ASYMPTOTIC PROPERTIES OF SOME SEQUENTIAL NONPARAMETRIC
ESTIMATORS IN SOME MULTIVARIATE LINEAR MODELS

by

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Asymptotic Properties of Some Sequential Nonparametric Estimators in Some Multivariate Linear Models

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1. INTRODUCTION

For a general multivariate linear model (which includes the one-sample and two-sample location models as special cases), robust sequential point as well as interval estimators based on suitable rank order statistics are proposed and studied. In a non-sequential set up, parallel procedures were considered by Sen and Puri [14]. Also, the sequential point estimation problem based on sample means (in the univariate case) has been studied earlier by Blum, Hanson and Rosenblatt [3], and later, in a more general set up, by Mogyorodi [10], among others. Finally, the sequential interval estimation procedures, based on the principles of Chow and Robbins [4], extends the univariate theory developed in Sen and Ghosh [13], and Ghosh and Sen [5, 6, 7] to the general multivariate case.

In Section 2, along with our basic model, we briefly sketch the problems. Preliminary notions and basic assumptions are then considered in Section 3. Section 4 is devoted to the study of the asymptotic properties of sequential point estimators based on robust rank order statistics. The problem of robust

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sequential interval estimation is then treated in Section 5. The last section is devoted to a comparison with the corresponding parametric procedures, and presents the allied asymptotic relative efficiency (ARE) results.

2. THE PROBLEMS

Consider a sequence \( \{X_i = (X_{1i}, \ldots, X_{pi})', i \geq 1\} \) of \( p(\geq 1) \)-variate stochastic vectors, defined on a probability space \((\Omega, \mathcal{A}, P)\), where \( X_i \) has an absolutely continuous cumulative distribution function (cdf) \( F_i(x), x \in \mathbb{R}^p \), the \( p \)-dimensional Euclidean space. It is assumed that

\[
F_i(x) = F(x - \alpha_i - \beta c_i), \quad i \geq 1, \tag{2.1}
\]

where \( \alpha = (\alpha_1, \ldots, \alpha_p)' \) and \( \beta = (\beta_1, \ldots, \beta_p)' \) are unknown parameters (vectors), and \( \{c_i, i \geq 1\} \) is a sequence of known (scalar) constants.

Robust point as well as interval estimators of \((\alpha, \beta)\) based on suitable rank order statistics when the sample size is large but non-random were studied in detail in Sen and Puri [14]. We are primarily concerned here with the following two sequential extensions of this theory.

Let \( \{N_\nu, \nu \geq 1\} \) be a sequence of non-negative inter-valued random variables, such that

\[
\nu^{-1/2} N_\nu \Rightarrow \lambda \text{, in probability, as } \nu \to \infty, \tag{2.2}
\]

where \( \lambda \) is a positive random variable having an arbitrary distribution

\[
H(u) = P(\lambda \leq u), \quad 0 < u < \infty, \tag{2.3}
\]

and defined on the same probability space \((\Omega, \mathcal{A}, P)\). Consider then an estimator \( (\hat{\alpha}_{N_\nu}, \hat{\beta}_{N_\nu}) \) of \((\alpha, \beta)\) based on \( X_1, \ldots, X_{N_\nu} \) through a general class of rank order statistics, to be precisely defined in section 3. Our first problem is to derive
(along the lines of Blum, Hanson and Rosenblatt [3], and Mogyorodi [10]) the asymptotic normality of \( \frac{1}{N}\left[ (\hat{\alpha}_{N\nu}, \hat{\beta}_{N\nu}, \hat{\gamma}_{N\nu}) \right] \) (as \( \nu \to \infty \)). This enables us to study various asymptotic properties of \((\hat{\theta}_{N\nu}, \hat{\beta}_{N\nu})\).

In the second problem, our sample size \( N \) remains a random variable, but so determined by a "stopping rule" that we have a simultaneous confidence interval for \((\alpha, \beta)\), with the property that the confidence coefficient is asymptotically equal to a predetermined \( 1-\varepsilon \): \( 0 < \varepsilon < 1 \), and the length of the interval for each component of \( \alpha \) (or \( \beta \)) is bounded above by \( 2d \) (or by a known multiple of \( 2d \)), where \( d > 0 \) is a predetermined (small) number. The theory is an extension of the corresponding univariate theory developed in Sen and Ghosh [13], and Ghosh and Sen [5, 6, 7]. It is also a sequential extension of the theory developed in Sen and Puri [14], and a nonparametric analogue of the theory developed in Gleser [8] and Albert [2].

3. PRELIMINARY NOTIONS AND BASIC ASSUMPTIONS

Let \( F_p \) be the class of all \( p \)-variate absolutely continuous cdf's with finite Fisher information matrix, and let \( F_p^0 \) be the subclass of \( F_p \) for which the distribution is diagonally symmetric about \( 0 \).

**Assumption I.** If we are only interested in \( \beta \), we assume that \( F \in F_p \), otherwise, we assume that \( F \in F_p^0 \), where \( F \) is defined in (2.1). For every \( \nu > 1 \), let

\[
\bar{c}_\nu = \nu^{-1} \sum_{i=1}^\nu c_i, \quad \sigma^2 = \sum_{i=1}^\nu (c_i - \bar{c}_\nu)^2.
\]  

(3.1)

We have then the following problems: (a) estimation of \( \alpha \) assuming \( \beta = 0 \); (b) estimation of \( \beta \) treating \( \alpha \) as a nuisance parameter; (c) simultaneous estimation of \((\alpha, \beta)\). For (a) no assumptions are needed on \( \{c_i, i \geq 1\} \); for (b) and (c) our assumptions are respectively (II, III) and (II, III', IV), where,
Assumption II. As $\nu \to \infty$,
\[
\max_{1 \leq i \leq \nu} (c_i - \bar{c})^2 / \bar{c}^2 = O(1);
\]  
(3.2)

Assumption III.
\[
\lim_{\nu \to \infty} \bar{c}_\nu^2 = \infty;
\]  
(3.3)

Assumption III'.
\[
\lim_{\nu \to \infty} \nu^{-1} \bar{c}_\nu^2 = \bar{c}^2 \quad (0 < \bar{c} < \infty);
\]  
(3.4)

Assumption IV.
\[
\lim_{\nu \to \infty} \bar{c}_\nu = \bar{c} \text{ (finite)}.
\]  
(3.5)

It is easy to verify that all these assumptions hold true for the multivariate one sample (where $c_i = 0$ \( \forall \ i \)) and two-sample (where $c_i$ is either 0 or 1) models.

For every $\nu \geq 1$, let $R_{\nu_i}^{(j)}$ (or $R_{\nu_i}^{(j)+}$) be the rank of $X_{1j}$ (or $|X_{1j}|$) among $X_{1j}, \ldots, X_{\nu j}$ (or $|X_{1j}|, \ldots, |X_{\nu j}|$) for $1 \leq i \leq \nu$, $1 \leq j \leq p$. To estimate $\tilde{\beta}$, consider the following linear (regression) rank statistics
\[
S_{\nu j} = \sum_{i=1}^{\nu} (c_i - \bar{c}) a_{\nu}^{(j)} (R_{\nu_i}^{(j)}), \quad j = 1, 2, \ldots, p,
\]  
(3.6)

\[
S_{\nu} = (S_{\nu 1}, \ldots, S_{\nu p})',
\]  
(3.7)

where the rank scores $a_{\nu}^{(j)}(i), \ 1 \leq i \leq \nu, \ j = 1, \ldots, p$ are defined by
\[
a_{\nu}^{(j)}(i) = \phi_j (U_{\nu i}) [\text{or} \ \phi_j (i/(\nu + 1))], \quad i = 1, \ldots, \nu;
\]  
(3.8)

$\phi_j(u)$ is non-decreasing and absolutely continuous inside $[0, 1]$, $U_{\nu 1} \leq \ldots \leq U_{\nu \nu}$ are the ordered random variable in a sample of size $\nu$ from the rectangular $[0, 1]$ distribution. Regarding the score functions $\phi_1, \ldots, \phi_p$ one assumes as in Ghosh and Sen [6] that for every $j = 1, \ldots, p$, 

\[\]
\[ |\phi_j(u)| \leq K[-\log(u(1-u))], \quad |\phi'_j(u)| \leq K[u(1-u)]^{-1}, \quad 0 < u < 1 \quad (3.9) \]

where \(0 < K \leq \infty\). This implies the existence of a \(t_0(>0)\) such that

\[ M_j(t) = \int_{-\infty}^{\infty} \exp(t\phi_j(u)) du < \infty \quad \text{for all } t: |t| \leq t_0, \quad (3.10) \]

for all \(j=1, \ldots, p\).

For estimating \(\gamma\), we need an alignment procedure and the following type of one sample rank order statistics

\[ T_{\gamma j} = \sum_{i=1}^{\nu} c(x_{ij}) a_\gamma (j)^*(R_{\nu i}^{(j)}), \quad j=1, \ldots, p, \quad (3.11) \]

\[ T_\gamma = (T_{\gamma 1}, \ldots, T_{\gamma p})', \quad (3.12) \]

where \(c(u) = 1, \frac{1}{2} \text{ or } 0\) according as \(u >, = \text{ or } < 0\),

\[ a_\gamma (j)^*(i) = E\phi_j^*(U_{\nu i}) \quad \text{[or } \phi_j^*(i/(\nu+1))], \quad 1 \leq i \leq \nu, \quad (3.13) \]

\[ \phi_j^*(u) = \phi_j\left(\frac{1+u}{2}\right) \text{ and assume that } \phi_j(u) + \phi_j(1-u) = 0. \quad (3.14) \]

Some well-known cases of \(S_\gamma\) and \(T_\gamma\) are the normal scores and the Wilcoxon scores statistics which relate respectively to \(\phi_j(u)\) as the inverse of the standard normal cdf and \(\phi_j(u) = 2u-1, \quad 0 < u < 1\). Let us also define for later use

\[ \Gamma = ((y_{j\ell})), \quad y_{j\ell} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_j(F_{[j]}(x)) \phi_\ell(F_{[\ell]}(y)) dF_{[j\ell]}(x, y) - \frac{u_{j\ell}}{4}, \quad (3.15) \]

for \(j, \ell=1, \ldots, p\), where \(F_{[j]}\) is the jth marginal cdf and \(F_{[j\ell]}\) is the bivariate \((j, \ell)\)th joint cdf in the joint cdf \(F\), and
\[ \mu_j = \int_0^1 \phi_j(u) du, \quad j = 1, \ldots, p. \] (3.16)

**Assumption V.** For every \( j = 1, \ldots, p \), the density function \( f^{[j]} = F'[j] \) and its first derivative \( f'[j] \) exist and are bounded for almost all \( x \) (a.a.x), and

\[ \lim_{x \to \pm \infty} |\phi_j'(F[j](x)) f'[j](x)| \text{ is finite.} \] (3.17)

Let us then denote by

\[ B_j = B(F[j], \phi_j) = \int_{-\infty}^\infty (d/dx)\phi_j(F[j](x)) dF[j](x), \quad j = 1, \ldots, p; \] (3.18)

\[ \Upsilon = ((\tau_{jl})) \quad \tau_{jl} = \gamma_{jl}/[B_j B_l]; \quad j, l = 1, \ldots, p \] (3.19)

Note that \( B_j > 0 \quad \forall j \), and \( \Upsilon \) is positive semi-definite.

4. **ASYMPTOTIC PROPERTIES OF ROBUST SEQUENTIAL POINT ESTIMATORS OF \((\alpha, \beta)\)**

We find it more convenient to consider separately the following three problems:

(I) Estimation of \( \alpha \) assuming that \( \beta = 0 \) (one-sample model),

(II) Estimation of \( \beta \) treating \( \alpha \) to be a nuisance parameter,

(III) Joint estimation of \((\alpha, \beta)\).

In the first problem, assume that \( F \in F_p^0 \), and denote by \( R_{v1}^{(j)+}(a_j) \), the rank of \( |X_{ij} - a| \) among \( |X_{ij} - a|, \ldots, |X_{vj} - a|, 1 \leq i \leq v, 1 \leq j \leq p \); the resulting rank statistics, defined in (3.11) are denoted by \( T_{v_j}(a), j = 1, \ldots, p \). Note that

\[ T_{v_j}(a) \text{ is } + \text{ in } a \text{ for all } j = 1, \ldots, p. \] (4.1)

Define for each positive integer \( v \),
\( \hat{\alpha}_{\nu j}^{(1)} = \sup \{ a : T_{\nu j}(a) > 0 \}, \quad \hat{\alpha}_{\nu j}^{(2)} = \inf \{ a : T_{\nu j}(a) < 0 \}; \)
\( \hat{\alpha}_{\nu j} = \frac{1}{2}(\hat{\alpha}_{\nu j}^{(1)} + \hat{\alpha}_{\nu j}^{(2)}), \quad j = 1, \ldots, p; \)
\( \hat{\alpha}_\nu = (\hat{\alpha}_{\nu 1}, \ldots, \hat{\alpha}_{\nu p})'. \)

We intend to study various asymptotic properties of \( \hat{\alpha}_N \), and towards this goal, we have the following.

**Theorem 4.1.** When \( \mathcal{F}_p \) and \( \beta = 0 \), under (2.1), (2.2), (2.3), (3.9), (3.13), (3.14) and (3.17), as \( \nu \to \infty \)
\( \mathcal{L}(\nu_{1/2} \hat{\alpha}_N - \alpha) \to N_p (0, T), \)

where \( T \) is defined by (3.19).

**Proof.** We use a recent powerful result of Mogyorodt [10] (Theorem 2), according to which we are only to show that for non-stochastic \( \nu \),
\( \mathcal{L}(\nu_{1/2} \hat{\alpha}_N - \alpha) \to N_p (0, T), \) as \( \nu \to \infty, \)

and for every \( \varepsilon > 0 \) and \( \eta > 0 \), there exists a \( \delta > 0 \) and an \( n_o = n_o(\varepsilon, \eta) \), such that for \( n \geq n_o \),
\( P\{ \max_{k} \left| n - k \right| \delta_n \sqrt{n} \left| \hat{\alpha}_k - \alpha_k \right| > \varepsilon \} < \eta, \)

where \( \left| \hat{x} \right| = \max_{1 \leq j \leq p} |x_j|, \ x = (x_1, \ldots, x_p)' \). Now, (4.6) has already been proved in Theorem 6.2.3 (on page 226) of Puri and Sen [11]. On the other hand, the left hand side of (4.7) is bounded above by
\( \sum_{j=1}^{p} P\{ \max_{k} \left| n - k \right| \delta_n \sqrt{n} \left| \hat{\alpha}_{k,j} - \alpha_{n,j} \right| > \varepsilon \}, \)
and hence, by the same technique as in Lemma 5.3 of Sen and Ghosh [13], it can be shown that (4.8) can be bounded by \( \eta(>0) \) but a proper choice of \( \delta(>0) \) and \( n \). For brevity, the proof is therefore omitted.

Since \( \lambda \), defined by (2.2), is a positive random variable, for every \( 0<\varepsilon<1 \), there exists a \( \lambda_\varepsilon(>0) \), such that \( P\{\lambda_\varepsilon(>0) \geq 1-\varepsilon \} \), and hence, \( N_\varepsilon \to \infty \), in probability, as \( \varepsilon \to \infty \). Consequently, by (4.5)

\[
\hat{\theta}_{N_\varepsilon} \to \theta, \text{ in probability, as } \varepsilon \to \infty. \tag{4.9}
\]

Consider now the problem of estimating \( \hat{\theta} \) treating \( \theta \) as a nuisance parameter. Assume that \( F\in F_p \) and that II and III hold. Let \( R_{v_1}^{(j)}(b) \) be the rank of \( X_{i,j}^b \) among \( X_{i,j}^b, \ldots, X_{n,j}^b \) \((1 \leq i \leq \nu; 1 \leq j \leq \nu)\), \( b \) real; the resulting rank statistics defined by (3.6) are then denoted by \( S_{v_j}(b) \) \((1 \leq j \leq \nu; \nu \geq 1)\). It follows from Sen ([12], section 6) that

\[
S_{v_j}(b) \text{ is } + \text{ in } b \text{ for all } j=1,\ldots,p. \tag{4.11}
\]

Define for each \( \nu \geq 1 \),

\[
\hat{\beta}_{v_j}^{(1)} = \sup \{b: S_{v_j}(b) > 0\}, \quad \hat{\beta}_{v_j}^{(2)} = \inf \{b: S_{v_j}(b) < 0\}, \quad 1 \leq j \leq p; \tag{4.12}
\]

\[
\hat{\beta}_{v_j} = \frac{1}{2}(\hat{\beta}_{v_j}^{(1)} + \hat{\beta}_{v_j}^{(2)}), \quad 1 \leq j \leq p; \tag{4.13}
\]

\[
\hat{\beta}_v = (\hat{\beta}_{v_1}, \ldots, \hat{\beta}_{v_p}). \tag{4.14}
\]

Then, parallel to theorem 4.1, we have the following.

**Theorem 4.2.** For \( F\in F_p \), when (2.1), (2.2), (2.3), (3.2), (3.3), (3.9), and (3.17) hold, as \( \nu \to \infty \),

\[
\xi(C_\nu \left[ \hat{\beta}_v - \beta \right]) \to N_p(0, \Sigma), \tag{4.15}
\]
where $T$ is defined by (3.19).

Proof. As in the proof of theorem 4.1, we require only to show that for non-stochastic $\nu$,

$$
\hat{\mathcal{L}}(C_{\nu}[\hat{\beta}_{\nu}]) \rightarrow \mathcal{N}_p(0, I) \text{ as } \nu \rightarrow \infty,
$$

(4.16)

and for every $\varepsilon > 0$ and $\eta > 0$, there exist a $\delta > 0$ and an $n_0 = n_0(\varepsilon, \eta)$, such that for $n > n_0$,

$$
P\{ \max_k \left| n_k \frac{C}{H} \left\| \hat{\beta}_k - \hat{\beta}_n \right\| \right| > \varepsilon \} < \eta.
$$

(4.17)

Now, (4.16) has already been proved in theorem 5.1 of Sen and Puri [14], while (4.17), by virtue of an inequality similar to that in (4.7)-(4.8), follows from Lemma 4.4 of Ghosh and Sen [6]. Hence, the details are omitted.

By (4.16), (3.3) and the discussion preceding (4.9), as $\nu \rightarrow \infty$,

$$
\hat{\beta}_{\nu} \rightarrow \beta, \text{ in probability.}
$$

(4.18)

Finally, consider the joint estimation of $(\alpha, \beta)$. Assume that $P \in F_0^0$ and assumptions II, III', IV and V hold. Define the estimators $\hat{\beta}_{\nu}$ as in (4.12)-(4.14), and then for estimating $\alpha$, consider the following aligned rank statistics.

Let $R_{V_1}^{(j)}(a)$ be the rank of $|X_{ij} - a - \hat{\beta}_{V_j} c_i|$ among $|X_{ij} - a - \hat{\beta}_{V_j} c_1|, \ldots, |X_{ij} - a - \hat{\beta}_{V_j} c_V|$, $(1 \leq i \leq V, 1 \leq j \leq p)$. The resulting one-sample rank-order statistics defined by (3.11) are denoted by $\tilde{T}_{V_j}(a)$, $(1 \leq j \leq p, \nu \geq 1)$. Define

$$
\tilde{\alpha}_{V_j}^{(1)} = \sup \{ a : \tilde{T}_{V_j}(a) > 0 \}, \tilde{\alpha}_{V_j}^{(2)} = \inf \{ a : \tilde{T}_{V_j}(a) < 0 \}, \nu \geq 1, 1 \leq j \leq p;
$$

(4.19)

$$
\tilde{\alpha}_{V_j} = \frac{1}{2}(\tilde{\alpha}_{V_j}^{(1)} + \tilde{\alpha}_{V_j}^{(2)}), \quad 1 \leq j \leq p, \nu \geq 1;
$$

(4.20)

$$
\tilde{\alpha} = (\tilde{\alpha}_{V_1}, \ldots, \tilde{\alpha}_{V_p}), \quad \nu \geq 1.
$$

(4.21)
For notational simplicity, let $\overline{\theta}=(\overline{\theta}_1', \overline{\theta}_2')'$ and $\widehat{\theta}_v=(\widehat{\theta}_1', \widehat{\theta}_2')'$. Then, we have the following theorem.

**Theorem 4.3.** Under (2.1)-(2.3), (3.9), (3.13), (3.14) and assumptions I, II, III', IV and V, as $v \to \infty$,

$$\mathbb{L}(\mathcal{N}(\overline{\theta}_v, \overline{\sigma}_v, \overline{\theta}_v)) \to \mathcal{N}_2(\overline{\theta}, \overline{\sigma}_T),$$  

(4.22)

where $T$ is defined by (3.19) and

$$\overline{\sigma}_T = \begin{bmatrix} 1 + \overline{c}^2 / C^2 & - \overline{c} / C^2 \\ - \overline{c} / C^2 & 1 / C^2 \end{bmatrix}$$

(4.23)

**Proof.** First note that by the same technique as in the proofs of results in section 7 of Sen and Puri [14] (who considered the particular case of $\overline{c}_v=0$ for all $v>1$), one gets,

$$\mathbb{L}(\nu^2(\overline{\theta}_v, \overline{\sigma}_v)) \to \mathcal{N}_2(\overline{\theta}, \overline{\sigma}_T).$$

(4.24)

Hence, similarly as in theorems 4.1 and 4.2, one needs to show that for every $\varepsilon>0$ and $\eta>0$, there exists a $\delta>0$ and an $n_0 = n_0(\varepsilon, \eta)$ such that for $n \geq n_0$,

$$\mathbb{P}(\max_{k-n<\delta n} \left| \overline{\theta}_k - \overline{\theta}_n \right|^2 \leq \varepsilon) < 2\eta.$$  

(4.25)

Now, the left hand side of (4.25) is bounded above by

$$\mathbb{P}(\max_{k-n<\delta n} \left| \overline{\theta}_k - \overline{\theta}_n \right|^2 \leq \varepsilon) + \mathbb{P}(\max_{k-n<\delta n} \left| \overline{\theta}_k - \overline{\theta}_n \right|^2 > \varepsilon)$$

(4.26)

By virtue of (4.17) and Bonferroni inequality, it suffices to show now that

$$\sum_{j=1}^{\nu} \mathbb{P}(\max_{k-n<\delta n} \left| \overline{\theta}_k - \overline{\theta}_n \right|^2 \leq \varepsilon) < \eta.$$  

(4.27)
For simplicity, instead of proving (4.27) we shall consider the following:

Let $\tilde{K}^{(j)+}$ be the rank of $|X_{ij} - \hat{\beta} \nu_j(c_i - \hat{c}_\nu)|$ among $|X_{ij} - \hat{\beta} \nu_j(c_i - c_\nu)|$, $1 \leq i \leq \nu$, $1 \leq j \leq p$. The resulting one sample rank order statistics defined by (3.11) will now be denoted by $\tilde{T}_{\nu j}(a)$ ($1 \leq j \leq p; \nu \geq 1$). Define

$$\hat{\delta}_{\nu j}^{(1)} = \sup\{a: \tilde{T}_{\nu j}(a) > 0\}, \quad \hat{\delta}_{\nu j}^{(2)} = \inf\{a: \tilde{T}_{\nu j}(a) < 0\}, \quad \nu \geq 1, \quad 1 \leq j \leq p; \quad (4.28)$$

$$\hat{\delta}_{\nu j} = \frac{1}{2}(\hat{\delta}_{\nu j}^{(1)} + \hat{\delta}_{\nu j}^{(2)}), \quad 1 \leq j \leq p, \quad \nu \geq 1; \quad (4.29)$$

$$\hat{\delta}_{\nu} = (\hat{\delta}_{\nu 1}, \ldots, \hat{\delta}_{\nu p})', \quad \nu \geq 1. \quad (4.30)$$

It follows from the results of Adichie [1] that

$$\hat{\delta}_{\nu j} = \tilde{\alpha}_{\nu j} + \tilde{\beta}_{\nu j} \tilde{c}_\nu \quad (1 \leq j \leq p; \nu \geq 1), \quad (4.31)$$

i.e.,

$$\hat{\delta}_{\nu} = \tilde{\alpha}_\nu + \tilde{\beta}_{\nu} \tilde{c}_\nu \quad (\nu \geq 1). \quad (4.32)$$

In view of (4.25)-(4.27) and (4.31)-(4.32) it now suffices to show that for every $\varepsilon > 0$ and $\eta > 0$, there exists a $\delta > 0$ and an $n_0 = n_0(\varepsilon, \eta)$ such that for $n \geq n_0$,

$$\sum_{j=1}^{p} \max_{k: |k-n| < \delta n} \|n^{1/2}(\hat{\delta}_k - \hat{\delta}_n)\| > \varepsilon < \eta. \quad (4.33)$$

To prove (4.33) we prove the following two lemmas. Since $\tilde{\alpha}_\nu$ and $\tilde{\beta}_\nu$ (and hence $\hat{\delta}_\nu$) are translation invariant for every $\nu$ (see [14]), we may, for proving these lemmas, assume that $\tilde{\alpha} = \tilde{\beta} = 0$.

**Lemma 4.4.** Under the assumptions of Theorem 4.3, for every $s > 0$, there exist positive constants $c^{(1)}_s$ and $c^{(2)}_s$ and a positive integer $\nu_s$ such that for $q = \beta = 0$, and all $\nu \geq \nu_s$, 


\[ P\left\{ \sup_{|a| \leq K_0 (\log \nu)^{k-\frac{1}{2}}} \nu^{-\frac{1}{2}} \left| \tilde{T}_{\nu_j}^{(1)}(a) - \tilde{T}_{\nu_j}(a) \right| > c_s^{(1)} \nu^{-\delta} (\log \nu)^{k+1} \right\} \leq c_s^{(2)} \nu^{-s}, \quad (4.34) \]

where \( K_0 \) is a positive constant, \( k \) any positive integer and \( \delta \) fixed \((0 < \delta < k)\).

Before proving the above lemma, we may note that taking \( s > 1 \) and on using the Borel-Cantelli Lemma, (4.34) implies that

\[ \sup_{|a| \leq K_0 \nu^{-\frac{1}{2}} (\log \nu)^k} \nu^{-\frac{1}{2}} \left| \tilde{T}_{\nu_j}(a) - \tilde{T}_{\nu_j}(a) \right| \rightarrow 0 \text{ a.s. as } \nu \rightarrow \infty. \quad (4.35) \]

The proof of the lemma is accomplished in several steps. First we show that for any real \( b \), defining \( T_{\nu_j}(a, b) \) as similar to \( \tilde{T}_{\nu_j}(a) \) with \( \hat{\nu}_j \) replaced by \( b \), for \( \nu > \nu_s^{(1)} \) (depending on \( s \)),

\[ P\left\{ \sup_{|a| \leq K_0 \nu^{-\frac{1}{2}} (\log \nu)^k} \sup_{|b| \leq K_1 \nu^{-\frac{1}{2}} (\log \nu)^k} \nu^{-\frac{1}{2}} \left| T_{\nu_j}(a, b) - \tilde{T}_{\nu_j}(a) \right| > c_s^{(3)} \nu^{-\delta} (\log \nu)^{k+1} \right\} < c_s^{(4)} \nu^{-s}, \quad (4.36) \]

where \( K_1 \) is a positive constant, \( c_s^{(3)} \), \( c_s^{(4)} \) are positive constants depending on \( s \). Next, in analogy to lemma 4.1 of Ghosh and Sen [6], one can show that for every \( s > 0 \), there exist positive constants \( c_s^{(5)} \) and \( c_s^{(6)} \) and a positive integer \( \nu_{s_2} \) such that for \( \nu > \nu_{s_2} \),

\[ P_{\beta=0} \left\{ C_\nu \hat{\nu}_j > c_s^{(5)} (\log \nu)^2 \right\} \leq c_s^{(6)} \nu^{-s}. \quad (4.37) \]

Defining now \( c_s^{(1)} \), \( c_s^{(2)} \) and \( \nu_{s} \) appropriately on the basis of \( c_s^{(i)} \) \((i = 3, 4, 5, 6)\), \( \nu_{s}^{(1)} \) and \( \nu_{s}^{(2)} \), one gets (4.34) from (4.35), (4.37) and (3.4). Let \( H_{\nu,j,a,b}(x) = \nu^{-1} \sum_{i=1}^{\nu} u(x - (X_{ij} - a - b(c_i - \tilde{c}_\nu))) \) be the sample df of \( X_{ij} - a - b(c_i - \tilde{c}_\nu)'s \), and let \( C_{\nu,j,a,b}(x) = \nu^{-1} \sum_{i=1}^{\nu} u(x - |X_{ij} - a - b(c_i - \tilde{c}_\nu)|) \) be the sample cdf for \( |X_{ij} - a - b(c_i - \tilde{c}_\nu)|'s \). The corresponding population cdf's are
denoted respectively by $\bar{F}_{\nu,j,a,b}(x) = \nu^{-1} \sum_{i=1}^{\nu} F_{[j]}(x+a+b(c_i-c_\nu))$ and $\bar{D}_{\nu,j,a,b}(x) = \bar{F}_{\nu,j,a,b}(x)-\bar{F}_{\nu,j,a,b}(-x)$. Writing $\phi_j^*(i/(\nu+1)) = a_j^*(j)^*(i)$ (1$\leq i \leq \nu$, \nu$\geq$1), one can now write

$$\nu^{-1}[T_{\nu,j}(a,b)-\bar{T}_{\nu,j}(a)] = \int_0^\infty \phi_j^*(\nu+1)G_{\nu,j,a,b}(x))dH_{\nu,j,a,b}(x)$$

$$- \int_0^\infty \phi_j^*(\nu+1)G_{\nu,j,a,o}(x))dH_{\nu,j,a,o}(x).$$

A result analogous to theorem 3.6.6 of Puri and Sen [11] give

$$\max_{1 \leq i \leq \nu} |\phi_j^*(i/(\nu+1))-\phi_j^*(i/(\nu+1))| = O(\nu^{-1/2}-\delta)$$

for some \(\delta > 0, j=1,2,\ldots,p\). Hence, one can write,

$$\nu^{-1}[T_{\nu,j}(a,b)-\bar{T}_{\nu,j}(a)] = I_{\nu j1}(a,b) + I_{\nu j2}(a,b) + O(\nu^{-1/2}-\delta), \hspace{1cm} (4.38)$$

where

$$I_{\nu j1}(a,b) = \int_0^\infty [\phi_j^*(\nu+1)G_{\nu,j,a,b}(x)) - \phi_j^*(\nu+1)G_{\nu,j,a,o}(x))dH_{\nu,j,a,b}(x), \hspace{1cm} (4.39)$$

$$I_{\nu j2}(a,b) = \int_0^\infty \phi_j^*(\nu+1)G_{\nu,j,a,o}(x))d[H_{\nu,j,a,b}(x)-H_{\nu,j,a,o}(x)]. \hspace{1cm} (4.40)$$

On integration by parts, one can write, using (3.9), (3.13) and (3.14),

$$I_{\nu j2}(a,b) = \int_0^\infty [H_{\nu,j,a,b}(x)-H_{\nu,j,a,o}(x)]\phi_j^*(\nu+1)G_{\nu,j,a,o}(x)) \frac{\nu}{\nu+1}G_{\nu,j,a,o}(x). \hspace{1cm} (4.41)$$

We shall now state a lemma. The proof follows the same line as lemma 4.1 of Sen and Ghosh [13] and theorem 3.1 of Ghosh and Sen [6]. For brevity, the details are omitted.
Lemma 4.5. For every $s(>0)$, there exist two positive constants $k^{(1)}_s$ and $k^{(2)}_s$, and a positive integer $\nu^*_s$ (all of which may depend on $s$) such that for $\nu \geq \nu^*_s$, $k > 1$ and $0 < \delta < \frac{1}{4}$,

\[
P\{ \sup_{-\infty < x < \infty} \sup_{a \leq K_0^{-\frac{1}{2}}(\log \nu)^k} |a| \leq K_0^{-\frac{1}{2}}(\log \nu)^k \ | \sup_{j \leq K_1^{-\frac{1}{2}}(\log \nu)^k} |H_{\nu, j, a, b}(x) - H_{\nu, j, a, o}(x)| - F_{\nu, j, a, b}(x) + F_{\nu, j, a, o}(x) | > k^{(1)}_s^{-\frac{1}{2}}(\log \nu)^k \} \leq k^{(2)}_s^{-\nu s}. \tag{4.42}
\]

Using also the fact that $F_{\nu, j, a, b}(x) - F_{\nu, j, a, o}(x) = \nu^{-1} \sum_{i=1}^{\nu} \left[ F(x+a+b(c_i-c_{\nu})) - F(x+a) \right] = 0(\nu^{-1}(\log \nu)^{2k})$, uniformly in $x$, $a$ and $|b| \leq K_1^{-\frac{1}{2}}(\log \nu)^k$, one gets,

\[
P\{ \sup_{-\infty < x < \infty} \sup_{a \leq K_0^{-\frac{1}{2}}(\log \nu)^k} |a| \leq K_0^{-\frac{1}{2}}(\log \nu)^k \ | \sup_{j \leq K_1^{-\frac{1}{2}}(\log \nu)^k} |H_{\nu, j, a, b}(x) - H_{\nu, j, a, o}(x)| > k^{(1)}_s^{-\frac{1}{2}}(\log \nu)^k \} \leq k^{(2)}_s^{-\nu s} \text{ for } \nu \geq \nu^{**}, \text{ say.} \tag{4.43}
\]

Thus, by (4.41) and (4.43), one gets by using (3.9), (3.13) and (4.14) that

\[
\sup_{-\infty < x < \infty} |a| \leq K_0^{-\frac{1}{2}}(\log \nu)^k \ | \sup_{j \leq K_1^{-\frac{1}{2}}(\log \nu)^k} I_{\nu j}(a, b) | \leq 0(\nu^{-\frac{1}{2}}(\log \nu)^k) \sum_{i=1}^{\nu} \frac{\nu}{\nu+1} K[1-i/\nu+1]^{-1} = 0(\nu^{-\frac{1}{2}}(\log \nu)^{k+1}), \tag{4.44}
\]

with probability $\geq 1 - k^{(2)}_s^{-\nu s}$, for $\nu \geq \nu^{**}$.

Again, write
\[ I_{\nu j}^1(a, b) = \frac{\nu}{\nu + 1} \int_0^\infty [G_{\nu, j, a, b}(x) - G_{\nu, j, a, o}(x)] \phi_j^*(\frac{\nu}{\nu + 1}) [\theta G_{\nu, j, a, b}(x) + (1 - \theta)G_{\nu, j, a, o}(x))] dh_{\nu, j, a, b}(x), \quad (0 < \theta < 1). \] (4.45)

Since \( G_{\nu, j, a, b}(x) - G_{\nu, j, a, o}(x) = [H_{\nu, j, a, b}(x) - H_{\nu, j, a, o}(x)] - [H_{\nu, j, a, b}(-x) - H_{\nu, j, a, o}(-x)] \), it follows from (4.43) that

\[
\sup_{-\infty < k < \infty} \left| a \right| \leq K_0 \nu^{-\frac{1}{2}} (\log \nu)^k \sup_{-\infty < k < \infty} \left| b \right| \leq K_0 \nu^{-\frac{1}{2}} (\log \nu)^k |G_{\nu, j, a, b}(x) - G_{\nu, j, a, o}(x)| \leq K_s^{1(1)} \nu^{-\frac{1}{2} - \delta} (\log \nu)^k
\]

with probability \( \geq 1 - K_s^{(2)} \nu^{-s} \) for large \( \nu \). Using arguments analogous to (4.20)-(4.27) of theorem 4.3 in Sen and Ghosh [13], one can prove now that

\[
\sup_{-\infty < k < \infty} \left| a \right| \leq K_0 \nu^{-\frac{1}{2}} (\log \nu)^k \sup_{-\infty < k < \infty} \left| b \right| \leq K_0 \nu^{-\frac{1}{2}} (\log \nu)^k |I_{\nu j}^1(a, b)| \leq K_s^{3(3)} \nu^{-\frac{1}{2} - \delta} (\log \nu)^{k+1}
\]

with probability \( \geq 1 - K_s^{(4)} \nu^{-s} \) for large \( \nu \). Hence, the lemma.

For proving (4.33) we need another lemma which we prove below. For proving this lemma, we take \( a^{(j)*}_\nu(i) = \phi_j^*(i/(\nu + 1)) = \phi_j^*(U_{\nu i}) \) \((1 \leq i \leq \nu, 1 \leq j \leq p)\).

**Lemma 4.6.** For \( \alpha = \gamma = 0 \), for every \( s > 0 \), there exist positive constants \( d_s^{(1)} \) and \( d_s^{(2)} \) and a positive integer \( \nu_{so} \) such that for \( \nu > \nu_{so} \),

\[
P\{ \nu_{\nu j}^* \geq d_s^{(1)} \nu^{-\frac{1}{2}} (\log \nu)^k \} \leq d_s^{(2)} \nu^{-s}. \] (4.46)

**Proof.** We prove only the case of \( P\{ \nu_{\nu j}^* \geq d_s^{(1)} \nu^{-\frac{1}{2}} (\log \nu)^k \} \) as the other case follows similarly. Note that
\[
P\{\hat{\delta}_{v_j} > d_s^{(1)} v^{-\frac{1}{2}} (\log v)^k\} \leq P\{\hat{\delta}_{v_j}^{(2)} > d_s^{(1)} v^{-\frac{1}{2}} (\log v)^k\}
\]
\[
= P\{\tilde{T}_{v_j} (d_s^{(1)} v^{-\frac{1}{2}} (\log v)^k) > 0\} = P\{v^{-\frac{1}{2}} \tilde{T}_{v_j} (d_s^{(1)} v^{-\frac{1}{2}} (\log v)^k) \geq 0\}. \tag{4.47}
\]

It follows from Lemma 4.5 that for every \(s > 0\), for large \(v\), with probability \(\geq 1 - c_s^{(2)} v^{-s}\),
\[
v^{-\frac{1}{2}} [\tilde{T}_{v_j} (d_s^{(1)} v^{-\frac{1}{2}} (\log v)^k) - \tilde{T}_{v_j} (d_s^{(1)} v^{-\frac{1}{2}} (\log v)^k)] \leq c_s^{(1)} v^{-\delta} (\log v)^k. \tag{4.48}
\]

Further, from theorem 4.3 of Sen and Ghosh [6], we have
\[
v^{-\frac{1}{2}} [\tilde{T}_{v_j} (d_s^{(1)} v^{-\frac{1}{2}} (\log v)^k) - \tilde{T}_{v_j} (0)] - d_s^{(1)} (\log v)^k \leq d_s^{(2)} v^{-\delta} (\log v)^k, \tag{4.49}
\]
with probability \(\geq 1 - c_s^{(2)} v^{-s}\), for large \(v\). Hence, from (4.47)-(4.49), it suffices to show that for large \(v\), for every \(s > 0\), there exist constants \(d_s^{(1)}\) and \(d_s^{(2)}\) such that
\[
P\{v^{-\frac{1}{2}} \tilde{T}_{v_j} (0) > d_s^{(1)} (\log v)^k\} \leq d_s^{(2)} v^{-\delta} (\log v)^k. \tag{4.50}
\]

Since, \(\alpha = \beta = 0\), for every \(v\), \(R^{(j)+}_{v_j} = (R^{(j)+}_{v_1}, \ldots, R^{(j)+}_{v_r})\), is independent of \(s_j^{(j)} = (s(X_{1j}), \ldots, s(X_{ij}))\), where \(s(u) = 2c(u) - 1\) i.e., \(s(u) = 1, 0\) or \(-1\), according as \(u > 0\), =, or \(< 0\). Now
\[
v^{-\frac{1}{2}} \tilde{T}_{v_j} (0) = v^{-\frac{1}{2}} \sum_{i=1}^{l} \frac{1+s(X_{ij})}{2} \mathbb{E} \phi_j^*(U R^{(j)+}_{v_i})
\]
\[
= \frac{1}{2} \int_{0}^{\frac{1}{2}} \phi_j^*(u) du + \frac{1}{2} v^{-\frac{1}{2}} \sum_{i=1}^{l} s(X_{ij}) \mathbb{E} \phi_j^*(U R^{(j)+}_{v_i})
\]

Since, (3.10) holds, we get from (3.13) and (3.14) the first term to be \(O(v)\).

Hence, (4.49) will be proved if one can show that for large \(v\)
\[ P\{v^{-\frac{1}{2}T_{v(j-1)}} > 2d_s^{(1)}(\log v)^{k} \} \leq d_s^{(2)}v^{-s}, \quad (4.51) \]

where
\[ T_{v(j-1)} = \sum_{i=1}^{v} s(X_{ij}) e_{ij}(U_{ij})^{(j+1)}, \quad 1 \leq j \leq p. \]

Writing \( g_v = 2d_v^{(1)}v^{\frac{1}{2}}(\log v)^{k} \), and using the Bernstein inequality, one gets,

\[ P\{T_{v(j-1)} > g_v \} \leq \inf_{t>0} E[\exp\{t(T_{v(j-1)} - g_v)\}] \quad (4.52) \]

Now,
\[ E[\exp\{t(T_{v(j-1)} - g_v)\}] = \exp(-tg_v)E[\exp(tT_{v(j-1)})]. \]

Again,
\[ E[\exp(tT_{v(j-1)})] = E[E[\prod_{i=1}^{v} \exp(ts(X_{ij})e_{ij}(U_{ij})^{(j+1)})|R_{v}^{(j)+}]] \]

Using the independence of \( s_{ij}^{(j)} \) and \( r_{ij}^{(j)+} \) and also the elementary inequality \( \frac{1}{2}(e^x + e^{-x}) \leq \exp(x^2/2) \), one gets,

\[ E[\exp(tT_{v(j-1)})] = E[\prod_{i=1}^{v} \{\frac{1}{2} \exp(t e_{ij}(U_{ij})^{(j+1)}) + \frac{1}{2} \exp(t e_{ij}(U_{ij})^{(j)+})\}] \]

\[ \leq E \prod_{i=1}^{v} \exp(\frac{t^2}{2} (e_{ij}(U_{ij})^{(j)+})^2) \]

\[ \leq E \prod_{i=1}^{v} \exp(\frac{t^2}{2} E_{j}^{(j)}(U_{ij}^{(j)+})) \]

\[ = E \exp(\frac{t^2}{2} \sum_{i=1}^{v} E_{j}^{(j)}(U_{ij}^{(j)+})) \]

\[ = E \exp(\frac{t^2}{2} \sum_{i=1}^{v} E_{j}^{*(j)}(U_{ij}^{(j)+})) = \exp(\frac{vt^2A_j^2}{2^l}), \]

where \( A_j^2 = \int_{0}^{1} \phi_{j}^{*2}(u)du. \) Thus, from (4.52),

\[ P\{T_{v(j-1)} > g_v \} \leq \inf_{t>0} \exp(-tg_v + \frac{vt^2A_j^2}{2^l}) = \exp(-\frac{g_v^2}{2vtA_j^2}) \]

\[ = \exp(-\frac{2d_s^{(1)}v^2}{A_j^2}(\log v)^{2k}), \]
and hence, (4.50) follows. Hence, the lemma.

It follows from Lemmas 4.4 and 4.6 that for large \( \nu \), \( \nu^{-2}[\tilde{T}_{v,j}(\hat{\delta}_v) - T_{v,j}(\hat{\delta}_v)] = O(\nu^{-s}(\log \nu)^k) \) with probability \( \geq 1-\text{const. } \nu^{-s} \). Again, it follows from theorem 4.3 of Sen and Ghosh [13] that \( \nu^{-2}[T_{v,j}(\hat{\delta}_v) - T_{v,j}(0)] + \nu^{2s}\delta B_j = 0(\nu^{-s}(\log \nu)^k) \) with probability \( \geq 1-\text{const. } \nu^{-s} \). Hence, with probability \( \geq 1-\text{const. } \nu^{-s} \), \( \nu^{-2}[T_{v,j}(\hat{\delta}_v) - T_{v,j}(0)] + \nu^{2s}\delta B_j = O(\nu^{-s}(\log \nu)^k) \). i.e., \( \nu^{2s}\delta B_j - \nu^{-2}T_{v,j}(0) = O(\nu^{-s}(\log \nu)^k) \), noting that \( \tilde{T}_{v,j}(\hat{\delta}_v) = 0 \). (4.33) now follows from theorem 4.5 of Sen and Ghosh [13]. Hence the theorem.

5. BOUNDED LENGTH (SEQUENTIAL) CONFIDENCE BANDS FOR \( \varrho \)

Parallel to problems (I)-(III) of section 4, we consider here the following three problems.

Problem I' Confidence estimation of \( \varrho \) assuming that \( \varrho = 0 \). More specifically we want a p-dimensional confidence rectangle for \( \varrho \) such that the length of each side \( \leq 2d \) \( (d>0, \text{ preassigned}) \) and the confidence coefficient \( \geq 1-\alpha \). This can be achieved by a direct extension of the results of Sen and Ghosh [13].

To see this, first note that under \( \varrho = \varrho = 0 \), \( T_{v,0} = (T_{v,10}, \ldots, T_{v,p0})' \), \( (T_{v,j0})'s \) defined after (4.51) has a distribution independent of \( F \) diagonally symmetric about \( 0 \). Hence, there exists a known constant \( T_{v,\varepsilon} \) such that

\[
P_{\varrho = \varrho = 0} \{ \max_{1 \leq j \leq p} |T_{v,j0}| \leq T_{v,\varepsilon} \} = 1-\varepsilon + 1-\varepsilon \text{ as } \nu \to \infty.
\]  

(5.1)

For large \( \nu \), \( \sqrt{\nu} T_{v,\varepsilon} + \chi_{p,\varepsilon}^* \) where \( \chi_{p,\varepsilon}^* \) is the upper 100\( \varepsilon \)% point of the distribution of the maximum of \( \gamma_1, \ldots, \gamma_p \) where \( \gamma = (\gamma_1, \ldots, \gamma_p)' \) is N(0, \( \nu \)). Define now

\[
\hat{\alpha}_{L,j,v} = \sup\{a: T_{v,j0}(a) > T_{v,\varepsilon}\},
\]  

(5.2)

\[
\hat{\alpha}_{U,j,v} = \inf\{a: T_{v,j0}(a) < -T_{v,\varepsilon}\},
\]  

(5.3)
where $T_{\nu j}(a)$ is defined in the same way as $T_{\nu j} = T_{\nu j}(0)$, replacing $X_i$'s by $X_1$'s $(1 \leq j \leq p, 1 \leq \nu \leq p)$. Then, $P_{\nu = \nu_0} = \{\alpha_L, j, \nu \leq \alpha_j \leq \alpha_U, j, \nu \forall 1 \leq j \leq p\} = P_{\nu = \nu_0} \leq T_{\nu j}(0, \nu, \nu) = 1 - \epsilon_{\nu} \to 1 - \epsilon$ as $\nu \to \infty$.

We define the stopping variable $N = N(d)$ to be the least positive integer $n(n_0)$ such that $\max_{1 \leq j \leq p} (\hat{\alpha}_U, j, n - \hat{\alpha}_L, j, n) \leq 2d$. Now, using Theorem 4.3 and Lemma 5.1 of Sen and Ghosh [13],

$$\sqrt{\nu} [T_{\nu j}(\hat{\alpha}_U, j, \nu) - T_{\nu j}(0) + 2\hat{\alpha}_U, j, \nu B_j] = O(\nu^{-\frac{1}{4}}(\log \nu)^{\frac{3}{4}}) \quad (5.4)$$

with probability $\geq 1 - \text{const.} \nu^{-s}$, for every $s > 0$, large $\nu$. Thus noting that when $a^{(j)}_{\nu}(i) = E[\phi^{(j)}_{\nu i}(\nu)]$, $1 \leq i \leq \nu$, $T_{\nu j}(a) = 2T_{\nu j}(a) - \int_0^1 \phi^{(j)}(u)du$, for all real $a$, it follows from (5.3) and (5.4) that

$$-\chi_{p, \nu}^* - \sqrt{\nu} T_{\nu j}(0) + \sqrt{\nu} \alpha_{U, j, \nu} B_j \to 0 \text{ a.s. as } \nu \to \infty.$$  

Similarly,

$$\chi_{p, \nu}^* - \sqrt{\nu} T_{\nu j}(0) + \sqrt{\nu} \alpha_{L, j, \nu} B_j \to 0 \text{ a.s. as } \nu \to \infty.$$  

Thus,

$$\sqrt{\nu} (\hat{\alpha}_{U, j, \nu} - \hat{\alpha}_{L, j, \nu}) \to \frac{2\chi_{p, \nu}^*}{B_j} \text{ a.s. as } \nu \to \infty.$$  

Hence,

$$\max_{1 \leq j \leq p} \sqrt{\nu} (\hat{\alpha}_{U, j, \nu} - \hat{\alpha}_{L, j, \nu}) \to \frac{2\chi_{p, \nu}^*}{\min_{1 \leq j \leq p} B_j} \text{ a.s. as } \nu \to \infty. \quad (5.5)$$

It follows now from the definition of $N$ that $\lim_{d \to 0} N(d)/s(d) = 1$ a.s., where $s(d) = \chi_{p, \nu}^2 / d^2 \min_{1 \leq j \leq p} B_j^2$, and as to the rate of convergence, we can make a similar statement as (5.4). Thus, generalizing the results of Sen and Ghosh [13], we get the following theorem.

**Theorem 5.1.** Under the assumptions $F \in \cal F^0_p$, (2.1)-(2.3), (3.9), (3.13)-(3.14) and
(3.17),

\[ N(\pi N(d)) \text{ is a non-increasing function of } d; \quad N(d) \leq 0 \]

with probability 1, \( EN(d) \leq 0 \) for all \( d > 0 \),

\[ \lim_{d \to 0} N(d) = 0 \text{ a.s., and } \lim_{d \to 0} EN(d) = 0. \]  \hspace{1cm} (5.6)

\[ \lim_{d \to 0} \frac{N(d)}{s(d)} = 1 \text{ a.s.} \]  \hspace{1cm} (5.7)

\[ \lim_{d \to 0} \mathbb{P} \left\{ \hat{\alpha}_L, j, N \leq \alpha_j \leq \hat{\alpha}_U, j, N \quad \forall 1 \leq j \leq p \right\} = 1 - \varepsilon. \]  \hspace{1cm} (5.8)

\[ \lim_{d \to 0} \frac{EN(d)}{s(d)} = 1. \]  \hspace{1cm} (5.9)

We now suggest an alternate procedure for the same problem. We find a confidence region \( R_N \) for \( \alpha \) such that the maximum diameter of \( R_N \leq 2d \). Our procedure is analogous to the one proposed by Srivastava [15].

We define

\[ \hat{\gamma}_{j,k}^{(n)} = \int_{-\infty}^{\infty} \phi_j \left( \frac{n}{n+1} F_{j,n}(x) \right) \phi_k \left( \frac{n}{n+1} \mathbb{F}^{[j]} n(y) \right) dF_{j,k}(x,y) - \mu_j \mu_k, \]  \hspace{1cm} (5.10)

for \( 1 \leq j \neq k \leq p \) where \( F_{j,n}(x) \) and \( F_{j,k}(x,y) \) are the empirical df's corresponding to the true df's \( F_{[j]}(x) \) and \( F_{[j,k]}(x,y) \) respectively, for \( j = k \), \( \hat{\gamma}_{j,j}^{(n)} = \gamma_{j,j} = \int_{-\infty}^{\infty} \phi_j^2(u) du - \mu_j^2 \), \( 1 \leq j \leq p \). Also, define \( \hat{B}_{j,n} \) as the estimator of \( B_j \) (\( 1 \leq j \leq p \)) as in Lemma 4.2 of Ghosh and Sen [6]. Define then

\[ \hat{T}_n = (\hat{\gamma}_{j,k}^{(n)}); \quad \hat{\gamma}_{j,k}^{(n)} = \hat{\gamma}_{j,k} / \hat{B}_{j,n}, \quad j, k = 1, \ldots, p. \]  \hspace{1cm} (5.11)

We denote by

\[ \hat{\lambda}_n = \text{max. ch. root of } \hat{T}_n; \quad \lambda = \text{max. ch. root of } T, \]  \hspace{1cm} (5.12)

where \( T \) is defined by (3.19); finally, \( \chi^2_{p, \varepsilon} \) is defined as the upper 100\( \varepsilon \)% point of the chi square distribution with \( p \) degrees of freedom. Our procedure is as follows.
Starting with an initial sample of size \( n_0 \) (\( >p \)), we continue drawing observations one at a time according to a stopping time \( N \) defined by

\[
N = N(d) = \text{smallest } n \geq n_0 \text{ such that } \frac{\hat{\lambda}_n - d^2 n}{\chi^2_{p, \alpha}} \leq \frac{d^2}{n}
\]  

(5.13)

When sampling is stopped at \( N = n \), construct the region \( R_n \) defined by

\[
R_n = \{ z : (\hat{\alpha}_n - z)'(\hat{\alpha}_n - z) \leq \frac{d^2}{n} \}
\]

(5.14)

Then, we have the following theorem.

**Theorem 5.2.** Under the assumption that \( 0 < \lambda < \infty \) and the hypothesis of Theorem 5.1, the results of Theorem 5.1 all hold for the stopping variable \( N(d) \), defined by (5.13) and \( R_n \), defined by (5.14), provided we replace \( s(d) \) in (5.7) and (5.9) by

\[
u(d) = \chi_{p, \alpha}^2 \lambda / d^2.
\]

(5.15)

**Proof.** Running down the proof of Srivastava [16], it suffices to show that \( \hat{\lambda}_n \rightarrow \lambda \) a.s., as \( n \rightarrow \infty \); by the Courant Theorem, it thus suffices to show that

\[
\hat{\lambda}_n \rightarrow T \text{ a.s., as } n \rightarrow \infty.
\]

(5.16)

Since, \( \hat{B}_{j,n} \), \( j=1, \ldots, p \), converge a.s. to \( B_{j} \), \( j=1, \ldots, p \) as \( n \rightarrow \infty \) (See [13]), it suffices to prove the following lemma.

**Lemma 5.3.** Under (3.4), (3.17), (3.18) and (3.19),

\[
\hat{\gamma}_{j,l}^{(n)} \rightarrow \gamma_{j,l} \text{ a.s., as } n \rightarrow \infty, \text{ for all } 1 \leq j \neq l \leq p.
\]

(5.17)

**Proof.** Since \( \phi_j(u) \) is assumed to be non-decreasing, absolutely continuous and square integrable inside \([0,1]\), by Lemma 5.1 of Hájek [9], we may write for \( 0 < u < 1 \),
\[ \phi_j(u) = \phi^{(1)}_j(u) - \phi^{(2)}_j(u) + \phi^{(3)}_j(u), \]  
(5.18)

where \( \phi^{(1)}_j(n) \) is a polynomial (i.e., has bounded second derivative) and

\[ \int_0^1 \{\phi^{(k)}_j(\mu)\}^2 d\mu < \frac{1}{2} \int_0^1 \phi^2_j(\mu) d\mu, \quad k=2,3, \]  
(5.19)

where \( \varepsilon > 0 \) is arbitrarily small. By (3.8), we may decompose the scores \( a^{(j)}_v(i), \quad 1 \leq i \leq v, \) also in three parts. On the first part, involving \( \phi^{(1)}_j \), almost sure convergence of \( F_j[n] \) and \( F_{j,\varepsilon}[n] \) to \( F_j \) and \( F_{j,\varepsilon} \) (respectively) implies the a.s. convergence of the corresponding component of \( \hat{\gamma}_j[\varepsilon] \) to that of \( \gamma_j[\varepsilon] \); on the other components, the Schwarz inequality and (5.19) imply that the same can be made arbitrarily small by proper choice of \( \varepsilon (> 0) \). Q.E.D.

Remark. In (5.14), we could have taken a region \( \{ z : \tilde{A}_n - \tilde{z} \tilde{A}^{-1}_n (\tilde{A}_n - \tilde{z}) \leq \tilde{d}^2 \} \),

where \( \tilde{A} \) is any given positive definite matrix. In that case, we need to define

\( \hat{\lambda}_n = \) max. ch. root of \( \tilde{A}^{-1}_n T_n \) and \( \lambda = \) max. ch. root of \( \tilde{A}^{-1}_n T \). The proofs follows on parallel lines.

Problem II'. Confidence band for \( \beta \) treating \( \alpha \) as a nuisance parameter

(1) Rectangular regions. Note that under \( \beta = 0 \), \( s_{\gamma j} \)'s have a completely specified distribution generated by \( (n!)^p \) equally likely realizations of the ranks. Hence, there exists a known \( s_{v,\varepsilon} \) such that

\[ P_{\beta = 0} \left\{ \max_{1 \leq j \leq p} |S_{\gamma j}| \leq S_{v,\varepsilon} \right\} = 1 - \varepsilon_v \to 1 - \varepsilon \text{ as } v \to \infty. \]

For large \( v, \sqrt{S}_{v,\varepsilon} \to x^*_p, \varepsilon \), the upper 100\% point of the distribution of the maximum of \( \gamma_1, \ldots, \gamma_p \) where \( \gamma = (\gamma_1, \ldots, \gamma_p)' \) is \( N(0,v) \). Define now

\[ \hat{\beta}_{L,v} = \sup\{b: S_{\gamma j}(b) > S_{v,\varepsilon}\} \]
\[ \hat{\beta}_{U,j,v} = \inf \{ b : S_{v,j}(b) < -s_{v,v}, \epsilon \} \]

Then,

\[ P_{\beta = 0} \{ \hat{\beta}_L,j,v \leq \beta_j \leq \hat{\beta}_U,j,v \ \forall 1 \leq j \leq p \} \]
\[ = 1 - \epsilon_v \rightarrow 1 - \epsilon \text{ as } v \rightarrow \infty. \] (5.20)

We define the stopping variable \( N = N(d) \) to be the least positive integer \( n(\geq n_0) \) such that \( \max_{1 \leq j \leq p} (\hat{\beta}_{U,j,n} - \hat{\beta}_{L,j,n}) \leq 2d. \) Using Lemma 4.2 of Ghosh and Sen [6], we can now prove the following theorem. The proof is omitted because of its obvious analogy to Theorem 5.1.

**Theorem 5.4.** If \( F \in F_p \), then under (2.1)-(2.3), (3.9) and (3.17), \( N(d) \) as defined above and the related confidence band for \( \beta \) satisfy the results of Theorem 5.1 provided we define

\[ s(d) = Q^{-1}(\chi_p^2/[d^2 \max_{1 \leq j \leq p} B_j^2]), \] (5.21)

where \( Q(n) = C_n^2 \) for \( n \geq 1 \) and is obtained by linear interpolation for non-integer \( t(>0). \)

(ii) Spherical or Ellipsoidal regions. Here, we start by taking \( n_0(\geq p) \) observations \( x_1, \ldots, x_{n_0} \) and continue sampling one observation at a time in accordance with the stopping variable

\[ N(d) = \text{smallest } n(\geq n_0) \text{ such that } \lambda_n \leq d^2 C_n^2/\chi_p^2, \]

where \( \lambda_n \) and \( \chi_p^2, \epsilon \) are defined in (5.12) and after that. When sampling is stopped at \( N=n \), we construct the region \( R_n \) defined by

\[ R_n = \{ \beta : (\hat{\beta}_n - \hat{\beta})'(\hat{\beta}_n - \beta) \leq d^2 \}, \] (5.22)

where \( \hat{\beta}_n \) is defined by (4.14). Then, we have the following.
Theorem 5.5. The conclusions of Theorem 5.2 holds for \( N(d) \) and \( R_n \), defined as above, provided we let
\[
\nu(d) = Q^{-1}(\lambda \chi^2_{p,e}/d^2).
\]

The proof follows along the same line as in Theorems 5.1 and 5.2.

Problem III'. Confidence bands for \( \theta \). Here also, we can have either a rectangular or an ellipsoidal region for \( \Theta = (\alpha, \beta) \). We need to change \( \chi^*_p, e \) and \( \chi^2_p, e \) to \( \chi^*_2p, e \) and \( \chi^2_{2p, e} \) respectively, and therefore, in view of the similarity with problems I' and II', the details are omitted.

REFERENCES


