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This work was completed while the author was at the University of North Carolina at Chapel Hill, Department of Statistics.

# ASYMPTOTIC EXPANSIONS OF THE DISTRIBUTIONS OF $|S_1S_2^{-1}|$ and $|S_2(S_1+S_2)^{-1}|$ FOR TESTING THE EQUALITY OF TWO COVARIANCE MATRICES

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1. INTRODUCTION. Let $p\times 1$ vectors $X_{i1}, X_{i2}, \ldots, X_{iN_i}$ be a random sample from p-variate normal distribution with mean $\mu_i$ and covariance matrix $\Sigma_i$ for $i = 1,2$. Put $S_i = \Sigma_{\alpha=1}^{N_i}(X_{i\alpha}-X_i)(X_{i\alpha}-X_i)'$ and $X_i = N^{-1}\Sigma_{\alpha=1}^{N_i} X_{i\alpha}$. Then the statistic $|S_1S_2^{-1}|$ or $|S_2(S_1+S_2)^{-1}|$ may be used to test the null hypothesis $H:\Sigma_1=\Sigma_2$ for unknown $\mu_i$ against one-sided alternatives $K: \gamma_i \geqq 1$ and $\Sigma_{i=1}^{p} \gamma_i > p$, where $\gamma_i$ means the i-th largest characteristic root of $\Sigma_1\Sigma_2^{-1}$. We can reject $H$, when the observed value of $|S_1S_2^{-1}|$ (or $|S_2(S_1+S_2)^{-1}|^{-1}$) is larger than a preassigned constant. The monotonicity of the power functions of these tests with respect to $\gamma_j$ among other tests, were proved by Anderson and Das Gupta [2] from a more general point of view.

Putting $n_i = N_i-1$ and $n_1+n_2 = n$, we shall derive, in this paper, asymptotic expansion of the non-null distribution of $\lambda_1 = \sqrt{n}\log|(n_2/n_1)S_1S_2^{-1}|$, as well as of the null distribution, according to Box [3] and then asymptotic expansions of the distributions of $\lambda_2 = -\sqrt{n}\log|(n/n_2)S_2(S_1+S_2)^{-1}|$ both under the null hypothesis and local alternatives, based on the hypergeometric function of matrix argument due to Constantine [4], James [7] and used by Fujikoshi [5], Sugiura and Fujikoshi [13], Sugiura [12]. The limiting distribution of $\lambda_2$ under

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$K$ is also derived. The asymptotic expansions are given in terms of the normal distribution function and its derivatives, which are different from the expansions of the distributions of the likelihood ratio criteria for similar problems in Sugiura [11], [12]. In fact, the null distributions of the likelihood ratio criteria were expanded in terms of the $\chi^2$ distributions with different degrees of freedom. Asymptotic formulas for the upper percentage points of $\lambda_1$ and $\lambda_2$ will also be given by applying the general inverse expansion formula of Hill and Davis [6], by which some numerical values of the power of these tests are computed and compared with the exact value obtained by Pillai and Jayachandran [9] for the test $\lambda_2$ with $p=2$.

In the following discussion, we shall always assume without loss of generality that, under $K$, $S_1$ has the Wishart distribution $W_p(n_1,\Gamma)$ and $S_2$ has $W_p(n_2,I)$, where $\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_p)$. Then the null hypothesis $H$ can be expressed by $\Gamma=I$. Put $n_i = \rho_i n$ with $\rho_1+\rho_2 = 1$ and let $n$ tend to infinity for fixed $\rho_i > 0$.

## 2. Asymptotic Distributions of $|S_1S_2^{-1}|$.

It is easy to see that the characteristic function of $\lambda_1 = \sqrt{n}\log|(\rho_2/\rho_1)S_1S_2^{-1}|$ under $K$, is given by

(2.1)

$$(\rho_2/\rho_1)^{\sqrt{n}pit}\,|\Gamma|^{\sqrt{n}it}\,\Gamma_p(\tfrac{1}{2}n_1+\sqrt{n}it)\Gamma_p(\tfrac{1}{2}n_2-\sqrt{n}it)/[\Gamma_p(\tfrac{1}{2}n_1)\Gamma_p(\tfrac{1}{2}n_2)]\ ,$$

where $\Gamma_p(x) = \pi^{p(p-1)/4}\,\Pi_{j=1}^{p}\,\Gamma(x-(j-1)/2)$. Applying the asymptotic formula of $\log\Gamma(x+h)$ for large $|x|$ with fixed $h$, to the above expression as in Box [3], Anderson ([1], p.204), Sugiura [12] and etc., we can rewrite (2.1) as

(2.2)

$$|\Gamma|^{\sqrt{n}it} \exp\Big[-(\rho_1^{-1} + \rho_2^{-1})pt^2 + \sum_{r=1}^{\infty} (2it)^r\, p\, n^{-r/2}$$

$$\Big\{2(it)^2(r+1)^{-1}(r+2)^{-1}\,[(-1)^r\, \rho_1^{-r-1} + \rho_2^{-r-1}]$$

$$+ (p+1)[(-1)^r\, \rho_1^{-r} + \rho_2^{-r}]/(4r)\Big\}$$

$$- \sum_{r=1}^{\infty} (-2/n)^r\, B_{r+1}\{r(r+1)\}^{-1}\,\{(\rho_1 + 2itn^{-\frac{1}{2}})^{-r}$$

$$+ (\rho_2 - 2itn^{-\frac{1}{2}})^{-r} - \rho_1^{-r} - \rho_2^{-r}\}\Big] ,$$

where $B_{r+1} = \Sigma_{j=1}^{p} B_{r+1}(-(j-1)/2)$ with the Bernoulli polynomial $B_r(h)$ of degree $r$. Specifying the sequence of alternatives $K$ as $K_\delta$: $\Gamma = I + n^{-\delta}\Theta$ $(\delta \geqq 0)$ for a fixed diagonal matrix $\Theta$ having non-negative diagonal elements and considering the first term of (2.2), we easily get

**THEOREM 2.1.** Under $K$, the statistics $\lambda_1$, $\lambda_1 - \mathrm{tr}\Theta$ and $\lambda_1 - \sqrt{n}\log|\Gamma|$, have asymptotically the same normal distribution with mean zero and variance $\tau^2 = 2p(\rho_1^{-1} + \rho_2^{-1})$, according as $\delta > \frac{1}{2}$, $\delta = \frac{1}{2}$ and $\delta < \frac{1}{2}$ respectively.

There is no singularity of the limiting distribution at the null hypothesis as was the case with the likelihood ratio tests for covariance matrix in Sugiura [11], [12]. We can write the first four terms of (2.2) as

$$(2.3)\qquad \exp\left[-\tfrac{1}{2}\tau^2 t^2 + it\sqrt{n}\log|\Gamma|\right]\cdot\left[1 + n^{-\frac{1}{2}}\{a_3(it)^3 + a_1(it)\} + n^{-1}\sum_{j=1}^{3} b_{2j}(it)^{2j} + n^{-3/2}\sum_{j=1}^{5} c_{2j-1}(it)^{2j-1}\right],$$

where $\tau^2 = 2p(\rho_1^{-1} + \rho_2^{-1})$ and

$$(2.4)\qquad \begin{aligned}
a_1 &= \tfrac{1}{2}p(p+1)(-\rho_1^{-1} + \rho_2^{-1}), \qquad a_3 = \tfrac{2}{3}p(-\rho_1^{-2} + \rho_2^{-2}),\\
b_2 &= \tfrac{1}{2}a_1^2 + \tfrac{1}{2}p(p+1)(\rho_1^{-2} + \rho_2^{-2}), \qquad b_4 = a_1a_3 + \tfrac{2}{3}p(\rho_1^{-3} + \rho_2^{-3}),\\
b_6 &= \tfrac{1}{2}a_3^2,\\
c_1 &= \tfrac{1}{12}p(2p^2 + 3p - 1)(-\rho_1^{-2} + \rho_2^{-2}),\\
c_3 &= p(p+1)\{\tfrac{2}{3}(-\rho_1^{-3} + \rho_2^{-3}) + \tfrac{1}{2}(\rho_1^{-2} + \rho_2^{-2})a_1\} + \tfrac{1}{6}a_1^3,\\
c_5 &= \tfrac{4}{5}p(-\rho_1^{-4}+\rho_2^{-4}) + \tfrac{1}{2}p(p+1)(\rho_1^{-2}+\rho_2^{-2})a_3 + \tfrac{2}{3}p(\rho_1^{-3}+\rho_2^{-3})a_1 + \tfrac{1}{2}a_3a_1^2,\\
c_7 &= \tfrac{1}{2}a_1a_3^2 + \tfrac{2}{3}p(\rho_1^{-3} + \rho_2^{-3})a_3, \qquad c_9 = \tfrac{1}{6}a_3^3.
\end{aligned}$$

Inverting the characteristic function (2.3) yields

**THEOREM 2.2.** Put $\lambda_1 = \sqrt{n}\log|(n_2/n_1)S_1S_2^{-1}|$ and $\tau = \{2p(\rho_1^{-1}+\rho_2^{-1})\}^{\frac{1}{2}}$. Then under alternatives $K$, we have

$$(2.5)\qquad P(\{\lambda_1 - \sqrt{n}\log|\Gamma|\}/\tau < z) = \Phi(z) - \frac{1}{\sqrt{n}}\{a_3\tau^{-3}\Phi^{(3)}(z) + a_1\tau^{-1}\Phi^{(1)}(z)\} + \frac{1}{n}\sum_{j=1}^{3} b_{2j}\tau^{-2j}\Phi^{(2j)}(z) - \frac{1}{n\sqrt{n}}\sum_{j=1}^{5} c_{2j-1}\tau^{-2j+1}\Phi^{(2j-1)}(z) + O(n^{-2}),$$

where the coefficients $a_j$, $b_j$, $c_j$, are given by (2.4) and $\Phi^{(j)}(z)$ means the j-th derivative of the standard normal distribution function $\Phi(z)$. If the null hypothesis H is true, the same formula (2.5) holds by simply putting $\log|\Gamma| = 0$.

Applying the general inverse expansion formula of Hill and Davis [6] to (2.5) with $\log|\Gamma| = 0$, we get the following asymptotic formula for the upper $\alpha\%$ point of $\lambda_1/\tau$, in terms of the upper $\alpha\%$ point u of the standard normal distribution function.

(2.6)

$$u + \frac{1}{\sqrt{n}}\{\tilde{a}_3 H_3 + \tilde{a}_1\} - \frac{1}{n}\left[\sum_{j=1}^{3} \tilde{b}_{2j} H_{2j} + (\tilde{a}_3 H_3 + \tilde{a}_1)u\{\tilde{a}_3(\tfrac{1}{2}H_3 - 2) + \tfrac{1}{2}\tilde{a}_1\}\right]$$

$$+ \frac{1}{n\sqrt{n}}\left[\sum_{j=1}^{5} \tilde{c}_{2j-1} H_{2j-1} + \sum_{j=1}^{3} \tilde{b}_{2j} H_{2j} u\{\tilde{a}_3(H_3 - 2) + \tilde{a}_1\}\right.$$

$$+ \frac{1}{3}(u^2 - 1)(\tilde{a}_3 H_3 + \tilde{a}_1)^3 - (3u^2 - 1)\tilde{a}_3(\tilde{a}_3 H_3 + \tilde{a}_1)^2$$

$$\left. + (\tilde{a}_3 H_3 + \tilde{a}_1)(4\tilde{a}_3^2 u^2 - \sum_{j=1}^{3} \tilde{b}_{2j} H'_{2j})\right] + O(n^{-2}),$$

where $\tilde{a}_j = a_j \tau^{-j}$, $\tilde{b}_j = b_j \tau^{-j}$, $\tilde{c}_j = c_j \tau^{-j}$ and $H'_j$ means the derivative of Hermite polynomial $H_j = H_j(u)$ defined by $\Phi^{(j)}(u) = H_j(u)\Phi^{(1)}(u)$. For $j = 1, 2, \ldots, 9$

(2.7)

$$H_1 = 1, \quad H_2 = -u, \quad H_3 = u^2 - 1, \quad H_4 = -u^3 + 3u,$$

$$H_5 = u^4 - 6u^2 + 3, \quad H_6 = -u^5 + 10u^3 - 15u,$$

$$H_7 = u^6 - 15u^4 + 45u^2 - 15, \quad H_8 = -u^7 + 21u^5 - 105u^3 + 105u,$$

$$H_9 = u^8 - 28u^6 + 210u^4 - 420u^2 + 105.$$

3. **ASYMPTOTIC DISTRIBUTION OF** $|S_2(S_1+S_2)^{-1}|$. Unlike the previous section, we cannot get a simple expression for the characteristic function of $\lambda_2 = -\sqrt{n}\log|(n/n_2)S_2(S_1+S_2)^{-1}|$ under $K$. However, by applying the lemma in Sugiura [11], which is a direct extension of Siotani and Hayakawa [10], to the case of several Wishart matrices, to $f(S_1,S_2) = \log|S_2| - \log|S_1+S_2|$, we get the limiting distribution of $\lambda_2$ under $K$.

**THEOREM 3.1.** Under alternatives $K$, the statistic $\lambda_2 - \sqrt{n}\log|\rho_1\Gamma + \rho_2 I|$ has asymptotically normal distribution with mean zero and variance $2(\rho_1^{-1} + \rho_2^{-1})\mathrm{tr}(I + \rho_2\rho_1^{-1}\Gamma^{-1})^{-2}$.

Now we shall consider the sequence of alternatives $K_\delta$: $\Gamma = I + n^{-\delta}\Theta(\delta \geqq 0)$, under which the characteristic function of $\lambda_2$ can be expressed for large $n$ such that all positive elements of diagonal matrix $n^{-\delta}\Theta$ are less than one, as

$$(3.1)\qquad \rho_2^{\sqrt{n}pit}\,\Gamma_p(\tfrac{1}{2}n_2 - \sqrt{n}it)\Gamma_p(\tfrac{1}{2}n)/\{\Gamma_p(\tfrac{1}{2}n_2)\Gamma_p(\tfrac{1}{2}n - \sqrt{n}it)\}$$
$$\cdot\ {}_2F_1(-\sqrt{n}it,\ \tfrac{1}{2}n_1;\ \tfrac{1}{2}n - \sqrt{n}it;\ I - \Gamma).$$

The above formula (3.1) can be obtained by the similar argument as in Sugiura ([12], Section 5), based on the hypergeometric function of matrix argument due to Constantine [4] and James [7]. The first factor gives the characteristic function under $H$, which can be expanded asymptotically in a similar way as in Section 2, having

(3.2)

$$\exp\Big[-p(\rho_2^{-1} - 1)t^2 + \sum_{r=1}^{\infty} (2it)^r p n^{-r/2} \{2(it)^2(r+1)^{-1}(r+2)^{-1}(\rho_2^{-r-1} - 1)$$

$$+ (p+1)(\rho_2^{-r} - 1)/(4r)\} - \sum_{r=1}^{\infty} (-2/n)^r B_{r+1} r^{-1}(r+1)^{-1}$$

$$\cdot \{1 - (1-2itn^{-\frac{1}{2}})^{-r} + (\rho_2 - 2itn^{-\frac{1}{2}})^{-r} - \rho_2^{-r}\}\Big] ,$$

the first four terms of which is expressed by (2.3) after putting $\log|\Gamma| = 0$ and replacing $\tau$, $a_j$, $b_j$, $c_j$ with $\tau' = \{2p(\rho_2^{-1} - 1)\}^{\frac{1}{2}}$ $a_j'$, $b_j'$, $c_j'$ respectively, given by

(3.3) $$a_1' = \tfrac{1}{2}p(p+1)(\rho_2^{-1} - 1), \qquad a_3' = \tfrac{2}{3}p(\rho_2^{-2} - 1),$$

$$b_2' = \tfrac{1}{2}a_1'^2 + \tfrac{1}{2}p(p+1)(\rho_2^{-2} - 1),$$

$$b_4' = a_1'a_3' + \tfrac{2}{3}p(\rho_2^{-3} - 1), \qquad b_6' = \tfrac{1}{2}a_3'^2 ,$$

$$c_1' = \tfrac{1}{12}p(2p^2 + 3p - 1)(\rho_2^{-2} - 1),$$

$$c_3' = p(p+1)\{\tfrac{2}{3}(\rho_2^{-3} - 1) + \tfrac{1}{2}(\rho_2^{-2} - 1)a_1'\} + \tfrac{1}{6}a_1'^3 ,$$

$$c_5' = \tfrac{4}{5}p(\rho_2^{-4}-1) + \tfrac{2}{3}p(\rho_2^{-3}-1)a_1' + \tfrac{1}{2}p(p+1)(\rho_2^{-2}-1)a_3' + \tfrac{1}{2}a_1'^2a_3' ,$$

$$c_7' = \tfrac{1}{2}a_1'a_3'^2 + \tfrac{2}{3}p(\rho_2^{-3} - 1)a_3' , \quad c_9' = \tfrac{1}{6}a_3'^3 .$$

Inverting the characteristic function, we get

THEOREM 3.2. Under the null hypothesis $H$, the distribution of $\lambda_2 = -\sqrt{n}\,\tau'^{-1} \log|(n/n_2)S_2(S_1+S_2)^{-1}|$ is expanded asymptotically by the formula (2.5) after replacing $\tau$, $a_j$, $b_j$, $c_j$ with $\tau' = \{2p(\rho_2^{-1}-1)\}^{\frac{1}{2}}$ $a_j'$, $b_j'$, $c_j'$ respectively.

Asymptotic formula for the upper α% point of $\lambda_2/\tau'$ is also given by (2.6) after replacing $\tilde{a}_j$, $\tilde{b}_j$, $\tilde{c}_j$ with $\tilde{a}'_j = a'_j/\tau'^j$, $\tilde{b}'_j = b'_j/\tau'^j$, $\tilde{c}'_j = c'_j/\tau'^j$ respectively.

When $\delta = \frac{1}{2}$, the second factor of (3.1) is written by

$$(3.4) \qquad {}_2F_1 = \sum_{k=0}^{\infty}\sum_{(\kappa)} (-\sqrt{n}it)_\kappa(\tfrac{1}{2}n_1)_\kappa C_\kappa(-n^{-\frac{1}{2}}\Theta)/\{(\tfrac{1}{2}n - \sqrt{n}it)_\kappa k!\} .$$

By Fujikoshi [5] and Sugiura [12], we know already that

(3.5)

$$(-\sqrt{n}it) = (-\sqrt{n}it)^k\left[1-\frac{a_1(\kappa)}{2it\sqrt{n}}+\frac{1}{24(it)^2n}\{3a_1(\kappa)^2-a_2(\kappa)+k\}+O(n^{-3/2})\right],$$

$$(\tfrac{1}{2}n_1)_\kappa = (\tfrac{1}{2}n\rho_1)^k\left[1+\frac{1}{n\rho_1}a_1(\kappa)+O(n^{-2})\right],$$

$$(\tfrac{1}{2}n-\sqrt{n}it)_\kappa = (\tfrac{1}{2}n)^k\left[1-\frac{2it}{\sqrt{n}}k+\frac{1}{n}\{a_1(\kappa)+2(it)^2k(k-1)\}+O(n^{-3/2})\right],$$

where $a_1(\kappa) = \Sigma_{\alpha=1}^p k_\alpha(k_\alpha-\alpha)$ and $a_2(\kappa) = \Sigma_{\alpha=1}^p k_\alpha(4k_\alpha^2-6\alpha k_\alpha+3\alpha^2)$. Hence the right-hand side of (3.4) is equal to

$$(3.6) \qquad \sum_{k=0}^{\infty}\sum_{(\kappa)} C_\kappa(\rho_1 it\Theta)(k!)^{-1}\Big[1+\frac{1}{\sqrt{n}}\{2itk-a_1(\kappa)/2it\}$$

$$+\frac{1}{n}\{\tfrac{1}{8}a_1(\kappa)^2(it)^{-2}+(\rho_1^{-1}-1+k)a_1(\kappa)-\tfrac{1}{24}a_2(\kappa)(it)^{-2}$$

$$+k(4(it)^2+\tfrac{1}{24}(it)^{-2})+2(it)^2k(k-1)\}+O(n^{-3/2})\Big].$$

It follows from the lemma by Sugiura and Fujikoshi [13] that

(3.7)

$${}_2F_1 = \{\mathrm{etr}(\rho_1 it\Theta)\}\Big[1+\frac{1}{\sqrt{n}}\{2\rho_1(it)^2\mathrm{tr}\Theta-\frac{1}{2}\rho_1^2(it)\mathrm{tr}\Theta^2\}$$

$$+ \frac{1}{n}\{2\rho_1^2(it)^4(tr\Theta)^2 + (it)^3(4\rho_1 tr\Theta - \rho_1^3 tr\Theta tr\Theta^2)$$

$$+ (it)^2(\rho_1[1-3\rho_1]tr\Theta^2 + \frac{1}{8}\rho_1^4[tr\Theta^2]^2) + \frac{1}{3}it\rho_1^3 tr\Theta^3\} + O(n^{-3/2})\Big] .$$

Multiplying (3.7) with the first factor of (3.1) given by (2.3) with $a_j'$, $b_j'$, $c_j'$, $\tau'$ instead of $a_j$, $b_j$, $c_j$, $\tau$, we finally get the asymptotic formula of the characteristic function of $\lambda_2$ under $K_\delta(\delta = \frac{1}{2})$ as

$$\text{(3.8)} \qquad \exp[-\tfrac{1}{2}\tau'^2 t^2 + \rho_1 it\Theta]\cdot[1 + \frac{1}{\sqrt{n}}\{a_3'(it)^3 + 2\rho_1(it)^2 tr\Theta$$

$$+ it(a_1' - \tfrac{1}{2}\rho_1^2 tr\Theta^2)\} + \frac{1}{n}\sum_{j=1}^{6} g_j(it)^j + O(n^{-3/2})] ,$$

where

$$\text{(3.9)} \qquad g_1 = -\frac{1}{3}\rho_1^3 tr\Theta^3 ,$$

$$g_2 = b_2' + \rho_1(1 - 3\rho_1 - \tfrac{1}{2}a_1'\rho_1)tr\Theta^2 + \frac{1}{8}\rho_1^4(tr\Theta^2)^2 ,$$

$$g_3 = \rho_1 tr\Theta(\rho_1^2 tr\Theta^2 - 2a_1' - 4) ,$$

$$g_4 = 2\rho_1^2(tr\Theta)^2 - \tfrac{1}{2}a_3'\rho_1^2 tr\Theta^2 + b_4' ,$$

$$g_5 = 2a_3'\rho_1 tr\Theta ,$$

$$g_6 = b_6'$$

and $a_j'$, $b_j'$ are given by (3.3). Hence we can get

THEOREM 3.3. Under the sequence of alternatives $K_\delta(\delta = \frac{1}{2})$: $\Gamma = I + n^{-\frac{1}{2}}\Theta$, the following asymptotic expansion of the distribution of $\lambda_2 = -\sqrt{n}\log|(n/n_2)S_2(S_1+S_2)^{-1}|$ holds, for large $n$ such that all positive elements of the diagonal matrix $n^{-\frac{1}{2}}\Theta$ are less than one.

$$(3.10) \qquad P(\{\lambda_2 - \rho_1 \mathrm{tr}\Theta\}/\tau' < z) = \Phi(z) + \frac{1}{\sqrt{n}}\{-a_3'\tau'^{-3}\Phi^{(3)}(z)$$

$$+ 2\rho_1 \mathrm{tr}\Theta\tau'^{-2}\Phi^{(2)}(z) + (\tfrac{1}{2}\rho_1^2 \mathrm{tr}\Theta^2 - a_1')\tau'^{-1}\Phi^{(1)}(z)\}$$

$$+ \frac{1}{n}\sum_{j=1}^{6} (-1/\tau')^j g_j \Phi^{(j)}(z) + O(n^{-3/2}),$$

where $\tau' = \{2p(\rho_2^{-1} - 1)\}^{\frac{1}{2}}$

From (3.7) and (3.5), it is easy to see that the limiting distribution of $\lambda_2$ under $K_\delta(\delta > \frac{1}{2})$ is normal with mean zero and variance $\tau'^2$. Hence the limiting distributions of $\lambda_1/\tau$ and $\lambda_2/\tau'$ under $k_\delta$ $(\delta > \frac{1}{2})$ are the same. Even when $\delta = \frac{1}{2}$, they are the same, because their asymptotic means given Theorem 2.1 and Theorem 3.3 are equal. The power of the test $\lambda_1$ will be almost the same as that of $\lambda_2$, when alternative $K$ is near to the null hypothesis. This is verified numerically in the following special cases.

## 4. NUMERICAL EXAMPLES.

We shall consider two cases, namely, Case 1: $p=2$ and $n_1=13$, $n_2=63$, Case 2: $p=5$ and $n_1=50$, $n_2=150$. Asymptotic formula (2.6) for the upper percentage points gives

| | 5% points of $\lambda_1/\tau$. | | 5% points of $\lambda_2/\tau'$. | |
|---|---|---|---|---|
| | Case 1 | Case 2 | Case 1 | Case 2 |
| first term | 1.6449 | 1.6449 | 1.6449 | 1.6449 |
| term of $O(n^{-\frac{1}{2}})$ | -0.3576 | -0.4020 | 0.2365 | 0.2450 |
| term of $O(n^{-1})$ | 0.0791 | 0.0409 | 0.0206 | 0.0272 |
| term of $O(n^{-3/2})$ | -0.0262 | -0.0195 | 0.0044 | 0.0051 |
| approx. value | 1.340 | 1.264 | 1.906 | 1.922 |

All these examples show that the powers of the $\lambda_1$-test and the $\lambda_2$-test are almost the same when alternative is near to the null hypothesis.

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