SOME INCOMPLETE AND BOUNDEDLY COMPLETE FAMILIES OF DISTRIBUTIONS*

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SUMMARY

SOME INCOMPLETE AND BOUNDEDLY COMPLETE FAMLIES OF DISTRIBUTIONS

Let $\mathcal{P}$ be a family of distributions on a measurable space such that (†) $\int u_i \, dP = c_i$, $i = 1, \ldots, k$, for all $P \in \mathcal{P}$, and which is sufficiently rich; for example, $\mathcal{P}$ consists of all distributions dominated by a σ-finite measure and satisfying (†) . It is known that when conditions (†) are not present, no nontrivial symmetric unbiased estimator of zero (s.u.e.z.) based on a random sample of any size n exists. Here it is shown that (I) if $g(x_1, \ldots, x_n)$ is a s.u.e.z. then there exist symmetric functions $h_i(x_1, \ldots, x_{n-1})$, $i = 1, \ldots, k$, such that

$$g(x_1, \ldots, x_n) = \sum_{i=1}^{k} \sum_{j=1}^{n} \{u_i (x_j) - c_i\} h_i(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) ;$$

and (II) if every nontrivial linear combination of $u_1, \ldots, u_k$ is unbounded then no bounded nontrivial s.u.e.z. exists. Applications to unbiased estimation and similar tests are discussed.

SOME INCOMPLETE AND BOUNDEDLY COMPLETE FAMILIES OF DISTRIBUTIONS[1]

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[Abbreviated title: Incomplete and boundedly complete families.]

1. Introduction and Statement of Results. Let A be a σ-field of subsets of a set X, and let P be a family of distributions (probability measures) P on (X,A) which satisfy the conditions

$$\int u_i \, dP = c_i \ , \quad i = 1, \ldots, k \ , \tag{1.1}$$

where k is a positive integer, $u_1, \ldots, u_k$ are given A-measurable functions, and $c_1, \ldots, c_k$ are given real numbers. Let $A^{(n)}$ be the σ-field of subsets of X^n generated by the (cylinder) sets in A^n, and let $P^{(n)} = \{P^n : P \in P\}$ denote the family of the n-fold product measures P^n on $(X^n, A^{(n)})$.

A family Q of distributions on $(X^n, A^{(n)})$ will be said to be complete relative to the permutation group if the condition that the $A^{(n)}$-measurable symmetric real-valued function g satisfies $\int g \, dQ = 0$ for all $Q \in Q$

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implies $g(x_1, \ldots, x_n) = 0$ a.e. $(\mathcal{Q})$. Here g is called symmetric if it is invariant under all permutations of its arguments. The family $\mathcal{Q}$ will be said to be boundedly complete relative to the permutation group if the same conclusion holds under the additional condition that g is bounded. Informally, $\mathcal{Q}$ is [boundedly] complete relative to the permutation group if there is no nontrivial [bounded] symmetric unbiased estimator of zero. (This definition relates to the well-known notion of a [boundedly] complete family [5] as follows. Let T be a maximal invariant under the permutation group and let $\mathcal{Q}^T = \{Q^T: Q \in \mathcal{Q}\}$ be the family of distributions of T induced by the distributions in $\mathcal{Q}$. Then $\mathcal{Q}$ is [boundedly] complete relative to the permutation group iff the family $\mathcal{Q}^T$ is [boundedly] complete.)

It is well known (Halmos 1946, Fraser 1953) that if the conditions (1.1) are absent and $\mathcal{P}$ is sufficiently rich then $\mathcal{P}^{(n)}$ is complete relative to the permutation group. This is not true in the presence of conditions (1.1) (unless the u_i and c_i are such that the conditions impose no restriction). Indeed, if $h_1, \ldots, h_k$ are any $\mathcal{A}^{(n-1)}$-measurable symmetric functions such that $\int |h_i| dP^{n-1} < \infty$, $i = 1, \ldots, k$, for all $P \in \mathcal{P}$, then the function g defined by

$$(1.2) \quad g(x_1, \ldots, x_n) = \sum_{i=1}^{k} \sum_{j=1}^{n} \{u_i(x_j) - c_i\} h_i(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$$

is a symmetric unbiased estimator of zero.

In this paper two theorems (each in two versions) are proved. The first theorem shows that if $\mathcal{P}$ is sufficiently rich then a symmetric unbiased estimator of zero is necessarily of the form (1.2). The second theorem shows that although $\mathcal{P}^{(n)}$ is not complete relative to the permutation group it is boundedly complete if all nontrivial linear combinations of $u_1, \ldots, u_k$ are unbounded.

To state the theorems, we introduce the following notation. If $\mathcal{A}$ contains the one-point sets, let $\mathcal{P}_0$ be the family of all distributions P concentrated on finite subsets of $\mathcal{X}$ which satisfy conditions (1.1). If μ is a σ-finite measure on $(\mathcal{X},\mathcal{A})$, let $\mathcal{P}_0(\mu)$ be the family of all distributions absolutely continuous with respect to μ whose densities $dP/d\mu$ are simple functions (finite linear combinations of indicator functions of sets in $\mathcal{A}$) and which satisfy conditions (1.1).

<u>Theorem 1A</u>: Let $\mathcal{A}$ contain the one-point sets and let $\mathcal{P}$ be a convex family of distributions on $(\mathcal{X},\mathcal{A})$ which satisfy conditions (1.1), such that $\mathcal{P}_0 \subset \mathcal{P}$. If g is a symmetric $\mathcal{A}^{(n)}$-measurable function such that $\int g\, dP^n = 0$ for all $P \in \mathcal{P}$ then there exist k symmetric $\mathcal{A}^{(n-1)}$-measurable functions $h_1, \ldots, h_k$ which are P^{n-1}-integrable for all $P \in \mathcal{P}$, such that (1.2) is satisfied for all $(x_1, \ldots, x_n) \in \mathcal{X}^n$.

<u>Theorem 2A</u>. If the conditions of Theorem 1A are satisfied and if g is bounded while every nontrivial linear combination of $u_1, \ldots, u_k$ is unbounded then $g(x_1, \ldots, x_n) = 0$ for all $(x_1, \ldots, x_n) \in \mathcal{X}^n$.

The following analogs of the two theorems hold for dominated families of distributions.

We shall say that a A-measurable function u is P-unbounded if for every real number c there is a P in P such that $P(|u(x)| > c) \neq 0$.

__Theorem 1B__. Let P be a convex family of distributions absolutely continuous with respect to a σ-finite measure μ on (X,A) which satisfy conditions (1.1), such that $P_0(\mu) \subset P$. If g is a symmetric $A^{(n)}$-measurable function such that $\int g \, dP^n = 0$ for all $P \in P$ then there exist k symmetric $A^{(n-1)}$-measurable functions $h_1, \ldots, h_k$ which are P^{n-1}-integrable for all $P \in P$, such that (1.2) holds a.e. $(P^{(n)})$.

__Theorem 2B__. If the conditions of Theorem 1B are satisfied and if g is bounded while every nontrivial linear combination of $u_1, \ldots, u_k$ is P-unbounded then $g(x_1, \ldots, x_n) = 0$ a.e. $(P^{(n)})$.

__Remark 1__. The assumption that the family P is convex is used only to prove that there are versions of the functions h_i that are integrable. Note that the families P_0, $P_0(\mu)$, and the family of all P which are absolutely continuous with respect to μ and satisfy conditions (1.1), are convex.

__Remark 2__. Theorems 1B and 2B remain true if $P_0(\mu)$ is defined as the family of all distributions absolutely continuous with respect to μ whose densities are finite linear combinations of indicator functions of sets in a ring which generates the σ-field A ; compare Fraser (1953).

__Remark 3__. If the assumptions of Theorems 1A or 1B are satisfied but conditions (1.1) are absent then the family $P^{(n)}$ is complete relative to the permutation group. Here the assumption that $\mathcal{P}$ is convex is not needed. This is essentially known (as noted above) and is easily seen from the proofs.

The theorems are proved in Sections 3 - 6. Section 2 contains lemmas that are used in the proofs.

This section is concluded with three examples of applications of Theorems 1 and 2.

Example 1. Let $X_1, \ldots, X_n$ be independent real-valued random variables with common probability density $p = dP/dx$ and suppose that the first k moments, $\int x^i p(x)dx = c_i$, $i = 1, \ldots, k$, are known. Nothing else is assumed. Let $\psi(P) = P^m\{(X_1, \ldots, X_m) \in A\}$ be the probability of some event involving $X_1, \ldots, X_m$, where $m \leq n$. It is reasonable to require that an estimator $\hat{\psi}$ of $\psi(P)$ satisfy $0 \leq \hat{\psi} \leq 1$. It is easy to see that a symmetric unbiased estimator, $\hat{\psi}_0$, with this property exists. Theorem 2B implies that it is unique. (Note that the moment conditions may restrict the range of $\psi(P)$ to a proper subset S of $[0,1]$, via Chebyshev-type inequalities. Typically, the values of $\hat{\psi}_0$ are not confined to S. In such a case the use of an unbiased estimator can not be recommended.)

Example 2. Let $X_1, \ldots, X_n$ be independent real-valued random variables with common distribution P whose variance is known. Consider testing the hypothesis $\int x\, dP = 0$ against the alternatives $\int x\, dP > 0$. For every $n \geq 1$, every $\alpha \in (0,1)$, and every $\varepsilon > 0$ there exists a strictly unbiased test of size α against the alternatives $\int x\, dP \geq \varepsilon$. (In Hoeffding (1956), p. 112, a test is exhibited which (after a suitable change in notation) is strictly unbiased against $\int x\,dP = \varepsilon$. This test can be shown to be strictly unbiased against $\int x\,dP \geq \varepsilon$.) Theorem 2 implies that against the alternatives $\int x\,dP > 0$ no nontrivial unbiased test exists. (One first shows that every unbiased test is similar; see [5]. We may assume that the test is symmetric. By Theorem 2 the only symmetric similar test of size α is trivial.)

Example 3. Let the assumptions of Theorem 1 (A or B) be satisfied. If $\psi(P)$ admits an unbiased estimator, then the difference of any two symmetric unbiased estimators is given by (1.2). We discuss only the simplest case, $n = 1$. Let $\psi(P) = \int w\,dP$. Then any unbiased estimator $t(x)$ is given by

$$t(x) = w(x) + \sum_{k=1}^{k} h_i\{u_i(x) - c_i\} ,$$

where $h_1, \dots, h_k$ are arbitrary constants. Suppose that $w, u_1, \dots, u_k$ have finite second moments. Then

$$\mathrm{var}_P(t) = \mathrm{var}_P(w) + 2\sum_{i=1}^{k} h_i C_i(P) + \sum_{i=1}^{k}\sum_{j=1}^{k} h_i h_j D_{ij}(P)$$

where $C_i(P) = \mathrm{cov}_P(w,u_i)$ and $D_{ij}(P) = \mathrm{cov}_P(u_i,u_j)$. It is straightforward to minimize $\mathrm{var}_P(t)$ with respect to $h_1, \dots, h_k$. Let Q be a distribution in $\mathcal{P}$ such that the matrix $(D_{ij}(Q))$ is nonsingular, and let $(D^{ij}(Q))$ be its inverse. Then the unbiased estimator which has minimum variance when the distribution is Q is $t(x)$ with $h_i = \Sigma_j\, D^{ij}(Q)C_j(Q)$, and its variance at $P = Q$ is $\mathrm{var}_Q(w) - \Sigma\Sigma D^{ij}(Q)C_i(Q)C_j(Q)$.

2. Lemmas. The following lemmas will be used in the proofs of the theorems.

Lemma 1: (Halmos). A homogeneous polynomial on R^n which is zero on a nondegenerate n-dimensional interval, is identically zero on R^n .

The proof is by induction on n , just as Halmos' proof in [2] for the case where the n-dimensional interval is the positive orthant.

Lemma 2: If $a_{ij}(i = 1, \dots ,m; j = 1, \dots ,n)$ are real numbers, there exists an index h $(1 \le h \le m)$ such that the inequalities

$$\sum_{j=1}^{n} (a_{hj} - a_{ij})p_j \ge 0 \ , \quad i = 1, \dots ,m \ , \tag{2.1}$$

are satisfied in a nondegenerate n-dimensional sub-interval of $\{(p_1, \dots ,p_n): p_1 > 0, \dots ,p_n > 0\}$.

Proof: The inequality (2.1) with $i = h$ is trivially true for all $p_1, \dots ,p_n$. If two rows in the matrix $A = (a_{ij})$ are equal, the corresponding two inequalities in (2.1) are equivalent. We may therefore assume that A has no equal rows. Under this assumption it will be shown that there exist an index h and strictly positive numbers $p_1, \dots ,p_n$ such that the inequalities (2.1) with $i \ne h$ are strict:

$$\sum_{j=1}^{n} (a_{hj} - a_{ij})p_j > 0 \ , \quad i = 1, \dots ,h - 1, h + 1, \dots ,m \ . \tag{2.2}$$

This implies, by continuity, that the same is true in a neighborhood of the point $(p_1, \dots ,p_n)$, and the lemma will be proved.

Let A have no equal rows. Then there exist an integer r , $1 \le r \le m$; integers $m(1), \ldots, m(r)$ such that

$$m = m(0) \ge m(1) > m(2) > \ldots > m(r) = 1 ;$$

and columns $C_1, \ldots, C_r$ of the matrix A such that column C_1 has exactly $m(1)$ maximal elements (the *specified* elements in column C_1) and, for $g = 1, \ldots, r - 1$, the $m(g)$ elements in column C_{g+1} which are in the same rows as the $m(g)$ specified elements in column C_g contain exactly $m(g + 1)$ maximal elements (the *specified* elements in column C_{g+1}).

We may assume that C_g is the g-th column of the matrix A and that the specified $m(g)$ elements in column C_g are the first $m(g)$ elements of that column, for $g = 1, \ldots, r$. Then

$$a_{1,g} = \ldots = a_{m(g),g} > a_{i,g} , \quad i = m(g) + 1, \ldots, m(g - 1) ,$$

for $g = 1, \ldots, r$. Hence if $i = m(g) + 1, \ldots, m(g - 1)$; $g = 1, \ldots, r$, then

$$\text{(2.3)} \qquad \sum_{j=1}^{n} (a_{1,j} - a_{i,j})p_j = \sum_{j=g}^{n} (a_{1j} - a_{ij})p_j ,$$

where $a_{1g} - a_{ig} > 0$. Thus if $p_{r+1}, \ldots, p_n$ are any fixed positive numbers and, having defined p_j for $j \ge g + 1$, we take $p_g > 0$ so large that the right side of (2.3) is strictly positive, with $g = r, r - 1, \ldots, 1$, then the inequalities (2.2) are satisfied with $h = 1$.

We denote by u the column vector with k components $u_1, \ldots, u_k$.

<u>Lemma 3A</u>: If, for $(x_1, \ldots, x_n) \in \mathcal{X}^n$,

$$(2.4)\quad g(x_1, \ldots, x_n) = \sum_{i=1}^{k} \sum_{j=1}^{n} u_i(x_j) h_i(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) ,$$

where each h_i is symmetric in its $n - 1$ arguments, and if $z_1, \ldots, z_k$ are k points in $\mathcal{X}$ such that the $k \times k$ matrix

$$(2.5)\qquad U = \left(u(z_1), \ldots, u(z_k)\right)$$

is nonsingular, then, for $(x_1, \ldots, x_n) \in \mathcal{X}^n$,

$$(2.6)\qquad g(x_1, \ldots, x_n) = \sum_{m=0}^{n-1} (-1)^{n-m-1}\, T_{n,m}(x_1, \ldots, x_n) ,$$

where

$$(2.7)\quad T_{n,m}(x_1, \ldots, x_n) = \sum\nolimits_{m,n-1} \sum_{i_1=1}^{k} \cdots \sum_{i_{n-m}=1}^{k} g(x_{j_1}, \ldots, x_{j_m}, z_{i_1}, \ldots, z_{i_{n-m}}) v_{i_1}(x_{j_{m+1}}) \cdots v_{i_{n-m}}(x_{j_n}) ,$$

$$(2.8)\qquad v(x) = U^{-1} u(x) ,$$

and $\sum_{m,n-m}$ denotes summation over those permutations $j_1, \ldots, j_n$ of the integers $1, \ldots, n$ for which $j_1 < \ldots < j_m$ and $j_{m+1} < \ldots < j_n$.

<u>Remark</u>: Note that representation (2.6) of $g(x_1, \ldots, x_n)$ does not involve the functions $h_1, \ldots, h_k$ which appear in (2.4).

<u>Proof</u>: From (2.4) and (2.8) we have

$$(2.9)\quad g(x_1, \ldots, x_n) = \sum_{i=1}^{k} \sum_{j=1}^{n} v_i(x_j) f_i(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) ,$$

where

$$(2.10) \qquad f(x_1, \dots ,x_{n-1}) = U^T h(x_1, \dots ,x_{n-1}) .$$

(The superscript T stands for transpose.) By (2.8),

$$(2.11) \qquad v_i(z_i) = 1 , \quad v_i(z_j) = 0 , \quad i \neq j .$$

Hence, for $1 \le i_r \le k$, $r = 1, \dots ,n - m$; $n - m = 1, 2, \dots ,n$,

$$(2.12) \quad g(x_1, \dots ,x_m, z_{i_1}, \dots ,z_{i_{n-m}})$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{m} v_i(x_j) f_i(x_1, \dots ,x_{j-1}, x_{j+1}, \dots ,x_m, z_{i_1}, \dots ,z_{i_{n-m}})$$

$$+ f_{i_1}(x_1, \dots ,x_m, z_{i_2}, \dots ,z_{i_{n-m}}) + \dots$$

$$+ f_{i_{n-m}}(x_1, \dots ,x_m, z_{i_1}, \dots ,z_{i_{n-m-1}}) .$$

In particular, with $m = n - 1$,

$$f_{i_1}(x_1, \dots ,x_{n-1}) = g(x_1, \dots ,x_{n-1}, z_{i_1})$$

$$- \sum_{i=1}^{k} \sum_{j=1}^{n-1} v_i(x_j) f_i(x_1, \dots ,x_{j-1}, x_{j+1}, \dots ,x_{n-1}, z_{i_1}) .$$

Inserted in (2.9) this gives

$$g(x_1, \dots ,x_n) = T_{n,n-1} (x_1, \dots ,x_n) - R_1 ,$$

where

$$R_1 = \sum_{i_1=1}^{k} \sum_{i_2=1}^{k} \sum\nolimits_{n-2,2} v_{i_1}(x_{j_{n-1}}) v_{i_2}(x_{j_n}) \{ f_{i_1}(x_{j_1}, \dots, x_{j_{n-2}}, z_{i_2}) + f_{i_2}(x_{j_1}, \dots, x_{j_{n-2}}, z_{i_1}) \} .$$

In general, by induction on m,

$$(2.13) \quad g(x_1, \dots, x_n) = T_{n,n-1} - T_{n,n-2} + \dots + (-1)^{m-1} T_{n,n-m} + (-1)^m R_m ,$$

$$m = 0, 1, \dots, n-1 ,$$

where $T_{n,r} = T_{n,r}(x_1, \dots, x_n)$, and R_m differs from $T_{n,n-m-1}$ only in that $g(\dots, z_{i_1}, \dots, z_{i_{m+1}})$ is replaced by $f_{i_1}(\dots, z_{i_2}, \dots, z_{i_{m+1}}) + \dots + f_{i_{m+1}}(\dots, z_{i_1}, \dots, z_{i_m})$. In particular, by (2.12) with $m = 0$, we have $R_{n-1} = T_{n,0}$, and (2.6) follows from (2.13).

<u>Lemma 3B</u>: Let ν be a finite measure on the measurable space (X, A), let g be a A^n-measurable function such that $\int |g| d\nu^n < \infty$, and let $u_1, \dots, u_k$ be A-measurable functions such that $\int |u_i| d\nu < \infty$, $i = 1, \dots, k$. If there exist symmetric A^{n-1}-measurable functions $h_1, \dots, h_k$ such that $g(x_1, \dots, x_n)$ can be represented in the form (2.4) for all $(x_1, \dots, x_n) \in X^n$, and if $B_1, \dots, B_k$ are k sets in A such that the $k \times k$ matrix

$$(2.14) \qquad U_\nu = \left(\int_{B_1} u \, d\nu, \dots, \int_{B_k} u \, d\nu \right)$$

is nonsingular, then, for all $(x_1, \dots, x_n) \in X^n$,

$$(2.15) \qquad g(x_1, \dots, x_n) = \sum_{m=0}^{n-1} (-1)^{n-m-1} T_{n,m}^{(\nu)}(x_1, \dots, x_n) ,$$

where $T^{(\nu)}_{n,m}(x_1, \ldots, x_n)$ is defined like $T_{n,m}(x_1, \ldots, x_n)$ in (2.7) but with $g(x_{j_1}, \ldots, x_{j_m}, z_{i_1}, \ldots, z_{i_{n-m}})$ replaced by

$$(2.16) \quad \int_{B_{i_1}} d\nu(y_1) \ldots \int_{B_{i_{n-m}}} d\nu(y_{n-m}) g(x_{j_1}, \ldots, x_{j_m}, y_1, \ldots, y_{n-m})$$

and $v(x) = \left(v_1(x), \ldots, v_k(x)\right)^T$ replaced by

$$(2.17) \qquad v^{(\nu)}(x) = U_\nu^{-1}\, u(x) \ .$$

The same is true with the phrase "for all $(x_1, \ldots, x_n) \in X^n$" replaced by " a.e. $\nu^{(n)}$ " in the two places where it occurs.

The *proof* of Lemma 3B closely parallels that of Lemma 3A. The only difference is that any substitution, in a function $f(\ldots, x_i, \ldots)$, of z_j for x_i in the proof of Lemma 3A is replaced by integration over B_j with respect to $d\nu(x_i)$.

Incidentally, Lemma 3B contains Lemma 3A.

The functions $h_1, \ldots, h_k$ in representation (2.4) of g are not, in general, uniquely determined by the functions $g, u_1, \ldots, u_k$. For example, if $h_1(x,y)$, $h_2(x,y)$ satisfy (2.4) with $k = 2$, $n = 3$, so do

$$H_1(x,y) = h_1(x,y) + w(y)u_2(x) + w(x)u_2(y) \ ,$$
$$H_2(x,y) = h_2(x,y) - w(y)u_1(x) - w(x)u_1(y) \ ,$$

where $w(x)$ is arbitrary. The following lemma records, for future reference, a certain version of the functions $h_1, \ldots, h_k$.

<u>Lemma 4</u>: Suppose there exist symmetric functions $h_1, \ldots, h_k$ such that g has the representation (2.4). Under the conditions of Lemma 3A, $h_1, \ldots, h_k$ can be so chosen that each $h_i(x_1, \ldots, x_{n-1})$ is a finite linear combination of terms of the form

$$(2.18)\quad g(x_{j_1}, \ldots, x_{j_m}, z_{i_1}, \ldots, z_{i_{n-m}})u_{r_1}(x_{j_{m+1}}) \ldots u_{r_{n-m-1}}(x_{j_{n-1}}) ,$$

where the subscripts $j_1, \ldots, j_{n-1}$ are all different. Under the conditions of Lemma 3B, each $h_i(x_1, \ldots, x_{n-1})$ can be chosen as a finite linear combination of terms of the form

$$(2.19)\quad \int_{B_{i_1}} d\nu(y_1) \ldots \int_{B_{i_{n-m}}} d\nu(g_{n-m})g(x_{j_1}, \ldots, x_{j_m}, y_1, \ldots, y_{n-m}) \\ u_{r_1}(x_{j_{m+1}}) \ldots u_{r_{n-m-1}}(x_{j_{n-1}}) ,$$

where $j_1, \ldots, j_{n-1}$ are all different.

<u>Proof</u>: Under the conditions of Lemma 3A, $g(x_1, \ldots, x_n)$ has the representation (2.6), where $T_{n,m}(x_1, \ldots, x_n)$ is defined in (2.7) and each $v_i(x)$ is a linear combination of $u_1(x), \ldots, u_k(x)$. Hence $g(x_1, \ldots, x_n)$ can be written as a linear combination of terms, each of which, for some i and some j , is of the form $u_i(x_j)$ times a product of the form (2.18) which does not involve x_j . This fact and the symmetry of $g(x_1, \ldots, x_n)$ imply the assertion of the lemma. Under the conditions of Lemma 3B the proof is analogous.

3. Proof of Theorem 1A. We may and shall assume that conditions (1.1) are satisfied with $c_1 = \dots = c_k = 0$. Thus $\mathcal{P}_0$ is the family of all distributions P concentrated on finite subsets of X which satisfy the conditions,

$$(3.1) \qquad \int u_i \, dP = 0 \,, \quad i = 1, \dots, k \,,$$

and $\mathcal{P}$ is a convex family of distributions P on (X,A) which satisfy (3.1), such that $\mathcal{P} \supset \mathcal{P}_0$. Let g be a symmetric A^n-measurable function such that $\int g \, dP^n = 0$ for all $P \in \mathcal{P}$. We must show that there exist symmetric A^{n-1}-measurable functions $h_1, \dots, h_k$ such that

$$(3.2) \qquad \int |h_i| \, d\, P^{n-1} < \infty \,, \quad i = 1, \dots, k \,, \quad \text{if } P \in \mathcal{P}$$

and, for all $(x_1, \dots, x_n) \in X^n$,

$$(3.3) \qquad g(x_1, \dots, x_n) = \sum_{i=1}^{k} \sum_{j=1}^{n} u_i(x_j) h_i(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) \,.$$

Since $\mathcal{P}_0 \subset \mathcal{P}$, we have that if N is a positive integer, $x_1, \dots, x_N$ are points in X , and $p_1, \dots, p_N$ are nonnegative numbers such that

$$(3.4) \qquad p_1 + \dots + p_N = 1 \,,$$

$$(3.5) \qquad u_i(x_1)p_1 + \dots + u_i(x_N)p_N = 0 \,, \quad i = 1, \dots, k \,,$$

then

$$(3.6) \qquad \sum_{i_1=1}^{N} \dots \sum_{i_n=1}^{N} g(x_{i_1}, \dots, x_{i_n}) p_{i_1} \dots p_{i_n} = 0 \,.$$

It will be convenient to identify points in R^k with the corresponding column vectors, $u = (u_1, \ldots, u_k)^T$. We write $u(x)$ for $(u_1(x), \ldots, u_k(x))'$ and 0 for $(0, \ldots, 0)^T$.

Let

$$\mathcal{U} = \{u(x): x \in X\} .$$

Conditions (3.4) and (3.5) show that the origin 0 is in the convex hull of $\mathcal{U}$.

First assume that 0 is in the interior of the convex hull of $\mathcal{U}$. Then there exist $k + 1$ points $y_1, \ldots, y_{k+1}$ in X such that 0 is in the interior of the polytope whose vertices are $u(y_1), \ldots, u(y_{k+1})$. Thus there are strictly positive numbers $\lambda_1, \ldots, \lambda_{k+1}$ such that

$$\lambda_1 + \ldots + \lambda_{k+1} = 1 , \tag{3.7}$$

$$u(y_1)\lambda_1 + \ldots + u(y_{k+1})\lambda_{k+1} = 0 .$$

The solution of this non-singular system of $k + 1$ equations for $\lambda_1, \ldots, \lambda_{k+1}$ is

$$\lambda_h = d_h/(d_1 + \ldots + d_{k+1}) , \quad h = 1, \ldots, k + 1 , \tag{3.8}$$

where d_h is the determinant

$$d_h = \left| \begin{matrix} \\ u(y_1), \end{matrix} \ldots , \begin{matrix} \\ u(y_{h-1}), \end{matrix} \begin{matrix} 1 \\ 0, \end{matrix} \begin{matrix} \\ u(y_{h+1}), \end{matrix} \ldots , \begin{matrix} \\ u(y_{k+1}) \end{matrix} \right| . \tag{3.9}$$

(The unspecified elements of the determinant do not affect its value.) Since $\lambda_h > 0$ for all h, the d_h are all of the same sign. We may and shall assume that

$$d_h > 0 \ , \quad h = 1, \ \ldots \ ,k + 1 \ . \tag{3.10}$$

For $1 \leq h \neq i \leq k + 1$ define $D_{i,h}(u)$ as the determinant obtained from d_h in (3.9) when $u(y_i)$ is replaced by u . Thus if $i < h$ then

$$D_{i,h}(u) = \left| u(y_1),\ldots,u(y_{i-1}),u,u(y_{i+1}),\ldots,u(y_{h-1}),\overset{1}{0},u(y_{h+1}),\ldots,u(y_{k+1}) \right| \tag{3.11}$$

We note that

$$D_{i,h}(u) = -D_{h,i}(u) \ , \tag{3.12}$$

$$D_{i,h}\big(u(y_i)\big) = d_h, \ D_{i,h}\big(u(y_h)\big) = -d_i, \ D_{i,h}\big(u(y_g)\big) = 0 \quad \text{if} \quad g \neq i,h \ . \tag{3.13}$$

Now let $x_1, \ \ldots \ ,x_n$ be any n points in X and let $z_1, \ \ldots \ ,z_k$ be any k of the $k + 1$ points $y_1, \ \ldots \ ,y_{k+1}$. Occasionally we shall use the alternative notation

$$(x_{n+1}, \ \ldots \ ,x_{n+k}) = (z_1, \ \ldots \ ,z_k) \ .$$

Suppose that there are $n + k$ nonnegative numbers $p_1, \ \ldots \ ,p_{n+k}$ such that $p_1 + \ldots + p_{n+k} = 1$ and

$$u(x_1)p_1 + \ldots + u(x_{n+1})p_{n+1} + \ldots + u(x_{n+k})p_{n+k} = 0 \ . \tag{3.14}$$

Then, by (3.4) - (3.6) with $N = n + k$,

$$\sum_{j_1=1}^{n+k} \cdots \sum_{j_n=1}^{n+k} g(x_{j_1}, \ \ldots \ ,x_{j_n})p_{j_1} \ldots p_{j_n} = 0 \ . \tag{3.15}$$

Note that the restriction $p_1 + \dots + p_{n+k} = 1$ is irrelevant since equations (3.14) and (3.15) are not changed if all p_j are multiplied with the same positive constant. Only the assumption that the p_j are nonnegative is needed for (3.14) to imply (3.15).

For $p_1, \dots, p_n$ held fixed, consider (3.14) as a system of k equations for $p_{n+1}, \dots, p_{n+k}$. Condition (3.10) implies that the matrix $U = \big(u(z_1), \dots, u(z_k)\big)$ is non-singular, and the solution of system (3.14) is

$$(3.16) \qquad p_{n+i} = -\sum_{j=1}^{n} v_i(x_j)p_j \ , \quad i = 1, \dots, k \ ,$$

where $v(x) = U^{-1}u(x)$.

Write $p_{n+i}^{(h)}$ for p_{n+i} when $(z_1, \dots, z_k) = (y_1, \dots, y_{h-1}, y_{h+1}, \dots, y_{k+1})$ where $h = 1, \dots, k+1$. Then, in the notation of (3.9) and (3.11),

$$(3.17) \qquad d_h\, p_{n+i}^{(h)} = \begin{cases} \displaystyle\sum_{j=1}^{n} D_{h,i}\big(u(x_j)\big)p_j \ , & i = 1, \dots, h-1 \ , \\ \displaystyle\sum_{j=1}^{n} D_{h,i+1}\big(u(x_j)\big)p_j \ , & i = h, \dots, k \ . \end{cases}$$

Since $d_h > 0$ for all h , the condition $p_{n+i}^{(h)} \geq 0$, $i = 1, \dots, k$, is equivalent to

$$(3.18) \qquad \sum_{j=1}^{n} D_{h,i}\big(u(x_j)\big)p_j \geq 0 \ , \quad i = 1, \dots, h-1, h+1, \dots, k+1 \ .$$

We now show that for every point $(x_1, \dots, x_n)$ in X^n there is an integer h $(1 \leq h \leq k+1)$ such that (3.18) is satisfied in a nondegenerate n-dimensional sub-interval of $\{(p_1, \dots, p_n)\colon p_1 > 0, \dots, p_n > 0\}$. It is enough to show that for every point $(u_1, \dots, u_n)$ in R^{kn} there is an h

such that the conditions

$$(3.19)\quad \sum_{j=1}^{n} D_{h,i}(u_j)p_j \geq 0 \;,\quad i = 1, \ldots ,h - 1, h + 1, \ldots ,k + 1 \;,$$

are satisfied in such an interval.

For every point u in R^k there are real numbers $b_1(u), \ldots ,b_{k+1}(u)$ such that

$$u = \sum_{j=1}^{k+1} b_j(u)\, u(y_j) \;.$$

By (3.11) and (3.13)

$$D_{h,i}(u) = \sum_{j=1}^{k+1} b_j(u) D_{h,i}\big(u(y_j)\big) = b_h(u)d_i - b_i(u)d_h \;.$$

Therefore conditions (3.19) are equivalent to

$$\sum_{j=1}^{n} \{b_h(u_j)/d_h - b_i(u_j)/d_i\}p_j \geq 0 \;,\quad i = 1, \ldots ,k + 1 \;.$$

(For $i = h$ this inequality is trivially satisfied.) The assertion in the preceeding paragraph now follows from Lemma 2 with $m = k + 1$ and

$a_{ij} = b_i(u_j)/d_i$.

The foregoing shows that the set X^n can be covered with $k + 1$ measurable subsets $E_1, \ldots ,E_{k+1}$ such that if $(x_1, \ldots ,x_n) \in E_h$ then inequalities (3.18) are satisfied in a nondegenerate sub-interval I of $\{(p_1, \ldots ,p_n)\colon p_1 > 0, \ldots ,p_n > 0\}$.

Let $(x_1, \ldots ,x_n)$ be a point in X^n . Then $(x_1, \ldots ,x_n) \in E_h$ for some h (which will be held fixed in what follows). Let $(z_1, \ldots ,z_k) = (y_1, \ldots ,y_{h-1}, y_{h+1}, \ldots ,y_{k+1})$. Then, for $(p_1, \ldots ,p_n) \in I$ the values $p_{n+1}, \ldots ,p_{n+k}$ defined in (3.16) are non-negative. We shall insert these values in the left side of (3.15). The resulting homogeneous polynomial in $p_1, \ldots, p_n$ is zero on I and hence, by Lemma 1, is identically zero.

The sum in (3.15) may be written as

$$(3.20)\quad \sum_{j_1=1}^{n+k} \cdots \sum_{j_n=1}^{n+k} g(x_{j_1}, \ldots, x_{j_n}) p_{j_1} \cdots p_{j_n} = \sum_{m\geq 0, m_1\geq 0, \ldots, m_k\geq 0}$$

$$\frac{n!}{m!m_1!\ldots m_k!} \sum_{j_1=1}^{n} \cdots \sum_{j_m=1}^{n} g(x_{j_1}, \ldots, x_{j_m}, z_1^{m_1}, \ldots, z_k^{m_k}) p_{j_1} \cdots p_{j_m} \prod_{i=1}^{k} p_{n+i}^{m_i} ,$$

where z^m stands for the vector with m equal components z. Identity (3.20) can be proved by induction on k.

On inserting the expressions (3.16) for $p_{n+1}, \ldots, p_{n+k}$ in (3.20) we obtain

$$(3.21)\quad \sum_{j_1=1}^{n+k} \cdots \sum_{j_n=1}^{n+k} g(x_{j_1}, \ldots, x_{j_m}) p_{j_1} \cdots p_{j_n} = \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} a_{j_1\ldots j_n} p_{j_1} \cdots p_{j_n} ,$$

where

$$(3.22)\quad a_{j_1\ldots j_n} = \sum_{\substack{m\geq 0, m_1\geq 0, \ldots, m_k\geq 0 \\ m+m_1+\ldots+m_k=n}} (-1)^{n-m} \frac{n!}{m!m_1!\ldots m_k!}$$

$$g(x_{j_1}, \ldots, x_{j_m}, z_1^{m_1}, \ldots, z_k^{m_k}) \prod_{i=1}^{k} \prod_{r=m+\ldots+m_{i-1}+1}^{m+\ldots+m_i} v_i(x_{j_r}) .$$

Let $a^*_{j_1,\ldots,j_n}$ denote the symmetric average of the values $a_{\pi(j_1,\ldots,j_n)}$ over the n! permutations $\pi(j_1,\ldots,j_n)$ of the integers $j_1, \ldots, j_n$. The condition that the homogeneous polynomial in $p_1, \dots, p_n$ on the right of (3.21) is identically zero is equivalent to the condition that the coefficients $a^*_{j_1 \dots j_n}$ are all zero. We can write

$$(3.23) \qquad a^*_{1,\dots,n} = \sum_{m=0}^{n} (-1)^{n-m} S_{n,m}(x_1, \dots, x_n) ,$$

where

$$(3.24) \quad S_{n,m}(x_1, \dots, x_n) = \sum_{\substack{m_1 \geq 0, \dots, m_k \geq 0 \\ m_1 + \dots + m_k = n-m}} \sum\nolimits_{m,m_1,\dots,m_k} g(x_{j_1}, \dots, x_{j_m}, z_1^{m_1}, \dots, z_k^{m_k}) \prod_{i=1}^{k} \prod_{r=m+\dots+m_{i-1}+1}^{m+\dots+m_i} v_i(x_{j_r}) ,$$

and $\sum_{m,m_1,\dots,m_k}$ denotes the sum over those permutations $j_1, \dots, j_n$ of the integers $1, \dots, n$ which satisfy

$$j_1 < \dots < j_m, \; j_{m+1} < \dots < j_{m+m_1}, \; \dots, j_{m+\dots+m_{k-1}+1} < \dots < j_n .$$

The condition $a^*_{1,\dots,n} = 0$ is equivalent to

$$(3.25) \qquad g(x_1, \dots, x_n) = \sum_{m=0}^{n-1} (-1)^{n-m-1} S_{n,m}(x_1, \dots, x_n) .$$

For $m = 0, 1, \dots, n-1$, each term in the sum in (3.24) is, for some i and some j, the product of $v_i(x_j)$ and a factor not involving x_j. Moreover, $S_{n,m}(x_1, \dots, x_n)$ is symmetric in $x_1, \dots, x_n$. These facts imply that the right side of (3.25) can be written in the form

$$\sum_{i=1}^{k} \sum_{j=1}^{n} v_i(x_j) f_i(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) ,$$

where the functions $f_1, \dots, f_k$ are symmetric. Expressing the v_i in terms of $u_1, \dots, u_k$, we obtain a representation of $g(x_1, \dots, x_n)$ in the form (3.3) with symmetric functions $h_1, \dots, h_k$.

We now show that the functions $h_1, \dots, h_k$ can be so chosen that they satisfy the integrability condition (3.2). Let P_0 be the distribution which assigns probabilities $\lambda_1, \dots, \lambda_{k+1}$ to the respective points $y_1, \dots, y_{k+1}$, as defined in (3.7) to (3.9). Let B_i denote the set which consists of the single point y_i, for $i = 1, \dots, k$. Since the λ_j are strictly positive, the matrix $\left(\int_{B_1} u \, d\, P_0, \dots, \int_{B_k} u \, d\, P_0 \right)$ is non-singular. The conditions of Lemma 3B with $\nu = P_0$ are satisfied. By Lemma 4, the functions $h_1, \dots, h_k$ in (3.3) can be so chosen that each $h_i(x_1, \dots, x_{n-1})$ is a linear combination of terms of the form

$$\int_{B_{i_1}} d\, P_0(t_1) \dots \int_{B_{i_{n-m}}} d\, P_0(t_{n-m}) g(x_{j_1}, \dots, x_{j_m}, t_1, \dots, t_{n-m})$$

$$u_{r_1}(x_{j_{m+1}}) \dots u_{r_{n-m-1}}(x_{j_{n-1}}) .$$

Let P be a distribution in $\mathcal{P}$. The u_i are P-integrable by assumption. Hence to show that the h_i are P^{n-1}-integrable it is sufficient to show that

$$\int_{X^n} |g| \, d(P_0^{n-m} P^m) < \infty \tag{3.26}$$

for $m = 0, 1, \dots, n - 1$ and all $P \in \mathcal{P}$.

By (3.7) the distribution P_0 is in $\mathcal{P}_0$ and hence in $\mathcal{P}$. If P is in $\mathcal{P}$, so is $Q = \frac{1}{2}(P_0 + P)$, due to the convexity of $\mathcal{P}$. Hence $\int |g| \, d\, Q^n < \infty$. But $\int |g| \, dQ^n$ can be written as a linear combination with positive coefficients of the integrals in (3.26). Thus (3.26) is true. This completes the proof under the assumption that the origin 0 is in the interior of the convex hull of $\mathcal{U}$.

Now suppose that the origin is a boundary point of the convex hull of $\mathcal{U}$. Then there are real numbers $b_1, \ldots, b_k$, not all zero, such that $b_1u_1(x) + \ldots + b_ku_k(x) = 0$ for all $x \in X$. Therefore one of the conditions (3.1) is implied by the others. In this way the problem can be reduced to one of these two: (I) a problem of the same structure, with k replaced by k' , $1 \leq k' < k$, such that the origin of k'-space is in the interior of the convex hull of the set corresponding to $\mathcal{U}$; (II) the same kind of problem but with no restrictions (3.1) present. In case (I), the conclusion of the theorem follows from the first part of the proof. In case II, equality (3.6) with $N = n$ and arbitrary $(x_1, \ldots, x_n) \in X^n$ holds for all positive $p_1, \ldots, p_n$, so that, by Lemma 1, $g(x_1, \ldots, x_n) = 0$. (This is, essentially, Halmos' Lemma 2 in [2].) Theorem 1A is proved.

4. Proof of Theorem 2A. Let the conditions of Theorem 1A be satisfied, and suppose that g is bounded while every nontrivial linear combination of $u_1, \dots, u_k$ is unbounded. We must show that $g(x_1, \dots, x_n) = 0$ for all $(x_1, \dots, x_n) \in X^n$.

We again assume that $c_1 = \dots = c_k = 0$. Since every nontrivial linear combination of $u_1, \dots, u_k$ is unbounded, there exist k points $z_1, \dots, z_k$ in X such that the $k \times k$ matrix $(u(z_1), \dots, u(z_k))$ is nonsingular. Hence, by Theorem 1A and Lemma 3A, we have for all $(x_1, \dots, x_n) \in X^n$

$$(4.1) \qquad g(x_1, \dots, x_n) = \sum_{m=0}^{n-1} (-1)^{n-m-1} T_{n,m}(x_1, \dots, x_n; g) ,$$

where (we now exhibit the dependence of $T_{n,m}$ on g)

$$(4.2) \quad T_{n,m}(x_1, \dots, x_n; g) = \sum\nolimits_{m,n-m} \sum_{i_1=1}^{k} \cdots \sum_{i_{n-m}=1}^{k} g(x_{j_1}, \dots, x_{j_m}, z_{i_1}, \dots, z_{i_{n-m}}) v_{i_1}(x_{j_{m+1}}) \cdots v_{i_{n-m}}(x_{j_n}) .$$

Here each of $v_1, \dots, v_k$ is a nontrivial linear combination of $u_1, \dots, u_k$ and hence is unbounded.

The theorem will be proved by induction on k and, for each k, by induction on n.

For $n = 1$ and k arbitrary we have, by Theorem 1A, $g(x) = h_1 u_1(x) + \dots + h_k u_k(x)$, where $h_1, \dots, h_k$ are constants. The right side is bounded only if $h_1 = \dots = h_k = 0$, so that the theorem is true in this case.

Now let $k = 1$. By (4.2),

$$(4.3)\quad T_{n,m}(x_1, \dots ,x_n; g) = \sum_{m,n-m} g(x_{j_1}, \dots ,x_{j_m}, z, \dots ,z) \\ v(x_{j_{m+1}}) \dots v(x_{j_n}) ,$$

where $z = z_1$, and $v = v_1$ is unbounded. There is a sequence $\{y_N\}$ in X such that

$$|v(y_N)| \to +\infty \quad \text{as} \quad N \to \infty .$$

Divide both sides of (4.3) by $v(x_n)$, set $s_n = y_N$ and let $N \to \infty$. The terms on the right of (4.3) with $j_m = n$, divided by $v(x_n) = v(y_N)$, converge to zero, and we obtain

$$\lim_{N\to\infty} T_{n,m}(x_1, \dots ,x_{n-1}, y_N; g)/v(y_N) = \sum_{m,n-1-m} g(x_{j_1}, \dots ,x_{j_m}, z, \dots ,z) \\ v(x_{j_{m+1}}) \dots v(x_{j_{n-1}}) \\ = T_{n-1,m}(x_1, \dots ,x_{n-1}; g^{(1)}) ,$$

where $g^{(1)}(x_1, \dots ,x_{n-1}) = g(x_1, \dots ,x_{n-1}, z)$, for $m = 0, \dots ,n - 2$. For $m = n - 1$, the limit is $g^{(1)}(x_1, \dots ,x_{n-1})$. Thus if we set $x_n = y_N$ in (4.1), divide by $v(y_N)$ and let $N \to \infty$, we obtain

$$g^{(1)}(x_1, \dots ,x_{n-1}) = \sum_{m=0}^{n-2} (-1)^{n-m-2}\, T_{n-1,m}(x_1, \dots ,x_{n-1}; g^{(1)}) .$$

It follows by induction on n that the theorem is true for $k = 1$.

Now let $k \geq 2$, and suppose that the theorem is true with k replaced by $k - 1$.

Since v_k is unbounded, there is a sequence $\{y_N\}$ in X such that $|v_k(y_N)| \to \infty$ as $N \to \infty$. There is a subsequence $\{y_{N'}\}$ of $\{y_N\}$ such that $v_1(y_{N'})/v_k(y_{N'})$ tends to a limit λ_1, $-\infty \le \lambda_1 \le \infty$. Repeating this argument, we see that there is a sequence $\{y_N\}$ in X such that $|v_k(y_N)| \to \infty$ and $v_i(y_N)/v_k(y_N) \to \lambda_i$, $i = 1, \ldots ,k$, as $N \to \infty$, where $-\infty \le \lambda_i \le \infty$ for $i = 1, \ldots ,k - 1$. Suppose that $\lambda_1, \ldots ,\lambda_{k-1}$ are not all finite, say $|\lambda_i| = \infty$ for $i = 1, \ldots ,r$; $|\lambda_i| < \infty$ for $i \ge r + 1$. Then $v_k(y_N)/v_r(y_N) \to 0$, hence $|v_r(y_N)| \to \infty$ and $v_i(y_N)/v_r(y_N) \to \lambda_i'$ with $\lambda_i' = 0$ or 1, for $i \ge r$. Also, there is a subsequence $\{y_{N'}\}$ of $\{y_N\}$ such that $v_i(y_{N'})/v_r(y_{N'}) \to \lambda_i'$, with $-\infty \le \lambda_i' \le \infty$, for $i \le r - 1$. It now follows by induction that there is an index j, $1 \le j \le k$, and a sequence $\{y_N\}$ in X such that $|v_j(y_N)| \to \infty$ and $v_i(y_N)/v_j(y_N) \to \lambda_i$, $i = 1, \ldots ,k$, where $\lambda_1, \ldots ,\lambda_k$ are all finite. We may assume that $j = k$, so that

$$(4.4) \quad \lim_{N\to\infty} |v_k(y_N)| = \infty, \ \lim_{N\to\infty} v_i(y_N)/v_k(y_N) = \lambda_i, \ |\lambda_i| < \infty, \ i = 1, \ldots ,k - 1 .$$

Divide both sides of (4.2) by $v_k(x_n)$, set $x_n = y_N$, and let $N \to \infty$. After this operation the right side of (4.2) is modified as follows. Each term with $j_i = n$ for some $i \le m$ is replaced by zero. In each of the remaining terms the factor $v_{i_g}(x_n)$ is replaced by λ_{i_g}. On rearranging terms and taking account of the symmetry of g, we obtain

$$\lim_{N\to\infty} T_{n,m}(x_1, \ldots ,x_{n-1}, y_N; g)/v_k(y_N) = T_{n-1,m}(x_1, \ldots ,x_{n-1}; g^{(1)}) ,$$

where

$$(4.6) \qquad g^{(1)}(x_1, \ldots ,x_{n-1}) = \sum_{i=1}^{k} \lambda_i \, g(x_1, \ldots ,x_{n-1}, z_i) .$$

If in (4.1) we divide by $v(x_n)$, set $x_n = y_N$, and let $N \to \infty$, the limit of the left side is 0, and since $T_{n-1,n-1}(x_1, \ldots, x_{n-1}; g^{(1)}) = g^{(1)}(x_1, \ldots, x_{n-1})$, we obtain

$$(4.7) \quad g^{(1)}(x_1, \ldots, x_{n-1}) = \sum_{m=0}^{n-2} (-1)^{n-m-2}\, T_{n-1,m}(x_1, \ldots, x_{n-1}; g^{(1)}) .$$

It follows in the same way that if we define $g^{(1)}, g^{(2)}, \ldots, g^{(n-1)}$ by (4.6) and

$$(4.8) \quad g^{(s+1)}(x_1, \ldots, x_{n-s-1}) = \sum_{i=1}^{k} \lambda_i g^{(s)}(x_1, \ldots, x_{n-s-1}, z_i) ,$$

$$s = 1, \ldots, n-2 ,$$

then

$$(4.9) \quad g^{(n-s)}(x_1, \ldots, x_s) = \sum_{m=0}^{s-1} (-1)^{s-1-m}\, T_{s,m}(x_1, \ldots, x_s; g^{(n-s)})$$

for $s = n-1, n-2, \ldots, 1$. In particular,

$$g^{(n-1)}(x) = T_{1,0}(x; g^{(n-1)}) = \sum_{i=1}^{k} g^{(n-1)}(z_i) v_i(x) .$$

Hence

$$(4.10) \quad g^{(n-1)}(x) = 0 , \text{ all } x \in X .$$

We now show that $g^{(n-s+1)}(x_1, \ldots, x_{s-1}) = 0$ for all $(x_1, \ldots, x_{s-1}) \in X^{s-1}$ implies $g^{(n-s)}(x_1, \ldots, x_s) = 0$ for all $(x_1, \ldots, x_s) \in X^s$, $s = 2, \ldots, n$. Suppose that

$$(4.11) \quad g^{(n-s+1)}(x_1, \ldots, x_{s-1}) = 0 \text{ for } (x_1, \ldots, x_{s-1}) \in X^{s-1} .$$

By (4.8)

$$g^{(n-s+1)}(x_1, \dots ,x_{s-1}) = \sum_{i=1}^{k} \lambda_i \, g^{(n-s)}(x_1, \dots ,x_{s-1}, z_i) .$$

Hence, as $\lambda_k = 1$,

$$(4.12) \quad g^{(n-s)}(x_1, \dots ,x_{s-1}, z_k) = -\sum_{i=1}^{k-1} \lambda_i \, g^{(n-s)}(x_1, \dots ,x_{s-1}, z_i) .$$

By (4.9), $g^{(n-s)}(x_1, \dots ,x_s)$ is a sum involving the terms

$$(4.13) \quad T_{s,m}(x_1, \dots ,x_s; g^{(n-s)}) = \sum\nolimits_{m,s-m} \sum_{i_1=1}^{k} \cdots \sum_{i_{s-m}=1}^{k}$$

$$g^{(n-s)}(x_{j_1}, \dots ,x_{j_m}, z_{i_1}, \dots ,z_{i_{s-m}}) \cdot v_{i_1}(x_{j_{m+1}}) \cdots v_{i_{s-m}}(x_{j_s})$$

with $m = 0, 1, \dots ,s - 1$.

Let

$$(4.14) \qquad w_i(x) = v_i(x) - \lambda_i v_k(x) \;, \quad i = 1, \dots ,k - 1 ,$$

and let $T^*_{n,m}(x_1, \dots ,x_n; g)$ be defined as $T_{n,m}(x_1, \dots ,x_n; g)$; but with k , $v_1(\cdot), \dots ,v_k(\cdot)$ replaced by $k - 1$, $w_1(\cdot), \dots ,w_{k-1}(\cdot)$. If we eliminate z_k from the right side of (4.13) by using (4.12), we obtain

$$(4.15) \quad T_{s,m}(x_1, \dots ,x_s; g^{(n-s)}) = T^*_{s,m}(x_1, \dots ,x_s; g^{(n-s)})$$

for $m = 0, 1, \dots ,s - 1$. Note that any nontrivial linear combination of $w_1, \dots ,w_{k-1}$ is unbounded. It now follows from (4.9), (4.15) and the induction hypothesis that $g^{(n-s)}(x_1, \dots ,x_s) = 0$ for all $(x_1, \dots ,x_s)$ in $\mathcal{X}^s$. Thus $g(x_1, \dots ,x_n) = 0$ for all $(x_1, \dots ,x_n) \in \mathcal{X}^n$.

5. <u>Proof of Theorem 1B</u>. We again assume that $c_1 = \dots = c_k = 0$. Let μ be a σ-finite measure on the measurable space (X,A) , let P be a convex family of distributions which are absolutely continuous with respect to μ and satisfy conditions (3.1), and let $P_0(\mu) \subset P$. Let g be a symmetric $A^{(n)}$-measurable real-valued function such that $\int g \, dP^n = 0$ for all $P \in P$. We must show that there exist k symmetric $A^{(n-1)}$-measurable real-valued functions $h_1, \dots, h_k$ such that

$$(5.1) \qquad \int |h_i| \, dP^{n-1} < \infty , \quad i = 1, \dots, k , \text{ if } P \in P$$

and

$$(5.2) \quad g(x_1, \dots, x_n) = \sum_{i=1}^{k} u_i(x_j) h_i(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) \text{ a.e. } (P^{(n)}) .$$

Let A_0 be the class of all sets A in A such that

$$(5.3) \qquad \mu(A) + \sum_{i=1}^{k} \int_A |u_i| \, d\mu < \infty .$$

Let N be a positive integer, $A_1, \dots, A_N$ be sets in A_0 , and $a_1, \dots, a_N$ be nonnegative numbers such that

$$(5.4) \qquad \sum_{j=1}^{N} a_j \, \mu(A_j) = 1 ,$$

$$(5.5) \qquad \sum_{j=1}^{N} a_j \int_{A_j} u_i \, d\mu = 0 , \quad i = 1, \dots, k .$$

Then $p(x) = \sum_{j=1}^{N} a_i I_{A_j}(x)$, where I_A denotes the indicator function of the set A , is the probability density with respect to μ of a distribution in $P_0(\mu)$ and therefore in P . Hence conditions (5.4) and (5.5) imply

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The following paragraph was omitted from page 29. It should follow the third displayed equation and precede the paragraph which begins: "Let $A_1, \ldots, A_n$ be any n sets ... ".

We first assume that 0 is in the interior of the convex hull of $\mathcal{U}(\mu)$. Then there exist $k + 1$ sets $B_1, \ldots, B_{k+1}$ in $\mathcal{A}_0$ of positive μ-measure such that 0 is an inner point of the polytype whose vertices are

$$\int_{B_j}, u \, d\mu/\mu(B_j), \quad j = 1, \ldots, k + 1 .$$

$$(5.6) \qquad \sum_{j_1=1}^{N} \cdots \sum_{j_n=1}^{N} a_{j_1} \cdots a_{j_n} G(A_{j_1}, \ldots, A_{j_n}) = 0 ,$$

where

$$(5.7) \qquad G(A_1, \ldots, A_n) = \int_{A_1 \times \ldots \times A_n} g \, d\mu^n .$$

(The existence of the integrals in (5.6) is guaranteed by the assumption that $\int g \, d \, P^n$ exists for $P \in \mathcal{P}$.)

Conditions (5.4) and (5.5) also imply that the origin 0 of R^k is in the convex hull of the set

$$\mathcal{U}(\mu) = \left\{ \int_A u \, d\mu / \mu(A) : A \in \mathcal{A}_0, \ \mu(A) > 0 \right\} .$$

Let $A_1, \ldots, A_n$ be any n sets in $\mathcal{A}_0$ and let $A_{n+1}, \ldots, A_{n+k}$ be any k of the sets $B_1, \ldots, B_{k+1}$. Then if $a_1, \ldots, a_{n+k}$ are nonnegative numbers such that conditions (5.4), (5.5) with $N = n + k$ are satisfied, condition (5.6) with $N = n + k$ is satisfied.

We now use the argument in the proof of Theorem 1A, with p_j, X, x_j, $u(x_j)$ and $g(x_1, \ldots, x_n)$ replaced by a_j, $\mathcal{A}_0$, A_j, $\int_{A_j} u \, d\mu$ and $G(A_1, \ldots, A_n)$ respectively, to infer that there exist symmetric real-valued functions $H_1, \ldots, H_k$ on $\mathcal{A}_0^{n-1}$ such that, for any $(A_1, \ldots, A_n) \in \mathcal{A}_0^n$,

$$(5.8) \quad G(A_1, \ldots, A_n) = \sum_{i=1}^{k} \sum_{j=1}^{n} \int_{A_j} u_i \, d\mu \, H_i(A_1, \ldots, A_{j-1}, A_{j+1}, \ldots, A_n) .$$

The matrix

$$(5.9) \qquad U_\mu = \left(\int_{B_1} u \, d\mu, \ldots, \int_{B_k} u \, d\mu \right)$$

is nonsingular. By Lemma 3A, with X replaced by $\mathcal{A}_0$, we have for

$(A_1, \ldots, A_n) \in \mathcal{A}_0^n$

$$(5.10) \qquad G(A_1, \ldots, A_n) = \sum_{m=0}^{n-1} (-1)^{n-m-1} T_{n,m}(A_1, \ldots, A_n) ,$$

where, with $V_i(A) = \int_A v_i \, d\mu$, $v(x) = U_\mu^{-1} u(x)$,

$$(5.11) \quad T_{n,m}(A_1, \ldots, A_n) = \sum\nolimits_{m,n-m} \sum_{i_1=1}^{k} \cdots \sum_{i_{n-m}=1}^{k} G(A_{j_1}, \ldots, A_{j_m}, B_{i_1}, \ldots, B_{i_{n-m}}) V_{i_1}(A_{j_{m+1}}) \cdots V_{i_{n-m}}(A_{j_n}) .$$

Hence if we define, for $(x_1, \ldots, x_n) \in X^n$,

$$(5.12) \quad t_{n,m}(x_1, \ldots, x_n) = \sum\nolimits_{m,n-m} \sum_{i_1=1}^{k} \cdots \sum_{i_{n-m}=1}^{k} g(x_{j_1}, \ldots, x_{j_m}, B_{i_1}, \ldots, B_{i_{n-m}}) v_{i_1}(x_{j_{m+1}}) \cdots v_{i_{n-m}}(x_{j_n}) ,$$

$$(5.13) \quad g(x_1, \ldots, x_m, A_1, \ldots, A_{n-m}) = \int_{A_1} d\mu(x_{m+1}) \cdots \int_{A_{n-m}} d\mu(x_n) \; g(x_1, \ldots, x_m, \ldots, x_n) ,$$

then

$$(5.14) \quad \int_C \left\{ g(x_1, \ldots, x_n) - \sum_{m=0}^{n-1} (-1)^{n-m-1} t_{n,m}(x_1, \ldots, x_n) \right\} d\mu^n = 0$$

for all sets $C = A_1 \times \ldots \times A_n$ in $\mathcal{A}_0^n$.

Let $w(x_1, \ldots, x_n)$ denote the integrand in (5.14). The integral $J(C) = \int_C w \, d\mu^n$ is zero for $C \in \mathcal{A}_0^n$. Let B be a set in $\mathcal{A}_0$. By a standard argument, $J(E \cap B^n) = 0$ for all E in the σ-field $\mathcal{A}^{(n)}$ and hence $w(x_1, \ldots, x_n) = 0$ a.e. (μ^n) on B^n .

Now let P be a distribution in $\mathcal{P}$, let p be a version of $dP/d\mu$, and let $B_\varepsilon = \{x: p(x) > \varepsilon\}$. Then

$$\infty > \int_{B_\varepsilon} (1 + \sum_{i=1}^{k} |u_i|)p \, d\mu \geq \varepsilon \int_{B_\varepsilon} (1 + \sum_{i=1}^{k} |u_i|)d\mu .$$

Thus $B_\varepsilon \in A_0$ for all $\varepsilon > 0$. Hence $w(x_1, \ldots ,x_n) = 0$ a.e. (μ^n) on $\cup_{\varepsilon>0} B_\varepsilon^n$, a set of P^n-measure one, that is

$$(5.15) \qquad g(x_1, \ldots ,x_n) = \sum_{m=0}^{n-1} (-1)^{n-m-1} t_{n,m} (x_1, \ldots ,x_n)$$

a.e. (P^n) , for all $P \in \mathcal{P}$. It is easily seen (compare the proof of Theorem 1A) that the sum in (5.15) can be written in the form (5.2) where each function h_i is symmetric and $A^{(n-1)}$-measurable.

We now show that the functions h_i can be so chosen that the integrability condition (5.1) is satisfied. With $B_1, \ldots ,B_{k+1}$ as above, there exist strictly positive numbers $\lambda_1, \ldots ,\lambda_{k+1}$ such that $\sum_{j=1}^{k+1} \lambda_j = 1$ and

$$\sum_{j=1}^{k+1} \lambda_j \int_{B_j} u \, d\mu/\mu(B_j) = 0 .$$

Let P_0 be the distribution whose density with respect to μ is

$$p_0(x) = \sum_{j=1}^{k+1} \lambda_j I_{B_j}(x)/\mu(B_j) .$$

Then P_0 is in $\mathcal{P}_0(\mu)$ and hence in $\mathcal{P}$. Since the matrix U_μ in (5.9) is nonsingular and the λ_j are strictly positive, the matrix

$$U_{P_0} = \left(\int_{B_1} u \, d P_0, \ldots , \int_{B_k} u \, d P_0 \right)$$

is also nonsingular. The conditions of Lemma 3B with $\nu = P_0$ are satisfied, except that the representation (2.4) of $g(x_1, \ldots, x_n)$ holds a.e. $(\mathcal{P}^{(n)})$. Hence (2.15), with $\nu = P_0$, holds a.e. $(\mathcal{P}^{(n)})$. By Lemma 4, each $h_i(x_1, \ldots, x_{n-1})$ can be chosen as a finite linear combination of terms of the form (2.19) with $\nu = \mathcal{P}_0$. The P^{n-1}-integrability of these functions follows in the same way as in the proof of Theorem 1A, making use of the convexity of $\mathcal{P}$.

Now suppose that the origin 0 is a boundary point of the convex hull of $\mathcal{U}(\mu)$. Then there are real numbers $b_1, \ldots, b_k$, not all zero, such that $\sum_{i=1}^{k} b_i \int_A u_i \, d\mu = 0$ for all $A \in \mathcal{A}_0$, and therefore $\sum_{i=1}^{k} b_i \int u_i \, dP = 0$ for all $P \in \mathcal{P}$. Thus one of the conditions $\int u_i \, dP = 0$, $i = 1, \ldots, k$, follows from the others. The rest of the proof is similar to that for Theorem 1A.

6. Proof of Theorem 2B. Let the conditions of Theorem 1B be satisfied, and suppose that g is bounded while every nontrivial linear combination of $u_1, \ldots, u_k$ is $\mathcal{P}$-unbounded. We must show that $g(x_1, \ldots, x_n) = 0$ a.e. $(\mathcal{P}^{(n)})$.

We again assume that $c_1 = \ldots = c_k = 0$.

First it will be shown that there exists a measure ν on $(X, \mathcal{A})$ which is (i) equivalent to the family $\mathcal{P}$, (ii) finite, and (iii) satisfies

$$\int |u_i| \, d\nu < \infty, \quad i = 1, \ldots, k.$$

Since the family $\mathcal{P}$ is dominated by a σ-finite measure, it contains a countable equivalent subset (Halmos and Savage (1949), Lemma 7). Let the sequence $P_1, P_2, \ldots$ of distributions in $\mathcal{P}$ be equivalent to $\mathcal{P}$ (so that $P_j(A) = 0$ for all j implies $P(A) = 0$ for all P in $\mathcal{P}$). Let $d_j = \sum_{i=1}^{k} \int |u_i| \, d P_j$, $b_j = 2^{-j}(1 + d_j)^{-1}$, $\nu = \sum_{j=1}^{\infty} b_j P_j$. The numbers b_j are strictly positive and $\sum b_j$ is finite. Hence ν is a finite measure equivalent to $\mathcal{P}$. Also, $\sum_{i=1}^{k} \int |u_i| d\nu = \sum_{j=1}^{\infty} b_j d_j < \sum_{j=1}^{\infty} 2^{-j} < \infty$, so that ν satisfies conditions (i), (ii), (iii).

Since ν is equivalent to $\mathcal{P}$, we have that if u is a nontrivial linear combination of $u_1, \ldots, u_k$ then $\nu\{|u(x)| > c\} \neq 0$ for all real c. Let $\mathcal{A}_+$ denote the class of sets A in $\mathcal{A}$ such that $\nu(A) \neq 0$. For $A \in \mathcal{A}_+$ define the set functions $U_1, \ldots, U_k$ by

$$U_i(A) = \int_A u_i \, d\nu / \nu(A), \quad i = 1, \ldots, k.$$

Then every nontrivial linear combination of $U_1, \ldots, U_k$ is unbounded on $\mathcal{A}_+$.

Hence there exist k set $B_1, \ldots, B_k$ in A_+ such that the matrix

$$U_\nu = \left[\int_{B_1} u \, d\nu, \ \ldots \ , \int_{B_k} u \, d\nu \right]$$

is nonsingular.

By Theorem 1B, the conditions of Lemma 3B (last paragraph) are satisfied. ($\int |g| \, d\nu^n$ is finite since g is bounded.) Hence the representation (2.15) of $g(x_1, \ldots, x_n)$ holds a.e. $(\nu^{(n)})$. Let $A_1, \ldots, A_n$ be n sets in A_+. Integrating both sides of (2.15) over the product set $A_1 \times \ldots \times A_n$ in A_+^n with respect to ν^n, we obtain

$$(6.1) \qquad G^\dagger(A_1, \ldots, A_n) = \sum_{m=0}^{n-1} (-1)^{n-m-1} \, T^\dagger_{n,m}(A_1, \ldots, A_n)$$

where

$$G^\dagger(A_1, \ldots, A_n) = \int_{A_1 \times \ldots \times A_n} g \, d\nu^n / \prod_{j=1}^{n} \nu(A_j) ,$$

$$T^\dagger_{n,m}(A_1, \ldots, A_n) = \sum\nolimits_{m,n-m} \sum_{i_1=1}^{k} \ldots \sum_{i_{n-m}=1}^{k} G^\dagger(A_{j_1}, \ldots, A_{j_m}, B_{i_1}, \ldots, B_{i_{n-m}})$$

$$V^\dagger_{i_1}(A_{j_{m+1}}) \ldots V^\dagger_{i_{n-m}}(A_{j_n}) ,$$

$$V^\dagger_i(A) = \nu(B_i) \int_A v_i \, d\nu / \nu(A) , \quad V(x) = U_\nu^{-1} u(x) .$$

The representation (6.1) of the set function $G^\dagger(A_1, \ldots, A_n)$ is strictly analogous to the representation (4.1) of $g(x_1, \ldots, x_n)$ .. Since g is bounded, $G^\dagger$ is bounded on A_+^n, and the $V^\dagger_i(A)$ are unbounded on A_+. Thus the proof of Theorem 2A implies that $G^\dagger(A_1, \ldots, A_n) = 0$ on A_+^n. Therefore

$$\int_C g \, d\nu^n = 0$$

for all cylinder sets $C = A_1 \times \dots \times A_n$ in A^n. Hence $g(x_1, \dots, x_r) = 0$ a.e. (ν^n), and thus a.e. $(P^{(n)})$.

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