A NOTE ON STEP-DOWN PROCEDURE IN MANOVA PROBLEM WITH UNEQUAL DISPERSION MATRICES

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In this paper we have considered step-down procedure in multivariate analysis of variance problem when dispersion matrices are different and unknown. The distribution problem of the test criterion has also been studied under the null hypothesis.

1. *Introduction*. Step-down procedure in standard MANOVA problem has been considered by J. Roy [7]. The essential feature of this procedure is that if on some a priori grounds the variates are arranged in descending order of importance, then the test procedure can be carried out sequentially by considering marginal and conditional distributions of the variates concerned. At each stage F-statistic can be used which are independently distributed under the null hypothesis so that the overall hypothesis can be tested by combining the component tests. Optimum properties of this procedure have been discussed by Roy [7] and Roy et al. [9].

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In this article the problem has been considered when the dispersion matrices are not equal. It has been shown that by transforming the original vector variables to Scheffé-vector-variables (Eqn. (2.2) below) Hotelling's T^2 test can be constructed at each stage of the step-down procedure and these T^2's are shown to be independently distributed. It may be noted that the distributions of the Scheffé-vector-variables chosen from the original vector variables at different stages (vide, Remark at the end of Sec. 2) may not, in general, be the same as those considered earlier. This is possibly true under certain restrictions (to be shown in Sec. 3), when the test constructions will be difficult in this situation. However, a reasonably satisfactory solution can be attained only in Bhargava's [3] procedure, which is a modification of Anderson's [2] procedure in generalized multivariate Behrens-Fisher problem.

2. Step-down procedure.

2.1. Hypothesis and Test Construction. Let $\underset{\sim}{X}_{\alpha}^{(t)}(1\times p)$ be the α-th observation vector in t-th population ($\alpha=1,\ldots,n_t$, $t=1,\ldots,m$) and distributed as $N_p(\underset{\sim}{\eta}^{(t)}, \underset{\sim}{\Sigma}_t)$, where $\underset{\sim}{\eta}^{(t)}(1\times p)$ is the mean vector and $\underset{\sim}{\Sigma}_t(p\times p)$ the dispersion matrix in t-th population. The problem is to test

$$(2.1) \qquad H_o[\underset{\sim}{\eta}^{(t)} = \ldots = \underset{\sim}{\eta}^{(m)}]$$

against the alternative of at least one inequality among $\underset{\sim}{\eta}^{(t)}$'s.

Anderson [2] proposed a Hotelling's T^2 test for (2.1) by assuming $n_1 \leq \ldots \leq n_m$ and transforming the variables $\underset{\sim}{X}_{\alpha}^{(t)}$ to Scheffé-vector-variables

$$(2.2)\qquad \underset{\sim}{U}_{\alpha}^{(r)} = \underset{\sim}{X}_{\alpha}^{(1)} - \sqrt{\frac{n_1}{n_r}}\,\underset{\sim}{X}_{\alpha}^{(r)} + \frac{1}{\sqrt{n_1 n_r}} \sum_{\beta=1}^{n_1} \underset{\sim}{X}_{\beta}^{(r)} - \bar{\underset{\sim}{X}}^{(r)}$$

for $r=2,\ldots,m$, $\alpha=1,\ldots,n_1$, where $\underset{\sim}{U}_{\alpha} = (\underset{\sim}{U}_{\alpha}^{(2)},\ldots,\underset{\sim}{U}_{\alpha}^{(m)})$ jointly follow a $p(m-1)$-variate normal distribution.

Now suppose the p variables in the vector $\underset{\sim}{X}_{\alpha}^{(t)}$ are arranged in descending order of importance and the ordering remains the same for each $t=1,\ldots,m$ and is given by

$$(2.3)\qquad \underset{\sim}{X}_{\alpha}^{(t)} = (x_{1\alpha}^{(t)},\ldots,x_{p\alpha}^{(t)})$$

After transforming the vectors $\underset{\sim}{X}_{\alpha}^{(t)}$ to (2.2) if we order the variables in $\underset{\sim}{U}_{\alpha}^{(r)}(1\times p)$ as $(U_{1\alpha}^{(r)},\ldots,U_{p\alpha}^{(r)})$, these new variables correspond to the ordered variables in (2.3) through this transformation. Now writing $\underset{\sim}{U}_{\alpha}(1\times p(m-1)) = (\underset{\sim}{U}_{1\alpha},\ldots,\underset{\sim}{U}_{p\alpha})$, where $\underset{\sim}{U}_{i\alpha} = (U_{i\alpha}^{(2)},\ldots,U_{i\alpha}^{(m)})$, we have the distribution of $\underset{\sim}{U}_{\alpha}$ as $N_{p(m-1)}(\underset{\sim}{\theta},\underset{\sim}{\Gamma})$, where for $r=2,\ldots,m$, $i=1,\ldots,p$

$$(2.4)\qquad \underset{\sim}{\theta}=(\underset{\sim}{\theta}_1,\ldots,\underset{\sim}{\theta}_p);\ \underset{\sim}{\theta}_i=(\theta_i^{(2)},\ldots,\theta_i^{(m)});\ \theta_i^{(r)}=\eta_i^{(1)}-\eta_i^{(r)}$$

$$(2.5)\qquad \underset{\sim}{\Gamma}(p(m-1)\times p(m-1)) = (\underset{\sim}{\Sigma}_{ij}),\ \underset{\sim}{\Sigma}_{ij}(m-1\times m-1) = \sigma_{ij}^{(1)}\underset{\sim}{J} + \operatorname{diag}\left(\frac{n_1}{n_2}\sigma_{ij}^{(2)},\ldots,\frac{n_1}{n_m}\sigma_{ij}^{(m)}\right)$$

where $i,j=1,\ldots,p$, $\sigma_{ij}^{(t)}$ is the (i,j)th element of Σ_t, $t=1,\ldots,m$, and $\underset{\sim}{J}(m-1\times m-1)$ is the matrix with all the elements unity.

Now if we consider j-th step-down procedure, then we are to consider the conditional distribution of $\underset{\sim}{U}_{j\alpha}(1\times m-1)$ for fixed $\underset{\sim}{U}_{(j-1)\alpha} = (\underset{\sim}{U}_{1\alpha},\ldots,\underset{\sim}{U}_{j-1\alpha})$, which is an $(m-1)$-variate normal distribution with mean vector $\underset{\sim}{\Phi}_j$ and residual dispersion matrix $\underset{\sim}{\Gamma}_{j.1,\ldots,j-1}$, where

(2.6) $\Phi_j = \underset{\sim}{\xi}_j + \underset{\sim}{U}_{(j-1)\alpha}\underset{\sim}{B}_{j-1}$; $\underset{\sim}{\xi}_j = \underset{\sim}{\theta}_j - \underset{\sim}{\theta}_{(j-1)}\underset{\sim}{B}_{j-1}$

(2.7) $\underset{\sim}{B}'_{j-1} = (\underset{\sim}{\beta}'_{j,1},\ldots,\underset{\sim}{\beta}'_{j,j-1})$

where $\underset{\sim}{\beta}_{j,s}$ is a matrix of order $(m-1)\times(m-1)$ and $\underset{\sim}{B}_{j-1}$ is the (j-1)th order step-down regression matrix.

Under this set up, the hypothesis (2.1) can be written as follows

(2.8) $H_o[\underset{\sim}{\eta}^{(1)}=\ldots=\underset{\sim}{\eta}^{(m)}] \equiv H_o:[\underset{\sim}{\theta}=\underset{\sim}{0}] \equiv \bigcap_{j=1}^{p} H_o^{(j)}[\underset{\sim}{\theta}_j=\underset{\sim}{0}|\underset{\sim}{\theta}_{(j-1)}=\underset{\sim}{0}] \equiv \bigcap_{j=1}^{p} H_o^{(j)}[\underset{\sim}{\xi}_j=\underset{\sim}{0}]$.

Thus the component hypothesis $H_o^{(j)}[\underset{\sim}{\xi}_j=\underset{\sim}{0}]$ can be tested from model (2.6) by Hotelling's T^2 (Anderson [1], Page 187), where

(2.9) $T_j^2 = (n_1-(m-1)(j-1)-1)n_1\hat{\underset{\sim}{\xi}}_j \underset{\sim}{S}_{U_j}^{-1}\hat{\underset{\sim}{\xi}}'_j$,

where

$$\hat{\underset{\sim}{\xi}}_j = \bar{\underset{\sim}{U}}_j - \bar{\underset{\sim}{U}}_{(j-1)}\hat{\underset{\sim}{B}}_{j-1},\quad \hat{\underset{\sim}{B}}'_{j-1} = (\hat{\underset{\sim}{\beta}}'_{j,1},\ldots,\hat{\underset{\sim}{\beta}}'_{j,j-1})$$

and

$$\hat{\underset{\sim}{\beta}}_{j,S} = [\sum_{\alpha=1}^{n}(\underset{\sim}{U}_{S_\alpha}-\bar{\underset{\sim}{U}}_S)'(\underset{\sim}{U}_{S_\alpha}-\bar{\underset{\sim}{U}}_S)]^{-1}\sum_{\alpha=1}^{n_1}(\underset{\sim}{U}_{S_\alpha}-\bar{\underset{\sim}{U}}_S)'\underset{\sim}{U}_{j\alpha},\quad (S=1,\ldots,j-1)$$

$$\underset{\sim}{S}_{U_j} = \sum_{\alpha=1}^{n_1}(\underset{\sim}{U}_{j\alpha}-\hat{\underset{\sim}{\xi}}_j-\underset{\sim}{U}_{(j-1)\alpha}\hat{\underset{\sim}{B}}_{j-1})'(\underset{\sim}{U}_{j\alpha}-\hat{\underset{\sim}{\xi}}_j-\underset{\sim}{U}_{(j-1)\alpha}\hat{\underset{\sim}{B}}_{j-1})$$

This $\underset{\sim}{S}_{U_j}$ is distributed as $W_{m-1}(n_1-(m-1)j,\ \underset{\sim}{\Gamma}_{j.1,\ldots,j-1})$ and $(n_1-(m-1)j)(m-1)^{-1}(n_1-(m-1)(j-1)-1)^{-1}T_j^2$ is distributed as $F(m-1,n_1-(m-1)j)$, $j=1,2,\ldots,p$.

2.2. Independence of $T_1^2,\ldots,T_p^2$. It is clear that for fixed j, $\hat{\underset{\sim}{\xi}}_j$ and $\underset{\sim}{S}_{U_j}$ are independently distributed (Anderson [1]). To prove that $T_1^2,\ldots,T_p^2$ are independently distributed under H_o, let us consider a vector $\underset{\sim}{\ell}(1\times m-1)$ of real elements so that $\underset{\sim}{\ell}\underset{\sim}{S}_{U_j}\underset{\sim}{\ell}'/\underset{\sim}{\ell}\underset{\sim}{\Gamma}_{j.1,\ldots,j-1}\underset{\sim}{\ell}'$ is distributed as $\chi_j^2(n_1-(m-1)j)$. Now following Roy et al. [9], Page 47, it follows that this χ_j^2 is distributed independently of $\underset{\sim}{U}_{(j-1)}\underset{\sim}{\ell}'$, whether $H_o^{(j)}$ is true or not. So that $\chi_1^2,\ldots,\chi_p^2$ are independently distributed. It follows, therefore, (Rao [6], Page 453) that $\underset{\sim}{S}_{U_j}$ for j=1,...,p are independently distributed. Also under $H_o^{(j)}$, $\hat{\underset{\sim}{\xi}}_j\underset{\sim}{\ell}'$ is distributed independently of $\underset{\sim}{U}_{(j-1)}\underset{\sim}{\ell}'$ and is true for every $\underset{\sim}{\ell}$, so that $\hat{\underset{\sim}{\xi}}_j$ is distributed independently of $\underset{\sim}{U}_{(j-1)}$. Since T_j^2 in (2.9) can be written

$$(2.10)\quad [n_1-(m-1)(j-1)-1]^{-1}T_j^2 = n_1\hat{\underset{\sim}{\xi}}_j\underset{\sim}{\Gamma}^{-1}_{j.1,\ldots,j-1}\hat{\underset{\sim}{\xi}}'_j(\hat{\underset{\sim}{\xi}}_j S^{-1}_{U_j}\hat{\underset{\sim}{\xi}}'_j/\hat{\underset{\sim}{\xi}}_j\underset{\sim}{\Gamma}^{-1}_{j.1,\ldots,j-1}\hat{\underset{\sim}{\xi}}'_j)$$

where the distribution of $\hat{\underset{\sim}{\xi}}_j\underset{\sim}{S}^{-1}_{U_j}\hat{\underset{\sim}{\xi}}'_j/\hat{\underset{\sim}{\xi}}_j\underset{\sim}{\Gamma}^{-1}_{j.1,\ldots,j-1}\hat{\underset{\sim}{\xi}}'_j$ does not depend on $\hat{\underset{\sim}{\xi}}_j$ (Rao [6], Page 458). Hence under $H_o^{(j)}$, the conditional distribution of T_j^2 for fixed $\underset{\sim}{U}_{(j-1)\alpha}$ does not depend on $\underset{\sim}{U}_{(j-1)\alpha}$ (j=2,...,p) and T_1^2 is distributed as $(n_1-m+1)^{-1}(m-1)$F-distribution with d.f. $(m-1,\ n_1-m+1)$. Hence unconditionally also $T_1^2,\ldots,T_p^2$ are independently distributed.

The test criterion for the overall hypothesis (2.8) can be constructed from the component tests either by using union-intersection principle (Roy [8]) or by considering the test criterion $\Lambda = \prod_{j=1}^{p}\Lambda^{(j)}$, where $\Lambda^{(j)} = \{1+(n_1-(m-1)(j-1)-1)^{-1}T^2\}^{-1}$, where $\Lambda^{(j)}$ is the product of (m-1) independent beta variables. The exact as well as asymptotic null-distribution of the statistic Λ are available (Chakravorti [4], eqn. 4.39 and 4.55 for zero non-centrality parameter).

REMARK. It may be noted that if we start with the marginal and conditional distributions of $x_{1\alpha}^{(t)}$, $x_{2\alpha}^{(t)}$ given $x_{1\alpha}^{(t)},\ldots,x_{p\alpha}^{(t)}$ given $x_{1\alpha}^{(t)},\ldots,x_{p-1\alpha}^{(t)}$ from the distribution of $\underset{\sim}{X}_{\alpha}^{(t)}$, and then choose the Scheffé-variables in each stage of the step-down procedure, the distributions of the resulting vector-variables will not be the same as those obtained from (2.2). Since these variables are correlated both variate-wise and group-wise, we are to impose a number of restrictions on the regression coefficient matrix $\underset{\sim}{B}_{j-1}$ to satisfy this requirement, in which case a satisfactory solution of the test construction is not easily available by Anderson's procedure.

However, if we assume that the regression coefficients of $x_j^{(t)}$ on $x_s^{(t)}$ $(s=1,2,\ldots,j-1)$ are same for $t=1,2,\ldots,m$, we can apply the procedure of Bhargava [3], which is the modification of Anderson's procedure.

3. Bhargava's procedure. Following Bhargava [3] let us assume that $n_1=(m-1)n'$, $n_2=\ldots=n_m=n'$ and consider the following transformation on $\underset{\sim}{X}_{\alpha}^{(t)}$,

$$(3.1)\qquad \underset{\sim}{U}_{\alpha}^{(r)} = \underset{\sim}{X}_{\alpha}^{(r)}-\underset{\sim}{X}_{\alpha+(r-2)n'}^{(1)},\qquad r=2,\ldots,m;\ \alpha=1,\ldots,n'.$$

Then under the set up considered in Sec. 2.1, we can choose the vector $\underset{\sim}{U}_{\alpha}(1\times p(m-1)) = (\underset{\sim}{U}_{1\alpha},\ldots,\underset{\sim}{U}_{p\alpha})$, which is distrubuted as $N_{p(m-1)}(\underset{\sim}{\theta},\underset{\sim}{\Gamma}^*)$, where

$$(3.2)\qquad \underset{\sim}{\theta}=(\underset{\sim}{\theta}_1,\ldots,\underset{\sim}{\theta}_p);\ \underset{\sim}{\theta}_i=(\theta_i^{(2)},\ldots,\theta_i^{(m)});\ \theta_i^{(r)}=\eta_i^{(r)}-\eta_i^{(1)}.$$

$$(3.3)\qquad \underset{\sim}{\Gamma}^*=(\underset{\sim}{\Sigma}_{ij}^*);\ \underset{\sim}{\Sigma}_{ij}^*=\sigma_{ij}^{(1)}\underset{\sim}{I}_{m-1} + \mathrm{diag}(\sigma_{ij}^{(2)},\ldots,\sigma_{ij}^{(m)});\ i,j=1,\ldots,p.$$

Then j-th step-down procedure will lead to the model (2.6), where $\underset{\sim}{\theta}_j$'s are defined according to (3.2) and submatrices of $\underset{\sim}{B}_{j-1}$ given by (2.7) are for s=1,...,j-1

(3.4) $$\underset{\sim}{\beta}_{j,s}(m-1\ m-1) = \mathrm{diag}(\beta_{j,s}^{(2,2)},\ldots,\beta_{j,s}^{(m,m)}).$$

When $\beta_{j,s}^{(r,r)}$ remains constant for $r=2,\ldots,m$, ($\beta_{j,s}$ (say)) the conditions stated in the remark of Sec. 2 are satisfied. Now since the components of $\underset{\sim}{U}_{j\alpha}=(U_{j\alpha}^{(2)},\ldots,U_{j\alpha}^{(m)})$ are independently distributed let us write the null hypothesis (2.8) as follows from (3.2),

(3.5) $$H_o[\underset{\sim}{\theta}=\underset{\sim}{0}] \equiv \bigcap_{j=1}^{p}\bigcap_{r=2}^{m} H_{oj}^{(r)}[\theta_j^{(r)}=0|\underset{\sim}{\theta}_{(j-1)}^{(r)}=\underset{\sim}{0}];\ \underset{\sim}{\theta}_{(j-1)}^{(r)}=(\theta_1^{(r)},\ldots,\theta_{j-1}^{(r)})$$

Thus we can construct test for $H_{oj}^{(r)}$ by using individual estimates to $\beta_{j,s}$ for $r=2,\ldots,m$ and hence the combined test for $H_{oj} = \bigcap_{r=2}^{m} H_{oj}^{(r)}$ by

(3.6) $$L_j = \prod_{r=2}^{m} [1+j(n'-j)^{-1}F_{jr}]^{-1}$$

where F_{jr} follows central F-distribution with $(j,n'-j)$ d.f. Obviously L_j is the product of (m-1) independent beta variables, the distribution of which is well-known (Anderson [1], Rao [5]). Test criterion of the overall hypothesis can therefore be constructed by considering $L = \prod_{j=1}^{p} L_j$. The independence of L_j for $j=1,\ldots,p$ can be verified by the similar arguments as in Sec. 2.2.

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REFERENCES

[1] ANDERSON, T.W. (1958). An Introduction to Multivariate Statistical Analysis. John Wiley & Sons, New York.

[2] ANDERSON, T.W. (1963). A test for equality of means when covariance matrices are unequal. Ann. Math. Statist. **34** 671-672.

[3] BHARGAVA, R.P. (1971). A test for equality of means of multivariate normal distributions when covariance matrices are unequal. Cal. Stat. Assoc. Bulletin **20** 153-156.

[4] CHAKRAVORTI, S.R. (1973). On some tests of growth curve model under Behrens-Fisher situation. Institute of Statistics Mimeo Series No. 870, University of North Carolina, Chapel Hill.

[5] RAO, C.R. (1951). An asymptotic expansion of the distribution of Wilks' Λ-criterion. Bull. Inst. Internat. Stat. **33** Part II 177-180.

[6] RAO, C.R. (1965). Linear Statistical Inference and Its Applications. John Wiley & Sons, New York.

[7] ROY, J. (1958). Step-down procedure in multivariate analysis. Ann. Math. Statist. **29** 1177-1187.

[8] ROY, S.N. (1953). On a heuristic method of test construction and its use in multivariate analysis. Ann. Math. Statist. **24** 220-238.

[9] ROY, S.N., GNANADESIKAN, R., SRIVASTAVA, J.N. (1971). Analysis and Design of Certain Quantitative Multiresponse Experiments. Pergamon Press, New York.