A NOTE ON THE P-ARY REPRESENTATION OF INTEGERS

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This article gives a new form of the p-ary representation of any non-negative integer for prime p, which is a generalization of Ikeda's formula for p = 2 [1].

For any non-negative integer N, the usual formula of the binary representation is given by

\[ N = \sum_{k=0}^{s} c_k \cdot 2^k \]

where \( c_k = \left[ \frac{N}{2^k} \right] - 2 \left[ \frac{N}{2^{k+1}} \right] \), \( k=0, 1, \ldots, s \); \( s = \max \{ h; 2^h \leq N \} \).

On the other hand, Ikeda [1] showed that N is expanded in the form

\[ N = \sum_{k=0}^{s} \binom{N}{2^k} \cdot 2^k \]

where, for any positive integer R, \( R_2 = 0 \) if R is even, and = 1 if R is odd.

This implies, by the uniqueness of the binary representation, that

\[ \left( \frac{N}{2^k} \right) = \binom{N}{2^k} \quad (\text{mod. } 2). \]
The purpose of this article is to prove the following identity.

(2) \[ \left[ \frac{N}{p^k} \right] - p \left[ \frac{N}{p^{k+1}} \right] = \left( \frac{N}{p^k} \right)_p, \]

or, equivalently,

(3) \[ \left[ \frac{N}{p^k} \right] = \left( \frac{N}{p} \right)^k \pmod{p}, \]

where \( N \) and \( k \) are non-negative integers, \( p \) is any given prime, \( [\cdot] \) is the ordinary Gauss symbol, and \( \left( \frac{N}{n} \right)_p \) denotes the remainder when \( \binom{N}{n} \) is divided by \( p \). Here, we use the convention that the number of \( n \)-combinations out of \( N \), \( \binom{N}{n} \), is equal to zero if \( n > N \).

**PROOF OF (2)**

(I) In the case \( 0 \leq N < p^k \), it is evident that

\[ \left[ \frac{N}{p^k} \right] - p \left[ \frac{N}{p^{k+1}} \right] = 0 = \left( \frac{N}{p^k} \right)_p \]

for any non-negative integer \( k \).

(II) In the case \( p^k \leq N < p^{k+1} \), \( N \) is expressed uniquely in the following form

\[ N = jp^k + i, \]

for some integers \( j \) and \( i \); \( 1 \leq j \leq p-1 \) and \( 0 \leq i \leq p-1 \).

Then, for the left hand side of (2) it is easy to see that

\[ \left[ \frac{jp^k + i}{p^k} \right] - p \left[ \frac{jp^k + i}{p^{k+1}} \right] = j \]

for any non-negative \( k \). Thus, putting

\[ I_{j,i}^k = \left( \frac{N}{p^k} \right)_p = \left( \frac{jp^k + i}{p} \right)_p \]

we have to show that the relation

(4) \[ I_{j,i}^k = j \]
holds true for any non-negative integer $k$.

In the first place, we shall prove the relation (4) when $i = 0$, that is,

$$I_{j,0}^k = j,$$

or equivalently,

$$\left( \frac{jp^k}{p^k} \right) = j \quad \text{(mod. } p\text{).}$$

Since

$$\left( \frac{jp^k}{p^k} \right) = j (\frac{jp^k}{p^k} - 1),$$

it suffices to prove that the relation

$$(5) \quad \left( \frac{jp^k}{p^k} - 1 \right) \equiv 1 \quad \text{(mod. } p\text{)}$$

holds true for any non-negative $k$. When $k = 0$, this relation is trivial.

Let us consider the expansion

$$\left( \frac{jp^k}{p^k} - 1 \right) = \prod_{h=1}^{p-1} \frac{jp^k - h}{h}$$

for any positive $k$.

In the above expansion, $h$ is expressed in general as

$$h = p^{m(h)} \cdot \delta(h)$$

where $\delta(h)$ is prime to $p$ and $m(h)$ is an integer such that $0 \leq m(h) \leq k-1$.

If $h$ is prime to $p$, then $m(h) = 0$ and hence $h = \delta(h)$.

Thus, it follows from (6) that

$$(7) \quad \left( \frac{jp^k}{p^k} - 1 \right) = \prod_{h=1}^{p-1} \frac{jp^{k-m(h)} - \delta(h)}{h}$$

Here, we have

$$\prod_{h=1}^{p-1} \left( \frac{jp^{k-m(h)} - \delta(h)}{h} \right) = p \cdot Q(p) + (-1)^{p-1} \prod_{h=1}^{p-1} \delta(h),$$

where $Q(p)$ is a polynomial in $p$. 
Thus, we obtain

\[
\frac{(j^p - 1)}{p^k - 1} = \frac{p \cdot Q(p)}{\prod_{h=1}^{p-1} \delta(h)} + (-1)^{p-1}.
\]

Recalling that \(p^k - 1\) is even when \(p \geq 3\), we thus have,

\[
\frac{(j^p - 1)}{p^k - 1} = \frac{p \cdot Q(p)}{\prod_{h=1}^{p-1} \delta(h)} + 1.
\]

Since \(\prod_{h=1}^{p-1} \delta(h)\) is prime to \(p\), it divides \(Q(p)\), and hence we have (5) for \(p \geq 3\).

When \(p = 2\), it holds that

\[
\frac{(j \cdot 2^k - 1)}{2^k - 1} = 2 \cdot \frac{Q(2)}{\prod_{h=1}^{2^k-1} \delta(h)} - 1 \equiv 1 \pmod{2}.
\]

Thus, the relation (5) is true for any prime \(p\).

In the second place, we shall prove the relations

(8) \[
\frac{1^k}{i,j,l} = \frac{1^k}{i,j,i-1}
\]

for \(i=1,2,\ldots,p^k-1\) and \(j=1,2,\ldots,p-1\), (\(k \geq 1\)).

Putting, as before, \(h = p^{m(h)} \cdot \delta(h)\) for \(h=1,2,\ldots,p^k-1\), we have

\[
\frac{(j^p + 1)}{p^k} - \frac{(j^p + 1 - h)}{p^k} = \prod_{h=1}^{p^k-1} \frac{j^{p-k} + 1 - h}{\delta(h)}
\]

\[
= p^{k-m(i)} \cdot \prod_{h=1, h \neq i}^{p^k-1} \frac{(j^{k-m(h)} + p^{m(i)-m(h)} \cdot \delta(i) - \delta(h))}{\prod_{h=1}^{p^k-1} \delta(h)}
\]
Since the product \( \prod_{h=1}^{p-1} \delta(h) \) is prime to \( p \) and \( 0 \leq m(i) \leq k-1 \), the second factor of the last expression of the above equalities should be a positive integer.

Hence

\[
\left( j p^k + 1 \right) \pmod{p^k} - \left( j p^k + 1 - 1 \right) = 0
\]

which proves (8).

This completes the proof of (4) in the case (II).

(III) In the case \( p^{k+1} \nmid N \), we may put for some \( q, j \) and \( i \),

\[
N = q p^{k+1} + j p^k + i,
\]

where \( q \) is a positive integer, \( j=0,1,...,p-1 \) and \( i=0,1,...,p^{k-1} \).

Let us put

\[
j_{j,i}^k = \left( q p^{k+1} + j p^k + i \right) p^k.
\]

Then, by a similar argument in the case (II), we have

\[
j_{j,0}^k = j_{j,1}^k = \ldots = j_{j,p-1}^k, \quad j=0,1,...,p-1
\]

for any \( k \) positive and

\[
j_{j,0}^k = j, \quad j=0,1,...,p-1
\]

for any non-negative \( k \). This implies that (2) is true in the case (III).

Hence, in all the cases, we proved the relations (2) for any prime number \( p \) and non-negative integer \( k \).

**Reference**