BILINEAR TRANSFORMATIONS BETWEEN DISCRETE- AND CONTINUOUS-TIME INFINITE-DIMENSIONAL LINEAR SYSTEMS [1]

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Abstract: We show that several system-theoretic properties are preserved under a bilinear transformation from a continuous-time linear system to a discrete-time one (and conversely). For example, strong stability, infinite-time admissibility of the control and observation operators and approximate controllability and observability are all preserved. On the other hand, exponential stability (stabilizability) is not. Moreover, we establish a 1-1 correspondence between the solutions of the discrete- and continuous-time Lyapunov and Riccati equations.

Key words: Bilinear transformations, Discrete-time infinite-dimensional linear systems, Continuous-time infinite-dimensional linear systems, Algebraic Riccati equation, Lyapunov equation, Spectral factorization.

1 Introduction

There is a well-known bilinear mapping (Möbius transformation) that maps the open right-half plane $\mathbb{C}^+ = \{s \in \mathbb{C} : \Re e(s) > 0\}$ into the exterior of the unit disc $\mathbb{D}^+ = \{z \in \mathbb{C} : |z| > 1\}$, and conversely,

$$z = \frac{1+s}{1-s}; \quad s = \frac{z-1}{z+1}. \tag{1}$$

In harmonic analysis, this is used to translate properties of function $G(s)$ for $s \in \mathbb{C}^+$ into properties of functions $G_d(z)$ for $z \in \mathbb{D}^+$ and vice versa (see Duren [5]). A natural application is to relate the transfer function $G(s)$ of a continuous-time system via the formula

$$G_d(z) = G(\frac{z-1}{z+1}) \tag{2}$$

to obtain a transfer function of a discrete-time system. The continuous-time transfer function can be recovered via

$$G(s) = G_d(\frac{1+s}{1-s}) \tag{3}$$

i.e., this establishes a 1-1 relationship between a continuous-time and a discrete-time system. This connection has been exploited to deduce properties of a continuous-time system from those of its discrete-time counterpart or vice-versa. For example, in the context of Riccati equations in Hitz and Anderson [12], Kondo and Furuta [13] and in the context of $\mathbf{H}_\infty$-control problems in Green [8], Green et. al. [9] and Kondo and Hara [14]. The relationship has also been used in the context of infinite-dimensional systems in Curtain [1], Ober and Montgomery-Smith [15] and in Curtain and Rodriguez [3]. Since the discrete-time system has bounded operators, in contrast to the unbounded operators in continuous-time systems, the transformation offers obvious technical advantages. On the other hand, the relationship is not so simple for infinite-dimensional systems. For example, if we start with the infinitesimal generator A of a C_0-semigroup on a Hilbert-space X, the corresponding discrete-time operator $A_d = (I+A)(I-A)^{-1}$ is always bounded. However, if A generates an exponentially stable semigroup, A_d will rarely be power stable (the latter property is equivalent to the spectral radius of A_d being less than 1). Power stability plays the role in discrete-time Riccati equation theory that exponential stability does in the continuous-time case, see Zabczyk [25], Przyluski [17]. However, if A_d is power stable, its continuous-time analogue $A = (I + A_d)^{-1}(A_d - I)$ is necessarily bounded. (see Curtain and Rodriguez [3]). This lack of symmetry has limited the use of the relationships between the continuous-

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and discrete-time pairs.

Recently, it has become clear that the relevant stability property for a general Riccati equation theory is not exponential stability, but a type of strong stability, introduced in Staffans [18, 19, 20, 21, 22, 23]. In this paper, we shall show that if, instead of concentrating on exponential and power stability, we examine strong stability, then it is possible to obtain a very nice symmetry between the system-theoretic properties of continuous- and discrete-time systems., which can be profitably exploited to obtain new results on the theory of existence and uniqueness of solutions of discrete-time Riccati equations. In an earlier paper [3], Curtain and Rodriguez used a similar approach to deduce existence results for a special J-spectral, discrete-time Riccati equation with a power stable generator from known results on the analogous continuous-time Riccati equation with a bounded generator. Now we can extend this relationship to strongly stable systems with unbounded generators.

In section 2 we consider continuous-time systems with bounded input and output operators and define their discrete-time analogues. Within this class we show that the system-theoretic properties of strong stability, approximate controllability and observability, infinite-time input and output admissibility are preserved under the bilinear transformation. We adopt the concept of strong stability (stabilizability, detectability) of a system from Staffans [19, 20] and show that these properties are also preserved under the bilinear transformation.

In section 3, we examine Riccati equations for discrete- and continuous-time systems, and show that there is a 1-1 correspondence between strongly stabilizing solutions. Section 4 deals with factorization problems associated with the Riccati equations of section 3; the key is the relationship between the continuous- and discrete-time Popov functions. In section 5, we use the results of sections 3 and 4 to derive conditions for existence and uniqueness of the strongly stabilizing solution of the discrete-time Riccati equation from existing results for the continuous-time case.

Finally, we remark that our approach carries over to the more general class of well-posed linear systems, which allows for unbounded input and output operators. All the results in this paper hold for this class, but as the proofs are of a more technical nature, this research will be reported elsewhere.

2 The continuous- and discrete-time analogues: system theoretic equivalences

In this paper, we let X, U, Y be separable Hilbert spaces and we define

- *The continuous-time system* $\Sigma(A, B, C, D)$, where A is the infinitesimal generator of a C_0-semigroup $T(t)$ on X, $B \in \mathcal{L}(U, X)$, $C \in \mathcal{L}(X, Y)$ and $D \in \mathcal{L}(U, Y)$. We denote the transfer function by $G(s) = D + C(sI - A)^{-1}B$.

- *The discrete-time system* $\Sigma_d(A_d, B_d, C_d, D_d)$, where $A_d \in \mathcal{L}(X)$, $B_d \in \mathcal{L}(U, X)$, $C_d \in \mathcal{L}(X, Y)$ and $D_d \in \mathcal{L}(U, Y)$. We denote the transfer function by $G_d(z) = D_d + C_d(zI - A_d)^{-1}B_d$.

We call Σ_d the discrete-time analogue of Σ and Σ the continuous-time analogue of Σ_d, if they are related in the following way (where we assume without loss of generality that $1 \notin \sigma(A)$, $-1 \notin \sigma(A_d)$).

$$\begin{aligned} A_d &= (I + A)(I - A)^{-1}, \\ B_d &= \sqrt{2}(I - A)^{-1}B, \\ C_d &= \sqrt{2}C(I - A)^{-1}, \\ D_d &= D + C(I - A)^{-1}B, \end{aligned} \tag{4}$$

$$\begin{aligned} A &= (I + A_d)^{-1}(A_d - I), \\ B &= \sqrt{2}(I + A_d)^{-1}B_d, \\ C &= \sqrt{2}C_d(I + A_d)^{-1}, \\ D &= D_d - C_d(I + A_d)^{-1}B_d. \end{aligned} \tag{5}$$

Note that the above relationships are reversible and that the transfer functions satisfy (2), (3). Thus, a stable, continuous-time transfer function $G(s)$ is holomorphic and bounded on $\mathbb{C}^+$ if and only if its analogue $G_d(z)$ is holomorphic and bounded on $\mathbb{D}^+$. If it happens that $1 \in \sigma(A)$, then one can define $A_d = (\mu I + A)(\mu I - A)^{-1}$ for some $\mu \notin \sigma(A)$ and modify (4), (5) accordingly.

As mentioned in the introduction, the stability properties of A and A_d are not so clearly linked. We compare the following concepts.

Definition 2.1 *The C_0-semigroup $T(t)$ is exponentially stable if there exist positive constants M, α such that*

$$\|T(t)\| \leq Me^{-\alpha t}$$

and it is strongly stable if $\|T(t)x\| \to 0$ as $t \to \infty$ for all $x \in X$. A_d is power stable if there exist positive constants M and γ, $\gamma < 1$ such that

$$\|A_d^k\| \leq M\gamma^k$$

and A_d is strongly stable if $A_d^k x \to 0$ as $k \to \infty$ for all $x \in X$.

Lemma 2.2 *a. $G(s)$ is holomorphic and bounded on $\mathbb{C}^+$ if and only if $G_d(z)$ is holomorphic and bounded on $\mathbb{D}^+$.*
b. If $1 \notin \sigma(A)$, $-1 \notin \sigma(A_d)$, then the relationship between the spectra is via the bilinear map $\lambda = \frac{1+\mu}{1-\mu}$:

$$\lambda \in \rho(A_d) \ \Leftrightarrow \ \mu = \frac{\lambda - 1}{\lambda + 1} \in \rho(A).$$

c. If $A_d \in \mathcal{L}(X)$ is power stable then its continuous-time analogue A is bounded and it generates an exponentially stable C_0-semigroup.
d. If A is bounded and e^{At} is exponentially stable, then its discrete-time analogue $A_d \in \mathcal{L}(X)$ is power stable.
e. If A is an unbounded infinitesimal generator of an exponentially stable C_0-semigroup, then $A_d \in \mathcal{L}(X)$ is not power stable; $\sigma(A_d)$ lies within the unit circle, except for $-1 \in \sigma(A_d)$.
f. If A generates a strongly stable C_0-semigroup then its discrete-time analogue A_d is strongly stable.
g. If A_d is strongly stable, then its continuous-time analogue is the infinitesimal generator of a strongly stable C_0-semigroup.

Proof. a. follows since the Möbius transformation maps $\mathbb{C}^+$ into $\mathbb{D}^+$.
b. c. d. e. are in Lemma 4.4 of Curtain and Rodriguez [3].
f. Suppose that A generates the strongly stable semigroup $T(t)$. Then A_d defined by (4) is bounded. Moreover, the following defines an equivalent norm on X (see Pazy [16])

$$|x|_H = \sup_{t \geq 0} \|T(t)x\|_X.$$

It induces a Hilbert space H under the inner product

$$\langle x, y \rangle_H = \sup_{t \geq 0} \langle T(t)x, T(t)y \rangle_X.$$

$T(t)$ is a strongly stable contraction semigroup on H and hence $\|A_d^k x\|_H \to 0$ as $k \to \infty$ (Fuhrmann [6], Theorem 10-19, p 158). Since $|\cdot|_H$ is equivalent to $\|\cdot\|_X$, we have that

$$\|A_d^k x\|_X \to 0 \text{ as } k \to \infty.$$

g. Conversely, suppose that A_d is strongly stable. Then the following defines an equivalent norm on X

$$|x|_H = \sup_{k \in \mathbb{N}} \|A_d^k x\|_X$$

and it induces a Hilbert space H under the inner product

$$\langle x, y \rangle_H = \sup_{k \in \mathbb{N}} \langle A_d^k x, A_d^k y \rangle_X.$$

Under this norm A_d is a contraction and its continuous analogue A generates a strongly stable contraction semigroup $T(t)$ on H (Fuhrmann [6], Theorem 10-19, p 158). $T(t)$ also defines a semigroup on X and, since the norms are equivalent, it is strongly stable. ∎

The equivalence between strong stability for the pair $T(t), A_d$ is well-known in the case that either is a contraction, but for the more general case, we were unable to find any known equivalence.

Next we compare the Lyapunov equations which are linked to the following concepts of infinite-time admissibility (See Hansen and Weiss [11], Grabowski [7]) and to controllability and observability concepts.

Definition 2.3 *1. Consider $\Sigma(A, B, C, D)$. We say that B is an* infinite-time admissible control operator *for $T(t)$ if the extended controllability map $\mathcal{B} \in \mathcal{L}(\mathbf{L}_2(0, \infty; U), X)$, where $\mathcal{B}$ is defined by*

$$\mathcal{B}u = \lim_{\tau \to \infty} \int_0^\tau T(t)Bu(t).$$

C is an infinite-time admissible observation operator *for $T(t)$ if the extended observability map $\mathcal{C} \in \mathcal{L}(X, \mathbf{L}_2(0, \infty; Y))$, where $\mathcal{C}$ is defined by*

$$\mathcal{C}x = CT(\cdot)x$$

for $x \in X$. Σ is approximately controllable if $\text{range}(\mathcal{B}) = X$ *and approximately observable if* $\ker(\mathcal{C}) = \{0\}$.

2. Consider $\Sigma_d(A_d, B_d, C_d, D_d)$. We say that B_d is an infinite-time admissible control operator *for A_d if the extended controllability map $\mathcal{B}_d \in \mathcal{L}(l_2(0, \infty; U), X)$, where $\mathcal{B}_d$ is defined by*

$$\mathcal{B}_d u = \lim_{N \to \infty} \sum_{k=0}^N A_d^k B_d u(k).$$

C_d *is an* infinite-time admissible observation operator *for* A_d *if the extended observability map* $\mathcal{C}_d \in \mathcal{L}(X, l_2(0,\infty;Y))$*, where* $\mathcal{C}_d$ *is defined by*

$$\mathcal{C}_d x = (C_d x, C_d A_d x, C_d A_d^2 x, \cdots).$$

for $x \in X$*.* Σ_d *is approximately controllable if* $\mathrm{Range}(\mathcal{B}_d) = X$ *and approximately observable if* $\ker(\mathcal{C}_d) = \{0\}$*.*

The controllability and observability properties are determined by the control and observation Lyapunov equations.

$$AL_B x + L_B A^* x = -BB^* x \quad \text{for } x \in D(A^*) \tag{6}$$

$$A^* L_C x + L_C A x = -C^* C x \quad \text{for } x \in D(A) \tag{7}$$

$$A_d L_B^d A_d^* - L_B^d = -B_d B_d^* \tag{8}$$

$$A_d^* L_C^d A_d - L_C^d = -C_d^* C_d \tag{9}$$

which have very nice inter-relationships with each other and the concepts from Definition 2.3

Theorem 2.4 *a. For* $\Sigma(A,B,C,D)$ *the following are equivalent statements:*

(i) B is an infinite-time admissible control operator for $T(t)$;

(ii) The controllability gramian $L_B \in \mathcal{L}(X)$ is well-defined by

$$\begin{aligned} L_B x &= \lim_{\tau\to\infty} \int_0^\tau T(t) B B^* T^*(t) x dt \\ &= \mathcal{B}\mathcal{B}^* x; \end{aligned}$$

(iii) There exists a solution to the control Lyapunov equation (6).

Moreover, if $T^*(t)$ is strongly stable, L_B is the unique solution of (6), and Σ is approximately controllable if and only if $L_B > 0$.

b. For $\Sigma(A,B,C,D)$ the following are equivalent statements:

(i) C is an infinite-time admissible control operator for $T(t)$;

(ii) The observability gramian $L_C \in \mathcal{L}(X)$ is well-defined by

$$\begin{aligned} L_C x &= \lim_{\tau\to\infty} \int_0^\tau T(t)^* C^* C T(t) x dt \\ &= \mathcal{C}^* \mathcal{C} x; \end{aligned}$$

(iii) There exists a solution to the observation Lyapunov equation (7).

Moreover, if $T(t)$ is strongly stable, L_C is the unique solution of (6) and Σ is approximately observable if and only if $L_C > 0$.

c. $\Pi \in \mathcal{L}(X)$ is a solution of (6) if and only if Π is a solution of (8).

d. $\Pi \in \mathcal{L}(X)$ is a solution of (7) if and only if Π is a solution of (9).

e. For $\Sigma_d(A_d, B_d, C_d, D_d)$ the following are equivalent statements:

(i) B_d is an infinite-time admissible control operator for A_d;

(ii) The controllability gramian $L_B^d \in \mathcal{L}(X)$ is well-defined by

$$\begin{aligned} L_B^d x &= \lim_{N\to\infty} \sum_{k=0}^{N} A_d^k B_d B_d^* (A_d^*)^k x dt \\ &= \mathcal{B}_d \mathcal{B}_d^* x; \end{aligned}$$

(iii) There exists a solution to the control Lyapunov equation (8).

Moreover, if A_d^* is strongly stable, L_B^d is the unique solution of (8) and Σ_d is approximately controllable if and only if $L_B^d > 0$.

f. For $\Sigma_d(A_d, B_d, C_d, D_d)$ the following are equivalent statements:

(i) C_d is an infinite-time admissible control operator for A_d;

(ii) The observability gramian $L_C^d \in \mathcal{L}(X)$ is well-defined by

$$\begin{aligned} L_C^d x &= \lim_{N\to\infty} \sum_{k=0}^{N} (A_d^*)^k C_d^* C_d A_d^k x dt \\ &= \mathcal{C}_d^* \mathcal{C}_d x; \end{aligned}$$

(iii) There exists a solution to the observation Lyapunov equation (9).

Moreover, if A_d is strongly stable, L_C^d is the unique solution of (9) and Σ_d is approximately observable if and only if $L_C^d > 0$.

g. B is an infinite-time admissible control operator for $T(t)$ if and only if B_d is an infinite-time admissible control operator for A_d, and the controllability gramians are identical.

h. C is an infinite-time admissible observation operator for $T(t)$ if and only if C_d is an infinite-time admissible observation operator for A_d, and the controllability gramians are identical *i.* Σ *is approximately controllable (observable) if and only if* Σ_d *is approximately controllable (observable).*

j. The time-domain Hankel operator $\Gamma = \mathcal{C}\mathcal{B}$ *for* Σ *is bounded (compact) if and only if the time-domain Hankel operator* $\Gamma_d = \mathcal{C}_d\mathcal{B}_d$ *for* Σ_d *is bounded (compact). If either* Γ_d *or* Γ *is compact, then so is the other and* Σ *and* Σ_d *have identical Hankel singular values.*

Many parts of the above theorem are known, for example a., b. in Grabowski [7] and Hansen and Weiss [11], c., d., e., and f. in Curtain and Zwart [4], exercises 4.29 and 4.30 and i. in Curtain [1]. However, the connection between discrete- and continuous-time systems assumed exponential and power stability, which is unnecessary.

Proof. a., b. See Grabowski [7] and Hansen and Weiss [11].

c., d. By direct computation, since

$\frac{1}{2}(I-A) \times (8) \times (I-A^*)$ equals (6)

and

$\frac{1}{2}(I+A_d) \times (6) \times (I+A_d^*)$ equals (8).

A similar computation establishes the relationship between (7) and (9).

e., f. Since these are dual statements, it suffices to prove only f. (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii): Suppose that (9) has a solution $\Pi \in \mathcal{L}(X)$. Multiplying successively from the left by A_d^* and from the right by A_d, we obtain

$$\begin{aligned}
C_d^*C_d &= \Pi - A_d^*\Pi A_d \\
A_d^*C_d^*C_dA_d &= A_d^*\Pi A_d - (A_d^*)^2\Pi A_d^2 \\
&\vdots \\
(A_d^*)^kC_d^*C_dA_d^k &= (A_d^*)^k\Pi A_d^k - (A_d^*)^{k+1}\Pi A_d^{k+1}.
\end{aligned}$$

whence,

$$\sum_{k=0}^{N}(A_d^*)^kC_d^*C_dA_d^k = \Pi - (A_d^*)^{N+1}\Pi A_d^{N+1} \leq \Pi,$$

since $\Pi > 0$. This proves that C_d is an infinite-time admissible control operator for A_d and so, by definition, $L_C^d = \mathcal{C}_d^*\mathcal{C}_d \in \mathcal{L}(X)$.
(ii) $\Rightarrow$ (iii): Suppose that

$$L_C^d = \mathcal{C}_d^*\mathcal{C}_d = \sum_{k=0}^{\infty}(A_d^*)^kC_d^*C_dA_d^k \in \mathcal{L}(X).$$

Direct verification shows then that L_C^d satisfies (9).

g. follows from a., c. and e.

h. follows from a., d. and f.

i. Now, $L_C - \mathcal{C}^*\mathcal{C} = \mathcal{C}_d^*\mathcal{C}_d$ shows that $\ker(L_C) = \ker(\mathcal{C}) = \{0\}$ if and only if $L_C > 0$. Hence Σ (Σ_d) is approximately observable if and only if $L_C > 0$. The rest follows by duality.

j. This follows as in Curtain [1]. ■

Recently, in Staffans [19] it became apparent that exponential stability was not so relevant for a comprehensive Riccati equation theory, but, instead, the following concept of stability.

Definition 2.5 $\Sigma(A,B,C,D)$ *is a* strongly stable continuous-time system *if*

1. *A generates a strongly stable C_0–semigroup $T(t)$;*
2. *C is an infinite-time admissible observation operator for $T(t)$;*
3. *B is an infinite-time admissible control operator for $T(t)$;*
4. *$D + C(sI-A)^{-1}B \in \mathbf{H}_\infty(\mathbb{C}^+;\mathcal{L}(U,Y))$.*

The corresponding discrete-time concept is

Definition 2.6 $\Sigma_d(A_d,B_d,C_d,D_d)$ *is a* strongly stable discrete-time system *if*

1. *A_d is strongly stable;*
2. *C_d is an infinite-time admissible observation operator for A_d;*
3. *B_d is an infinite-time admissible control operator for A_d;*
4. *$D_d + C_d(zI-A_d)^{-1}B_d \in \mathbf{H}_\infty(\mathbb{D}^+;\mathcal{L}(U,Y))$.*

¿From the results of Theorem 2.4 we have the following corollary.

Corollary 2.7 *$\Sigma(A,B,C,D)$ is a strongly stable continuous-time system if and only if its discrete-time analogue $\Sigma_d(A_d,B_d,C_d,D_d)$ is a strongly stable discrete-time system.*

The concept of strongly stable system leads naturally to the following definition of strong stabilizability and detectability (See Staffans [22]).

Definition 2.8 *1. The continuous-time system $\Sigma(A,B,C,D)$ is*

a. *strongly stabilizable if there exists a* $F \in \mathcal{L}(X,U)$ *such that* $\Sigma(A+BF,B,F,D)$ *is a strongly stable continuous-time system.*

b. *strongly detectable if there exists a* $H \in \mathcal{L}(Y,X)$ *such that* $\Sigma(A+HC,H,C,D)$ *is a strongly stable continuous-time system.*

2. *The discrete-time system* $\Sigma_d(A_d,B_d,C_d,D_d)$ *is*

a. *strongly stabilizable if there exists a* $F_d \in \mathcal{L}(X,U)$ *such that* $\Sigma_d(A_d + B_dF_d, B_d, F_d, D_d)$ *is a strongly stable discrete-time system.*

b. *strongly detectable if there exists a* $H_d \in \mathcal{L}(Y,X)$ *such that* $\Sigma_d(A_d + H_dC_d, H_d, C_d, D_d)$ *is a strongly stable discrete-time system.*

While the more usual concepts of exponential stabilizability and detectability are not retained under the bilinear transformation, we can show that the "strong" versions are.

Theorem 2.9 *Suppose that* $\Sigma(A,B,C,D)$ *and* $\Sigma_d(A_d,B_d,C_d,D_d)$ *are continuous- and discrete-time analogues. Then* $\Sigma(A,B,C,D)$ *is strongly stabilizable (detectable) if and only if* $\Sigma_d(A_d,B_d,C_d,D_d)$ *is strongly stabilizable (detectable).*

Proof. a. Suppose that $\Sigma(A,B,C,D)$ is strongly stabilizable by the feedback $F \in \mathcal{L}(X,U)$;i.e.,$\Sigma(A+BF,B,F,D)$ is a strongly stable system. Corollary 2.7 shows that the corresponding discrete-time system $\Sigma_d^F = (A_d^F, B_d^F, F_d^F, D_d^F)$ is also strongly stable, where

$$\begin{aligned}
A_d^F &= (I+A+BF)(I-A-BF)^{-1},\\
B_d^F &= \sqrt{2}(I-A-BF)^{-1}B,\\
F_d^F &= \sqrt{2}F(I-A-BF)^{-1},\\
D_d^F &= D + F(I-A-BF)^{-1}B.
\end{aligned}$$

Simple computations show that (writing F_d for F_d^F)

$$\begin{aligned}
A_d^F &= A_d + B_dF_d,\\
B_d &= B_d^F(I - \tfrac{1}{\sqrt{2}}FB_d).
\end{aligned}$$

We show that F_d is also a stabilizing feedback for Σ_d. Now $A_d + B_dF_d$ is strongly stable, since it equals A_d^F which is. Further

$$\sum_{k=0}^{\infty}(A_d^F)^kB_d = \left(\sum_{k=0}^{\infty}(A_d^F)^kB_d^F\right)(I - \frac{1}{\sqrt{2}}FB_d),$$

which shows that $\sum_{k=0}^{\infty}(A_d^F)^kB_d$ is bounded if $\sum_{k=0}^{\infty}(A_d^F)^kB_d^F$ is; the latter holds since Σ_d^F is strongly stable. Noting that $F_d^F = F_d$, and

$$\begin{aligned}
&D_d + F_d(zI - A_d - B_dF_d)^{-1}B_d\\
&\quad = D_d + F_d(zI - A_d^F)^{-1}B_d^F(I - \frac{1}{\sqrt{2}}FB_d)
\end{aligned}$$

completes the proof that $\Sigma_d(A_d + B_dF_d, B_d, C_d, D_d)$ is strongly stable.

b. Conversely, suppose that $\Sigma_d(A_d,B_d,F_d,D_d)$ is strongly stabilizable by $F_d \in \mathcal{L}(X,U)$, i.e. $\Sigma_d^F(A_d + B_dF_d, B_d, F_d, D_d)$ is a strongly stable system. Then Corollary 2.7 shows that the corresponding continuous-time system $\Sigma^F(A^F,B^F,F^F,D^F)$ is also strongly stable, where

$$\begin{aligned}
A^F &= (I + A_d + B_dF_d)^{-1}(A_d + B_dF_d - I),\\
B^F &= \sqrt{2}(I + A_d + B_dF_d)^{-1}B_d\\
F^F &= \sqrt{2}F_d(I + A_d + B_dF_d)^{-1},\\
D^F &= D_d - C_d(I + A_d + B_dF_d)^{-1}B_d.
\end{aligned}$$

Simple computations reveal that (writing F for F^F)

$$\begin{aligned}
A^F &= A + BF,\\
B &= B^F(I - \tfrac{1}{\sqrt{2}}F_dB).
\end{aligned}$$

That F is also a stabilizing feedback for Σ follows from the identities

$$\begin{aligned}
&\int_0^{t_1} T^F(s)Bu(s)ds\\
&\quad = \int_0^{t_1} T^F(s)B^F(I + \frac{1}{\sqrt{2}}F_dB)u(s)ds,
\end{aligned}$$

and

$$\begin{aligned}
&F(sI - A - BF)^{-1}B =\\
&\quad F(sI - A - BF)^{-1}B^F(I - \frac{1}{\sqrt{2}}F_dB).
\end{aligned}$$

The equivalence of the detectability properties follows by a duality argument. ■

3 Relationships between Riccati equations

In this section, we examine the connection between the following Riccati equations,

$$\begin{aligned}
&A^*Xx + XAx + C^*QCx - (B^*X + D^*QC)^* \cdot\\
&\quad (D^*QD)^{-1}(B^*X + D^*QC)x = 0 \qquad (10)
\end{aligned}$$

for $x \in D(A)$, and

$$\begin{aligned}
&A_d^*XA_d - X + C_d^*QC_d \qquad (11)\\
&\quad = K_d^*(B_d^*XB_d + D_d^*QD_d)^{-1}K_d,
\end{aligned}$$

where $K_d = B_d^* X A_d + D_d^* Q C_d$ and $Q = Q^* \in \mathcal{L}(U)$. The connection becomes more transparent if we reformulate (10), (11) into forms that are closer to the spectral factorizations associated with these Riccati equations (see Ionescu and Halanay [10], Weiss [24], Green [8] and Curtain and Rodriguez [3]). For the continuous-time Riccati equation (10) we consider

$$\begin{bmatrix} XA + A^*X + C^*QC & XB + C^*QD \\ D^*QC + B^*X & D^*QD \end{bmatrix} = \begin{bmatrix} L^* \\ V^* \end{bmatrix} Q \begin{bmatrix} L & V \end{bmatrix} \tag{12}$$

and for the discrete-time Riccati equation (11)

$$\begin{bmatrix} A_d^* & C_d^* \\ B_d^* & D_d^* \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} A_d & B_d \\ C_d & D_d \end{bmatrix} = \begin{bmatrix} I & L_d^* \\ 0 & V_d^* \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} I & 0 \\ L_d & V_d \end{bmatrix} \tag{13}$$

The advantage of the forms (11) and (13) is that they remain valid formulations even for the singular Riccati equations, e.g. D^*QD not invertible. As in Curtain and Rodriguez [3], it is fairly straightforward to show that $X \in \mathcal{L}(X)$ solves (12) if and only if X solves (13), where $L, L_d \in \mathcal{L}(U, X)$ and $V, V_d \in \mathcal{L}(U)$ are related by

$$\begin{aligned} L_d &= \sqrt{2} L (I - A)^{-1}, \\ V_d &= V + L(I - A)^{-1} B \end{aligned} \tag{14}$$

$$\begin{aligned} L &= \sqrt{2} L_d (I + A_d)^{-1}, \\ V &= V_d - L_d (I + A_d)^{-1} B_d. \end{aligned} \tag{15}$$

The proof is by sequential verification of the following, using results from the preceding step where appropriate.

(i) $\frac{1}{2}(I - A^*) \times$[(1,1)-block of (13)]$\times (I - A)$ equals the (1,1)-block of (12). Conversely, $\frac{1}{2}(I + A_d^*) \times$[(1,1)-block of (12)]$\times (I + A_d)$ equals the (1,1)-block of (13).

(ii) $\frac{1}{\sqrt{2}}(I - A^*) \times$[(1,2)-block of (13)] equals the (1,2)-block of (12). Conversely, $\frac{1}{\sqrt{2}}(I + A_d^*) \times$[(1,2)-block of (12)] equals the (1,2)-block of (13).

(iii) the (2,2)-block of (13) equals the (2,2)-block of (12).

Note that (14) shows that if V is invertible, then V_d is invertible and

$$V_d^{-1} = V^{-1} - V^{-1} L (I - A + BV^{-1}L) BV^{-1}. \tag{16}$$

The converse follows from (15) and so V is invertible if and only if V_d is. If either V or V_d is invertible, then the candidate for the continuous-time stable generator associated with (10) is

$$\begin{aligned} A_X &= A - B(D^*QD)^{-1}(B^*X + D^*QC) \\ &= A - BV^{-1}L \end{aligned} \tag{17}$$

and its discrete-time analogue is

$$\begin{aligned} A_{dX} &= A_d - B_d V_d^{-1} L_d \\ &= A_d - B_d (D_d^* Q D_d + B_d^* X B_d)^{-1} K_d, \end{aligned} \tag{18}$$

which corresponds with the usual stable generator associated with (11) (see the proof in Curtain and Rodriguez [3], Lemma 4.7). So we have the following nice equivalence between the solutions of (10), (11).

Theorem 3.1 *$X \in \mathcal{L}(X)$ is a solution of (12) if and only if it is a solution of (13). D^*QD is invertible if and only if $D_d^*QD_d + B_d^*XB_d$ is and in this case (10) is equivalent to (12) and (11) to (13). Moreover, A_X generates a strongly stable semigroup if and only if A_{dX} is strongly stable, and the continuous-time system $\Sigma(A_X, B, F, D)$ is strongly stable if and only if its discrete-time analogue $\Sigma_d(A_{dX}, B_d, F_d, D_d)$ is. In this case, X is the unique, self adjoint, strongly stabilizing solution of (10) and of (11).*

Proof. It only remains to prove the assertion about the equivalence of strong stability for the two systems, but this follows from Theorem 2.9, where the feedback $F = -(D^*QD)^{-1}(B^*X + D^*QC) = -V^{-1}L$, and $F_d = \sqrt{2}F(I - A - BF)^{-1} = -\sqrt{2}V^{-1}L(I - A - BV^{-1}L)^{-1} = -V_d^{-1}L_d$. ∎

Since it may not be immediately obvious that (12), (13) cover a wealth of different types of Riccati equations, including singular ones, we specify a few special cases.

Example 3.2 (The Positive Case) Defining

$$Q = \begin{bmatrix} Q_1 & N^* \\ N & R \end{bmatrix}, \quad D = V = \begin{bmatrix} 0 \\ I \end{bmatrix},$$

$$C = \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad L = \begin{bmatrix} 0 \\ L_0 \end{bmatrix},$$

yields the more familiar terms $C^*QC = Q_1$, $D^*QC = N$, $D^*QD = R$ that appear in Weiss [24]. The corresponding continuous-time Riccati equation is

$$\begin{aligned} &A^*Xz + XAz + Q_1 z \\ &\quad = (B^*X + N)^* R^{-1} (B^*X + N) z, \end{aligned}$$

for $z \in D(A)$.

Example 3.3 (The J-spectral Case) Now we take Q equal to

$$J_{lm}(\gamma) = \begin{bmatrix} I_l & 0 \\ 0 & -\gamma^2 I_m \end{bmatrix},$$

where l, m are positive integers and I_m denotes the $m \times m$ identity matrix and γ is a real number. The resulting continuous-time and discrete-time Riccati equations are exactly the equations for the condition for J-spectral factorizations for the systems $\Sigma(A, B, C, D)$ and $\Sigma_d(A_d, B_d, C_d, D_d)$, respectively. These Riccati equations are (10) and (11), respectively, with Q replaced by $J_{lm}(\gamma)$.

Example 3.4 (The $\mathbf{H_\infty}$-Case) The theory of H_∞-control uses Riccati equations corresponding to the choice

$$B = \begin{bmatrix} B_1 & B_2 \end{bmatrix},$$
$$C = \begin{bmatrix} \alpha_1 C_1 \\ \alpha_2 C_1 \end{bmatrix}, D = V = I,$$
$$Q = \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix}, L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix},$$

where γ is a constant assumed to be greater than one and

$$\alpha_1^2 = \frac{\gamma^2}{\gamma^2 - 1}, \alpha_2 = \frac{\alpha_1}{\gamma^2}.$$

The case for $\gamma \leq 1$ can be deduced from the above case by scaling.
The corresponding H_∞ Riccati equation is

$$\begin{aligned} &A^* X z + X A z + C_1^* C_1 z \\ &\quad = X B_1 B_1^* X z - \frac{1}{\gamma^2} X B_2 B_2^* X z, \end{aligned}$$

where $z \in D(A)$.

4 Factorizations of the Popov function

Let us first consider the continuous-time case and define the Popov function associated with the Riccati identity (12)

Definition 4.1 *The Popov function* $\Pi(j\omega) : j\mathbb{R} \to \mathcal{L}(U)$ *associated with the Riccati identity (12) is defined for* $\omega \in \mathbb{R}$ *by*

$$\Pi(j\omega) = G(j\omega)^* Q G(j\omega), \tag{19}$$

where

$$G(s) = D + C(sI - A)^{-1} B. \tag{20}$$

It is then an easy exercise to verify the following result.

Lemma 4.2 *If the Riccati identity (12) holds, then the following factorization is valid for* $\omega \in \mathbb{R}$

$$\begin{aligned} \Pi(j\omega) &= (V + L(j\omega I - A)^{-1} B)^* \cdot \\ &\quad Q(V + L(j\omega I - A)^{-1} B). \end{aligned} \tag{21}$$

¿From this general factorization result it is straightforward to recover the well known factorizations.

Example 4.3 (The Positive Case) The choice

$$Q = \begin{bmatrix} Q_1 & N^* \\ N & R \end{bmatrix}, \quad D = V = \begin{bmatrix} 0 \\ I \end{bmatrix},$$
$$C = \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad L = \begin{bmatrix} 0 \\ L_0 \end{bmatrix},$$

yields

$$\begin{aligned} \Pi(j\omega) &= G_0(j\omega)^* Q G_0(j\omega) \\ &= W_0(j\omega)^* R W_0(j\omega), \end{aligned}$$

where

$$G_0(s) = \begin{bmatrix} (sI - A)^{-1} B \\ I \end{bmatrix},$$

and

$$\begin{aligned} W_0(s) &= (I + L_0(sI - A)^{-1} B) \\ &= I + R^{-1}(N + B^* X)(sI - A)^{-1} B. \end{aligned}$$

Special cases of this are:

- the positive-real case:
  $Q_1 = 0$ yields the factorization

  $$G_1(j\omega)^* + G_1(j\omega) = W_0(j\omega)^* R W_0(j\omega),$$

  where $G_1(s) = D_1 + N(sI - A)^{-1} B$ and $R = D_1 + D_1^*$.

- the standard LQ case:
  $N = 0, Q_1 = C_0^* C_0$ yields the factorization

  $$\begin{aligned} &R + B^*(-j\omega - A^*)^{-1} C_0^* C_0 (j\omega - A)^{-1} B \\ &\quad = W_0(j\omega)^* R W_0(j\omega). \end{aligned}$$

- the factorization

  $$\gamma^2 I - G^*(j\omega) G(j\omega) = W_0^*(j\omega) R W_0(j\omega)$$

  follows by choosing $Q_1 = -C^* C, N = -C^* D$, and $R = \gamma^2 I - D^* D$ and $G(s) = D + C(sI - A)^{-1} B$.

Example 4.4 (J-Spectral Factorization) As in Example 3.3, we take $Q = J_{lm}(\gamma)$ and identify $R = J_{qm}(\gamma)$ to obtain

$$G(j\omega)^* J_{lm}(\gamma) G(j\omega) = W_0(j\omega)^* J_{qm}(\gamma) W_0(j\omega),$$

where

$$W_0(s) = V + L(sI - A)^{-1} B,$$

$$L = J_{qm}^{-1} V^{-*}(D^* J_{lm}(\gamma) C + B^* X)$$

and V is any square invertible matrix solution of

$$V^* J_{qm}(\gamma) V = D^* J_{lm}(\gamma) D.$$

We are now in a position to relate these results to their discrete-time counterparts.

Definition 4.5 *The discrete-time Popov function $\Pi_d : e^{j\theta} \to \mathcal{L}(U)$ associated with the Riccati identity (13) is defined for $\theta \in (-\pi/2, \pi/2]$ by*

$$\Pi_d(e^{j\theta}) = G_d(e^{j\theta})^* Q G_d(e^{j\theta}), \tag{22}$$

where,

$$G_d(z) = D_d + C_d(zI - A_d)^{-1} B_d. \tag{23}$$

It is now an algebraic exercise to verify the following result.

Lemma 4.6 *If the Riccati identity (13) holds, then the following factorization is valid for $\theta \in (-\pi/2, \pi/2]$*

$$\begin{aligned} \Pi_d(e^{j\theta}) \quad &= \quad (V_d + L_d(e^{j\theta} I - A_d)^{-1} B_d)^* \cdot \\ &\qquad Q(V_d + L_d(e^{j\theta} I - A_d)^{-1} B_d). \end{aligned} \tag{24}$$

Finally, using the relationships between the continuous- and discrete-time systems established in Section 2, we have the following.

Lemma 4.7 *The Popov functions are related by*

$$\Pi_d(z) = \Pi(\frac{z-1}{z+1})$$

$$\Pi(s) = \Pi_d(\frac{1+s}{1-s}).$$

The factorization (21) holds if and only if the factorization (24) holds.

5 New Results for Riccati Equations and Factorizations

Using the relationships established in this paper, it is fairly straightforward to obtain new results on the existence and uniqueness of strongly stabilizing solutions of discrete-time Riccati equations from the recent results on continuous-time Riccati equations from Staffans [18, 19, 20, 21, 22, 23]. In the case of bounded B and C operators it is possible to obtain equivalence with the existence of factorizations of the Popov function (see Curtain and Oostveen [2]). As an illustration, we consider the case that $\Sigma(A, B, C, D)$ is a strongly stable, continuous-time system and the Popov function satisfies a coercivity condition. First we quote recent results from Curtain and Oostveen [2]

Theorem 5.1 *Let $\Sigma(A, B, C, D)$ be a strongly stable, continuous-time system and suppose that $D^* Q D \geq \mu I$ for some $\mu > 0$. Then the Riccati equation (10) possesses a unique self-adjoint, strongly stabilizing solution if and only if the associated Popov function Π defined by (19) is coercive, i.e., $\Pi(\omega) \geq \varepsilon I$ for some positive constant ε and almost all $\omega \in \mathbb{R}$*

We remark that in the above theorem, a strongly stabilizing solution X is one that satisfies the Riccati equation and is such that

$$\Sigma(A_{F_X}, B, \begin{bmatrix} F_X \\ C \end{bmatrix}, D)$$

is a strongly stable system where $F_X = -(D^* Q D)^{-1}(B^* X + D^* Q C)$ and $A_{F_X} = A + BF_X$.

Appealing to Lemma 4.6 and Theorem 3.1 we can conclude the following new result for discrete-time Riccati equations.

Theorem 5.2 *Let $\Sigma_d(A_d, B_d, C_d, D_d)$ be the discrete-time counterpart of a strongly stable continuous-time system. Then the Riccati equation (11) possesses a unique self-adjoint, strongly stabilizing solution if and only if its associated Popov function $\Pi_d(e^{j\theta})$ defined by (20) is coercive for almost all $\theta \in (-\pi/2, \pi/2]$.*

In the above theorem, by strongly stabilizing solution is meant a bounded linear operator X which satisfies the Riccati equation (11) and is such that

$$\Sigma(A_{F_{dX}}, B_d, \begin{bmatrix} F_{dX} \\ C_d \end{bmatrix}, D_d)$$

is a strongly stable discrete-time system where $F_{dX} = -(B_d^* X B_d + D_d^* Q D_d)^{-1}(B_d^* X A_d + D_d^* Q C_d)$ and $A_{F_{dX}} = A_d + B_d F_{dX}$.

In a similar manner, we can derive other new results on Riccati equations for discrete-time systems from known results on continuous-time ones. However, in this paper we are restricted to the rather small class of discrete-time systems that corresponds to continuous-time systems with bounded B and C operators. In order to obtain results for general discrete-time Riccati equations, we need to extend the results in this paper to the more general class of well-posed linear systems. This is indeed feasible and research is being carried out in this direction.

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