Identification of Hysteretic Control Influence Operators Representing Smart Actuators: Convergent Approximations

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Abstract

In a previous paper, the authors investigated the lowersemicontinuity properties of two generalizations of the classical Preisach operator; the smoothed Preisach operator and the Krasnoselskii / Pokrovskii (KP) integral hysteresis operators. In particular, it was demonstrated that the output least squares identification problem for the KP operator is well-posed over compact subsets of the Preisach plane. The identification of the hysteretic control influence operator was shown to be equivalent to the identification of a measure in the space of probability measures taken with the weak-* topology. In this paper, a consistent and convergent approximation scheme is introduced for this class of integral hysteresis operator. The Galerkin approximation scheme is shown to be function space parameter convergent. A numerical example is presented that illustrates aspects of the theory derived in this paper.

1 Introduction and Motivation

While there has been unprecedented interest in utilizing the class of so-called "smart materials" for control actuation in diverse engineering applications (see for example [6] or Chapter 1 of [11] for surveys), numerous fundamental modeling issues remain open when these actuators are employed in realistic scenarios. Perhaps one of the most significant challenges remaining for those interested in developing refined models for smart actuators is the derivation of accurate, low dimensional representations of hysteresis inherent in these devices. For example, while it is accepted that linear models provide reasonable representations of piezoceramic actuators operated at low electric field strengths, it is desirable to routinely operate this class of devices at high field strengths where they exhibit non-neglible hysteretic effects [11], [13], [7]. Similarly, shape memory alloy (SMA) actuating mechanisms exhibit profound hysteresis in their thermomechanical response [11], [15], [4], [5]. Similar observations have been made by Wereley in [22] for the recently introduced class of magnetorheological fluids (MRF).

General theories for the mathematical representation of hysteresis operators are relatively new. The seminal work of Krasnoselskii and Pokrovskii [14] has only recently been translated to the English language. Significant recent contributions to the general theory of hysteresis operators also includes the volume by Visintin [21] and Mayergoyz [17]. The research detailed below, and in the Part I of this paper [8], has relied on these excellent treatments of hysteresis modeling. Given the widespread interest in smart actuation devices in general, it is inevitable that some research that treats hysteresis in control has appeared in the literature and research. For example, Ge in [7] employs the formulation of Mayergoyz in [17] to implement a simple PID control scheme that considers hysteresis in piezoceramic actuation. Hughes and Wen derive linearizing compensators for both piezoceramic and SMA actuators in [13]. Other significant efforts for modeling hysteresis operators as they appear in classes of partial differential equations are described in [19] and [20]. One particularly interesting paper by Kokotovic and Tao in [18] describes an adaptive control strategy for a class of systems exhibiting a "piecewise linear" hysteresis loop. Andronikou, Bekey and Hadaegh in [1] study the identifiability of systems that incorporate normalized bilinear hysteretic elements.

In a previous paper [8], we investigated the (lower) semicontinuity properties of a class of output least squares identification problems associated with a class of distributed parameter, second order evolution equations with hysteretic input operators. In particular, we showed that when the input hysteresis operator is defined in terms of the Krasnoselskii-Pokrovskii (KP) kernel, and we consider measures that define the operators to be endowed with the the weak* topology, the identification problem is well-posed. In this paper, we derive consistent and convergent approximation methods for the measures that characterize the KP integral hysteresis operators discussed in a previous paper [8].

2 The Governing Dynamical Equations

In this section we present the parameter-dependent, weak form of the equations governing the vibration of a beam subject to a control input operator that can be represented as a hysteretic input operator. For a physical motivation, we can consider the governing equations to be a idealized version of the governing equations in [4] or [8] in which the lateral deflection of a beam is controlled by a shape memory alloy wire. In actual fact, the equations herein are simpler in form in that they do not account for the geometric nonlinearity associated with the "follower force" exerted by the SMA wire when the beam undergoes large deflections. This model only accounts for nonlinearity due to hystereis in the actuating wire. We do, however, generalize the class of systems considered in this paper in comparison to those of [8]. In particular, the dynamical model in this paper includes a wider class of damping operators than that treated in [8].

Following the framework in [11], [10], we let V, V_2 and H be real Hilbert spaces such that the imbeddings

$$V \hookrightarrow V_2 \hookrightarrow H \approx H^* \hookrightarrow V_2^* \hookrightarrow V^*$$

are dense and continuous. We identify H with H^* via the Riesz mapping to induce a pivot space formulation. As is customary, we denote duality products by $\langle \cdot, \cdot \rangle_{V^* \times V}$ and $\langle \cdot, \cdot \rangle_{V_2^* \times V_2}$ respectively. The intermediate space V_2 is selected so as to accomodate a wide variety of damping operators, as discussed in detail in [11], [10]. The governing equations are assumed to be of the form

$$\ddot{w}(t) \quad + \quad A_2(q)\dot{w}(t) + A_1(q)w(t) = B_\mu(u, f) \qquad \text{in } V^* \tag{1}$$

where the initial conditions are given by

$$w(0) = w_0, \qquad \dot{w}(0) = w_1, \tag{2}$$

and $B_\mu(u, f)$ is the generalized control input operator developed in part (I) (i.e., [8]) of this paper and summarized in §2 below. This equation depends parametrically on the unknowns $(q, \mu) \in \mathbb{Q} \times \mathbb{P}$ where $\mathbb{Q}$ and $\mathbb{P}$ are compact metric spaces (to be discussed more fully below) with metrics d and ρ, respectively. As is conventional, the operators $A_2(q)$ and $A_1(q)$ are induced via the bilinear forms $a_2(q)(\cdot,\cdot)$ and $a_1(q)(\cdot,\cdot)$ defined on $V_2 \times V_2$ and $V \times V$, respectively. That is, we have $A_1(q) : V \longrightarrow V^*$ given by

$$\langle A_1(q)w, \xi\rangle_{V^* \times V} \equiv a_1(q)(w, \xi) \qquad \forall w, \xi \in V \tag{3}$$

and $A_2(q) : V_2 \longrightarrow V_2^*$

$$\langle A_2(q)v, \eta\rangle_{V_2^* \times V_2} \equiv a_2(q)(v, \eta) \qquad \forall v, \eta \in V_2. \tag{4}$$

This representation, of course, originates from the continuity conditions usually assumed for a_1, a_2. Specifically, as in [11], [10] and [9], we make the following assumptions on the bilinear form representing the stiffness operator $A_1(q)$

(A1) The bilinear form $a_1(q)(\cdot,\cdot)$ is $V \times V$ symmetric, that is

$$a_1(q)(u, v) = a_1(q)(v, u) \qquad \forall u, v \in V.$$

(A2) The bilinear form $a_1(q)(\cdot,\cdot)$ is parametrically continuous in the sense that

$$|a_1(q_1)(u, v) - a_1(q_2)(u, v)| \leq c_1 d(q_1, q_2) \|u\|_V \|v\|_V$$

for all $u, v \in V$, for all $q_1, q_2 \in \mathbb{Q}$, and for some constant $c_1 \geq 0$ independent of $(u, v, q_1, q_2) \in V \times V \times \mathbb{Q} \times \mathbb{Q}$.

(A3) The bilinear form $a_1(q)(\cdot,\cdot)$ is $\mathbb{Q}$-uniformly strictly V-coercive

$$|a_1(q)(u, v)| \geq k_1 \|u\|_V^2$$

for some constant $k_1 > 0$ independent of $u, q \in V \times \mathbb{Q}$.

In a similar manner, the operator $A_2(q)$, or bilinear form $a_2(q)(\cdot,\cdot)$, is assumed to satisfy the somewhat weaker assumptions:

(A4) The bilinear form $a_2(q)(\cdot,\cdot)$ is parametrically continuous in the sense that

$$|a_2(q_1)(u, v) - a_2(q_2)(u, v)| \leq c_2 d(q_1, q_2) \|u\|_{V_2} \|v\|_{V_2} \qquad \forall u, v \in V_2$$

for some constant $c_2 \geq 0$ independent of $(u, v, q_1, q_2) \in V_2 \times V_2 \times \mathbb{Q} \times \mathbb{Q}$.

(A5) The bilinear form $a_2(q)(\cdot,\cdot)$ satisfies a V_2 coercivity condition ($\mathbb{Q}$-uniformly)

$$a_2(q)(u,v) + \lambda_1 \|u\|_H^2 \geq k_2 \|u\|_{V_2}^2 \qquad \forall u \in V_2$$

for constants $\lambda_1 \in \mathbb{R}$ and $k_2 > 0$ independent of $(u,q) \in V_2 \times \mathbb{Q}$.

With these assumptions, the governing equation (1) can be expressed in equivalent weak or variational form (see [11], [10]),

$$\begin{aligned}\langle \ddot{w}(t), \phi\rangle_{V^* \times V} &+ a_2(q)(\dot{w}(t),\phi) + a_1(q)(w(t),\phi) \\ &= \langle [B_\mu(u,f)](t), \phi\rangle_{V_2^* \times V_2} \qquad \forall \phi \in V\end{aligned} \tag{5}$$

subject to the initial conditions

$$w(0) = w_0 \qquad \ddot{w}(0) = w_1 \tag{6}$$

Within the framework established thus far, the governing systems are well-posed with only minor restrictions on the structure of the hysteretic control influence operator $B_\mu(u,f)$. In fact, for each $q \in \mathbb{Q}$ we have the following result.

Theorem 2.1. *Suppose that $q \in \mathbb{Q}$ is fixed and $t \to B_\mu(u(t),f) \in L_2((0,T),V^*)$ and $w_0 \in V$, $w_1 \in V_2$. If the bilinear forms $a_1(q)(\cdot,\cdot)$ and $a_2(q)(\cdot,\cdot)$ satisfy assumptions (A1) through (A5), there is a unique solution $(w,\dot{w})$ of the system (5), (6) (equivalently (1), (2)) such that $w \in C_2([0,T],V)$, $\dot{w} \in L_2((0,T),V_2) \cap C([0,T];H)$, $\ddot{w} \in L_2((0,T),V^*)$.*

The map $(w_0, w_1, B_\mu(u,f)) \mapsto (w,\dot{w})$ is continuous from $V \times H \times L_2((0,T),V_2^*)$ to $L_2((0,T),V) \times L_2((0,T),V_2)$ for each $q \in \mathbb{Q}$.

The validity of this theorem follows directly from results given in [11], [10]. We note that the well-posedness results of this theorem still hold in the case of light damping (i.e., a non V_2- coercive bilinear form a_2) or even no damping ($a_2 = 0$). See Remark 4.4, p. 104 of [11] for a precise statement.

3 Integral Hysteretic Control Influence Operators

In [8], we introduced two classes of hysteretic control influence operators: (1) the smoothed Preisach operator, and (2) the KP integral operator. Each of these control influence operators have the same general form. They are constructed via integration of a kernel $k_s(u,\xi) : C[0,T] \times \mathcal{I} \longrightarrow C[0,T]$, where $\mathcal{I}$ is the set of admissable initial states for the kernel. Integration is carried out with respect to a measure $\mu \in \mathbb{M}$ defined over the *closed* Preisach plane $\overline{\mathcal{S}} = \{(s_1, s_2) : s_2 \geq s_1\}$

$$[B_\mu(u,f)](t) \equiv [P_\mu(u,f)](t) \cdot g \quad \equiv \quad \int_{\overline{\mathcal{S}}} [k_s(u, f(s))](t) d\mu(s) \cdot g \tag{7}$$

where $f : \bar{S} \rightarrow \mathcal{I}$ is a family of initial states, $g \in V_2^*$ and $\mathbb{M}$ is the class of finite signed Borel measures on $\bar{S}$. We investigated the continuity properties of the integral operators, as well as the lowersemicontinuity properties of the mapping from input to output error. A primary conclusion of the paper was that the identification problem for the control influence operator represented by the KP kernel was well-posed for the class of all probability measures over bounded, closed (i.e., compact) sets $\bar{\mathcal{S}}_\Delta$ Preisach plane. This point proves to be particularly useful in deriving consistent and convergent approximation schemes for the identification problem. Hence, we focus on the KP integral operator in the remainder of this paper.

To summarize the properties of the KP control influence operator, let us first recall that the operator developed in [8] constitutes a special class of Krasnoselskii's and Pokrovskii's notion of generalized play. Our construction is based on a single Lipschitz continuous ridge function $r(x)$ as depicted in Figure 1. By constructing s_1 and s_2 translates of $r(x)$,

$$r_{s_1} = r(x - s_1) \tag{8}$$

$$r_{s_2} = r(x - s_2) \tag{9}$$

for $s_2 > s_2$, we obtain an envelope of admissable paths that are output by the KP kernel. We note that the KP kernel of [8] is defined as a mapping

$$k_s(u, \xi) : C[0,T] \times \mathcal{I} \longrightarrow C[0,T]$$

for each fixed $s \in \overline{\mathcal{S}}$, and the definition of the kernel proceeds in three steps:

(i) Define the action of the kernel on monotone input functions contained in $C[0,T]$,

(ii) Define the action of the kernel on piecewise monotone input functions contained in $C[0,T]$, and

(iii) Extend the definition in (ii) above by density for an arbitrary function in $C[0,T]$.

For any monotone function $u_m(t) \in C[0,T]$ define the monotone output operator according to

$$[\mathcal{M}(u_m,\xi)](t) \equiv \begin{cases} max\,\{\xi, r(u(t)-s_2)\} & \text{if } u_m \text{ is non-decreasing} \\ min\,\{\xi, r(u(t)-s_1)\} & \text{if } u_m \text{ is non-increasing} \end{cases}$$

Now, suppose we are given the piecewise monotone input function $u_{pm} \in C[0,T] \bigcap S_{1,j}[0,T]$ where $S_{1,j}[0,T]$ is the set of piecewise linear splines with j knots in $[0,T]$. The definition in this case is defined recursively on each subinterval. First, we set $\mathcal{M}_0 \equiv \xi$ and subsequently define

$$[k_s(u_{pm})](t) = \begin{cases} [\mathcal{M}(u_{pm}, \mathcal{M}_{k-1})](t) & t \in [t_{k-1}, t_k] \\ \mathcal{M}_k = \mathcal{M}(u_{pm}, \mathcal{M}_{k-1})(t_k) & k = 1 \ldots j. \end{cases} \tag{10}$$

Obviously, for any piecewise linear discretization of the input, equation (10) suffices for computational purposes. The extension of this definition to arbitrary functions $u \in C[0,T]$ now follows standard density arguments as noted in [14] or [21]. With this definition in place, we now recall Theorem 2.2 of [8]. This theorem, whose proof was given in [8], plays an essential role in establishing the well-posedness of the identification procedure and the convergence of the approximation schemes.

Theorem 3.1. *Let k_s be the KP kernel defined in equations (7) through (10).*

(i) For each $u \in C[0,T]$, $\xi \in \mathcal{I}$ and $t \in [0,T]$, the map

$$s \mapsto [k_s(u,\xi)](t)$$

is continuous from $\overline{\mathcal{S}}$ to $\mathbb{R}$.

(ii) If the control influence operator B_μ is defined as in equation (7) with

$$B_\mu(u,f) \equiv P_\mu(u,f) \cdot g$$

where $g \in V_2^*$ and $\mu \in \mathbb{M}$, then

$$B_\mu(u,f) \in L_2((0,T), V_2^*)$$

4 Estimation and Approximation

Given the form of the governing equations discussed in the last section, the actual method for estimating the unknown parameters $(q,\mu) \in \mathbb{Q} \times \mathbb{P}$

depends to a large extent on the form of the measurements that can be made in practice. That is, we cannot, generally speaking, measure the states $(w, \dot{w}) \in V \times V_2$ directly. We will assume that the measure of output error between the prediction of our model in equation (5) and physically available quantities can be expressed as

$$\begin{aligned} \mathcal{J}(q,\mu) &\equiv \int_0^T j([w(q,\mu)](\tau), [\dot{w}(q,\mu)](\tau), t) d\tau \\ &= J(w(q,\mu), \dot{w}(q,\mu)) \end{aligned} \tag{11}$$

where the integrand satisfies the following assumptions:

(J1) The mapping $j : V \times V_2 \times \mathbb{R} \longrightarrow \mathbb{R}$ is nonnegative and measurable a.e. in $t \in [0, T]$.

(J2) $j(\cdot, \cdot, t)$ is lower semicontinuous on $V \times V_2$ a.e. in $t \in [0, T]$.

The conditions (J1) and (J2) are motivated by a prototypical least squares output error functional having the form

$$\mathcal{J}(q,\mu) \equiv \int_0^T \left\{ \|C_1[w(q,\mu)](t) - y_1\|_{Z_1}^2 + \|C_2[\dot{w}(q,\mu)](t) - y_2\|_{Z_2}^2 \right\} d\nu(t)$$

where $C_1 : V \longrightarrow Z_1$, $C_2 : V_2 \longrightarrow Z_2$ are bounded linear operators, y_1 and y_2 are measurements and Z_1, Z_2 are Banach spaces of measurements. Now our identification problem can be expressed succinctly in terms of the governing equations (5) and the output error functional $\mathcal{J}$ defined in equation (11).

Problem 4.1. *Find $(q_0, \mu_0) \in \mathbb{Q} \times \mathbb{P}$ such that*

$$\mathcal{J}(q_0, \mu_0) = \inf_{(q,\mu) \in \mathbb{Q} \times \mathbb{P}} \mathcal{J}(q, \mu)$$

where $\mathcal{J}(q,\mu)$ is defined in equation (11) and the pair $(w(q,\mu), \dot{w}(q,\mu))$ satisfies the governing equation (5).

As is usual in the consideration of distributed parameter systems, the identification problem (ID) is not directly solvable, except in a few special cases, because both the state space $V \times H$ and metric space $\mathbb{Q} \times \mathbb{P}$ of parameters are infinite dimensional. We introduce a sequence of finite dimensional subspaces $\{X_N\}_{N=1}^{\infty}$ of $V \subseteq H$ and denote the H-orthogonal projections $P_N : H \longrightarrow X_N$.

In addition, we assume that we can construct finite dimensional parameter sets $\mathbb{Q}_\ell \times \mathbb{P}_m$ that approximate the set of admissible parameters $\mathbb{Q} \times \mathbb{P}$

in a sense to be made precise shortly. Following conventional approximation frameworks in [11] [2], [10], we assume that the subspaces X_N and parameter sets $\mathbb{Q}_l \times \mathbb{P}_m$ satisfy the following hypotheses

(H1) The sequence of approximating parameter sets $\mathbb{Q}_l \times \mathbb{P}_m$ are finite dimensional compact subsets of the compact metric space $\mathbb{Q} \times \mathbb{P}$ with $\tilde{d} = d \times \rho$ the product metric topology.

(H2) For any $(q, \mu) \in \mathbb{Q} \times \mathbb{P}$, there exists a sequence $\{(q_\ell, \mu_m)\}$ with $(q_\ell, \mu_m) \in \mathbb{Q}_\ell \times \mathbb{P}_m$ such that $(q_\ell, \mu_m) \to (q, \mu)$ in $\mathbb{Q} \times \mathbb{P}$. Moreover, if $\{(q_\ell, \mu_m)\}$ is *any* sequence with $(q_\ell, \mu_m) \in \mathbb{Q}_\ell \times \mathbb{P}_m$, then there exists $(q^*, \mu^*) \in \mathbb{Q} \times \mathbb{P}$ and a subsequence $\{(q_{\ell_k}, \mu_{m_k})\}$ with $(q_{\ell_k}, \mu_{m_k}) \to (q^*, \mu^*)$ in $\mathbb{Q} \times \mathbb{P}$.

(H3) The sets X_N approximate V in the sense that

$$\|v - P_N v\|_V \longrightarrow 0 \text{ as } j \longrightarrow \infty \quad \text{for } v \in V.$$

(H4) The sets X_N approximate V_2 in the sense that,

$$\|w - P_N w\|_{V_2} \longrightarrow 0 \text{ as } N \longrightarrow \infty \quad \text{for } w \in V_2.$$

As noted earlier, this identification problem is complicated by the fact that the parameter sets and state space are infinite dimensional. We will frequently need to refer to the mapping from $\mathbb{Q} \times \mathbb{P}$ to $X_N \times X_N \subseteq V \times V_2$ that results from the discretization of the state variable only. That is, for $(q, \mu) \in \mathbb{Q} \times \mathbb{P}$, $w^N(q, \mu)$ and $\dot{w}^N(q, \mu)$ will refer to solutions of the equations

$$\begin{aligned} \langle \ddot{w}^N(t), \phi \rangle_{V^* \times V} &+ a_2(q)(\dot{w}^N(t), \phi) + a_1(q)(w^N(t), \phi) \\ &= \langle [B_\mu(u, f)](t), \phi \rangle_{V_2^* \times V_2} \qquad \text{for all } \phi \in X_N \quad (12) \end{aligned}$$

$$w^N(0) = P_N w_0, \qquad \dot{w}^N(0) = P_N w_1. \tag{13}$$

The reader should carefully note that this set of equations cannot be solved in general because the parameter sets are infinite dimensional, although the state space is of fixed dimension dependent on N.

For any particular choice of parameters $(q_l, \mu_m) \in \mathbb{Q}_l \times \mathbb{P}_m \subset \mathbb{Q} \times \mathbb{P}$ we can define (and compute) the Galerkin approximations $w^N_{\ell,m} = w^N(q_\ell, \mu_m)$ using (12), (13).

It is important to note that the cost functional changes as we vary the dimension of the space used to approximate the state variable. The discrete cost functional is defined to be

$$\mathcal{J}^N(q, \mu) \equiv \int_0^T j([w^N(q, \mu)](t), [\dot{w}^N(q, \mu)](t), t)dt \tag{14}$$

Now, we can define a collection of finite dimensional minimization problems $(ID)_{l,m}^N$ that will serve to approximate the infinite dimensional optimization problem of Problem 4.1.

Problem 4.2. *Find* $(q_l^N, \mu_m^N) \in \mathbb{Q}_l \times \mathbb{P}_m$ *such that*

$$\mathcal{J}^N(q_l^N, \mu_m^N) = \inf_{(q,\mu)\in\mathbb{Q}_l\times\mathbb{P}_m} \mathcal{J}^N(q,\mu)$$

where $\mathcal{J}^N(\cdot,\cdot)$ *is defined in (14) and* $w^N(q,\mu), \dot{w}^N(q,\mu)$ *satisfies equations (12) and (13).*

We have assumed that the approximate parameter spaces $\mathbb{Q}_l \times \mathbb{P}_m$ are contained in $\mathbb{Q} \times \mathbb{P}$. This is not necessary and one can readily treat cases where $\mathbb{Q}_\ell, \mathbb{P}_m$ are only approximations to $\mathbb{Q}, \mathbb{P}$ respectively (see [2] for this tedious but rather easy generalization). If we recall that the metric space $\mathbb{Q}$ represents parameters characterizing the differential operator of the bilinear forms $a_2(q)(\cdot,\cdot)$ and $a_1(q)(\cdot,\cdot)$ on $q \in \mathbb{Q}$, it is clear that there are an enormous number of possibilites for the selection of the spaces $\mathbb{Q}$ and $\mathbb{Q}_l$. These choices depend, of course, on what we require of the damping or stiffness operators, for instance. Many of the choices have been studied elsewhere, and the interested reader is referred to [2] and the references therein.

The choice of topology for the metric spaces $\mathbb{P}$ and $\mathbb{P}_m$ is of central importance to this paper. The metric spaces $\mathbb{P}$ and $\mathbb{P}_m$ contain the allowable probability measures used to represent the KP integral operator introduced in part (I) of this paper, and summarized in Section 3. Because the KP integral operator is defined on $\overline{\mathcal{S}}$, the closure of the Preisach half-plane

$$\mathcal{S} = \left\{(s_1, s_2) \in \mathbb{R}^2 : s_2 > s_1\right\},$$

we choose to let $\mathbb{P} \equiv \mathbb{P}(\bar{\mathcal{S}}_\Delta)$ denote the set of Borel probability measures on the separable, compact metric space $\bar{\mathcal{S}}_\Delta \equiv \{(s_1, s_2) \in R^2 | \underline{s} \le s_1 \le s_2 \le \bar{s}\}$ for fixed upper and lower bounds $\bar{s}, \underline{s}$, respectively on the set of switching parameters. The topology of weak* convergence of measures, or convergence in distribution, is obtained by defining

$$\mu_n \longrightarrow \mu \text{ in } \mathbb{P}(\bar{\mathcal{S}}_\Delta)$$

if and only if

$$\int_{\bar{\mathcal{S}}_\Delta} f d\mu_n \longrightarrow \int_{\bar{\mathcal{S}}_\Delta} f d\mu$$

for all bounded, uniformly continuous, real-valued functions $f \in C(\bar{\mathcal{S}}_\Delta)$. Because $\mathbb{P}(\bar{\mathcal{S}}_\Delta) \subseteq (C(\bar{\mathcal{S}}_\Delta))^*$, this corresponds to

$$\mu_n \longrightarrow \mu \text{ in } (C(\bar{\mathcal{S}}_\Delta))^*$$

and is referred to as weak* convergence. It is well-known that the space $\mathbb{P}(\bar{\mathcal{S}}_\Delta)$ is metrizable and the Prohorov metric ρ yields the same topology on $\mathbb{P}(\bar{\mathcal{S}}_\Delta)$. In fact, $\mathbb{P}(\bar{\mathcal{S}}_\Delta)$ is a compact metric space (see [3] for details). The following theorem will be crucial in our derivation of convergent approximation schemes in this section and the next.

Theorem 4.1. *Let $\bar{\mathcal{S}}_\Delta$ be the compact set defined above and let $\mathcal{R}_m$ be a set of m rational points in $\bar{\mathcal{S}}_\Delta$ chosen so that $\cup_{m=1}^{\infty}\mathcal{R}_m$ is dense in $\bar{\mathcal{S}}_\Delta$. Then defining*

$$\mathbb{P}_m \equiv \left\{ \sum_{i=1}^{m} \alpha_i \delta_{s_i} : \alpha_i \textit{ nonegative and rational}, \sum_{i=1}^{m} \alpha_i = 1, s_i \in \mathcal{R}_m \right\},$$

we have that $\cup_{m=1}^{\infty}\mathbb{P}_m$ is dense in $\mathbb{P}(\bar{\mathcal{S}}_\Delta)$ in the Prohorov metric ρ. Here δ_{s_i} is the Dirac measure with atom at s_i.

Proof: These results follow directly from Theorem 4, p. 237 of [3] along with application of the arguments given on p. 239 of [3].

In practice, the sets $\mathcal{R}_m$ will be generated via a sequence of rational lattice points over the Preisach plane whose characterizing mesh size $h \to 0$.

5 Function Space Parameter Estimation Convergence

Many identification frameworks have been introduced in the literature [16]. We choose to formulate the abstract identification problem in terms of *Function Space Parameter Estimation Convergence*, or FSPEC, presented in detail in [2].

Definition 5.1. *The sequence of finite dimensional identification problems $(ID)_{l,m}^N$ in Problem 4.2 is said to be function space parameter estimation convergent to the identification problem (ID) in Problem 4.1 provided that*

(i) For each $N \in \mathbb{N}$, there is a solution $(q_l^N, \mu_m^N) \in \mathbb{Q}_l \times \mathbb{P}_m$ of $(ID)_{l,m}^N$,

(ii) Every convergent subsequence $\left\{(q_{l_k}^{N_k}, \mu_{m_k}^{N_k})\right\}_{k=1}^{\infty}$ *of* $\left\{(q_l^N, \mu_m^N)\right\}_{l,m,N=1}^{\infty}$ *converges to a solution* $(q,\mu) \in \mathbb{Q} \times \mathbb{P}$ *of* (ID)*, that is,*

$$\begin{aligned} q_{l_k}^{N_k} &\longrightarrow q \quad \text{as} \quad l_k, N_k \longrightarrow \infty \\ \mu_{m_k}^{N_k} &\longrightarrow \mu \quad \text{as} \quad m_k, N_k \longrightarrow \infty, \end{aligned}$$

(iii) For each convergent subsequence $\left\{(q_{l_k}^{N_k}, \mu_{m_k}^{N_k})\right\}_{k=1}^{\infty}$ *of* $\left\{(q_l^N, \mu_m^N)\right\}_{l,m,N=1}^{\infty}$*, we have*

$$\mathcal{J}^{N_k}(q_{l_k}^{N_k}, \mu_{m_k}^{N_k}) \longrightarrow \mathcal{J}(q,\mu),$$

(iv) We have the convergence in state space

$$\left\| w_{l_k,m_k}^{N_k}(t) - w(q,\mu)(t) \right\|_V^2 + \left\| \dot{w}_{l_k,m_k}^{N_k}(t) - \dot{w}(q,\mu)(t) \right\|_{V_2}^2 \longrightarrow 0$$

uniformly in $t \in [0,T]$*, and*

(v) there exists at least one subsequence satisfying properties (ii) through (iv) above.

Theorem 5.1. *Under (J1), (J2), (H1)-(H4), (A1)-(A5), the sequence of finite dimensional identification problems* $(ID)_{l,m}^N$ *described in Problem 4.2 is function space parameter estimation convergent to the identification problem described in Problem 4.1.*

We will need several preliminary results to establish this theorem. First, let us recall a fundamental continuity result established in [9] (see also Theorem 5.2 of [11]).

Proposition 5.1. *Suppose that the approximating subspaces* X_N *satisfy hypotheses* $(H3)$ *and* $(H4)$*. Assume that the bilinear forms* $a_2(q)(\cdot,\cdot)$ *and* $a_1(q)(\cdot,\cdot)$ *satisfy assumptions* $(A1)$ *through* $(A5)$*. Let* (q_k,μ_k) *be an arbitrary convergent sequence in* $\mathbb{Q} \times \mathbb{P}$ *such that* $(q_k,\mu_k) \longrightarrow (q,\mu)$ *in* $\mathbb{Q} \times \mathbb{P}$*. Then for* $t > 0$*, we have*

$$\begin{aligned} \left\| [w^k(q_k,\mu_k)](t) - [w(q,\mu)](t) \right\|_V &\longrightarrow 0 \\ \left\| [\dot{w}^k(q_k,\mu_k)](t) - [\dot{w}(q,\mu)](t) \right\|_{V_2} &\longrightarrow 0 \end{aligned}$$

as $k \longrightarrow \infty$ *where* $w^k(q_k,\mu_k), \dot{w}^k(q_k,\mu_k)$ *are the solutions of equations (12), (13) for* $N = k$ *and* $(q,\mu) = (q_k,\mu_k)$ *and* $w, \dot{w}$ *are the solutions to equations (5), (6).*

Proof. The proof follows as a special case of Theorem 2, p. 878 of [9]. We simply identify the product parameter space $\mathbb{Q} \times \mathbb{P}$ with the parameter space utilized in [9]. In our case, we note that Theorem 2.1 guarantees the continuous dependence of the right hand side of the governing equations with respect to $\mu \in \mathbb{P}$, as required in equation (20) of [9]. Hypothesis $(H5)$ and $(H6)$ are special cases of the conditions in equations (18) and (19) in [9].

Proposition 5.2. *For each fixed $N \in \mathbb{N}$, the map*

$$(q, \mu) \mapsto \mathcal{J}^N(q, \mu)$$

is lower semicontinuous from $\mathbb{Q}_l \times \mathbb{P}_m$ to $\mathbb{R}$.

Proof. Recall that $\mathcal{J}^N(q, \mu) \equiv J(w^N(q, \mu), \dot{w}^N(q, \mu))$ where $w^N(q, \mu)$, $\dot{w}^N(q, \mu)$ is the unique solution to the finite dimensional, approximate variational system (12),(13), for some fixed $N \in \mathbb{N}$ and $(q, \mu) \in \mathbb{Q}_l \times \mathbb{P}_m$. But $\mathcal{J}^N(q, \mu)$ can be written as the composition of two maps, T_1 and T_2,

$$T_1 : (q, \mu) \mapsto (w^N(q, \mu), \dot{w}^N(q, \mu))$$

as defined in equation (12), (13), and the map T_2

$$T_2 : (u, v) \mapsto J(u, v)$$

as defined in equation (11). Now, T_1 is a continuous mapping from $\mathbb{Q}_l \times \mathbb{P}_m \longrightarrow L_2(0, T; X_N) \times L_2(0, T; X_N) \subseteq L_2(0, T; V) \times L_2(0, T; V_2)$. This fact follows from standard finite dimensional results on continuous dependence of solutions on (12), (13) on parameters. In addition, T_2 is lower semicontinuous from $L_2(0, T; V) \times L_2(0, T; V_2)$ to $\mathbb{R}$. Hence, the composition is lower semicontinuous, and the result follows.

Proof of Theorem 5.1: The proof of Theorem 5.1 follows rather directly from special cases of the arguments given for the proof of Theorem 1.1 in [2], p. 63. By virtue of the lower semicontinuity of the map

$$(q, \mu) \mapsto \mathcal{J}^N(q, \mu)$$

from $\mathbb{Q}_l \times \mathbb{P}_m$ to $\mathbb{R}$ proven in Proposition 5.2, and the compactness of $\mathbb{Q}_l \times \mathbb{P}_m$, we know that for each fixed $n \in \mathbb{N}$ there exists

$$(\bar{q}_l^N, \bar{\mu}_m^N) \in \mathbb{Q}_l \times \mathbb{P}_m \text{ such that } \mathcal{J}^N(\bar{q}_l^N, \bar{\mu}_m^N) = \inf_{(q,\mu) \in \mathbb{Q}_l \times \mathbb{P}_m} \mathcal{J}^N(q, \mu).$$

By hypothesis $(H2)$ on the approximating spaces, we can find a subsequence $\left\{(\bar{q}_{l_k}^{N_k}, \bar{\mu}_{m_k}^{N_k})\right\}$ that converges to some $(q^*, \mu^*) \in \mathbb{Q} \times \mathbb{P}$.

Applying Proposition 5.1, and assuming that the cost functional $\mathcal{J}$ of (11) satisfies (J1), (J2), we then have that

$$\mathcal{J}^{N_k}(\bar{q}_{l_k}^{N_k}, \bar{\mu}_{m_k}^{N_k}) \to \mathcal{J}(q^*, \mu^*) \text{ as } k \longrightarrow \infty.$$

Again using (H2) for *any* $(q, \mu) \in \mathbb{Q} \times \mathbb{P}$, we have a sequence $\{(q_l, \mu_m)\}$ with (q_l, μ_m) in $\mathbb{Q}_l \times \mathbb{P}_m$ such that $(q_l, \mu_m) \to (q, \mu)$. But $\mathcal{J}^{N_k}(\bar{q}_{l_k}^{N_k}, \bar{\mu}_{m_k}^{N_k}) \leq \mathcal{J}^{N_k}(q_{l_k}, \mu_{m_k})$. Passing to the limit we obtain $\mathcal{J}(q^*, \mu^*) \leq \mathcal{J}(q, \mu)$ so that (q^*, μ^*) is a solution to Problem 4.1. It follows that (i)-(v) of Definition 5.1 hold and hence we have established Theorem 5.1.

6 Computational Implementation: General Case

In this section, we discuss the details of numerical implementation of the finite dimensional optimization problems of Problem 4.1. As noted earlier, to approximate the state equations, we require a sequence of finite dimensional spaces $X_N \subset V$ satisfying the conditions (H3), (H4).

Let $\{\phi_i^N\}_1^N$ be a specific basis for X_N, that is,

$$X_N = span\left\{\phi_1^N, \dots, \phi_N^N\right\}.$$

We note that one simple way (not the only one, of course) to create such a sequence is to choose a sequence of elements $\{\phi_i\}_{i=1}^{\infty}$ that is total in V, truncate the sequence to the N first elements, and choose those as the basis for X_N. We denote any approximation derived from the finite dimensional subspace X_N by a superscript, as in

$$w^N(t) \equiv \sum_{j=1}^{N} \phi_j^N \xi_j^N(t). \tag{15}$$

When this finite dimensional representation is substituted into the governing equations (12), (13), we obtain a well-posed system. However, this system is still not tractable for calculation, due to the fact that the collection of probability measures is infinite dimensional. To achieve a finite dimensional system of evolution equations, we use a compact subset $\mathcal{S}_\triangle \subseteq \mathcal{S}$ of the Preisach plane as discussed in Section 4 and take as our approximation measure sets the sets $\mathbb{P}_m$ of Theorem 4.1.

For a given $\mu \in \mathbb{P}_m$, we can readily evaluate the integral hysteretic control influence operator. By definition, the control influence operator is

$$[B_\mu(u,f)](t) \equiv \int_{\bar{\mathcal{S}}_\triangle} [k_s(u,f(s))](t)d\mu(s) \cdot g \quad g \in V^*.$$

For $\mu_m = \sum_{i=1}^m \alpha_i^m \delta_{s_i}$ in $\mathbb{P}_m$, we have

$$\begin{aligned} [B_{\mu_m}(u,f)](t) &\equiv \int_{\bar{\mathcal{S}}_\triangle} [k_s(u,f(s))](t)d\mu_m(s) \cdot g \\ &= \sum_{i=1}^m [k_{s_i}(u,f(s_i)](t)\alpha_i^m \cdot g \\ &= \sum_{i=1}^m B_i^m(t)\alpha_i^m \cdot g. \end{aligned} \tag{16}$$

The system of finite dimensional evolution equations, appropriate for numerical simulation, are thus obtained by substituting the expression for the control influence operator into the system of ordinary differential equation (12).

$$\begin{aligned} \langle \ddot{w}^N(t), \phi\rangle_{V^*\times V} &+ a_1(\dot{w}^N(t),\phi) + a_0(w^N(t),\phi) \\ &= \langle g, \phi\rangle_{V_2^*\times V_2} \sum_{i=1}^m B_i^m(t)\alpha_i^m \end{aligned} \tag{17}$$

for $\phi \in X_N$.

When we substitute the explicit expansion for the finite dimensional solution w^N, and choose $\phi = \phi_l$, we obtain the N dimensional system

$$\begin{aligned} &\sum_{n=1}^N \left\{ \langle \phi_n, \phi_l\rangle_H \ddot{\xi}_n^N(t) + a_1(\phi_n,\phi_l)\dot{\xi}_n^N(t) + a_0(\phi_n,\phi_l)\xi_n^N(t) \right\} \\ &\qquad = \langle g, \phi_l\rangle_{V_2^*\times V_2} \sum_{i=1}^m B_i^m(t)\alpha_i^m, \quad l = 1,2,\ldots,N. \end{aligned} \tag{18}$$

In terms of a compact matrix notation, we can write the above system of ordinary differential equations

$$M^N\ddot{\xi}^N(t) + C^N\dot{\xi}^N(t) + K^N\xi^N(t) = B^m(t)\alpha^m \tag{19}$$

subject to the initial conditions

$$\xi^N(0) = \xi_0^N, \quad \dot{\xi}^N(0) = \xi_1^N \tag{20}$$

where ξ_0^N, ξ_1^N are determined using (13) and (15). Here M^N, C^N, K^N are matrices with elements $\langle \phi_n, \phi_l \rangle, a_1(\phi_n, \phi_l), a_0(\phi_n, \phi_l)$, respectively, and $B^m(t)$, α^m are the vectors with components B_i^m, α_i^m respectively.

To obtain the fully discrete version of the finite dimensional identification problem, we express the finite dimensional cost functional (using the prototypical least squares output of Section 4):

$$\begin{aligned} J^N(\xi^N(\alpha^m)) &= \int_0^T \left\{ \|C_1 \Phi^N \xi^N(t) - y_1\|_{Z_1}^2 \right. \\ &+ \left. \|C_2 \Phi^N \dot{\xi}^N(t) - y_2\|_{Z_2}^2 \right\} dt. \end{aligned} \tag{21}$$

Here we have collected the basis elements ϕ_i^N in vector form

$$\Phi^N = [\phi_1^N \dots \phi_N^N]$$

so that in vector notation we have

$$w^N(t) = \sum_{i=1}^N \phi_i^N \xi_i^N(t) = \Phi^N \xi^N(t).$$

7 Numerical Example

We have tested the computational methods developed in this paper in specific numerical examples, one of which we report on here. In this example, we consider the axial deflections of an undamped rod subject to a distributed hysteretic control influence operator. It is well-known that the *infinitesimal* deflection w of a clamped elastic rod can be described by

$$\frac{\partial^2 w}{\partial t^2}(t,x) = \frac{\partial^2 w}{\partial x^2}(t,x) + p(t,x) \tag{22}$$

subject to the boundary conditions

$$w(t,0) = 0, w(t,1) = 0 \tag{23}$$

and to the initial conditions

$$w(0,x) = w_0(x), \dot{w}(0,x) = w_1(x). \tag{24}$$

For simplicity all of the physical parameters (e.g. AE, ρ, ... etc.) and lengths have been normalized to unity and damping has been ignored for this example. We assume that the distributed load $p(t,x)$ is given by a KP integral hysteresis operator

$$p(t,x) = B_\mu(u,f)(t,x) \tag{25}$$

where

$$B_\mu(u,f)(t,x) = \int_{\bar{\mathcal{S}}_\triangle} [k_s(u,f(s))](t) d\mu(s) \cdot g(x) \tag{26}$$

for some $g \in L^2(0,1)$. Define $H = L^2(0,1)$, $V = H_0^1(0,1)$. Then since $g \in H$, this system can be written in the weak form (5),(6) by

$$\begin{aligned} \langle \frac{\partial^2 w}{\partial t^2}(t), \phi \rangle_{V^* \times V} + a(w(t),\phi) &= \langle [B_\mu(u,f)](t), \phi \rangle \quad \text{for } \phi \in V \\ w(0) = w_0, \qquad\qquad & \dot{w}(0) = w_1, \end{aligned} \tag{27}$$

where $\langle \cdot,\cdot \rangle$ is the inner product in H and a_1 is the V-continuous, V-elliptic symmetric sesquilinear form given by $a_1(\phi,\psi) = \langle \phi', \psi' \rangle$. The well posedness results of Section 2 can be applied to obtain existence, uniqueness and continuous dependence of solutions of (22)-(24) in the weak or variational sense (e.g., see Theorem 2.1).

One can consider an equivalent (see [10] or Chapter 4 of [11]) formulation in terms of semigroups which in this special example allows for a rather explicit characterization of solutions. We first rewrite (27) in first order form in operator notation. The precise and rigorous details for this approach are given in [10, 11].

We define an operator A in H to be the restriction of

$$\langle -A\phi, \psi \rangle_{V^*,V} = a_1(\phi,\psi) = \langle \phi', \psi' \rangle, \psi \in V$$

to dom $A = \{\phi \in V | A\phi \in H\} = H^2(0,1) \cap H_0^1(0,1)$. In this case we find $A\phi = \phi''$ for $\phi \in$ dom A. Defining

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$$

on dom $A \times V$ in $\mathcal{H} = V \times H$, we readily argue that $\mathcal{A}$ is the infinitesimal generator of a strongly continuous semigroup $T(t)$ on $\mathcal{H}$. Morover, the variational solution of (27) guaranteed by Theorem 2.1 can be written (again see Chapter 4 of [11]) in terms of this semigroup in a standard variation-of-parameters representation. Letting $z(t) = (w(t), \dot{w}(t))^T$, $z_0 = (w_0, w_1)^T$, and $F(t) = (0, B_\mu(u, f)(t))^T$, we have

$$z(t) = T(t)z_0 + \int_0^t T(t-\tau)F(\tau)\, d\tau. \tag{28}$$

Using the complete orthonormal family $\{\phi_n\}$ in $L^2(0,1)$ given by $\phi_n(x) = \sin n\pi x$, we can give an explicit series characterization for the semigroup $T(t)$ and hence the solution $z(t) = (w(t), \dot{w}(t))^T$ given in (28). Referring to the standard results given in Chapter 2 of [12], we have (in particular, see (2.27))

$$T(t)z_0 = \begin{pmatrix} \sum_{n=1}^{\infty} 2\{\langle w_0, \phi_n\rangle \cos n\pi t + \frac{1}{n\pi}\langle w_1, \phi_n\rangle \sin n\pi t\}\phi_n \\ \sum 2\{-n\pi\langle w_0, \phi_n\rangle \sin n\pi t + \langle w_1, \phi_n\rangle \cos n\pi t\}\phi_n \end{pmatrix} \tag{29}$$

which yields immediately a solution for the homogeneous version (22)-(24) or (27). This representation can also be used in (28) to characterize solutions to the nonhomogeneous equation. In particular, if we consider (27) with $z_0 = 0$, we obtain

$$z(t) = \int_0^t \begin{pmatrix} \sum_{n=1}^{\infty} \frac{2}{n\pi}\langle B_\mu, \phi_n\rangle \sin(n\pi(t-\tau))\phi_n \\ \sum_{n=1}^{\infty} 2\langle B_\mu, \phi_n\rangle \cos(n\pi(t-\tau))\phi_n \end{pmatrix} d\tau. \tag{30}$$

Thus for any distribution $g \in H$, the axial displacement corresponding to $z_0 = 0$ can be written

$$w(t, x; \mu) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\langle g, \phi_n\rangle \int_0^t \sin(n\pi(t-\tau)) \cdot [P_\mu(u, f)](\tau) d\tau \sin(n\pi x). \tag{31}$$

We note that in terms of the approximations of Section 6, we may choose $X_N = span\{\phi_1, \ldots, \phi_N\}$, $\phi_N(x) = \sin N\pi x$, and then w^N of (15) is given simply by the termination (at N terms) of the series in (31).

In the *special case* that $g(x) = \sin \pi x$, the series reduces to one term, which, using the double angle identities, we can write

$$\begin{aligned} w(t, x; \mu) \;\; = \;\; & \frac{2k_1}{\pi}\left\{\sin(\pi t)\int_0^t \cos(\pi\tau)[P_\mu(u, f)](\tau) d\tau \right. \\ & \left. - \cos(\pi t)\int_0^t \sin(\pi\tau)[P_\mu(u, f)](\tau) d\tau\right\} \sin(\pi x) \end{aligned} \tag{32}$$

where $k_1 \equiv \langle g, \phi_1 \rangle$ and μ is arbitrary in $\mathbb{P}$.

The corresponding expression for any μ_m in $\mathbb{P}_m$ given as in (16) is

$$w(t,x;\mu_m) = \frac{2k_1}{\pi} \sum_{i=1}^{m} \left\{ \sin(\pi t) \int_0^t \cos(\pi\tau)[k_{s_i}(u, f(s_i)](\tau) d\tau \right.$$
$$\left. - \cos(\pi t) \int_0^t \sin(\pi\tau)[k_{s_i}(u, f(s_i)](\tau) d\tau \right\} \alpha_i^m \sin(\pi x) \quad (33)$$

We used the analytic aspects of the example discussed above to illustrate the convergence guaranteed by Theorem 5.1. Since the parameter dependence ($q \in Q$) of such problems are well understood (e.g., see extensive examples and numerical results in [2, 11]), we assumed that q was fixed throughout and concentrated on the convergences $w^N(t;\bar{\mu}_m^N) \to w^*(t;\mu^*)$, $\bar{\mu}_m^N \to \mu^*$, where $\bar{\mu}_m^N$ is the solution of the corresponding problem $(ID)_m^N$ of Problem 4.2 involving minimization of $\mathcal{J}^N(\mu)$ over $\mathbb{P}_m$ of Theorem 4.1.

We first tested our algorithms for the special case of $g(x) = \sin \pi x$ so that the "approximate" solutions $w^N(t;\mu)$, $w^N(t;\mu_m)$ were given by (32), (33) for *all* N. This allowed us to verify the convergence $w^N(t;\bar{\mu}_m^N) = w(t;\bar{\mu}_m^N) \to w(t;\mu)$ as a special case of Theorem 5.1. The algorithm performed quite as expected and we turned to the more general cases where $\langle g, \phi_n \rangle$ was not the same for all n and hence the solution w corresponding to a distribution μ was given by the infinite series (31).

We report details here for the specific example with g given by

$$g(x) = \begin{cases} 2x & 0 \le x \le .5 \\ 2(1-x) & .5 \le x \le 1, \end{cases}$$

where again $X_N = span\{\phi_n\}_{n=1}^N$, $\phi_n(x) = \sin n\pi x$. In this case solutions of the full system are given by (31) with

$$\langle g, \phi_n \rangle = \frac{4}{n^2\pi^2} \sin \frac{n\pi}{2}.$$

For the ridge function r in our KP kernels (see [7, 8] and Figure 1) we chose

$$r(x) = \begin{cases} -1 & x < 0 \\ -1 + \frac{2}{5}x & 0 \le x \le 5 \\ 1 & x > 5. \end{cases}$$

We constructed an "exact" solution $w^*(t;\mu^*)$ to use as simulated data in the least squares cost functionals of Problems 4.1, 4.2 in testing numerically our inverse algorithm. The distribution μ^* was a discretized distribution of a modified Gaussian. Let $\bar{\mathcal{S}}_\Delta[20,60] \equiv \{(s_1,s_2) \in R^2 | \underline{s} = 20 \le s_1 \le s_2 \le \bar{s} = 60\}$ be a section of the Preisach plane as depicted in Figure 2. We discretize this by defining grid points (s_{1i}, s_{2j}) with

$$\begin{aligned} s_{1i} &= 20 + K(i-1) \\ s_{2j} &= 20 + K(j=1), i,j = 1,2,\ldots,K,\ j \ge i. \end{aligned}$$

For our exact measure μ^*, we chose $K = 31$ (so that we had $K(K+1)/2 = 496$ grid points) and constructed a distribution with weights α_{ij} associated with (s_{1i}, s_{2j}) given by

$$\begin{aligned} \alpha_{ij} &= \gamma_{ij}/(\sum_{i,j} \gamma_{ij}) \\ \gamma_{ij} &\equiv exp[-250((s_{1i}-40)^2 + s_{2j}^2]. \end{aligned}$$

To calculate the "exact" solution we use this μ^* in a truncated form of the series (31), i.e.,

$$w^N(t,x;\mu^*) = \\ \sum_{n=1}^{N} \frac{8}{n^3\pi^3} \sin(\frac{n\pi}{2}) \sin n\pi x \int_0^t \sin n\pi(t-\tau)[P_{\mu^*}(u,f)](\tau)d\tau. \quad (34)$$

We chose $f(s) \equiv -1$ for all $s \in \bar{\mathcal{S}}_\Delta[20,60]$ and the control (thermal) input u was taken as

$$u(t) = \begin{cases} 20 + 40\sin\frac{\pi}{4}t & 0 < t < 2 \\ 45 + 15\cos\frac{\pi}{2}(t-2) & 2 < t < 6 \\ 40 + 20\cos\frac{\pi}{2}(t-6) & t > 6. \end{cases}$$

We used this in the truncated expression (34) with $N = 20$ for our "data" in the cost functional. That is, $w^*(t) = w^*(t;\mu^*) \equiv w^{20}(t;\mu^*)$ where w^{20} is given by (34). The cost functional (see (21)) was taken (for $\mu_m \in \mathbb{P}_m$) to be

$$J^N(\mu_m) = \sum_{i=1}^{50} |w^N(t_i,.8;\mu_m) - w^*(t_i,.8)|^2 \quad (35)$$

which is a discretized version of (21) with $T = 10$, $t_{i+1} = t_i + .2$. We minimized over $\mathbb{P}_m$ as defined in Theorem 4.1 where the rational sets $\mathcal{R}_m$ of Theorem 4.1 were chosed to be the $m = K(K+1)/2$ grid points defined above and depicted in Figure 2.

In Figure 3 we plot the solutions $w^N(t, .8; \bar{\mu}_m^N)$, $m = K(K+1)/2$ corresponding to the identified measures $\bar{\mu}_m^N$ obtained from minimizing (35). These are compared for several values of N and m (or K) with the "data" $w^*(t, .8)$ (labeled as "true"and represented by the solid line). The corresponding minimizing measures $\bar{\mu}_m^N$ are given in Figure 4. It can be seen that the method performs quite satisfactorily when using "exact data" (i.e., "data" that was generated by the model). However, as discussed in [2] and [11], these computations do not constitute an adequate test of the suitability of the algorithm's use with experimental data (which always contain some error or "noise" even in the rare case where the model being used exactly describes the physical process).

To further test our methodology, we added noise (in the form of relative noise) to the "data" w^*. The noisy data has the form $\hat{w}(t_i) = w^*(t_i)[1 + \epsilon_i]$ where ϵ_i is obtained via a random number generator (with uniform distribution) varying in $[.0\delta, .0\delta]$ for $\delta = 1, 2, 5$ and 10 representing relative noise of 1%, 2%, 5% and 10% respectively. These data sets were used to identify $\bar{\mu}_m^N$ for $N = 11$ and $m = 91$ (or $K = 13$). In Figure 5 we plot the results comparing the exact data curve $w^*(t)$ - labeled as "true" and represented by the solid line - with discrete points from $w^N(t, .8; \bar{\mu}_m^N)$, $N = 11$, $m = 91$ for the several levels of noise 1%, 2%, 5% and 10%. The corresponding estimated measures $\bar{\mu}_m^N$ are given in Figure 6.

It is readily seen that the algorithm performs quite well for data with 1% or 2% relative noise. It also yields reasonable results for noisy data at the 5% level, but at the 10% noise level the performance has degraded to an unacceptable level. These results suggest acceptable robustness with respect to noise in data for our methodology when compared with typical results for algorithms in inverse or identification problems (e.g., see the examples in [2] and [11].

The test simulations reported on above are based on fits-to-data involving one spatial observation of displacements in time. While this is quite likely a reasonable formulation in many experimental situations, it provides a most severe challenge for our algorithms and the results above can be interpreted, in some sense, as a "worst case" scenario. In further simulation tests (involving observations made simultaneously at two or three spatial locations on the structure), the inverse algorithms exhibited enhanced robustness with respect to error in measurements.

8 Conclusion

In this paper we have presented a theoretical foundation for estimation of hysteresis operators of the Preisach class. Well-posedness of the associated inverse problems is established and rigorous convergence results for approximations are given. The theory is illustrated with computational examples which suggest the proposed methodology enjoys reasonable robustness with respect to noise in the data or observations of the system driven by the hysteresis input operators.

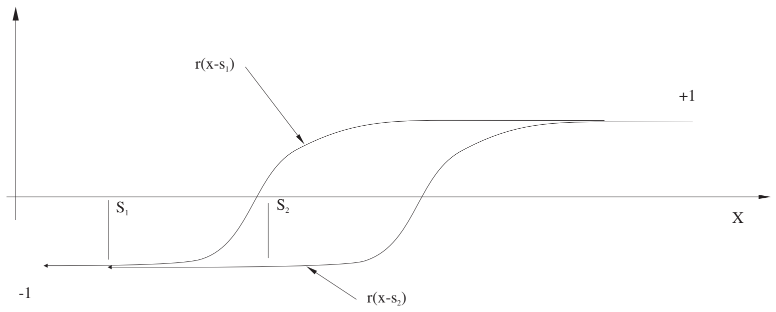

Figure 1: Prototypical hysteresis kernel

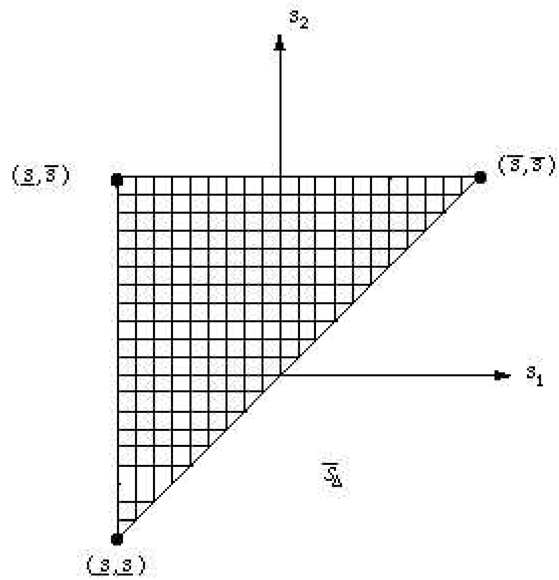

Figure 2: Discrete measure lattice on Preisach plane

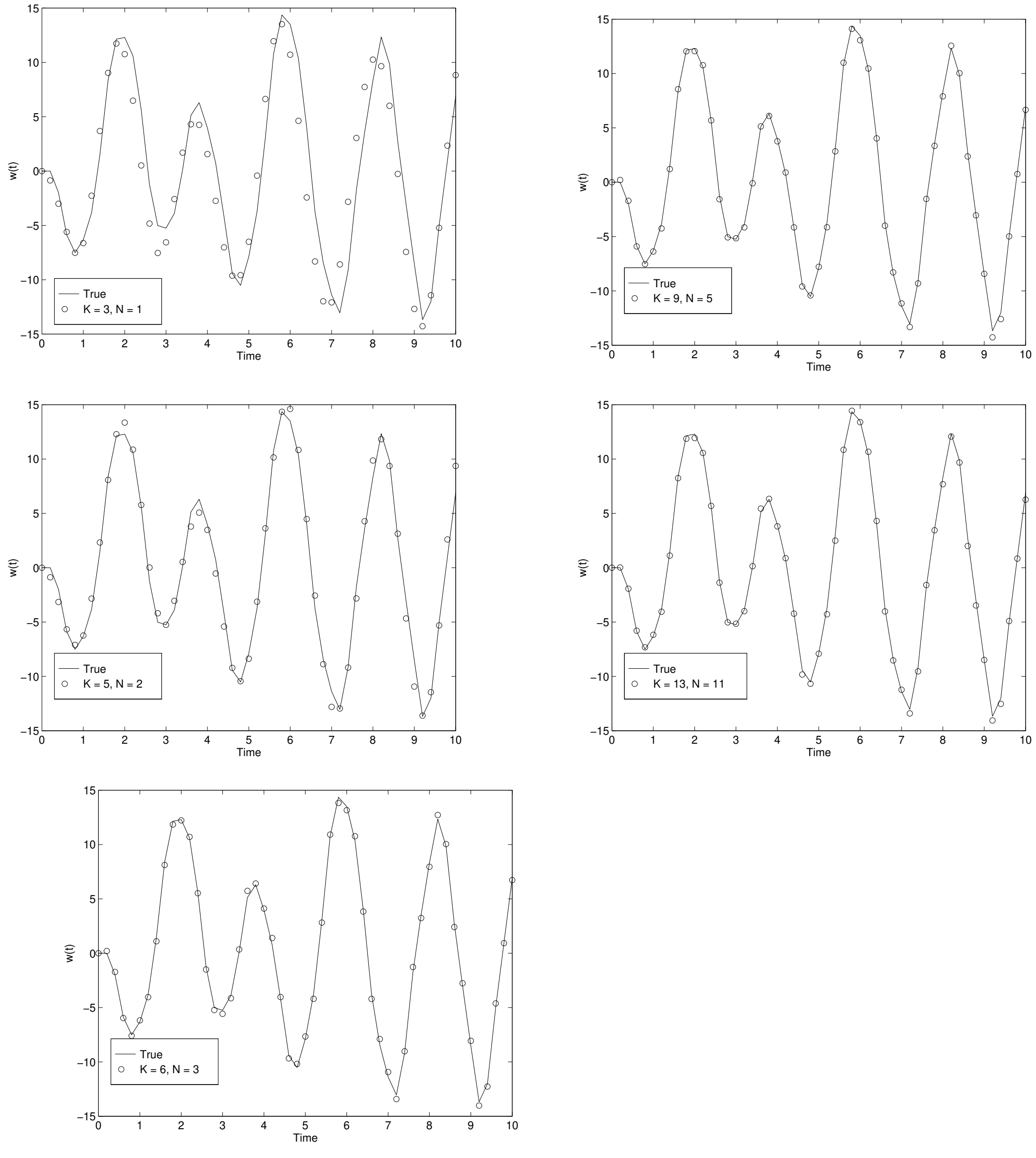

Figure 3: Trajectories ($w(t) = w(t, 0.8)$) for fits to noise free data

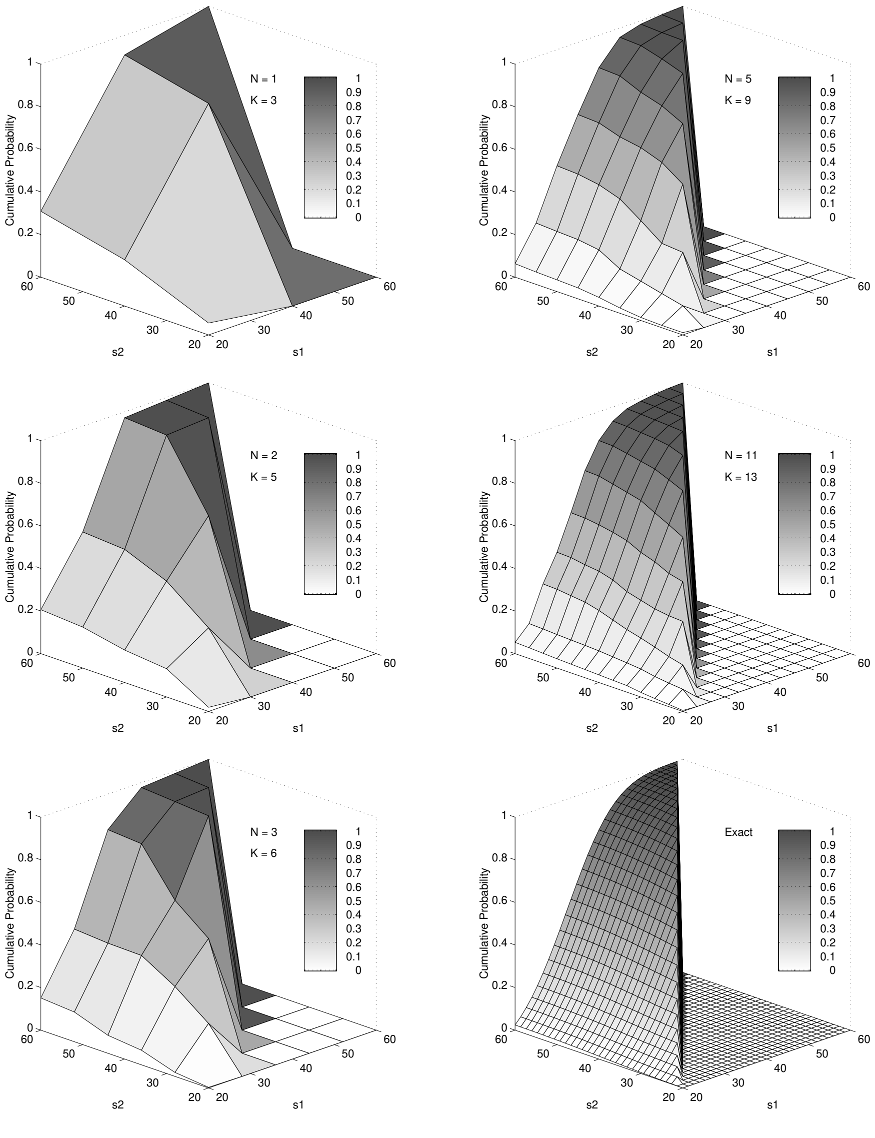

Figure 4: Estimated distributions – noise free data

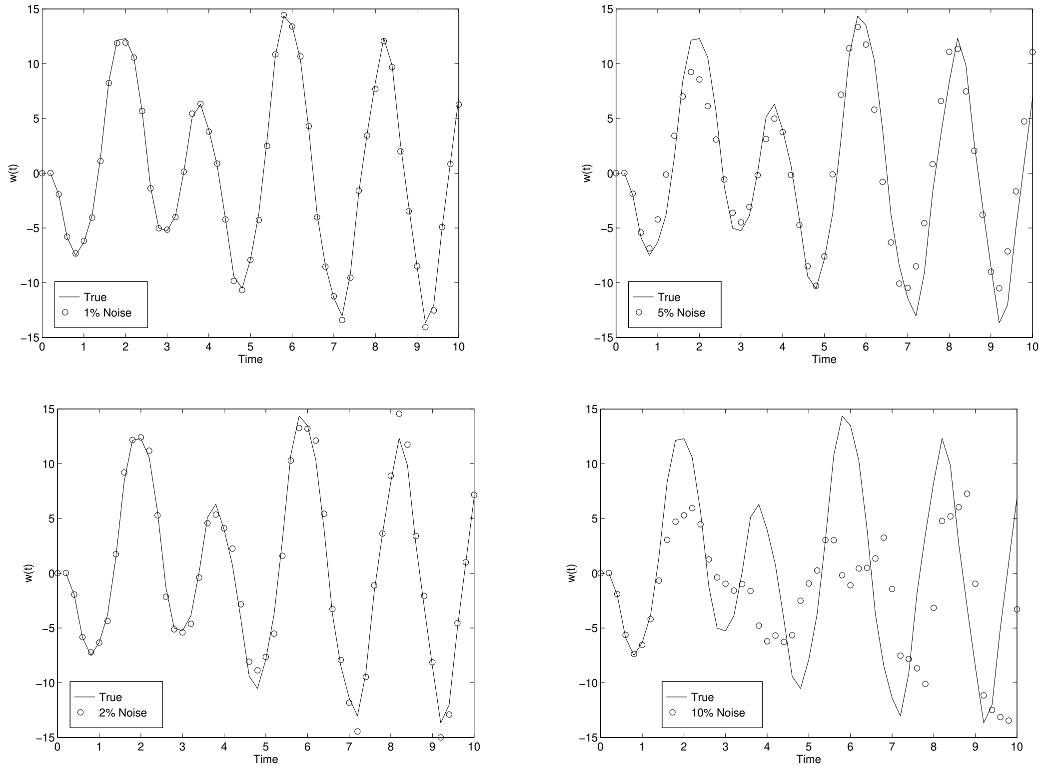

Figure 5: Trajectories ($w(t) = w(t, 0.8)$) for fits to noisy data

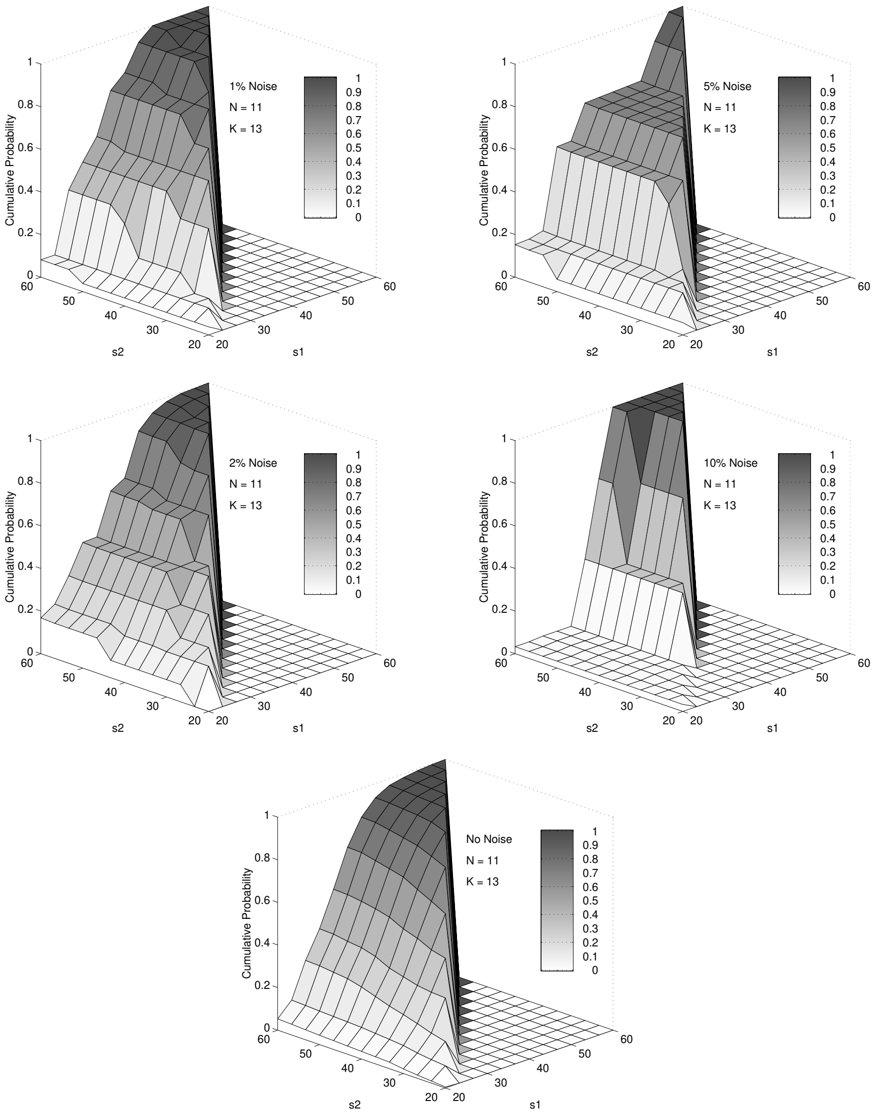

Figure 6: Estimated distributions – noisy data

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