A TIME SERIES APPROACH TO THE LIFE TABLE

by

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1. Introduction

"Statistics" was historically the name given to the science which collects facts about the state. One of the most useful and best-known collection of facts about the state is the life table. In this paper, we will present a time series approach to estimating the life table.

There are two general forms of the life table: the cohort life table and the current life table. The former follows the fortunes of a cohort or group of individuals born at the same time throughout their lifetimes until the death of the last-surviving member. The latter version of the life table describes the mortality experiences of an entire population at a fixed point in time.

In either case, $l_x$ is the number of people surviving at age $x$. The $l_x$ are usually related by the equations

\begin{align}
(1.1) \quad l_x &= p_{0,x} l_0 \\
(1.2) \quad l_x &= p_{x-1,x} l_{x-1}
\end{align}

In these equations, $p_{ij}$ may be interpreted as the conditional probability

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of an individual surviving to age \( j \) given that he was alive at age \( i \). If the function \( \ell_x \) is normalized so that \( \sum_x \ell_x = 1 \), it is referred to as the population distribution. An extensive discussion of the life table and the related biometric functions may be found in Keyfitz, (1968).

As early as 1907, Lotka had enunciated the idea of the stationary population distribution and suggested the form

\[
\ell(x) = p(x)\lambda e^{-(\lambda - \mu)x}.
\]

Here \( \ell(x) \) is meant to be a continuous (density) version of \( \ell_x \), \( p(x) \) is the probability of an individual surviving to age \( x \), and \( \lambda \) and \( \mu \) are respectively the crude birth and death rates. If we assume an individual has been exposed to a constant death rate, \( \mu \), throughout his \( x \) years, then

\[
p(x) = e^{-\mu x}
\]

so that (1.3) becomes

\[
\ell(x) = \lambda e^{-\lambda x}.
\]

In general, this suggests that \( \ell_x \) decreases like a negative exponential. This is approximately true although such a parametric form is generally not satisfactory to explain all details appearing in real data. A description of Lotka's work can be found in Lotka, (1956).

The fact that \( \ell_x \) is a monotone decreasing function is used to advantage by Grenander, (1956). He gives a non-parametric maximum likelihood estimate of \( \ell_x \). Grenander's type of estimate may be used to estimate any monotone probability density and has generated more work along these lines.

Chiang, (1960) demonstrates that \( \{\ell_x, x = 0,1,2,...\} \) is a Markov Stochastic Process. We will take advantage of this fact in order to use time series methods to study \( \{\ell_x, x = 0,1,2,...\} \). If \( \{\ell_x, x = 0,1,2,...\} \) is
a stochastic process, it is reasonable to expect that (1.2) still holds except that the relation is perturbed by an error term, $\varepsilon_x$. Hence we assume

\begin{equation}
\ell_x = p_{x-1,x} \ell_{x-1} + \varepsilon_x, \quad x = 1, 2, 3, \ldots
\end{equation}

where $\varepsilon_x$ is a sequence of uncorrelated zero mean random variables and where $\varepsilon_x$ is uncorrelated with $\ell_{x'}$, $x' < x$. Let us also, henceforth, write $p_x$ for $p_{x-1,x}$. A process satisfying equation (1.4) is known as a first-order autoregressive process. The relation (1.4) is sometimes referred to as a simple Markov relation. We shall devote the next section to outlining some facts about first order autoregressive processes. In the third section, we shall carry through some computations using the 1961 Indian Census data to illustrate our methods.

2. AUTOREGRESSIVE AND RELATED PROCESSES

In this section, let us write (1.4) as

\begin{equation}
X_t = p(t) X_{t-1} + \varepsilon_t \quad t = 1, 2, \ldots
\end{equation}

in order to keep the notation more consistent with conventional time series notation. We will give, in this section, the mean and covariance structure of $\{X_t: \ t = 1, 2, \ldots\}$ and propose estimates of $p(t)$.

We shall denote by $\text{EX}_t$ the expected value of $X_t$. From (2.1), it is easy to see

$$\text{EX}_t = p(t) \text{EX}_{t-1}, \quad t = 1, 2, \ldots$$

Using this formula recursively, we obtain

\begin{equation}
\text{EX}_t = \left[ \prod_{j=1}^{t} p(j) \right] \cdot \text{EX}_0 \quad t = 1, 2, \ldots
\end{equation}
If $X_0$ is given as an initial condition, the conditional expectation of $X_t$ given $X_0$ is

$$E(X_t | X_0) = \left[ \prod_{j=1}^{t} p(j) \right] \cdot X_0, \quad t = 1, 2, \ldots$$

Again using (2.1), we may study the covariance structure. Multiplying both sides of (2.1) by $X_{t-s}$, we have

$$X_t \cdot X_{t-s} = p(t) \cdot X_{t-1} \cdot X_{t-s} + \epsilon_t \cdot X_{t-s}$$

so that by taking the expectation, we have

$$E(X_t \cdot X_{t-s}) = p(t) \cdot E(X_{t-1} \cdot X_{t-s}).$$

Subtracting the product of the appropriate means,

$$(2.3) \quad \text{cov}(X_t, X_{t-s}) = p(t) \cdot \text{cov}(X_{t-1}, X_{t-s}).$$

Applying (2.3) recursively,

$$(2.4) \quad \text{cov}(X_t, X_{t-s}) = p(t) \cdot p(t-1) \ldots p(t-s+1) \cdot \text{var}(X_{t-s}).$$

Thus, the system of covariances, depends on the variances. Applying (2.1) recursively leads to

$$X_t = \left[ \prod_{j=1}^{t} p(j) \right] \cdot X_0 + \sum_{j=1}^{t-1} p(t) \ldots p(t-j+1) \epsilon_{t-j} + \epsilon_t.$$ 

Subtracting (2.2) from this yields

$$(X_t - E(X_t)) = \left[ \prod_{j=1}^{t} p(j) \right] \cdot (X_0 - E(X_0)) + \sum_{j=1}^{t-1} p(t) \ldots p(t-j+1) \epsilon_{t-j} + \epsilon_t.$$ 

Squaring, taking expected values and recalling $\epsilon_t$ is uncorrelated with $\epsilon_{t'}$, $t \neq t'$ and with $X_{t''} \leq t$, we have

$$(2.5) \quad \text{var}(X_t) = \left[ \prod_{j=1}^{t} p(j) \right]^2 \cdot \text{var}(X_0) + \left[ p^2 + \sum_{j=1}^{t-1} [p(t) \ldots p(t-j+1)]^2 \right] \cdot \text{var}.$$
Here, $\sigma_j^2 = \text{var}(\epsilon_j)$. In the special case $p(t)$ is a constant, say $p$, and $\text{var}(\epsilon_j)$ is a constant, say $\sigma^2$, (2.5) becomes

$$\text{var}(X_t) = p^{2t} \text{var}(X_0) + \sigma^2 \sum_{j=0}^{t-1} p^{2j}.$$  

Using the formula for a finite geometric sum

$$\text{var}(X_t) = p^{2t} \text{var}(X_0) + \frac{\sigma^2 (1-p^{2t})}{1-p^2}.$$  

If $X_0$ is known, then $\text{var}(X_0) = 0$ and the conditional variance is

$$\text{var}(X_t | X_0) = \frac{\sigma^2 (1-p^{2t})}{1-p^2}.$$  

Finally, the conditional covariance is

$$\text{cov}(X_t, X_{t-s} | X_0) = \frac{p^s \sigma^2 (1-p^{2t})}{1-p^2}.$$  

In certain cases, particularly in connection with census data, the errors appear not to have a constant variance, but rather to be proportional, roughly speaking, to population size. That is to say, there appears to be a constant percentage error. Thus, it is reasonable to let $\epsilon_t = X_{t-1} \eta_t$ where $\eta_t$ is uncorrelated with $\eta_{t'}$, $t \neq t'$ and with $X_{t'}$, $t' < t$ and where $\eta_t$ has zero mean and variance $\sigma^2$. In this case, (2.1) becomes

$$(2.6) \quad X_t = p(t) X_{t-1} + \eta_t X_{t-1}.$$  

For an equation of the form (2.6), (2.2) and (2.4) are still true. However, (2.5) is no longer true. Solving (2.6) recursively leads to

$$X_t = \prod_{j=1}^{t} (p(j)+\eta_j)X_0.$$
Squaring and taking expected values,

\[ \text{EX}^2_t = \prod_{j=1}^{t} (p^2(j) + \sigma^2) \cdot \text{EX}_0^2. \]

From the second moments and equation (2.2), the variance of \( X_t \) may be calculated,

\[ \text{var}(X_t) = \left[ \prod_{j=1}^{t} (p^2(j) + \sigma^2) \right] \text{EX}_0^2 - \left[ \prod_{j=1}^{t} p^2(j) \right] \{\text{EX}_0\}^2. \]

If \( p(t) \) is a constant,

\[ \text{var}(X_t) = (p^2 + \sigma^2)t \cdot \text{EX}_0^2 - p^2 t \cdot \{\text{EX}_0\}^2. \]

If \( X_0 \) is known,

\[ \text{var}(X_t|X_0) = X_0^2((p^2 + \sigma^2)t - p^2t). \]

In both (2.1) and (2.6), it is of considerable interest to estimate \( p(t) \).

If we parameterize \( p(t) \) as an \( \ell \)-th degree polynomial, we may use standard least squares procedures to determine the coefficients. In (2.1), let us assume \( \epsilon_t \) has a constant variance and let us also assume we have \( n \) observations \( x_0, \ldots, x_{n-1} \). We then wish to minimize

\[ \sum_{t=1}^{n-1} \left( x_t - \left( \sum_{j=0}^{\ell} a_j t^j \right) x_{t-1} \right)^2. \]

Taking the partial derivative with respect to \( a_k \) and equating to zero, we have

\[ \sum_{t=1}^{n-1} t^k x_t x_{t-1} = \sum_{j=0}^{\ell} a_j \left( \sum_{t=1}^{n-1} t^{j+k} x_t \right), \quad k = 0, \ldots, \ell. \]

Solving (2.7) simultaneously gives an estimate of \( a_j, j = 0, \ldots, \ell \) and, hence, an estimate of \( p(t) \). If \( p(t) \) is assumed to be a constant, then the estimate of \( a_0 = p \) is \( \hat{p} = \left( \sum_{t=1}^{n-1} x_t x_{t-1} \right) \cdot \left( \sum_{t=1}^{n-1} x_t^2 \right)^{-1}. \)
Let us rewrite (2.6) as

\[ x_t x_{t-1}^{-1} = p(t) + \eta_t. \]

Here we wish to minimize

\[ \sum_{t=1}^{n-1} (x_t x_{t-1}^{-1} - \left( \sum_{j=0}^{\ell} a_j t^j \right))^2. \]

Taking partial derivatives with respect to \( a_k \) and equating to zero, we have

\[ \sum_{t=1}^{n-1} x_t x_{t-1}^{-1} t^k = \sum_{j=0}^{\ell} a_j \sum_{t=1}^{n-1} t^{j+k}. \]

Again, solving (2.8) simultaneously gives the estimate of \( p(t) \). In the special case \( p(t) \) is constant, the estimate of \( a_0 = p \) is \( \hat{p} = (n-1)^{-1} \sum_{t=1}^{n} x_t x_{t-1}^{-1} \). An interesting feature of \( \hat{p} \) is that it is an unbiased estimate of \( p \). In addition, \( \text{var}(\hat{p}) = (n-1)^{-1} \sigma^2 \) where \( \sigma^2 \) is the variance of \( \eta_t \).

We close this section by noting that we describe the fitting of a polynomial \( p(t) = \sum_{j=0}^{\ell} a_j t^j \). The set of functions \( \{t^j\} \) are linearly independent but not orthonormal. It is frequently advisable to investigate the fitting of an orthonormal series, rather than one which is merely linearly independent.

### 3. Analysis of the Indian Population Count by Age

The data to which we shall refer in this section is taken from papers published in India on the 1961 Indian census. See Census of India, (1963). We shall present our technique for arriving at the population distribution, \( l_x \), but first we would like to mention the procedure followed by the Indian Census officials.
The population count of India by the stated age of the respondent is found in Table 1. Perhaps the most noticeable facet of this census data is the marked preference shown for the ages divisible by 5. This age heaping effect is well-known and represents a considerable distortion of the true situation. To eliminate this peaking, the Indian census officials smoothed their data by an eleven term moving average. They next formed 5 year totals (with some special adjustments for the very young and the very old), smoothed the result with a weighted 3 term moving average and finally interpolated by the well-known fifth difference osculatory interpolation formulae due to Kozakiewicz.

We have several objections to these procedures. First of all, from the point of view of time series an eleven term moving average is a low pass filter, filtering out variations with period smaller than 11 years. Since we are primarily interested in filtering the peaks which occur at 5 year intervals, a five term moving average is sufficient. (Clearly, one wants to use the filter which least changes the data, but still removes the unwanted periodicities.) A second objection is that the ordinary arithmetic moving average and the procedure of forming 5 years totals are both procedures which tend to reshape the data so that the altered data will have a tendency to cluster along a straight line. In Section 1, we observed, in the case of a stationary population, the population distribution tends to decrease as a negative exponential, hence, very much in a non-linear fashion. (In the case of a nonstationary population where the total population size increases, the population distribution will drop off even more quickly than in the stationary case.) In view of this situation, the data analyst should avoid as much as possible procedures which linearize the data.* We believe that the two

* We have experimented with 11- and 5- term moving averages, using these to filter a curve known to decrease exponentially. The filtered data, in both cases, fell above the original curve. The percentage difference was respectively about 8% and about 1% for the 11 term and the 5 term filters.
### Table 1
Population count (1961) of India by stated age of respondent

<table>
<thead>
<tr>
<th>Age</th>
<th>Population</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>13,551,710</td>
</tr>
<tr>
<td>1</td>
<td>11,673,441</td>
</tr>
<tr>
<td>2</td>
<td>13,427,444</td>
</tr>
<tr>
<td>3</td>
<td>14,129,333</td>
</tr>
<tr>
<td>4</td>
<td>13,242,893</td>
</tr>
<tr>
<td>5</td>
<td>15,227,496</td>
</tr>
<tr>
<td>6</td>
<td>13,966,354</td>
</tr>
<tr>
<td>7</td>
<td>12,268,094</td>
</tr>
<tr>
<td>8</td>
<td>13,655,781</td>
</tr>
<tr>
<td>9</td>
<td>9,471,091</td>
</tr>
<tr>
<td>10</td>
<td>14,366,679</td>
</tr>
<tr>
<td>11</td>
<td>7,115,477</td>
</tr>
<tr>
<td>12</td>
<td>12,729,548</td>
</tr>
<tr>
<td>13</td>
<td>6,922,327</td>
</tr>
<tr>
<td>14</td>
<td>8,094,839</td>
</tr>
<tr>
<td>15</td>
<td>8,403,095</td>
</tr>
<tr>
<td>16</td>
<td>8,184,236</td>
</tr>
<tr>
<td>17</td>
<td>4,641,026</td>
</tr>
<tr>
<td>18</td>
<td>10,225,422</td>
</tr>
<tr>
<td>19</td>
<td>4,370,035</td>
</tr>
<tr>
<td>20</td>
<td>13,596,192</td>
</tr>
<tr>
<td>21</td>
<td>4,536,574</td>
</tr>
<tr>
<td>22</td>
<td>9,314,331</td>
</tr>
<tr>
<td>23</td>
<td>4,427,801</td>
</tr>
<tr>
<td>24</td>
<td>5,395,409</td>
</tr>
<tr>
<td>25</td>
<td>17,076,873</td>
</tr>
</tbody>
</table>
procedures mentioned above have distorted the shape of their curve sufficiently so that their estimate of $\ell_x$ is virtually linear from age 20 to age 70. See for example Census of India, (1963, page 4).

If the moving average is modified by forming a geometric moving average, i.e.

$$Y_t = \left( \frac{1}{5} \right) (X_{t-2} \cdot X_{t-1} \cdot X_t \cdot X_{t+1} \cdot X_{t+2})$$

instead of the arithmetic moving average

$$Y_t = \left( \frac{1}{5} \right) (X_{t-2} + X_{t-1} + X_t + X_{t+1} + X_{t+2})$$

two advantages result. First, if $\{X_t: t = 0, 1, 2, \ldots\}$ truely is an exponential curve, then $\{Y_t: t = 2, 3, \ldots\}$ will be the same exponential curve, and second, it is well-known that geometric averages are less affected by extreme values than arithmetic averages, hence data filtered geometrically is smoother than data filtered through a corresponding arithmetic filter. We have used both the geometric and arithmetic filter and the results may be found in Figures 1 and 2 respectively. Both have a similar shape, but the curve of the arithmetically filtered data tends to lie above the curve of the geometrically filtered data.

Before we move on to the fitting of an autoregressive process, we should point out that the procedures followed by the Indian census officials will tend to keep the total population count constant throughout the whole graduation process whereas the geometric filter most assuredly will not. However, we believe it is the shape of the curve that is of interest and not the total population count.

We should now like to estimate $\ell_x$ based on the filtered data. (We shall illustrate the results for the geometrically filtered data only, but we shall give, in Table 2, results for the arithmetically filtered data also.)
Recall that our autoregressive model is

\[ l_x = p_x l_{x-1} + \varepsilon_x, \quad x = 1, 2, \ldots \]

To give us an idea of the form of \( p_x \), we can plot \( l_x \) verses \( l_{x-1} \). See Figure 3. It is evident from Figure 3, that \( p_x \) is probably a constant and that the errors \( \varepsilon_x \) are fairly small. Fitting higher degree polynomials to \( p_x \) confirms that \( p_x \) is effectively a constant.**

We fitted models both of the form (2.1) with \( \varepsilon_x \) having constant variance \( \sigma^2 \) and of the form (2.6). An analysis of the residuals indicates that the model of (2.6) is more appropriate. Figure 4 is the plot of the residuals for the fitting of this model. If one assumes that the errors are normally distributed†, then one may compute the marginal likelihood product as a function of \( p \). The ratio of this likelihood product as a function of \( p \) to the likelihood product for the least squares (maximum likelihood) estimate for model (2.6) is given in Figure 5. The sharpness of the plot of this ratio lends a good deal of confidence to this model and to the estimate of \( p, \hat{p} \). The results of both models using data filtered both ways is summarized in Table 2.

Recall from (2.2), we have

\[ E_l_x = p E_l_0. \]

** That \( p_x \) should be a constant in a human population is indeed remarkable. In developed countries, one expects the probability of surviving to the next year to be constantly decreasing. Perhaps, this effect is due to relatively harsh conditions in India, where only the very hardy can survive to an old age. Being hardy, the probability of dying in the next year is not significantly different from that of the generally less hardy younger generations.

† We constructed a histogram of the residuals and it appears as though the errors are probably not normally distributed, rather tending more towards a lighter tailed distribution. Nonetheless, the likelihood ratio will be at least approximately correct. In the case that we fitted (2.6) to the data filtered by the arithmetic filter, the normal assumption is much more believable.
<table>
<thead>
<tr>
<th>Model (2.1)</th>
<th>Model (2.6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\epsilon_x$ has variance $\sigma^2$</td>
<td>Estimated $p_x = 0.9571073391$</td>
</tr>
<tr>
<td>Estimated $p_x = 0.9626476654$</td>
<td>Estimated variance of $\eta_x = 0.7596421223$</td>
</tr>
<tr>
<td>Estimated $\sigma^2 = 19420364.46$</td>
<td>Ratio of the Sums of Squares = 78.18</td>
</tr>
<tr>
<td>Ratio of the Sums of Squares = 135.44</td>
<td></td>
</tr>
<tr>
<td>Estimated $p_x = 0.9702683796$</td>
<td>Estimated variance of $\eta_x = 0.763863102$</td>
</tr>
<tr>
<td>Estimated $\sigma^2 = 21267288.58$</td>
<td>Ratio of the Sums of Squares = 81.69</td>
</tr>
<tr>
<td>Ratio of the Sums of Squares = 156.32</td>
<td></td>
</tr>
</tbody>
</table>
Thus $E_l^x \approx p^x$. If we normalize so that $\sum x E_l^x = 1$, we have the expected population distribution. Using our estimate $\hat{\beta}$ from (2.6) and the geometrically filtered data, we may plot the estimated expected population distribution. This is done in Figure 6 along with the original data similarly normalized.

A direct statistical comparison of our methods with commonly used procedures is difficult to make since many commonly used procedures tend to arise from considerations other than statistical ones. We believe, however, that the statistical evidence gives a good deal of confidence in our procedures, particularly because of the relatively close agreement found in Table 2 in spite of the variations in the procedures. This statistical evidence together with the relative simplicity of our procedure and the previously mentioned objections to other procedures, certainly indicates that the time series approach merits consideration.

REFERENCES


APPENDIX

Legends for Figures

Figure 1. Data in Table 1 filtered by a 5 term geometric filter. Scale on vertical axis is \( x10^5 \).

Figure 2. Data in Table 1 filtered by a 5 term arithmetic filter. Scale on vertical axis is \( x10^5 \).

Figure 3. Scatter diagram \( L_x \) vs. \( L_{x-1} \) using data filtered by a five term geometric filter. Scale on both axes is \( x10^5 \).

Figure 4. Residuals using model of (2.6) fitted to data filtered by geometric filter. The arithmetic filter yields similar results.

Figure 5. Ratio of marginal likelihood products using model of (2.6) fitted to data filtered by geometric filter. The arithmetic filter again yields similar results.

Figure 6. Original data and expected curve from the model of (2.6) using data filtered geometrically. Both are normalized so that the total population is 1. Scale on the vertical axis is \( x10^{-2} \).