{j}^{-1}] \\
\]

Under \( H_{o} \), \( E(\chi_{o}^{2}) = (m-1)p \) and \( V(\chi_{o}^{2}) = 2(m-1)p \), is the expected result. Thus, by equating (3.11) and (3.12) with mean and variance of \( c\chi^{2}(f) \), we obtain c
and \( f \) as

\[
c = V/2E \quad \text{and} \quad f = 2E^2/V
\]

(3.13)

where \( E \) and \( V \) are the mean and variance of \( \xi^2 \) given by (3.11) and (3.12) respectively.

These \( c \) and \( f \) are obtained under the assumption that \( Z_{ij}^{(j)} \)'s are held fixed. Now let us assume that \( Z_{ij}^{(j)} \)'s are random vectors, distributed as \( N_q(p, \xi Z_j) \) for \( j=1,2,\ldots,m \). Then from assumption (i) and from (3.2),

\[
S_Z/(n-m) \sim \xi Z_j \quad \text{so that}
\]

\[
S_Z^{-1} (\xi Z_t^{(j)} - \xi Z_t^{(j)})' (\xi Z_t^{(j)} - \xi Z_t^{(j)}) \otimes \xi j Z_j^{-1}
\]

\[
= \sum_j \left( \frac{S_Z}{n-m} \right)^{-1} \frac{1}{n-m} \sum_j (\xi Z_t^{(j)} - \xi Z_t^{(j)})' (\xi Z_t^{(j)} - \xi Z_t^{(j)}) \otimes \xi j Z_j^{-1}
\]

(3.14)

Also since, \( S_Z^{-1} \Sigma j\xi (\xi Z_t^{(j)} - \xi Z_t^{(j)})' (\xi Z_t^{(j)} - \xi Z_t^{(j)}) = \left( \frac{S_Z}{n-m} \right)^{-1} \frac{1}{n-m} \Sigma j\xi (\xi Z_t^{(j)} - \xi Z_t^{(j)})' (\xi Z_t^{(j)} - \xi Z_t^{(j)}) \]

\[
P_j (n-m)^{-1} \xi Z_j^{-1} (1-r_j) \xi Z_j \quad \text{we have}
\]

\[
S_Z^{-1} \xi (\xi Z_t^{(j)} - \xi Z_t^{(j)})' (\xi Z_t^{(j)} - \xi Z_t^{(j)}) \otimes \xi j Z_j^{-1}
\]

\[
= (S_Z + \Sigma j\xi (\xi Z_t^{(j)} - \xi Z_t^{(j)}))^{-1} (\Sigma (\xi Z_t^{(j)} - \xi Z_t^{(j)})' (\xi Z_t^{(j)} - \xi Z_t^{(j)}) + \Sigma j\xi (\xi Z_t^{(j)} - \xi Z_t^{(j)})) \otimes \xi j Z_j^{-1}
\]

(3.15)

Now let us further make the following

**Assumption (iii)** - We conceive of a sequence \( \{H_n\} \) of usual Pitman type alternatives where

\[
H_n: \xi_{ij}^{(j)} = \xi_{ij}^{(j)} = n^{j} \xi p^{(j)} \quad (j=1,\ldots,m)
\]

(3.16)
where \( \varphi^{(j)} \)'s are vectors of finite constants. Then from assumptions (i) and (iii)

\[
\Sigma \sum_{j} \varphi^{(j)} \varphi^{(j)\prime} = \Sigma r_{j} \varphi^{(j)} \varphi^{(j)\prime}
\]

(3.17)

and for large \( n \), \( \sqrt{n} (\bar{Z} - \bar{Z})^{\prime} \varphi \propto 0 \), so that \( \vartheta \) defined in (3.9) becomes

\[
\vartheta = [I + (n - m)^{-1} \Sigma j (1 - r_{j}) \Sigma j \bar{Z} \bar{Z}^{-1}]^{-1} \sqrt{n - m} \Sigma j \sqrt{n - m} \Sigma j (\bar{Z} - \bar{Z})^{\prime} \varphi
\]

(3.18)

Also under this assumption (iii), (3.14) and (3.15) are asymptotically equal. Hence, unconditionally, the expressions for \( c \) and \( f \) defined by (3.13) can be obtained asymptotically, where using (3.14), (3.15), (3.17) and (3.18) in \( E \) and \( V \) involved in \( c \) and \( f \), we have

\[
E(\hat{\chi}^{2}_o) = \text{tr} \left[ \Sigma (1 - r_{j}) \bar{Z} \bar{Z}^{-1} + \Sigma r_{j} \varphi^{(j)\prime} \varphi^{(j)\prime} \right]
\]

(3.19)

\[
V(\hat{\chi}^{2}_o) = 2 \text{tr} \left[ \Sigma (1 - 2r_{j}) (\bar{Z} \bar{Z}^{-1})^2 + \Sigma r_{j} \varphi^{(j)\prime} \varphi^{(j)\prime} \right]
\]

(3.20)

Thus, under the assumption (iii), the mean and variance of \( \hat{\chi}^{2}_o \) obtained in (3.19) and (3.20) are the same as those obtained by Ito and Schull in ordinary MANOVA model under heteroscedasticity assumption. This establishes the robustness of \( T^{2}_o \)-test in multivariate analysis of covariance model under heteroscedasticity assumption.

It may be noted that asymptotic performance of likelihood ratio criterion being the same as that of \( T^{2}_o \)-test (see Hsu [8] and Ito [10]), we conclude that the existing test criteria are asymptotically robust under the set up we have considered.
4. SOME EXACT TEST PROCEDURES

So far, the exact test procedures of (2.2) in ordinary MANOVA model under Behrens-Fisher situation are available in some particular situations. Two samples problem leads to the well-known Bennett's [3] test, which is the generalization of Scheffe's [19] procedure, where the usual Hotelling's $T^2$-test can be applied. Anderson [2] extended this procedure to the case of more than two samples problem. With a modification of Anderson's procedure Bhargava [4] arrived at a new solution of the problem, where the usual likelihood ratio criterion can be applied.

We shall make an attempt to apply these two test procedures in our problem.

4.1. Anderson's procedure in testing (2.2) under the model (2.1). Let us assume that the sample sizes $n_1, n_2, \ldots, n_m$ are such that $n_1 \leq n_2 \leq \ldots \leq n_m$. Then we define the Scheffe-variables in terms of $X^{(j)}_{\alpha}$, the $\alpha$-th observation in $j$-th sample ($\alpha=1,2,\ldots,n_j; j=1,\ldots,m$) as

$$V^{(r)}_{\alpha} = \frac{X^{(1)}_{\alpha}}{\sqrt{n_1}} - \frac{1}{\sqrt{n_1 n_r}} \sum_{\beta=1}^{n_1} \frac{X^{(r)}_{\beta}}{\sqrt{n_1}} - \bar{X}^{(r)}$$

for $r=2,3,\ldots,m$, $\alpha=1,2,\ldots,n_1$. Then from (2.1) we have

$$V^{(r)}_{\alpha} = \bar{\eta}^{(r)}_{\alpha} + \bar{\psi}^{(r)}_{\alpha} + \bar{\zeta}^{(r)}_{\alpha},$$

where

$$\bar{\eta}^{(r)}_{\alpha} = \frac{\zeta^{(1)}_{\alpha}}{\sqrt{n_1}} - \frac{\bar{\zeta}^{(r)}_{\alpha}}{\sqrt{n_1}},$$

$$\bar{\psi}^{(r)}_{\alpha} = \frac{Z^{(1)}_{\alpha}}{\sqrt{n_1 n_r}} - \frac{1}{\sqrt{n_1 n_r}} \sum_{\beta=1}^{n_1} \frac{Z^{(r)}_{\beta}}{\sqrt{n_1}} - \bar{Z}^{(r)},$$

$$\bar{\zeta}^{(r)}_{\alpha} = \frac{\varepsilon^{(r)}_{\alpha}}{\sqrt{n_1 n_r}} - \frac{1}{\sqrt{n_1 n_r}} \sum_{\beta=1}^{n_1} \frac{\varepsilon^{(r)}_{\beta}}{\sqrt{n_1}} - \bar{\varepsilon}^{(r)}.$$
Then \( V^{(r)}_{\alpha} (1 \times p) \) is distributed as a \( p \)-variate normal distribution with

\[
E(V^{(r)}_{\alpha}) = \eta^{(r)} + \psi^{(r)}_{\alpha} B
\]

and

\[
E(V^{(r)}_{\alpha} - EV^{(r)}_{\alpha}, (V^{(r)}_{\alpha} - EV^{(r)}_{\alpha})' = \Sigma_1 + \frac{n_1}{n_r} \Sigma_r
\]

so that the vector \( W_{\alpha} = (V^{(2)}_{\alpha}, \ldots, V^{(m)}_{\alpha}) \) of order \( 1 \times (m-1)p \) follows a \( (m-1) \)
\( p \)-variate normal distribution with

\[
E(W_{\alpha}) = \eta + \psi_{\alpha} (I_{m-1} \otimes B)
\]

and dispersion matrix

\[
\Sigma^*_{\alpha} = \begin{pmatrix}
\Sigma_1 + \frac{n_1}{n_2} \Sigma_2 & \Sigma_1 & \ldots & \Sigma_1 \\
\Sigma_1 & \Sigma_1 + \frac{n_1}{n_3} \Sigma_3 & \ldots & \Sigma_1 \\
\vdots & \vdots & \ddots & \vdots \\
\Sigma_1 & \ldots & \ldots & \Sigma_1 + \frac{n_1}{n_m} \Sigma_m
\end{pmatrix}
\]

Under this set up the hypothesis (2.2) reduces to \( H_0 [\eta = 0] \). Since the explicit solutions of the estimates of \( \eta \) and \( B \) of the model (4.6) are difficult to obtain, either by generalized least squares or by maximum likelihood method, the test procedure cannot be constructed easily.

However, for two samples problem the usual \( T^2 \)-test can be obtained explicitly.

4.1.1. **Test procedure in two samples problem.** From (4.1) and (4.2) in case \( m = 2 \), we define the Scheffe-variable as

\[
V_{\alpha} = X^{(1)}_{\alpha} - \sqrt{\frac{n_1}{n_2}} X^{(2)}_{\alpha} + \frac{1}{\sqrt{n_1 n_2}} \sum \limits_{\beta=1}^{n_1} X^{(2)}_{\beta} - \bar{X}^{(2)}
\]

\[
= \eta + \psi_{\alpha} B + \zeta_{\alpha}
\]
where \( \eta = \xi^{(1)} - \xi^{(2)} \) and \( \psi_\alpha \) and \( \zeta_\alpha \) are obtained from (4.2) putting \( r=2 \).

Thus \( V_\alpha \) is distributed as a \( p \)-variate normal distribution with mean \( \eta + \psi_\alpha \beta \)
and dispersion matrix \( \Sigma^* = \Sigma_1 + \frac{n_1}{n_2} \Sigma_2 \). From (4.8), let us write the model

\[
E(V_\alpha) = U_\alpha \beta
\]

(4.9)

where \( U = [1; \psi_\alpha] \), a vector of order \((1\times l+q-p)\) and \( \beta = \begin{bmatrix} \eta \\ \beta \end{bmatrix} \), a matrix or order \((l+q-p\times p)\). Under this set up, the maximum likelihood estimate of \( \beta \) is given by

\[
\hat{\beta}_\Omega = A^{-1} \zeta
\]

(4.10)

where
\[
A = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}, \quad \text{a matrix of order } (l+q-p\times l+q-p)
\]
\[
\zeta = \begin{bmatrix} \zeta_\alpha \\ \zeta_\alpha \end{bmatrix}, \quad \text{a matrix}
\]
of order \((l+q-p\times p)\), where
\[
A_{11} = n_1, \quad A_{21} = A_{22} = n_1 \bar{V}, \quad A_{22} = \Sigma_{\psi_\alpha} \psi_\alpha
\]
\[
C_1 = n_1 \bar{V}, \quad C_2 = \Sigma_{\psi_\alpha} \psi_\alpha
\]
\[
A_{11} \cdot 2 = A_{11}^{-1} A_{12} A_{22}^{-1} A_{21} = n_1 (1-n_1 \bar{V} (\Sigma_{\psi_\alpha} \psi_\alpha)^{-1} \bar{V}),
\]
\[
A_{22} \cdot 1 = A_{22}^{-1} A_{21} A_{11} A_{12} = \Sigma (\psi_\alpha \bar{V}) (\psi_\alpha - \bar{V})
\]

(4.11)

Thus the estimate of \( \eta \) can be obtained as the first row of (4.10), given by,

\[
\hat{\eta}_\Omega = A_{11}^{-1} \cdot 2 C_1 - A_{11}^{-1} A_{12} A_{22}^{-1} \cdot 1 C_2
\]

\[
= \bar{V} - \bar{V} \beta_\Omega
\]

(4.12)

and

\[
\hat{\beta}_\Omega = S_{\psi_\alpha}^{-1} \Sigma (\psi_\alpha \bar{V}) (\psi_\alpha - \bar{V})
\]

(4.13)

where

\[
S_{\psi_\alpha} = \Sigma (\psi_\alpha \bar{V}) (\psi_\alpha - \bar{V})
\]

Thus, following Anderson [1], the hypothesis \( H_o [\eta = \xi^{(1)} - \xi^{(2)} = 0] \) can be
tested by the usual likelihood ratio criterion

\[ \Lambda = \left| S_W \right| / \left| S_W + \hat{n}_\Omega A_{11} \cdot 2 \hat{n}_\Omega \right| \] (4.14)

where \( S_W = n_1 \hat{\Sigma}_W^\alpha = \Sigma (V_\alpha - U_\alpha \hat{\beta}_\alpha) (V_\alpha - U_\alpha \hat{\beta}_\alpha)' \), which is distributed as a central Wishart distribution with \((n_1 - l - q + p)\) d.f., while \( \hat{n}_\Omega A_{11} \cdot 2 \hat{n}_\Omega \) is distributed as a central Wishart distribution with \(p\) d.f. when \( H_0 \) is true and a non-central Wishart distribution under alternative hypothesis, with non-centrality parameter \( \lambda = \text{tr} A_{11} \cdot 2 \tilde{n}_\Omega \hat{\Sigma}_W^{-1} \), which, therefore, depends on \( \psi_\alpha \) in the form of \( A_{11} \cdot 2 \). From (4.14), we have,

\[ \Lambda = (1 + A_{11} \cdot 2 \tilde{n}_\Omega S_W^{-1} \hat{n}_\Omega)^{-1} \] (4.15)

which shows that Hotelling's \( T^2 \)-test can also be used to this situation, where

\[ (n_1 - l - q + p)^{-1} T^2 = A_{11} \cdot 2 \hat{n}_\Omega S_W^{-1} \hat{n}_\Omega. \] (4.16)

Now \((n_1 - q) T^2 / (n_1 - l - q + p)\) follows central F-distribution with d.f. \((p, n_1 - q)\), under \( H_0 \), while a non-central F distribution, under alternative hypothesis, with same d.f.'s and n.c.p. \( \lambda = \text{tr} A_{11} \cdot 2 \tilde{n}_\Omega \hat{\Sigma}_W^{-1} \). Obviously, this is the conditional non-central distribution of \( T^2 \) defined by (4.16) for fixed \( \psi_\alpha \). Thus the unconditional non-central distribution of \( T^2 \) can be obtained by integrating out \( \psi_\alpha \) from the joint distribution of \( \psi_\alpha \) and \( T^2 \). Hence we have the following.

**THEOREM 1.** The unconditional non-central distribution of \( T^2 \) (defined by (4.16)) is given by
\[
\begin{align*}
\frac{1}{B\left(\frac{q-p}{2}, \frac{n_1-q+p}{2}\right)} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (-1)^{r+s} \Delta^{r+s} B\left(\frac{q-p}{2}, \frac{n_1-q+p}{2} + r+s\right) B\left(\frac{p}{2}, \frac{n_1-q}{2}, \frac{p}{2} + r, \frac{n_1-q}{2}\right) \\
\left(\frac{p}{n_1-q}\right)^{p-r-1} \left(1 + \frac{p}{n_1-q}\right)^{-1} \left(n_1-q+p+2r\right) \frac{d}{dn_1-q}
\end{align*}
\]

where

\[
\Delta = \text{tr} \ n_1 \eta_1' \eta_2^{-1}, \quad \Sigma^* = \Sigma_1 + \frac{n_1}{n_2} \Sigma_2
\]

**Proof.** From (4.11), it is easy to show that

\[
A_{11,2} = n_1 \left(1 + n_1 \Psi^{-1}_S \Psi^{-1}_L\right)^{-1}
\]

where \(\Sigma_{\Psi}\) is defined in (4.13). Now if we assume that the covariables \(Z_{\Psi}\)'s in the model (2.1) are random vectors distributed as \(N_{q-p}(0, \Sigma_{Z(1)}\Sigma_{Z(2)})\), then we observe that \(\Sigma_{\Psi}\) (defined in (4.8)), being a linear function of \(Z_{\Psi}\) in the model (2.1) and \(Z_{\Psi}\), is distributed as \(N_{q-p}(0, \Sigma_{Z(1)} + \frac{n_1}{n_2} \Sigma_{Z(2)} = \Sigma_{\Psi})\). Hence from (4.18)

\[
n_1 \Psi^{-1}_S \Psi^{-1}_L = (n_1-1)^{-1} T^2_{\Psi}
\]

where \(T^2_{\Psi}\) is a Hotelling's \(T^2\), distributed as a central \(F\) with d.f. \((q-p, n_1-q+p)\). Thus the conditional distribution of \(T^2\) defined by (4.16) is a non-central \(F\) distribution which involves the variables \(Z_{\Psi}\) only in the non-centrality parameter in the form of (4.18). Now the n.c.p. \(\lambda\) is given by

\[
\lambda = \left[1 + (n_1-1)^{-1} T^2_{\Psi}\right]^{-1} \Delta; \quad \Delta = n_1 \text{tr} \ n_1 \eta_1' \Sigma^*^{-1}
\]

Hence the unconditional distribution of \(T^2\) is obtained as

\[
f(T^2 | \Delta) = \int f(T^2 | T^2_{\Psi}) f(T^2_{\Psi}) dT^2_{\Psi}.
\]

Since
\[ f(T^2 | T^2_\psi) = e^{-\lambda/2} \sum_{r=0}^{\infty} \frac{(\lambda/2)^r}{r!} \phi_{p+2r, n_1 - q}(F) \]  

and \[ f(T^2_\psi) = \phi_{q-p, n_1 - q+p}(F) \]

where

\[ \psi_{m, n}(F) = [B(m, n)]^{-1} \left( \frac{m}{n} \right) F^2 - 1 \left( 1 + \frac{m}{n} \right) \frac{2}{2} \]

the result (4.17) follows by direct integration. Hence the theorem.

4.2. Bhargava's procedure in testing (2.2) under the model (2.1). Following Bhargava [4], let us assume that the sample sizes \( n_1, n_2, \ldots, n_m \) are of the type, \( n_1 = (m-1)n', n_2 = \ldots = n_m = n' \). Then under the usual randomization procedure the new set of variables are chosen, on the basis of \( X_{\alpha}^{(j)}(1 \times p), j = 1, \ldots, n_j, j = 1, \ldots, m \), as follows

\[ \begin{align*}
\psi_{\alpha}^{(2)} &= X_{\alpha}^{(2)} - X_{\alpha}^{(1)} \\
\psi_{\alpha}^{(3)} &= X_{\alpha}^{(3)} - X_{\alpha}^{(1)} \\
&\vdots \\
\psi_{\alpha}^{(m)} &= X_{\alpha}^{(m)} - X_{\alpha}^{(1)} \\
&\quad - X_{\alpha+(m-2)n'}
\end{align*} \]  

(4.24)

Then from (2.1)

\[ \psi_{\alpha}^{(r)} = X_{\alpha}^{(r)} - X_{\alpha+(r-2)n'} = \eta_{\alpha}^{(r)} + \phi_{\alpha}^{(r)} B + \theta_{\alpha}^{(r)} \]  

(4.25)

where

\[ \begin{align*}
\eta_{\alpha}^{(r)} &= \xi_{\alpha}^{(r)} - \xi_{\alpha}^{(1)} \\
\phi_{\alpha}^{(r)} &= \zeta_{\alpha}^{(r)} - \zeta_{\alpha+(r-2)n'} \\
\theta_{\alpha}^{(r)} &= \varepsilon_{\alpha}^{(r)} - \varepsilon_{\alpha+(r-2)n'}
\end{align*} \]  

(4.26)

for \( \alpha = 1, 2, \ldots, n', r = 2, 3, \ldots, n \). Hence \( \psi_{\alpha}^{(r)} \) is distributed as a p-variate normal distribution with mean vector and dispersion matrix

\[ E \psi_{\alpha}^{(r)} = \eta_{\alpha}^{(r)} + \phi_{\alpha}^{(r)} B, \]  

(4.27)
\[ \Sigma^* = \Sigma_T + \Sigma_1, \] (4.28)

and \( \gamma_2, \ldots, \gamma_m \) are independently distributed.

It should be noted here that Anderson's variables chosen in (4.1) are not independently distributed for \( r = 2, \ldots, m \), which, perhaps, does not lead to a satisfactory solution of the estimates of the parameters.

Thus, under the model (4.25), the hypothesis (2.2) can be written

\[ H_0 = \bigcap_{r=2}^m H_0^{(r)}; \quad H_0^{(r)}: [\gamma_i^{(r)} = \gamma_i^{(1)} = 0] \] (4.29)

Hence, knowing the test procedure for component hypotheses we can construct an overall test for (4.29).

Now for each of the component hypotheses, we can follow the same procedure as shown in Sec. 4.1.1, by which we can construct likelihood ratio criterion

\[ \Lambda^{(r)} = \left| S_W^{(r)} \right| / \left| S_W^{(r)} + \gamma_i^{(r)} \right| \] (4.30)

for \( r = 2, 3, \ldots, m \), where from (4.25)-(4.27), \( S_W^{(r)} = n'\Sigma^* \) and

\[ A_{11.2}^{(r)} = n'(1-n'\phi)(\gamma_i^{(r)} - 1) - \phi^{(r)} \phi^{(r)}' \]. Obviously,

\[ \Lambda^{(r)} = \left[ 1 + (n'-1-q+p)^{-1}T_r^2 \right]^{-1} \] (4.31)

where \( (n'-1-q+p)^{-1}T_r^2 = A_{11.2}^{(r)} S_W^{(r)} - \gamma_i^{(r)} \).

It may be noted here that while constructing these test criteria, we have given separate estimates of \( B \) (the common matrix of regression coefficients) for \( r = 2, 3, \ldots, m \) from (4.25), for which we get \( \Lambda^{(2)}, \ldots, \Lambda^{(m)} \) independently distributed.

Now under \( H_0^{(r)} \), \( \Lambda^{(r)} \) is distributed as a central beta distribution with parameters \( \left( \frac{r}{2}, \frac{n'-q}{2} \right) \). Hence under \( H_0 \), the conditional and unconditional
distributions of $\Lambda^{(r)}$ (for $r=2,\ldots,m$) remain the same. When, however, $H_0^{(r)}$'s are not true, these $\Lambda^{(r)}$'s will have non-central beta distributions, which involve $\phi^{(r)}_\alpha$'s only in the non-centrality parameter

$$
\lambda_r = \text{tr} \, A_{11\cdot 2}^{(r)} \, \eta^{(r)} \sum_r^{\times -1}.
$$

(4.32)

Under this set up, the hypothesis (4.29) can be tested by combining these component tests using union-intersection principle. We can, however, consider an overall test by considering the test criterion

$$
\Lambda = \prod_{r=2}^{m} \Lambda^{(r)}
$$

(4.33)

where $\Lambda^{(r)}$ for $r=2,\ldots,m$ are independent beta variables, distributed as central or non-central beta distributions according as component hypotheses are true or not.

4.2.1. **Exact (unconditional) non-central distribution of $\Lambda$.** Under $H_0$, $\Lambda$ is the product of $(m-1)$ independent beta variables, the exact as well as asymptotic distribution of which are well-known (see V. V. N. Rao [17], Rao [16], Roy [18], Anderson [1]) and since under $H_0$, the distributions of $\Lambda^{(r)}$'s are independent of $\phi^{(r)}_\alpha$'s, the conditional and unconditional distributions remain the same.

Under alternative hypothesis, however, the distributions of $\Lambda^{(r)}$'s depend on $\phi^{(r)}_\alpha$'s in the form of $A_{11\cdot 2}^{(r)}$'s only in the non-centrality parameters $\lambda_r$'s defined by (4.32). Here the exact unconditional non-central distribution of $\Lambda$ is obtained in case of three groups (i.e., $m=3$), which can be easily extended for $m>3$. Hence we have the following

**THEOREM 2.** The (unconditional) non-central distribution of $\Lambda = \Lambda^{(2)} \cdot \Lambda^{(3)}$, where $\Lambda^{(2)}$ and $\Lambda^{(3)}$ are independently distributed as non-central beta variables
with parameters \((\frac{p}{2}, \frac{n'-q}{2})\) and n.c.p. \(\lambda_r = A_{11,2}^{(r)} \text{tr} \eta_{r} \eta_{r}^{\prime} \sum_{r}^{-1} (r=2,3)\), is given by

\[
C \cdot \sum_{s} \sum_{a} \sum_{\beta} \frac{(-1)^{s} \alpha s + \alpha \beta}{2^{s+\alpha+\beta}} \frac{B\left(\frac{q-p}{2}, \frac{n'-q+p}{2} + \alpha+\beta\right)}{B\left(\frac{n'-q}{2}, \frac{p}{2} + \alpha\right)} \frac{B\left(\frac{p}{2} + \alpha, \frac{p}{2} + \ell\right) \lambda^{2} - 1}{(1-\lambda)^{p+\alpha+\ell-1}}
\]

\[
\cdot 2F_1\left(\frac{p}{2} + \ell, \frac{p}{2} + \alpha, \frac{p+\alpha+\ell}{2}, 1-\lambda\right) d\lambda
\]

(4.34)

where, \(\Delta_r = n' \text{tr} \eta_{r} \eta_{r}^{\prime} \sum_{r}^{-1} (r=2,3)\), and \(C = B\left(\frac{q-p}{2}, \frac{n'-q+p}{2}\right)^{-2}\).

**Proof.** Arguing exactly in the same way as in the proof of Theorem 1 for each \(r=2,3, \ldots, m\), we have from (4.32)

\[
\lambda_r = \left(1 + \frac{n'-q}{r} \phi_{\alpha}^{(r)}(r)-1 \frac{\phi_{\alpha}^{(r)}}{r}(r)^{-1} \Delta_r\right)^{-1}
\]

(4.35)

when \(\phi_{\alpha}^{(r)}\)'s are random vectors having the distributions, \(N_{q-p}(0, \Sigma_{\phi}(r))\), \(\lambda_r\) can be written as

\[
\lambda_r = \left(1 + \left(n'-q\right)^{-1} \phi_{\alpha}^{2}(r)\right)^{-1} \Delta_r
\]

(4.36)

where \(\frac{n'-q+p}{q-p} \phi_{\alpha}^{2}(r)\) follows a central F distribution with d.f. \((q-p, n'-q+p)\).

The conditional distribution of \(\Lambda^{(r)}\) for fixed \(\phi_{\alpha}^{(r)}\)'s is a non-central beta distribution with \(\left(\frac{p}{2}, \frac{n'-q}{2}\right)\) and n.c.p. \(\lambda_r\) is given by

\[
dP(\Lambda^{(r)} | T_{\phi}^{2}(r)) = e^{-\lambda_{r}/2} \sum_{s} \frac{\lambda_{r/2}^{s} \lambda_{2}^{2} \left(\frac{r}{2}\right)^{n'-q} - 1}{B\left(\frac{p}{2} + s, \frac{n'-q}{2}\right)} d\lambda^{(r)}
\]

(4.37)

Then following the same line of proof as in Theorem 1, the unconditional
non-central distribution of $\Lambda^{(r)}$ is

\[
\frac{1}{\mathcal{B}\left(\frac{q-p}{2}, \frac{n'-q+p}{2}\right)} \sum_{s=0}^{\ell} \frac{(-1)^s \binom{\ell}{s} \frac{\ell+s}{2} + \ell + s}{\ell! s!} \mathcal{B}\left(\frac{q-p}{2}, \frac{n'-q+p + \ell + s}{2}\right) \times 
\]

\[
\frac{(\Lambda^{(r)} - 1)}{2} \left(1 - \Lambda^{(r)}\right)^{\frac{p}{2} + s - 1} d\Lambda^{(r)} \tag{4.38}
\]

Since $\Lambda^{(2)}$ and $\Lambda^{(3)}$ are independently distributed, the unconditional joint distribution of $\Lambda^{(2)}$ and $\Lambda^{(3)}$ is the product of their respective distributions which are of the type (4.38). Hence, the distribution of $\Lambda = \Lambda^{(2)} \cdot \Lambda^{(3)}$ can easily be obtained by simple transformations (see Chakravorti [6]). Hence the theorem.

The central distribution of $\Lambda$ follows from (4.34), when $\Delta_2 = \Delta_3 = 0$ and is given by

\[
\frac{\mathcal{B}\left(\frac{p}{2}, \frac{p}{2}\right)}{\left[\mathcal{B}\left(\frac{p}{2}, \frac{n'-q}{2}\right)\right]^2} \left(\Lambda^{2} - 1\right)^{p-1} \frac{n'-q}{2} \mathcal{P}_{1}\left(\frac{p}{2}, \frac{p}{2}, p, 1-\Lambda\right) d\Lambda \tag{4.39}
\]

As stated earlier, Theorem 2 can be extended for $m>3$, exactly in the same way. The resulting distribution will, however, be very much complicated in form. From practical point of view, the unconditional non-central distribution of $\Lambda$, defined by (4.33), can be obtained in the asymptotic series form, following Sugiura and Fujikoshi [20].

4.2.2. Asymptotic (unconditional) non-central distribution of $\Lambda$. We shall make an attempt to derive the asymptotic (unconditional) non-central distribution of $\Lambda$ (given by (4.33)) in the line of Box [5], Anderson [1] and Sugiura and Fujikoshi [20].
Let us, therefore, consider the asymptotic expansion of the non-null distribution $P[-2\rho \log \Lambda < Z]$ up to the order $n^{-1}$, where by substituting in Sugiura and Fujikoshi [20] $N-S = n'-q$, $b=p$, $p=1$, $m=m'$, we have

$$m' = \rho n' = n'-q+(p-2)/2$$  \hspace{1cm} (4.40)

and

$$\rho = m'/n' = 1-(2q-p+2)/2n'$$  \hspace{1cm} (4.41)

Obviously $\rho$ remains the same for all $r=2,3,\ldots,m$.

Further, introducing the assumption (iii) of Sec. 3; we assume

$$\eta_r^{(r)} = n'-\bar{\delta}^{(r)}$$

where $\bar{\delta}^{(r)}$ is finite constant. Then the non-centrality parameter $\lambda_r$ given by (4.35) becomes

$$\lambda_r = (1+n'_{\bar{\delta}}^{(r)}S^{(r)}_{\bar{\delta}} - 1)^{-1}$$  \hspace{1cm} (4.42)

where $\Omega_r = \text{tr} \frac{\bar{\delta}^{(r)'}\bar{\delta}^{(r)}}{\bar{\delta}^{(r)'}\bar{\delta}^{(r)}}(r)/r$. Under this set up, the $h$-th moment of the conditional distribution of $\Lambda^{(r)}$, given by (4.37), is

$$E(\Lambda^{(r)}h|\bar{\delta}^{(r)}) = \frac{B\left[p, \frac{n'-q}{2}\right]}{B\left[p, \frac{n'-q}{2}\right]} \text{F}_1\left(h, \frac{n'-q+p}{2} + h, -\frac{\lambda_r}{2}\right)$$  \hspace{1cm} (4.43)

Then the $h$-th moment of the unconditional distribution of $\Lambda^{(r)}$ can be obtained by taking expectation over $\bar{\delta}^{(r)}$'s involved in $\lambda_r$. Now using the identity

$$\text{F}_1(h, (n'-q+p+2h)/2, -\lambda_r/2) = \sum_{s=0}^{\infty} \frac{(h)_{s}(-1/2)^{s}}{(s!)}$$

and taking expectation over $\bar{\delta}^{(r)}$'s, (where the expression $n'_{\bar{\delta}}^{(r)}S^{(r)}_{\bar{\delta}} - 1^{(r)}\lambda_r$) is
distributed as a Hotelling's $T^2$, we have

$$
E_{\phi} \sum_{s=0}^{\infty} \frac{(h)^s}{((n'-q+p+2h)/2)^s} \frac{((n'-q+p)/2)^s}{(n'^{1/2})^s} \frac{(-z^2 \Delta_r)^s}{s!}
$$

$$= \sum_{s=0}^{\infty} \frac{H_2(h, (n'-q+p)/2, (n'-q+p+2h)/2, n'/2, -z^2 \Delta_r)}{s!}
$$

$$= \sum_{s=0}^{\infty} \frac{2F_2(h, (n'-q+p)/2, (n'-q+p+2h)/2, n'/2, -z^2 \Delta_r)}{s!}
$$

$$= \sum_{s=0}^{\infty} \frac{2F_2(h, (n'-q+p)/2, (n'-q+p+2h)/2, n'/2, -z^2 \Delta_r)}{s!}
$$

(4.44)

Hence, inserting (4.44) in (4.43) and writing $\frac{1}{2} \Delta_r = \Omega_r$, the $h$-th moment of the unconditional distribution of $\Lambda(r)$ is

$$E(\Lambda(r)^h) = \sum_{s=0}^{\infty} \frac{2F_2(h, (n'-q+p)/2, (n'-q+p+2h)/2, n'/2, -\Omega_r)}{s!}
$$

(4.45)

Now let $C(t)$ be the characteristic function of $-2p \log \Lambda$. Since $\Lambda^{(2)}, \ldots, \Lambda^{(m)}$ are independently distributed, we have

$$C(t) = \prod_{r=2}^{m} C_r(t)
$$

(4.46)

where $C_r(t)$ is the c.f. of $-2p \log \Lambda^{(r)}$. Let us, therefore, derive the asymptotic expansion of $C_r(t)$. From (4.45) we have (following Sugiura and Fujikoshi [20], 946)

$$C_r(t) = \frac{\Gamma(\frac{1}{2}m' + (1-2it) - \frac{p-2}{2}) \Gamma(\frac{1}{2}m' + \frac{p+2}{4})}{\Gamma(\frac{1}{2}m' + (1-2it) + \frac{2p-4}{2}) \Gamma(\frac{1}{2}m' + \frac{p+2}{4})}
$$

$$\cdot \sum_{s=0}^{\infty} \frac{2F_2((-1)^s m', \frac{1}{2}m', \frac{1}{2}m' + \frac{p+2}{4}, \frac{1}{2}m' + \frac{p+2}{4}, -\Omega_r)}{s!}
$$

$$= C_r^{(1)}(t) \cdot C_r^{(2)}(t)
$$

(4.47)

Now the asymptotic expansion of $C_r^{(1)}(t)$, which is the c.f. of $-2p \log \Omega^{(r)}$ when $H_0^{(r)}$ is true, is well-known (Box [5], Anderson [1]) and for large $m'$,
it is given by

\[ C_r^{(1)}(t) = (1-2it)^{-p/2} [1 + O(m^{-2})] \]  

(4.48)

Consider the asymptotic expansion of \( C_r^{(2)}(t) \), where from (4.47)

\[
C_r^{(2)}(t) = \sum_{s=0}^{\infty} \frac{(-itm')^s}{(\frac{p+2}{4})_s} \frac{(\frac{2q-p+2}{4})_s}{s!} \frac{(\frac{q}{4})_s}{s!} \frac{(-\Omega)^s}{s!}
\]

(4.49)

Then following the notations of Sugiura and Fujikoshi in case \( p=1 \), where

\[ a_1(s) = s(s-1), \quad a_2(s) = s(4s^2-6s+3) \]

\[ e^{x\Omega} = \sum_{s=0}^{\infty} \frac{x^s\Omega^s}{s!}, \quad \sum_{s=1}^{\infty} \frac{\Omega^s}{s(s-1)!} = \Omega e^\Omega, \quad \sum_{s=2}^{\infty} \frac{\Omega^s}{s(s-1)!} = \Omega^2 e^\Omega \]

(4.50)

we have

\[
(-itm')_s = (-itm')^s [1 - a_1(s)(2itm')^{-1} + O(m'^{-2})]
\]

\[
(\frac{q}{4})(1-2it)_s = (\frac{q}{4})(1-2it)^s [1 - ((p+2)s + 2a_1(s))(2m'(1-2it))^{-1} + O(m'^{-2})]
\]

\[
(\frac{q}{4})(1-2it)_s = (\frac{q}{4})(1-2it)^s [1 - ((p+2)s + 2a_1(s))(2m')^{-1} + O(m'^{-2})]
\]

\[
(\frac{q}{4})(1-2it)_s = (\frac{q}{4})(1-2it)^s [1 - ((2q-p+2)s + 2a_1(s))(2m')^{-1} + O(m'^{-2})]
\]

Therefore,

\[
\frac{(-itm')_s}{(\frac{q}{4})(1-2it)_s} = \frac{(-2it)_s}{(1-2it)_s} \left[ 1 - \frac{1}{m'} \left( (p+2)s(2(1-2it))^{-1} + a_1(s)(2it(1-2it))^{-1} + O(m'^{-2}) \right) \right]
\]

(4.51)
\[
\frac{(2m' + p+2)}{4} = \left[ 1 + \frac{1}{2m'} ((p+2)s + a_1(s)) + O(m'^{-2}) \right]
\]
\[
\cdot \left[ 1 - \frac{1}{2m'} ((2q-p+2)s + a_1(s)) + O(m'^{-2}) \right]
\]
\[
= 1 - \frac{1}{2m'} (2(q-p)s) + O(m'^{-2}) \]  \tag{4.52}

Hence substituting (4.51) and (4.52) in (4.49), asymptotic expansion of \( C_r^{(2)}(t) \) is

\[
C_r^{(2)}(t) = \sum_{s=0}^{\infty} \frac{(2it)^s (1-2it)^{-s} \Omega_r^s}{s!}
\]

\[
\cdot \left[ 1 - \frac{1}{m'} ((p+2)s(2(1-2it))^{-1}a_1(s)(2it(1-2it))^{-1} + (q-p)s + O(m'^{-2}) \right]
\]

\[
= e^{2it(1-2it)^{-1} \Omega_r} \left[ 1 - \frac{1}{m'} ((p+2)it(1-2it)^{-2} \Omega_r + 2it(1-2it)^{-3} \Omega_r^2 \\
+ (q-p)it(1-2it)^{-1} \Omega_r \right] + O(m'^{-2}) \]  \tag{4.53}

(using results of (4.50)).

Hence from (4.48) and (4.53), the asymptotic expansion of \( C_r(t) \), given by (4.47), is

\[
C_r(t) = (1-2it)^{-p/2} e^{2it(1-2it)^{-1} \Omega_r}
\]

\[
\cdot \left[ 1 + \frac{1}{2m'} ((p+2)(1-2it)^{-1} \Omega_r - (1-2it)^{-2} ((p+2) \Omega_r - 2 \Omega_r^2) \\
- 2(1-2it)^{-3} \Omega_r^2 - 2(q-p)((1-2it)^{-1} - 1) \Omega_r \right] + O(m'^{-2}). \]  \tag{4.54}

Thus the asymptotic expansion of \( C(t) \) defined in (4.46) is obtained from (4.54) as
\begin{align*}
C(t) &= (1-2it)^{-(m-1)p/2} e^{2it(1-2it)^{-1} \Omega} \\
&\quad \cdot \prod_{r=2}^{m} \left[ 1 + \frac{1}{2m^r} \{(p+2)(1-2it)^{-1} \Omega - (1-2it)^{-2}((p+2)\Omega - 2\Omega^2) \right. \\
&\quad - 2(1-2it)^{-3} \Omega^2 - 2(q-p)((1-2it)^{-1} - 1)\Omega_r + o(m^{-2}) \} \\
&= (1-2it)^{-(m-1)p/2} e^{2it(1-2it)^{-1} \Omega} \\
&\quad \cdot \left[ 1 + \frac{1}{2m^r} \{(p+2)(1-2it)^{-1} \Omega^{(1)} - (1-2it)^{-2}((p+2)\Omega^{(1)} - 2\Omega^{(2)}) \right. \\
&\quad - 2(1-2it)^{-3} \Omega^{(2)} - 2(q-p)((1-2it)^{-1} - 1)\Omega^{(1)} + o(m^{-2}) \}
\end{align*}

where $\Omega^{(k)} = \sum_{r=2}^{m} \Omega^{k}_r$, $k=1,2$.

Thus by inverting the characteristic function using the fact that

\((1-2it)^{-f/2} e^{2it\delta^2/(1-2it)}\) is the c.f. of the non-central chi-square distribution with \(f\) d.f. and n.c.p. \(\delta^2\), we obtain the following

**THEOREM 3.** The asymptotic (unconditional) non-central distribution of \(\Lambda\) (defined by (4.33)) is given by

\[ P[-2 \rho \log < z] = P[\chi^2_{(m-1)p}(\Omega^{(1)} < z) \\
+ \frac{1}{2m^r} \{(p+2)\Omega^{(1)}(P[\chi^2_{(m-1)p+2}(\Omega^{(1)} < z) - ((p+2)\Omega^{(1)} - 2\Omega^{(2)}) \right. \\
- 2(q-p)\Omega^{(1)}(P[\chi^2_{(m-1)p+4}(\Omega^{(1)} < z) - 2\Omega^{(2)}P[\chi^2_{(m-1)p+6}(\Omega^{(1)} < z) \\
- 2(q-p)\Omega^{(1)}(P[\chi^2_{(m-1)p+2}(\Omega^{(1)} < z) - P[\chi^2_{(m-1)p}(\Omega^{(1)} < z)] \\
+ o(m^{-2}) \right) \right. \right. \]  \tag{4.55}

where $\Omega^{(1)} = \sum_{r=2}^{m} \Omega^{1}_r = \frac{1}{2} tr \sum_{r=2}^{m} \delta^{(r)} \delta^{(r)} \sum_{r=2}^{m} \Omega^{1}_r$.
It should be noted that in two samples problem, Anderson's procedure (discussed in Sec. 4.1.1) leads to a better test than that of Bhargava, since in the later case the variable $v^{(2)}_{\alpha}$ defined in (4.24) becomes Bartlett's variable. For more than two samples problem, we have arrived at a solution in Bhargava's case, under the restriction that the common matrix of regression coefficients have been estimated for each $r=2,3,\ldots,m$ separately.

The efficiency of Bhargava's test can be judged with that of Anderson's test procedure if the solution of the later test is available in general.

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REFERENCES


