ASYMPTOTIC PROPERTIES OF SOME SEQUENTIAL NONPARAMETRIC ESTIMATORS IN SOME MULTIVARIATE LINEAR MODELS

by

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Asymptotic Properties of Some Sequential Nonparametric Estimators in Some Multivariate Linear Models

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1. INTRODUCTION

For a general multivariate linear model (which includes the one-sample and two-sample location models as special cases), robust sequential point as well as interval estimators based on suitable rank order statistics are proposed and studied. In a non-sequential set up, parallel procedures were considered by Sen and Puri [14]. Also, the sequential point estimation problem based on sample means (in the univariate case) has been studied earlier by Blum, Hanson and Rosenblatt [3], and later, in a more general set up, by Mogyorodi [10], among others. Finally, the sequential interval estimation procedures, based on the principles of Chow and Robbins [4], extends the univariate theory developed in Sen and Ghosh [13], and Ghosh and Sen [5, 6, 7] to the general multivariate case.

In Section 2, along with our basic model, we briefly sketch the problems. Preliminary notions and basic assumptions are then considered in Section 3. Section 4 is devoted to the study of the asymptotic properties of sequential point estimators based on robust rank order statistics. The problem of robust

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sequential interval estimation is then treated in Section 5. The last section is devoted to a comparison with the corresponding parametric procedures, and presents the allied asymptotic relative efficiency (ARE) results.

2. THE PROBLEMS

Consider a sequence $\{\underset{\sim}{X}_i = (X_{i1},\ldots,X_{ip})', i\geq 1\}$ of $p(\geq 1)$-variate stochastic vectors, defined on a probability space (Ω,A,P), where $\underset{\sim}{X}_i$ has an absolutely continuous cumulative distribution function (cdf) $F_i(\underset{\sim}{x})$, $\underset{\sim}{x}\in R^p$, the p-dimensional Euclidean space. It is assumed that

$$F_i(\underset{\sim}{x}) = F(\underset{\sim}{x}-\underset{\sim}{\alpha}-\underset{\sim}{\beta}c_i), \quad i\geq 1, \tag{2.1}$$

where $\underset{\sim}{\alpha}=(\alpha_1,\ldots,\alpha_p)'$ and $\underset{\sim}{\beta}=(\beta_1,\ldots,\beta_p)'$ are unknown parameters (vectors), and $\{c_i, i\geq 1\}$ is a sequence of known (scalar) constants.

Robust point as well as interval estimators of $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$ based on suitable rank order statistics when the sample size is large but non-random were studied in detail in Sen and Puri [14]. We are primarily concerned here with the following two sequential extensions of this theory.

Let $\{N_\nu, \nu\geq 1\}$ be a sequence of non-negative inter-valued random variables, such that

$$\nu^{-1}N_\nu \to \lambda, \text{ in probability, as } \nu\to\infty, \tag{2.2}$$

where λ is a positive random variable having an arbitrary distribution

$$H(u) = P\{\lambda\leq u\}, \quad 0<u<\infty, \tag{2.3}$$

and defined on the same probability space (Ω,A,P). Consider then an estimator $(\hat{\underset{\sim}{\alpha}}_{N_\nu}, \hat{\underset{\sim}{\beta}}_{N_\nu})$ of $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$ based on $\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_{N_\nu}$ through a general class of rank order statistics, to be precisely defined in section 3. Our first problem is to derive (along the lines of Blum, Hanson and Rosenblatt [3], and Mogyorodi [10]) the asymptotic normality of $N_\nu^{\frac{1}{2}}[(\hat{\alpha}_{N_\nu}-\alpha),(\hat{\beta}_{N_\nu}-\beta)]$ (as $\nu\to\infty$). This enables us to study various asymptotic properties of $(\hat{\alpha}_{N_\nu},\hat{\beta}_{N_\nu})$.

In the second problem, our sample size N_ν remains a random variable, but so determined by a "stopping rule" that we have a simultaneous confidence interval for (α,β), with the property that the confidence coefficient is asymptotically equal to a predetermined $1-\varepsilon$: $0<\varepsilon<1$, and the length of the interval for each component of α (or β) is bounded above by 2d (or by a known multiple of 2d), where $d>0$ is a predetermined (small) number. The theory is an extension of the corresponding univariate theory developed in Sen and Ghosh [13], and Ghosh and Sen [5, 6, 7]. It is also a sequential extension of the theory developed in Sen and Puri [14], and a nonparametric analogue of the theory developed in Gleser [8] and Albert [2].

3. PRELIMINARY NOTIONS AND BASIC ASSUMPTIONS

Let $\mathcal{F}_p$ be the class of all p-variate absolutely continuous cdf's with finite Fisher information matrix, and let $\mathcal{F}_p^o$ be the subclass of $\mathcal{F}_p$ for which the distribution is diagonally symmetric about 0.

Assumption I. If we are only interested in β, we assume that $F\varepsilon\mathcal{F}_p$, otherwise, we assume that $F\varepsilon\mathcal{F}_p^o$, where F is defined in (2.1). For every $\nu\geq 1$, let

$$\bar{c}_\nu = \nu^{-1}\sum_{i=1}^{\nu} c_i, \qquad C_\nu^2 = \sum_{i=1}^{\nu}(c_i-\bar{c}_\nu)^2. \tag{3.1}$$

We have then the following problems: (a) estimation of α assuming $\beta=0$; (b) estimation of β treating α as a nuisance parameter; (c) simultaneous estimation of (α,β). For (a) no assumptions are needed on $\{c_i,\ i\geq 1\}$; for (b) and (c) our assumptions are respectively (II, III) and (II, III', IV), where, Assumption II. As $\nu\to\infty$,

$$\max_{1\le i\le\nu} \nu(c_i-\bar{c}_\nu)^2/C_\nu^2 = O(1); \tag{3.2}$$

Assumption III.

$$\lim_{\nu\to\infty} C_\nu^2 = \infty; \tag{3.3}$$

Assumption III'.

$$\lim_{\nu\to\infty} \nu^{-1}c_\nu^2 = C^2 \quad (0<C<\infty); \tag{3.4}$$

Assumption IV.

$$\lim_{\nu\to\infty} \bar{c}_\nu = \bar{c} \text{ (finite)}. \tag{3.5}$$

It is easy to verify that all these assumptions hold true for the multivariate one sample (where $c_i=0$ $\forall_i$) and two-sample (where c_i is either 0 or 1) models.

For every $\nu\ge1$, let $R_{\nu i}^{(j)}$ (or $R_{\nu i}^{(j)+}$) be the rank of X_{ij} (or $|X_{ij}|$) among $X_{1j},\ldots,X_{\nu j}$ (or $|X_{1j}|,\ldots,|X_{\nu j}|$) for $1\le i\le\nu$, $1\le j\le p$. To estimate $\underset{\sim}{\beta}$, consider the following linear (regression) rank statistics

$$S_{\nu j} = \sum_{i=1}^{\nu}(c_i-\bar{c}_\nu)a_\nu^{(j)}(R_{\nu i}^{(j)}), \quad j=1,2,\ldots,p, \tag{3.6}$$

$$\underset{\sim}{S}_\nu = (S_{\nu 1},\ldots,S_{\nu p})', \tag{3.7}$$

where the rank scores $a_\nu^{(j)}(i)$, $1\le i\le\nu$, $j=1,\ldots,p$ are defined by

$$a_\nu^{(j)}(i) = E\phi_j(U_{\nu i}) \ [\text{or } \phi_j(i/(\nu+1))], \quad i=1,\ldots,\nu; \tag{3.8}$$

$\phi_j(u)$ is non-decreasing and absolutely continuous inside [0,1], $U_{\nu 1}\le\cdots<U_{\nu\nu}$ are the ordered random variable in a sample of size ν from the rectangular [0,1] distribution. Regarding the score functions $\phi_1,\ldots,\phi_p$ one assumes as in Ghosh and Sen [6] that for every $j(=1,\ldots,p)$,

$$|\phi_j(u)| \leq K[-\log(u(1-u))], \quad |\phi_j'(u)| \leq K[u(1-u)]^{-1}, \quad 0<u<1 \tag{3.9}$$

where $0<K<\infty$. This implies the existence of a $t_o(>0)$ such that

$$M_j(t) = \int_{-\infty}^{\infty} \exp(t\phi_j(u))du < \infty \text{ for all } t: \ |t| \leq t_o, \tag{3.10}$$

for all $j=1,\ldots,p$.

For estimating $\underset{\sim}{\alpha}$, we need an alignment procedure and the following type of one sample rank order statistics

$$T_{\nu j} = \sum_{i=1}^{\nu} c(X_{ij}) a_\nu^{(j)*}(R_{\nu i}^{(j)+}), \quad j=1,\ldots,p, \tag{3.11}$$

$$\underset{\sim}{T}_\nu = (T_{\nu 1},\ldots,T_{\nu p})', \tag{3.12}$$

where $c(u) = 1$, $\frac{1}{2}$ or 0 according as $u >, =$ or <0,

$$a_\nu^{(j)*}(i) = E\phi_j^*(U_{\nu i}) \ [\text{or } \phi_j^*(i/(\nu+1))], \quad 1\leq i\leq \nu, \tag{3.13}$$

$$\phi_j^*(u) = \phi_j\left(\frac{1+u}{2}\right) \text{ and assume that } \phi_j(u)+\phi_j(1-u) = 0. \tag{3.14}$$

Some well-known cases of $\underset{\sim}{S}_\nu$ and $\underset{\sim}{T}_\nu$ are the normal scores and the Wilcoxon scores statistics which relate respectively to $\phi_j(u)$ as the inverse of the standard normal cdf and $\phi_j(u) = 2u-1$, $0<u<1$. Let us also define for later use

$$\underset{\sim}{\Gamma} = ((\gamma_{j\ell})), \ \gamma_{j\ell} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_j(F_{[j]}(x))\phi_\ell(F_{[\ell]}(y))dF_{[j\ell]}(x,y)-\mu_j\mu_\ell, \tag{3.15}$$

for $j,\ell=1,\ldots,p$, where $F_{[j]}$ is the jth marginal cdf and $F_{[j\ell]}$ is the bivariate (j,ℓ)th joint cdf in the joint cdf F, and

$$\mu_j = \int_0^1 \phi_j(u)du, \quad j=1,\ldots,p. \tag{3.16}$$

<u>Assumption V</u>. For every $j(=1,\ldots,p)$, the density function $f_{[j]}=F'_{[j]}$ and its first derivative $f'_{[j]}$ exist and are bounded for almost all x (a.a.x), and

$$\lim_{x\to\pm\infty} |\phi'_j(F_{[j]}(x))f_{[j]}(x)| \text{ is finite.} \tag{3.17}$$

Let us then denote by

$$B_j = B(F_{[j]},\phi_j) = \int_{-\infty}^{\infty} (d/dx)\phi_j(F_{[j]}(x))dF_{[j]}(x), \quad j=1,\ldots,p; \tag{3.18}$$

$$\underset{\sim}{T} = ((\tau_{j\ell})), \quad \tau_{j\ell} = \gamma_{j\ell}/[B_jB_\ell]; \quad j,\ell=1,\ldots,p \tag{3.19}$$

Note that $B_j>0$ $\forall j$, and $\underset{\sim}{T}$ is positive semi-definite.

4. ASYMPTOTIC PROPERTIES OF ROBUST SEQUENTIAL POINT ESTIMATORS OF $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$

We find it more convenient to consider separately the following three problems:

(I) Estimation of $\underset{\sim}{\alpha}$ assuming that $\underset{\sim}{\beta}=\underset{\sim}{0}$ (one-sample model),

(II) Estimation of $\underset{\sim}{\beta}$ treating $\underset{\sim}{\alpha}$ to be a nuisance parameter,

(III) Joint estimation of $(\underset{\sim}{\alpha},\underset{\sim}{\beta})$.

In the first problem, assume that $F\varepsilon\mathcal{F}_p^o$, and denote by $R_{\nu i}^{(j)+}(a_j)$, the rank of $|X_{ij}-a|$ among $|X_{1j}-a|,\ldots,|X_{\nu j}-a|$, $1\leq i\leq\nu$, $1\leq j\leq p$; the resulting rank statistics, defined in (3.11) are denoted by $T_{\nu j}(a)$, $j=1,\ldots,p$. Note that

$$T_{\nu j}(a) \text{ is } \downarrow \text{ in a for all } j=1,\ldots,p. \tag{4.1}$$

Define for each positive integer ν,

$$\hat{\alpha}_{\nu j}^{(1)} = \sup\{a: T_{\nu j}(a)>0\}, \quad \hat{\alpha}_{\nu j}^{(2)} = \inf\{a: T_{\nu j}(a)<0\}; \tag{4.2}$$

$$\hat{\alpha}_{\nu j} = \tfrac{1}{2}(\hat{\alpha}_{\nu j}^{(1)} + \hat{\alpha}_{\nu j}^{(2)}), \quad j=1,\ldots,p; \tag{4.3}$$

$$\hat{\underset{\sim}{\alpha}}_{\nu} = (\hat{\alpha}_{\nu 1},\ldots,\hat{\alpha}_{\nu p})'. \tag{4.4}$$

We intend to study various asymptotic properties of $\hat{\underset{\sim}{\alpha}}_{N_\nu}$, and towards this goal, we have the following.

__Theorem 4.1__. __When__ $F\varepsilon F_p^o$ __and__ $\underset{\sim}{\beta}=\underset{\sim}{0}$, __under__ (2.1), (2.2), (2.3), (3.9), (3.13), (3.14) __and__ (3.17), __as__ $\nu\to\infty$

$$\mathcal{L}(N_\nu^{\frac{1}{2}}[\hat{\underset{\sim}{\alpha}}_{N_\nu}-\underset{\sim}{\alpha}]) \to N_p(\underset{\sim}{0},\underset{\sim}{T}), \tag{4.5}$$

__where__ $\underset{\sim}{T}$ __is defined by__ (3.19).

__Proof__. We use a recent powerful result of Mogyorodi [10] (Theorem 2), according to which we are only to show that for non-stochastic ν,

$$\mathcal{L}(\nu^{\frac{1}{2}}[\hat{\underset{\sim}{\alpha}}_{\nu}-\underset{\sim}{\alpha}]) \to N_p(\underset{\sim}{0},\underset{\sim}{T}), \text{ as } \nu\to\infty, \tag{4.6}$$

and for every $\varepsilon>0$ and $\eta>0$, there exists a $\delta>0$ and an $n_o = n_o(\varepsilon,\eta)$, such that for $n\geq n_o$,

$$P\{\max_{k:\ |n-k|<\delta n} |\sqrt{n}|\,\|\hat{\underset{\sim}{\alpha}}_k-\hat{\underset{\sim}{\alpha}}_n\| > \varepsilon\} < \eta, \tag{4.7}$$

where $\|\underset{\sim}{x}\| = \max_{1\leq j\leq p}|x_j|$, $\underset{\sim}{x}=(x_1,\ldots,x_p)'$. Now, (4.6) has already been proved in Theorem 6.2.3 (on page 226) of Puri and Sen [11]. On the other hand, the left hand side of (4.7) is bounded above by

$$\sum_{j=1}^{p} P\{\max_{k:\ |k-n|<\delta n} \sqrt{n}|\hat{\alpha}_{k,j}-\hat{\alpha}_{n,j}| > \varepsilon\}, \tag{4.8}$$

and hence, by the same technique as in Lemma 5.3 of Sen and Ghosh [13], it can be shown that (4.8) can be bounded by $\eta(>0)$ but a proper choice of $\delta(>0)$ and n. For brevity, the proof is therefore omitted.

Since λ, defined by (2.2), is a positive random variable, for every $0<\varepsilon<1$, there exists a $\lambda_\varepsilon(>0)$, such that $P\{\lambda\geq\lambda_\varepsilon\}\geq 1-\varepsilon$, and hence, $N_\nu\to\infty$, in probability, as $\nu\to\infty$. Consequently, by (4.5)

$$\hat{\underset{\sim}{\alpha}}_{N_\nu}\to\underset{\sim}{\alpha}\text{, in probability, as }\nu\to\infty. \tag{4.9}$$

Consider now the problem of estimating $\underset{\sim}{\beta}$ treating $\underset{\sim}{\alpha}$ as a nuisance parameter. Assume that $F\varepsilon\mathcal{F}_p$ and that II and III hold. Let $R_{\nu i}^{(j)}(b)$ be the rank of $X_{ij}-bc_i$ among $X_{1j}-bc_1,\ldots,X_{nj}-bc_n$ ($1\leq i\leq\nu$; $1\leq j\leq p$), b real; the resulting rank statistics defined by (3.6) are then denoted by $S_{\nu j}(b)$ ($1\leq j\leq p$; $\nu\geq 1$). It follows from Sen ([12], section 6) that

$$S_{\nu j}(b)\text{ is }\downarrow\text{ in b for all }j=1,\ldots,p. \tag{4.11}$$

Define for each $\nu\geq 1$,

$$\hat{\beta}_{\nu j}^{(1)}=\sup\{b:\ S_{\nu j}(b)>0\},\qquad \hat{\beta}_{\nu j}^{(2)}=\inf\{b:\ S_{\nu j}(b)<0\},\quad 1\leq j\leq p; \tag{4.12}$$

$$\hat{\beta}_{\nu j}=\tfrac{1}{2}(\hat{\beta}_{\nu j}^{(1)}+\hat{\beta}_{\nu j}^{(2)}),\quad 1\leq j\leq p; \tag{4.13}$$

$$\hat{\underset{\sim}{\beta}}_\nu=(\hat{\beta}_{\nu 1},\ldots,\hat{\beta}_{\nu p})' \tag{4.14}$$

Then, parallel to theorem 4.1, we have the following.

Theorem 4.2. For $F\varepsilon\mathcal{F}_p$, when (2.1), (2.2), (2.3), (3.2), (3.3), (3.9), and (3.17) hold, as $\nu\to\infty$,

$$\mathcal{L}(C_{N_\nu}[\hat{\underset{\sim}{\beta}}_{N_\nu}-\underset{\sim}{\beta}])\to N_p(\underset{\sim}{0},\underset{\sim}{T}), \tag{4.15}$$

where $\underset{\sim}{T}$ is defined by (3.19).

Proof. As in the proof of theorem 4.1, we require only to show that for non-stochastic ν,

$$\mathcal{L}(C_\nu[\hat{\underset{\sim}{\beta}}_\nu - \underset{\sim}{\beta}]) \to N_p(0, \underset{\sim}{T}) \text{ as } \nu\to\infty, \tag{4.16}$$

and for every $\varepsilon>0$ and $\eta>0$, there exist a $\delta>0$ and an $n_o = n_o(\varepsilon,\eta)$, such that for $n \geq n_o$,

$$P\{\max_{k:\ |n-k|<\delta n} C_n \| \hat{\underset{\sim}{\beta}}_k - \hat{\underset{\sim}{\beta}}_n \| > \varepsilon\} < \eta. \tag{4.17}$$

Now, (4.16) has already been proved in theorem 5.1 of Sen and Puri [14], while (4.17), by virtue of an inequality similar to that in (4.7)-(4.8), follows from Lemma 4.4 of Ghosh and Sen [6]. Hence, the details are omitted.

By (4.16), (3.3) and the discussion preceding (4.9), as $\nu\to\infty$,

$$\hat{\underset{\sim}{\beta}}_{N_\nu} \to \underset{\sim}{\beta}, \text{ in probability.} \tag{4.18}$$

Finally, consider the joint estimation of $(\underset{\sim}{\alpha}, \underset{\sim}{\beta})$. Assume that $F \varepsilon \mathcal{F}_p^o$ and assumptions II, III', IV and V hold. Define the estimators $\hat{\underset{\sim}{\beta}}_\nu$ as in (4.12)-(4.14), and then for estimating $\underset{\sim}{\alpha}$, consider the following aligned rank statistics.

Let $\tilde{R}_{\nu i}^{(j)+}(a)$ be the rank of $|X_{ij} - a - \hat{\beta}_{\nu j} c_i|$ among $|X_{1j} - a - \hat{\beta}_{\nu j} c_1|, \ldots, |X_{\nu j} - a - \hat{\beta}_{\nu j} c_\nu|$, $(1 \leq i \leq \nu,\ 1 \leq j \leq p)$. The resulting one-sample rank-order statistics defined by (3.11) are denoted by $\tilde{T}_{\nu j}(a)$, $(1 \leq j \leq p,\ \nu \geq 1)$. Define

$$\tilde{\alpha}_{\nu j}^{(1)} = \sup\{a: \tilde{T}_{\nu j}(a) > 0\},\ \tilde{\alpha}_{\nu j}^{(2)} = \inf\{a: \tilde{T}_{\nu j}(a) < 0\},\ \nu \geq 1,\ 1 \leq j \leq p; \tag{4.19}$$

$$\tilde{\alpha}_{\nu j} = \tfrac{1}{2}(\tilde{\alpha}_{\nu j}^{(1)} + \tilde{\alpha}_{\nu j}^{(2)}), \quad 1 \leq j \leq p,\ \nu \geq 1; \tag{4.20}$$

$$\tilde{\underset{\sim}{\alpha}} = (\tilde{\alpha}_{\nu 1}, \ldots, \tilde{\alpha}_{\nu p})', \quad \nu \geq 1. \tag{4.21}$$

For notational simplicity, let $\underset{\sim}{\theta}=(\underset{\sim}{\alpha}',\underset{\sim}{\beta}')'$ and $\hat{\theta}_\nu=(\tilde{\underset{\sim}{\alpha}}_\nu',\hat{\underset{\sim}{\beta}}_\nu')'$. Then, we have the following theorem.

Theorem 4.3. Under (2.1)-(2.3), (3.9), (3.13), (3.14) and assumptions I, II, III', IV and V, as $\nu\to\infty$,

$$\mathcal{L}(N_\nu^{\frac{1}{2}}(\tilde{\underset{\sim}{\theta}}_{N_\nu}-\underset{\sim}{\theta})) \to N_{2p}(\underset{\sim}{0},\ \underset{\sim}{\Delta}\otimes\underset{\sim}{T}), \tag{4.22}$$

where $\underset{\sim}{T}$ is defined by (3.19) and

$$\underset{\sim}{\Delta} = \begin{pmatrix} 1+\bar{c}^2/C^2 & -\bar{c}/C^2 \\ -\bar{c}/C^2 & 1/C^2 \end{pmatrix} \tag{4.23}$$

Proof. First note that by the same technique as in the proofs of results in section 7 of Sen and Puri [14] (who considered the particular case of $\bar{c}_\nu=0$ for all $\nu\geq1$), one gets,

$$\mathcal{L}(\nu^{\frac{1}{2}}(\tilde{\underset{\sim}{\theta}}_\nu-\tilde{\underset{\sim}{\theta}})) \to N_{2p}(\underset{\sim}{0},\ \underset{\sim}{\Delta}\otimes\underset{\sim}{T}). \tag{4.24}$$

Hence, similarly as in theorems 4.1 and 4.2, one needs to show that for every $\varepsilon>0$ and $\eta>0$, there exists a $\delta>0$ and an $n_o=n_o(\varepsilon,\eta)$ such that for $n\geq n_o$,

$$P\{\max_{k:\ |k-n|<\delta n} \| n^{\frac{1}{2}}(\tilde{\underset{\sim}{\theta}}_k-\tilde{\underset{\sim}{\theta}}_n)\| > \varepsilon\} < 2\eta. \tag{4.25}$$

Now, the left hand side of (4.25) is bounded above by

$$P\{\max_{k:\ |k-n|<\delta n}\| n^{\frac{1}{2}}(\tilde{\underset{\sim}{\alpha}}_k-\tilde{\underset{\sim}{\alpha}}_n)\|>\varepsilon\} + P\{\max_{k:\ |k-n|<\delta n}\| n^{\frac{1}{2}}(\hat{\underset{\sim}{\beta}}_k-\hat{\underset{\sim}{\beta}}_n)\|>\varepsilon\} \tag{4.26}$$

By virtue of (4.17) and Bonferroni inequality, it suffices to show now that

$$\sum_{j=1}^{p} P\{\max_{k:\ |k-n|<\delta n}\| n^{\frac{1}{2}}(\tilde{\underset{\sim}{\alpha}}_k-\tilde{\underset{\sim}{\alpha}}_n)\|>\varepsilon\} < \eta \tag{4.27}$$

For simplicity, instead of proving (4.27) we shall consider the following: Let $\tilde{\tilde{R}}_{\nu i}^{(j)+}$ be the rank of $|X_{ij}-a-\hat{\beta}_{\nu j}(c_i-\bar{c}_\nu)|$ among $|X_{1j}-a-\hat{\beta}_{\nu j}(c_1-\bar{c}_\nu)|,\ldots,$ $|X_{\nu j}-a-\hat{\beta}_{\nu j}(c_\nu-\bar{c}_\nu)|$, $1\le i\le\nu$, $1\le j\le p$. The resulting one sample rank order statistics defined by (3.11) will now be denoted by $\tilde{\tilde{T}}_{\nu j}(a)$ ($1\le j\le p$; $\nu\ge 1$). Define

$$\hat{\delta}_{\nu j}^{(1)} = \sup\{a: \tilde{\tilde{T}}_{\nu j}(a)>0\}, \quad \hat{\delta}_{\nu j}^{(2)} = \inf\{a: \tilde{\tilde{T}}_{\nu j}(a)<0\}, \quad \nu\ge 1,\ 1\le j\le p; \tag{4.28}$$

$$\hat{\delta}_{\nu j} = \tfrac{1}{2}(\hat{\delta}_{\nu j}^{(1)} + \hat{\delta}_{\nu j}^{(2)}), \quad 1\le j\le p,\ \nu\ge 1; \tag{4.29}$$

$$\hat{\underset{\sim}{\delta}}_\nu = (\hat{\delta}_{\nu 1},\ldots,\hat{\delta}_{\nu p})', \quad \nu\ge 1. \tag{4.30}$$

It follows from the results of Adichie [1] that

$$\hat{\delta}_{\nu j} = \tilde{\alpha}_{\nu j} + \hat{\beta}_{\nu j}\bar{c}_\nu \quad (1\le j\le p;\ \nu\ge 1), \tag{4.31}$$

i.e.,

$$\hat{\delta}_\nu = \tilde{\underset{\sim}{\alpha}}_\nu + \hat{\underset{\sim}{\beta}}_\nu\bar{c}_\nu \quad (\nu\ge 1). \tag{4.32}$$

In view of (4.25)-(4.27) and (4.31)-(4.32) it now suffices to show that for every $\varepsilon>0$ and $\eta>0$, there exists a $\delta>0$ and an $n_o = n_o(\varepsilon,\eta)$ such that for $n\ge n_o$,

$$\sum_{j=1}^{p} P\{\max_{k:\ |k-n|<\delta n}\|n^{\frac{1}{2}}(\hat{\underset{\sim}{\delta}}_k-\hat{\underset{\sim}{\delta}}_n)\|>\varepsilon\} < \eta. \tag{4.33}$$

To prove (4.33) we prove the following two lemmas. Since $\tilde{\underset{\sim}{\alpha}}_\nu$ and $\hat{\underset{\sim}{\beta}}_\nu$ (and hence $\hat{\underset{\sim}{\delta}}_\nu$) are translation invariant for every ν (see [14]), we may, for proving these lemmas, assume that $\underset{\sim}{\alpha}=\underset{\sim}{\beta}=\underset{\sim}{0}$.

Lemma 4.4. Under the assumptions of Theorem 4.3, for every $s>0$, there exist positive constants $c_s^{(1)}$ and $c_s^{(2)}$ and a positive integer ν_s such that for $\underset{\sim}{\alpha}=\underset{\sim}{\beta}=\underset{\sim}{0}$, and all $\nu\ge\nu_s$,

$$P\{\sup_{|a|\le K_o(\log \nu)^k \nu^{-\frac{1}{2}}} \nu^{-\frac{1}{2}}|\tilde{\tilde{T}}_{\nu_j}(a)-\tilde{T}_{\nu j}(a)|>c_s^{(1)}\nu^{-\delta}(\log \nu)^{k+1}\} \le c_s^{(2)}\nu^{-s}, \tag{4.34}$$

where K_o is a positive constant, k any positive integer and δ fixed $(0<\delta<\frac{1}{4})$.

Before proving the above lemma, we may note that taking $s>1$ and on using the Borel-Cantelli Lemma, (4.34) implies that

$$\sup_{|a|\le K_o\nu^{-\frac{1}{2}}(\log \nu)^k} \nu^{-\frac{1}{2}}|\tilde{\tilde{T}}_{\nu j}(a)-\tilde{T}_{\nu j}(a)| \to 0 \text{ a.s. as } \nu\to\infty. \tag{4.35}$$

The proof of the lemma is accomplished in several steps. First we show that for any real b, defining $T_{\nu j}(a,b)$ as similar to $\tilde{\tilde{T}}_{\nu j}(a)$ with $\hat{\beta}_\nu$ replaced by b, for $\nu\ge\nu_s^{(1)}$ (depending on s),

$$P\{\sup_{|a|\le K_o\nu^{-\frac{1}{2}}(\log \nu)^k} \sup_{|b|\le K_1\nu^{-\frac{1}{2}}(\log \nu)^k} \nu^{-\frac{1}{2}}|T_{\nu j}(a,b)-\tilde{T}_{\nu j}(a)| > c_s^{(3)}\nu^{-\delta}(\log \nu)^{k+1}\} < c_s^{(4)}\nu^{-s}, \tag{4.36}$$

where K_1 is a positive constant, $c_s^{(3)}$, $c_s^{(4)}$ are positive constants depending on s. Next, in analogy to lemma 4.1 of Ghosh and Sen [6], one can show that for every $s(>0)$, there exist positive constants $c_s^{(5)}$ and $c_s^{(6)}$ and a positive integer ν_{s2} such that for $\nu\ge\nu_{s2}$,

$$P_{\beta=0}\ \{C_\nu|\hat{\beta}_{\nu j}| > c_s^{(5)}(\log \nu)^2\} \le c_s^{(6)}\nu^{-s}. \tag{4.37}$$

Defining now $c_s^{(1)}$, $c_s^{(2)}$ and ν_s appropriately on the basis of $c_s^{(i)}$ $(i=3,4,5,6)$, $\nu_s^{(1)}$ and $\nu_s^{(2)}$, one gets (4.34) from (4.35), (4.37) and (3.4). Let $H_{\nu,j,a,b}(x) = \nu^{-1}\sum_{i=1}^{\nu} u(x-(X_{ij}-a-b(c_i-\bar{c}_\nu)))$ be the sample df of $X_{ij}-a-b(c_i-\bar{c}_\nu)$'s, and let $G_{\nu,j,a,b}(x) = \nu^{-1}\sum_{i=1}^{\nu} u(x-|X_{ij}-a-b(c_i-\bar{c}_\nu)|) = H_{\nu,j,a,b}(x)-H_{\nu,j,a,b}(-x-)$ be the sample cdf for $|X_{ij}-a-b(c_i-\bar{c}_\nu)|$'s. The corresponding population cdf's are denoted respectively by $\bar{F}_{\nu,j,a,b}(x) = \nu^{-1} \sum_{i=1}^{\nu} F_{[j]}(x+a+b(c_i-\bar{c}_\nu))$ and $\bar{D}_{\nu,j,a,b}(x) = \bar{F}_{\nu,j,a,b}(x)-\bar{F}_{\nu,j,a,b}(-x-)$. Writing $\phi^*_{\nu j}(i/(\nu+1)) = a_\nu^{(j)*}(i)$ $(1\leq i\leq \nu,\ \nu\geq 1)$, one can now write

$$\nu^{-1}[T_{\nu,j}(a,b)-\tilde{T}_{\nu j}(a)] = \int_0^\infty \phi^*_{\nu j}(\tfrac{\nu}{\nu+1}G_{\nu,j,a,b}(x))\,dH_{\nu,j,a,b}(x)$$

$$- \int_0^\infty \phi^*_{\nu j}(\tfrac{\nu}{\nu+1}\, G_{\nu,j,a,o}(x))\,dH_{\nu,j,a,o}(x).$$

A result analogous to theorem 3.6.6 of Puri and Sen [11] give

$$\max_{1\leq i\leq \nu}|\phi^*_{\nu j}(i/(\nu+1))-\phi^*_j(i/(\nu+1))| = O(\nu^{-\frac{1}{2}-\delta})$$

for some $\delta>0$, $j=1,2,\ldots,p$. Hence, one can write,

$$\nu^{-1}[T_{\nu,j}(a,b)-\tilde{T}_{\nu j}(a)] = I_{\nu j1}(a,b) + I_{\nu j2}(a,b) + O(\nu^{-\frac{1}{2}-\delta}), \tag{4.38}$$

where

$$I_{\nu j1}(a,b) = \int_0^\infty [\phi^*_j(\tfrac{\nu}{\nu+1}G_{\nu,j,a,b}(x))-\phi^*_j(\tfrac{\nu}{\nu+1}G_{\nu,j,a,o}(x))]\,dH_{\nu,j,a,b}(x), \tag{4.39}$$

$$I_{\nu j2}(a,b) = \int_0^\infty \phi^*_j(\tfrac{\nu}{\nu+1}G_{\nu,j,a,o}(x))\,d[H_{\nu,j,a,b}(x)-H_{\nu,j,a,o}(x)]. \tag{4.40}$$

On integration by parts, one can write, using (3.9), (3.13) and (3.14),

$$I_{\nu j2}(a,b) = \int_0^\infty [H_{\nu,j,a,b}(x)-H_{\nu,j,a,o}(x)]\phi^*_j(\tfrac{\nu}{\nu+1}G_{\nu,j,a,o}(x))\tfrac{\nu}{\nu+1}G_{\nu,j,a,o}(x). \tag{4.41}$$

We shall now state a lemma. The proof follows the same line as lemma 4.1 of Sen and Ghosh [13] and theorem 3.1 of Ghosh and Sen [6]. For brevity, the details are omitted.

Lemma 4.5. For every $s(>0)$, there exist two positive constants $K_s^{(1)}$ and $K_s^{(2)}$, and a positive integer ν_s^* (all of which may depend on s) such that for $\nu \geq \nu_s^*$, $k \geq 1$ and $0<\delta<\frac{1}{4}$

$$P\{\sup_{-\infty<x<\infty} \sup_{|a|\leq K_o\nu^{-\frac{1}{2}}(\log \nu)^k} \sup_{|b|\leq K_1\nu^{-\frac{1}{2}}(\log \nu)^k} |H_{\nu,j,a,b}(x)-H_{\nu,j,a,o}(x) - \bar{F}_{\nu,j,a,b}(x) + \bar{F}_{\nu,j,a,o}(x)| > K_s^{(1)}\nu^{-\frac{1}{2}-\delta}(\log \nu)^k\} \leq K_s^{(2)}\nu^{-s}. \tag{4.42}$$

Using also the fact that $\bar{F}_{\nu,j,a,b}(x)-\bar{F}_{\nu,j,a,o}(x) = \nu^{-1}\sum_{i=1}^{\nu}[F(x+a+b(c_i-\bar{c}_\nu)) - F(x+a)] = O(\nu^{-1}(\log \nu)^{2k})$, uniformly in x, a and $|b|\leq K_1\nu^{-\frac{1}{2}}(\log \nu)^k$, one gets,

$$P\{\sup_{-\infty<x<\infty} \sup_{|a|\leq K_o\nu^{-\frac{1}{2}}(\log \nu)^k} \sup_{|b|\leq K_1\nu^{-\frac{1}{2}}(\log \nu)^k} |H_{\nu,j,a,b}(x)-H_{\nu,j,a,o}(x)| > K_s^{(1)}\nu^{-\frac{1}{2}-\delta}(\log \nu)^k\} \leq K_s^{(2)}\nu^{-s} \text{ for } \nu \geq \nu_s^{**}, \text{ say.} \tag{4.43}$$

Thus, by (4.41) and (4.43), one gets by using (3.9), (3.13) and (4.14) that

$$\sup_{|a|\leq K_o\nu^{-\frac{1}{2}}(\log \nu)^k} \sup_{|b|\leq K_1\nu^{-\frac{1}{2}}(\log \nu)^k} |I_{\nu j2}(a,b)| \leq [O(\nu^{-\frac{1}{2}-\delta}(\log \nu)^k)] \frac{\nu}{\nu+1} \sum_{i=1}^{\nu} K[1-i/(\nu+1)]^{-1} = O(\nu^{-\frac{1}{2}-\delta}(\log \nu)^{k+1}), \tag{4.44}$$

with probability $\geq 1-K_s^{(2)}\nu^{-s}$, for $\nu \geq \nu_s^{**}$.

Again, write

$$I_{\nu j1}(a,b) = \frac{\nu}{\nu+1}\int_0^\infty [G_{\nu,j,a,b}(x)-G_{\nu,j,a,o}(x)]\phi_j^{*\prime}(\frac{\nu}{\nu+1}[\theta G_{\nu,j,a,b}(x) + (1-\theta)G_{\nu,j,a,o}(x)])dH_{\nu,j,a,b}(x), \quad (0<\theta<1). \tag{4.45}$$

Since $G_{\nu,j,a,b}(x)-G_{\nu,j,a,o}(x) = [H_{\nu,j,a,b}(x)-H_{\nu,j,a,o}(x)]-[H_{\nu,j,a,b}(-x)-H_{\nu,j,a,o}(-x)]$, it follows from (4.43) that

$$\sup_{-\infty<k<\infty}\ \sup_{|a|\le K_o\nu^{-\frac{1}{2}}(\log\nu)^k}\ \sup_{|b|\le K_o\nu^{-\frac{1}{2}}(\log\nu)^k} |G_{\nu,j,a,b}(x)-G_{\nu,j,a,o}(x)| \le K_s^{(1)}\nu^{-\frac{1}{2}-\delta}(\log\nu)^k$$

with probability $\ge 1-K_s^{(2)}\nu^{-s}$ for large ν. Using arguments analogous to (4.20)-(4.27) of theorem 4.3 in Sen and Ghosh [13], one can prove now that

$$\sup_{|a|\le K_o\nu^{-\frac{1}{2}}(\log\nu)^k}\ \sup_{|b|\le K_1\nu^{-\frac{1}{2}}(\log\nu)^k} |I_{\nu j1}(a,b)| \le K_s^{(3)}\nu^{-\frac{1}{2}-\delta}(\log\nu)^{k+1},$$

with probability $\ge 1-K_s^{(4)}\nu^{-s}$ for large ν. Hence, the lemma.

For proving (4.33) we need another lemma which we prove below. For proving this lemma, we take $a_\nu^{(j)*}(i) = \phi_{\nu j}^*(i/(\nu+1)) = E\phi_j^*(U_{\nu i})$ $(1\le i\le\nu,\ 1\le j\le p)$.

Lemma 4.6. For $\underset{\sim}{\alpha}=\underset{\sim}{\beta}=\underset{\sim}{0}$, for every $s>0$, there exist positive constants $d_s^{(1)}$ and $d_s^{(2)}$ and a positive integer ν_{so} such that for $\nu\ge\nu_{so}$,

$$P\{|\hat{\delta}_{\nu j}| \ge d_s^{(1)}\nu^{-\frac{1}{2}}(\log\nu)^k\} \le d_s^{(2)}\nu^{-s}. \tag{4.46}$$

Proof. We prove only the case of $P\{\hat{\delta}_{\nu j} \ge d_s^{(1)}\nu^{-\frac{1}{2}}(\log\nu)^k\}$ as the other case follows similarly. Note that

$$P\{\hat{\delta}_{\nu j}\geq d_s^{(1)}\nu^{-\frac{1}{2}}(\log\nu)^k\} \leq P\{\hat{\delta}_{\nu j}^{(2)}\geq d_s^{(1)}\nu^{-\frac{1}{2}}(\log\nu)^k\}$$

$$= P\{\tilde{T}_{\nu j}(d_s^{(1)}\nu^{-\frac{1}{2}}(\log\nu)^k)\geq 0\} = P\{\nu^{-\frac{1}{2}}\tilde{T}_{\nu j}(d_s^{(1)}\nu^{-\frac{1}{2}}(\log\nu)^k)\geq 0\}\ . \quad (4.47)$$

It follows from Lemma 4.5 that for every $s>0$, for large ν, with probability $\geq 1-c_s^{(2)}\nu^{-s}$,

$$\nu^{-\frac{1}{2}}[\tilde{T}_{\nu j}(d_s^{(1)}\nu^{-\frac{1}{2}}(\log\nu)^k)-T_{\nu j}(d_s^{(1)}\nu^{-\frac{1}{2}}(\log\nu)^k)] \leq c_s^{(1)}\nu^{-\delta}(\log\nu)^k. \quad (4.48)$$

Further, from theorem 4.3 of Sen and Ghosh [6], we have

$$\nu^{-\frac{1}{2}}[T_{\nu j}(d_s^{(1)}\nu^{-\frac{1}{2}}(\log\nu)^k) - T_{\nu j}(0)] - d_s^{(1)}(\log\nu)^k \leq d_s^{(2)}\nu^{-\delta}(\log\nu)^k, \quad (4.49)$$

with probability $\geq 1-c_s^{(2)}\nu^{-s}$, for large ν. Hence, from (4.47)-(4.49), it suffices to show that for large ν, for every $s>0$, there exist constants $d_s^{(1)}$ and $d_s^{(2)}$ such that

$$P\{\nu^{-\frac{1}{2}}T_{\nu j}(0) > d_s^{(1)}(\log\nu)^k\} \leq d_s^{(2)}\nu^{-\delta}(\log\nu)^k. \quad (4.50)$$

Since, $\underset{\sim}{\alpha}=\underset{\sim}{\beta}=\underset{\sim}{0}$, for every ν, $\underset{\sim}{R}_\nu^{(j)+} = (R_{\nu 1}^{(j)+},\ldots,R_{\nu\nu}^{(j)+})'$ is independent of $\underset{\sim}{s}_\nu^{(j)} = (s(X_{1j}),\ldots,s(X_{\nu j}))$, where $s(u) = 2c(u)-1$ i.e., $s(u) = 1$, 0 or -1, according as $u>$, $=$, or <0. Now

$$\nu^{-\frac{1}{2}}T_{\nu j}(0) = \nu^{-\frac{1}{2}}\sum_{i=1}^{\nu}\frac{1+s(X_{ij})}{2}\,E\,\phi_j^*(U_\nu R_{\nu i}^{(j)+})$$

$$= \tfrac{1}{2}\nu^{\frac{1}{2}}\int_0^1\phi_j^*(u)du + \tfrac{1}{2}\nu^{-\frac{1}{2}}\sum_{i=1}^{\nu} s(X_{ij})E\phi_j^*(U_\nu R_{\nu i}^{(j)+})$$

Since, (3.10) holds, we get from (3.13) and (3.14) the first term to be $O(\nu)$. Hence, (4.49) will be proved if one can show that for large ν

$$P\{\nu^{-\frac{1}{2}}T_{\nu jo}(0)>2d_s^{(1)}(\log \nu)^k\} \le d_s^{(2)}\nu^{-s}, \tag{4.51}$$

where

$$T_{\nu jo}(0) = \sum_{i=1}^{\nu} s(X_{ij})E\phi_j^*(U_\nu R_{\nu i}^{(j)+}), \quad 1\le j\le p.$$

Writing $g_\nu = 2d_\nu^{(1)}\nu^{\frac{1}{2}}(\log \nu)^k$, and using the Bernstein inequality, one gets,

$$P\{T_{\nu jo}(0)\ g_\nu\} \le \inf_{t>0} E[\exp\{t(T_{\nu jo}(0)-g_\nu)\}] \tag{4.52}$$

Now,

$$E[\exp\{t(T_{\nu jo}(0)-g_\nu)\}] = \exp(-tg_\nu)E[\exp(tT_{\nu jo}(0))].$$

Again,

$$E[\exp(t\ T_{\nu jo}(0))] = E\ E[\Pi_{i=1}^{\nu}\exp(ts(X_{ij})E\phi_j^*(U_\nu R_{\nu i}^{(j)+}))|\underset{\sim}{R}_\nu^{(j)+}]$$

Using the independence of $\underset{\sim}{s}_\nu^{(j)}$ and $\underset{\sim}{R}_\nu^{(j)+}$ and also the elementary inequality $\frac{1}{2}(e^x + e^{-x}) \le \exp(x^2/2)$, one gets,

$$\begin{aligned}
E[\exp(tT_{\nu jo}(0))] &= E[\Pi_{i=1}^{\nu}\{\tfrac{1}{2}\exp(tE\phi_j^*(U_\nu R_{\nu i}^{(j)+})) + \tfrac{1}{2}\exp(tE\phi_j^*(U_\nu R_{\nu i}^{(j)+}))\}] \\
&\le E\ \Pi_{i=1}^{\nu}\exp(\frac{t^2}{2}(E\phi_j^*(U_\nu R_{\nu i}^{(j)+}))^2) \\
&\le E\ \Pi_{i=1}^{\nu}\exp(\frac{t^2}{2}E\phi_j^{*2}(U_\nu R_{\nu i}^{(j)+})) \\
&= E\ \exp(\frac{t^2}{2}\sum_{i=1}^{\nu}E\phi_j^{*2}(U_\nu R_{\nu i}^{(j)+})) \\
&= E\ \exp(\frac{t^2}{2}\sum_{i=1}^{\nu}E\phi_j^{*2}(U_{\nu i})) = \exp(\frac{\nu t^2 A_j^2}{2}),
\end{aligned}$$

where $A_j^2 = \int_0^1 \phi_j^{*2}(u)du$. Thus, from (4.52),

$$\begin{aligned}
P\{T_{\nu jo}(0)>g_\nu\} &\le \inf_{t>0}\exp(-tg_\nu + \frac{\nu t^2 A_j^2}{2}) = \exp(-\frac{g_\nu^2}{2\nu A_j^2}) \\
&= \exp(-\frac{2d_s^{(1)^2}}{A_j^2}(\log \nu)^{2k}),
\end{aligned}$$

and hence, (4.50) follows. Hence, the lemma.

It follows from Lemmas 4.4 and 4.6 that for large ν, $\nu^{-\frac{1}{2}}[\tilde{T}_{\nu j}(\hat{\delta}_\nu) - T_{\nu j}(\hat{\delta}_\nu)] = O(\nu^{-\delta}(\log \nu)^k)$ with probability $\geq$ 1-const. ν^{-s}. Again, it follows from theorem 4.3 of Sen and Ghosh [13] that $\nu^{-\frac{1}{2}}[T_{\nu j}(\hat{\delta}_\nu)-T_{\nu j}(0)] + \nu^{\frac{1}{2}}\hat{\delta}_\nu B_j = O(\nu^{-\delta}(\log \nu)^k)$ with probability $\geq$ 1-const. ν^{-s}. Hence, with probability $\geq$ 1-const. ν^{-s}, $\nu^{-\frac{1}{2}}[\tilde{T}_{\nu j}(\hat{\delta}_\nu)-T_{\nu j}(0)] + \nu^{\frac{1}{2}}\hat{\delta}_\nu B_j = O(\nu^{-s}(\log \nu)^k)$. i.e., $\nu^{\frac{1}{2}}\hat{\delta}_\nu B_j - \nu^{-\frac{1}{2}}T_{\nu j}(0) = O(\nu^{-\delta}(\log \nu)^k)$, noting that $\tilde{T}_{\nu j}(\hat{\delta}_\nu) = 0$. (4.33) now follows from theorem 4.5 of Sen and Ghosh [13]. Hence the theorem.

5. BOUNDED LENGTH (SEQUENTIAL) CONFIDENCE BANDS FOR $\underset{\sim}{\theta}$

Parallel to problems (I)-(III) of section 4, we consider here the following three problems.

Problem I' Confidence estimation of $\underset{\sim}{\alpha}$ assuming that $\underset{\sim}{\beta}=\underset{\sim}{0}$. More specifically we want a p-dimensional confidence rectangle for $\underset{\sim}{\alpha}$ such that the length of each side $\leq$ 2d (d>0, preassigned) and the confidence coefficient $\geq 1-\alpha$. This can be achieved by a direct extension of the results of Sen and Ghosh [13].

To see this, first note that under $\underset{\sim}{\alpha}=\underset{\sim}{\beta}=\underset{\sim}{0}$, $\underset{\sim}{T}_{\nu o} = (T_{\nu 10},\ldots,T_{\nu po})'$, ($T_{\nu jo}$'s defined after (4.51)) has a distribution independent of F diagonally symmetric about $\underset{\sim}{0}$. Hence, there exists a known constant $T_{\nu,\varepsilon}$ such that

$$P_{\underset{\sim}{\alpha}=\underset{\sim}{\beta}=\underset{\sim}{0}}\{\max_{1\leq j\leq p} |T_{\nu jo}| \leq T_{\nu,\varepsilon}\} = 1-\varepsilon_\nu \to 1-\varepsilon \text{ as } \nu\to\infty. \tag{5.1}$$

For large ν, $\sqrt{\nu}\, T_{\nu,\varepsilon} \to \chi^*_{p,\varepsilon}$ where $\chi^*_{p,\varepsilon}$ is the upper 100ε% point of the distribution of the maximum of $\gamma_1,\ldots,\gamma_p$ where $\underset{\sim}{\gamma} = (\gamma_1,\ldots,\gamma_p)'$ is $N(\underset{\sim}{0},\underset{\sim}{\nu})$. Define now

$$\hat{\alpha}_{L,j,\nu} = \sup\{a: T_{\nu jo}(a) > T_{\nu,\varepsilon}\}, \tag{5.2}$$

$$\hat{\alpha}_{U,j,\nu} = \inf\{a: T_{\nu jo}(a) < -T_{\nu,\varepsilon}\}, \tag{5.3}$$

where $T_{\nu jo}(a)$ is defined in the same way as $T_{\nu jo} = T_{\nu jo}(0)$, replacing X_i's by X_i-a's ($1\le j\le p$, $1\le i\le \nu$). Then, $P_{\underset{\sim}{\alpha}=\underset{\sim}{\beta}=\underset{\sim}{0}}\{\hat{\alpha}_{L,j,\nu} \le \alpha_j \le \hat{\alpha}_{U,j,\nu} \forall 1\le j\le p\} =$ $P_{\underset{\sim}{\alpha}=\underset{\sim}{\beta}=\underset{\sim}{0}}\{-T_{\nu,\varepsilon} \le T_{\nu jo} \le T_{\nu,\varepsilon} \forall 1\le j\le p\} = 1-\varepsilon_\nu \to 1-\varepsilon$ as $\nu\to\infty$.

We define the stopping variable $N = N(d)$ to be the least positive integer $n(\ge n_o)$ such that $\max_{1\le j\le p} (\hat{\alpha}_{U,j,n}-\hat{\alpha}_{L,n,n}) \le 2d$. Now, using Theorem 4.3 and Lemma 5.1 of Sen and Ghosh [13],

$$\sqrt{\nu}\ [T_{\nu j}(\hat{\alpha}_{U,j,\nu}) = T_{\nu j}(0) + \tfrac{1}{2}\hat{\alpha}_{U,j,\nu}B_j] = O(\nu^{-\frac{1}{4}}(\log \nu)^4) \tag{5.4}$$

with probability $\ge$ 1-const. ν^{-s}, for every $s>0$, large ν. Thus noting that when $a_\nu^{(j)*}(i) = E[\phi_j^*(U_{\nu i})]$, $1\le i\le \nu$, $T_{\nu jo}(a) = 2T_{\nu j}(a) - \int_0^1 \phi_j^*(u)du$, for all real a, it follows from (5.3) and (5.4) that

$$-\chi^*_{p,\varepsilon} - \sqrt{\nu}\ T_{\nu jo}(0) + \sqrt{\nu}\ \hat{\alpha}_{U,j,\nu}B_j \to 0 \text{ a.s. as } \nu\to\infty.$$

Similarly,

$$\chi^*_{p,\varepsilon} - \sqrt{\nu}\ T_{\nu jo}(0) + \sqrt{\nu}\ \hat{\alpha}_{L,j,\nu}B_j \to 0 \text{ a.s. as } \nu\to\infty$$

Thus,

$$\sqrt{\nu}\ (\hat{\alpha}_{U,j,\nu}-\hat{\alpha}_{L,j,\nu}) \to \frac{2\chi^*_{p,\varepsilon}}{B_j} \text{ a.s. as } \nu\to\infty.$$

Hence,

$$\max_{1\le j\le p} \sqrt{\nu}\ (\hat{\alpha}_{U,j,\nu}-\hat{\alpha}_{L,j,\nu}) \to \frac{2\chi_{p,\varepsilon}}{\min_{1\le j\le p} B_j} \text{ a.s. as } \nu\to\infty. \tag{5.5}$$

It follows now from the definition of N that $\lim_{d\to 0} N(d)/s(d) = 1$ a.s., where $s(d) = \chi^{*2}_{p,\varepsilon}/d^2 \min_{1\le j\le p} B_j^2$, and as to the rate of convergence, we can make a similar statement as (5.4). Thus, generalizing the results of Sen and Ghosh [13], we get the following theorem.

<u>Theorem 5.1</u>. <u>Under the assumptions</u> $F\varepsilon\mathcal{F}_p^o$, (2.1)-(2.3), (3.9), (3.13)-(3.14) <u>and</u>

(3.17),

$$N(=N(d)) \text{ \underline{is a non-increasing function of} } d;\ N(d)<\infty \text{ \underline{with probability} } 1,\ EN(d)<\infty \text{ \underline{for all} } d>0, \lim_{d\to 0} N(d)=\infty \text{ a.s., \underline{and} } \lim_{d\to 0} EN(d)=\infty. \tag{5.6}$$

$$\lim_{d\to 0} N(d)/s(d) = 1 \text{ a.s.} \tag{5.7}$$

$$\lim_{d\to 0} P_{\underset{\sim}{\alpha}}\{\hat{\alpha}_{L,j,N} \le \alpha_j \le \hat{\alpha}_{U,j,N}\ \forall\ 1\le j\le p\} = 1-\varepsilon. \tag{5.8}$$

$$\lim_{d\to 0} EN(d)/s(d) = 1. \tag{5.9}$$

We now suggest an alternate procedure for the same problem. We find a confidence region R_N for $\underset{\sim}{\alpha}$ such that the maximum diameter of $R_N \le 2d$. Our procedure is analogous to the one proposed by Srivastava [15].

We define

$$\hat{\gamma}_{j\ell}^{(n)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_j\left(\frac{n}{n+1}F_{[j]n}(x)\right)\phi_\ell\left(\frac{n}{n+1}F_{[\ell]n}(y)\right)dF_{[j,\ell]n}(x,y)-\mu_j\mu_\ell, \tag{5.10}$$

for $1\le j\ne\ell\le p$ where $F_{[j]n}(x)$ and $F_{[j,\ell]n}(x,y)$ are the empirical df's corresponding to the true df's $F_{[j]}(x)$ and $F_{[j,\ell]}(x,y)$ respectively, for $j=\ell$, $\hat{\gamma}_{jj}^{(n)} = \gamma_{jj} = \int_0^1 \phi_j^2(u)du-\mu_j^2$, $1\le j\le p$. Also, define $\hat{B}_{j,n}$ as the estimator of B_j $(1\le j\le p)$ as in Lemma 4.2 of Ghosh and Sen [6]. Define then

$$\hat{\underset{\sim}{T}}_n = ((\hat{\tau}_{j\ell}^{(n)}));\quad \hat{\tau}_{j\ell}^{(n)} = \hat{\gamma}_{j\ell}^{(n)}/\hat{B}_{j,n}\hat{B}_{\ell,n},\quad j,\ell=1,\dots,p. \tag{5.11}$$

We denote by

$$\hat{\lambda}_n = \text{max. ch. root of } \hat{\underset{\sim}{T}}_n;\quad \lambda = \text{max. ch. root of } \underset{\sim}{T}, \tag{5.12}$$

where $\underset{\sim}{T}$ is defined by (3.19); finally, $\chi_{p,\varepsilon}^2$ is defined as the upper 100ε% point of the chi square distribution with p degrees of freedom. Our procedure is as follows.

Starting with an initial sample of size $n_o(>p)$, we continue drawing observations one at a time according to a stopping time N defined by

$$N[=N(d)] = \text{smallest } n \geq n_o \text{ such that } \hat{\lambda}_n \leq d^2 n/\chi^2_{p,\alpha} \tag{5.13}$$

When sampling is stopped at N=n, construct the region R_n defined by

$$R_n = \{\underset{\sim}{z}: \ (\hat{\underset{\sim}{\alpha}}_n - \underset{\sim}{z})'(\hat{\underset{\sim}{\alpha}}_n - \underset{\sim}{z}) \leq d^2\} \tag{5.14}$$

Then, we have the following theorem.

Theorem 5.2. _Under the assumption that_ $0<\lambda<\infty$ _and the hypothesis of Theorem 5.1, the results of Theorem 5.1 all hold for the stopping variable_ N(d), _defined by_ (5.13) _and_ R_N, _defined by_ (5.14), _provided we replace_ s(d) _in_ (5.7) _and_ (5.9) _by_

$$\nu(d) = \chi^2_{p,\alpha}\,\lambda/d^2. \tag{5.15}$$

Proof. Running down the proof of Srivastava [16], it suffices to show that $\hat{\lambda}_n \to \lambda$ a.s., as $n\to\infty$; by the Courant Theorem, it thus suffices to show that

$$\hat{\underset{\sim}{T}}_n \to \underset{\sim}{T} \text{ a.s., as } n\to\infty. \tag{5.16}$$

Since, $\hat{B}_{j,n}$, j=1,...,p, converge a.s. to B_j, j=1,...,p as $n\to\infty$ (See [13]), it suffices to prove the following lemma.

Lemma 5.3. _Under_ (3.4), (3.17), (3.18) _and_ (3.19),

$$\hat{\gamma}^{(n)}_{j\ell} \to \gamma_{j\ell} \text{ a.s., as } n\to\infty, \text{ for all } 1\leq j\neq\ell\leq p. \tag{5.17}$$

Proof. Since $\phi_j(u)$ is assumed to be non-decreasing, absolutely continuous and square integrable inside [0,1], by Lemma 5.1 of Hájek [9], we may write for $0<u<1$,

$$\phi_j(u) = \phi_j^{(1)}(n) - \phi_j^{(2)}(u) + \phi_j^{(3)}(u), \tag{5.18}$$

where $\phi_j^{(1)}(n)$ is a polynomial (i.e., has bounded second derivative) and

$$\int_0^1 \{\phi_j^{(k)}(u)\}^2 du < \tfrac{1}{2}\,[\int_0^1 \phi_j^2(u)du], \quad k=2,3, \tag{5.19}$$

where $\varepsilon>0$ is arbitrarily small. By (3.8), we may decompose the scores $a_\nu^{(j)}(i)$, $1\le i\le\nu$, also in three parts. On the first part, involving $\phi_j^{(1)}$, almost sure convergence of $F_{[j]n}$ and $F_{[j,\ell]n}$ to $F_{[j]}$ and $F_{[j,\ell]}$ (respectively) implies the a.s. convergence of the corresponding component of $\hat{\gamma}_{j\ell}^{(n)}$ to that of $\gamma_{j\ell}$; on the other components, the Schwarz inequality and (5.19) imply that the same can be made arbitrarily small by proper choice of $\varepsilon(>0)$. Q.E.D.

Remark. In (5.14), we could have taken a region $\{\underset{\sim}{z}: (\hat{\underset{\sim}{\alpha}}_n-\underset{\sim}{z})'\underset{\sim}{A}^{-1}(\hat{\underset{\sim}{\alpha}}_n-\underset{\sim}{z}) \le d^2\}$, where $\underset{\sim}{A}$ is any given positive definite matrix. In that case, we need to define $\hat{\lambda}_n$ = max. ch. root of $\underset{\sim}{A}^{-1}\underset{\sim}{T}_n$ and λ = max. ch. root of $\underset{\sim}{A}^{-1}\underset{\sim}{T}$. The proofs follows on parallel lines.

Problem II'. Confidence band for $\underset{\sim}{\beta}$ treating $\underset{\sim}{\alpha}$ as a nuisance parameter

(i) Rectangular regions. Note that under $\underset{\sim}{\beta}=\underset{\sim}{0}$, $s_{\nu j}$'s have a completely specified distribution generated by $(n!)^p$ equally likely realizations of the ranks. Hence, there exists a known $s_{\nu,\varepsilon}$ such that

$$P_{\underset{\sim}{\beta}=\underset{\sim}{0}}\{\max_{1\le j\le p}|S_{\nu j}| \le S_{\nu,\varepsilon}\} = 1-\varepsilon_\nu \to 1-\varepsilon \text{ as } \nu\to\infty.$$

For large ν, $\sqrt{S}_{\nu,\varepsilon} \to \chi^*_{p,\varepsilon}$, the upper $100\varepsilon\%$ point of the distribution of the maximum of $\gamma_1,\ldots,\gamma_p$ where $\underset{\sim}{\gamma} = (\gamma_1,\ldots,\gamma_p)'$ is $N(\underset{\sim}{0},\underset{\sim}{\nu})$. Define now

$$\hat{\beta}_{L,j,\nu} = \sup\{b: \; S_{\nu j}(b) > S_{\nu,\varepsilon}\}$$

$$\hat{\beta}_{U,j,\nu} = \inf\{b: \; S_{\nu j}(b) < -S_{\nu,\varepsilon}\}$$

Then,

$$P_{\underset{\sim}{\beta}=\underset{\sim}{0}}\{\hat{\beta}_{L,j,\nu} \leq \beta_j \leq \hat{\beta}_{U,j,\nu} \; \forall \; 1\leq j\leq p\}$$
$$= 1-\varepsilon_\nu \to 1-\varepsilon \text{ as } \nu\to\infty. \tag{5.20}$$

We define the stopping variable N=N(d) to be the least positive integer $n(\geq n_o)$ such that $\max_{1\leq j\leq p} (\hat{\beta}_{U,j,n}-\hat{\beta}_{L,j,n}) \leq 2d$. Using Lemma 4.2 of Ghosh and Sen [6], we can now prove the following theorem. The proof is omitted because of its obvious analogy to Theorem 5.1.

<u>Theorem 5.4</u>. <u>If</u> $F\varepsilon\mathcal{F}_p$, <u>then under</u> (2.1)-(2.3), (3.9) <u>and</u> (3.17), N(d) <u>as defined above and the related confidence band for</u> $\underset{\sim}{\beta}$ <u>satisfy the results of Theorem 5.1 provided we define</u>

$$s(d) = Q^{-1}(\chi^{*2}_{p,\varepsilon}/[d^2 \max_{1\leq j\leq p} B_j^2]), \tag{5.21}$$

<u>where</u> $Q(n) = C_n^2$ <u>for</u> $n\geq 1$ <u>and is obtained by linear interpolation for non-integer</u> t(>0).

(ii) <u>Sprerical or Ellipsoidal regions</u>. Here, we start by taking $n_o(\geq p)$ observations $\underset{\sim}{X}_1,\ldots,\underset{\sim}{X}_{n_o}$ and continue sampling one observation at a time in accordance with the stopping variable

$$N(d) = \text{smallest } n(\geq n_o) \text{ such that } \lambda_n \leq d^2C_n^2/\chi^2_{p,\varepsilon},$$

where $\hat{\lambda}_n$ and $\chi^2_{p,\varepsilon}$ are defined in (5.12) and after that. When sampling is stopped at N=n, we construct the region R_n defined by

$$R_n = \{\underset{\sim}{\beta}: \; (\underset{\sim}{\beta}_n-\underset{\sim}{\beta})'(\beta_n-\underset{\sim}{\beta}) \leq d^2\}, \tag{5.22}$$

where $\hat{\underset{\sim}{\beta}}_n$ is defined by (4.14). Then, we have the following.

Theorem 5.5. The conclusions of Theorem 5.2 *holds for* N(d) *and* R_n, *defined as above, provided we let*

$$\nu(d) = Q^{-1}(\lambda\, \chi^2_{p,\varepsilon}/d^2).$$

The proof follows along the same line as in Theorems 5.1 and 5.2.

Problem III'. Confidence bands for $\underset{\sim}{\theta}$. Here also, we can have either a rectangular or an ellipsoidal region for $\underset{\sim}{\theta}=(\underset{\sim}{\alpha},\underset{\sim}{\beta})$. We need to change $\chi^*_{p,\varepsilon}$ and $\chi^2_{p,\varepsilon}$ to $\chi^*_{2p,\varepsilon}$ and $\chi^2_{2p,\varepsilon}$ respectively, and therefore, in view of the similarity with problems I' and II', the details are omitted.

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