SEQUENTIAL RANK TESTS FOR REGRESSION

By

MALAY GHOSH AND PRANAB KUMAR SEN

Indian Statistical Institute, Calcutta

and

Department of Biostatistics
University of North Carolina at Chapel Hill

Institute of Statistics Mimeo Series No. 948

AUGUST 1974
SEQUENTIAL RANK TESTS FOR REGRESSION

By MALAY GHOSH

Indian Statistical Institute, Calcutta

and

PRANAB KUMAR SEN

University of North Carolina, Chapel Hill

SUMMARY

Asymptotic theory of sequential rank order tests (SROT) for the regression coefficient in a simple linear regression model is considered here. The proposed SROT terminates with probability one. For local alternatives, the OC function of the proposed SROT is the same as that of the Wald SPRT (sequential probability ratio test). Finally, the asymptotic ASN of the proposed SROT is compared with that of the SPRT.


2) The work was completed while the author was visiting the University of North Carolina, Chapel Hill.

3) Currently visiting at the Institute for Mathematical Statistics, Albert-Ludwig University, Freiburg i. Br., W. Germany.
1. INTRODUCTION

Sen and Ghosh (1974) have developed sequential rank tests for the location parameter in the classical one sample location problem. Their results also apply to the two-sample location problem when observations are taken in pairs. Motivated by the above, we have developed here sequential rank tests for the regression coefficient in a simple linear regression model. As in Sen and Ghosh (1974), the underlying principle essentially germinates in the work of Bartlett (1946) and Cox (1963) on asymptotic sequential likelihood ratio tests (SLRT). The proposed sequential rank order test (SROT) terminates with probability one under fairly general regularity conditions. The asymptotic OC and ASN functions of the test are derived under stronger conditions. In this context, certain almost sure convergence results on linear rank statistics and the derived estimates of regression coefficients, established earlier by Ghosh and Sen (1972) and Sen and Ghosh (1972), are used. Finally, the asymptotic ASN of the proposed SROT is compared with the corresponding ones of SLRT or the Wald sequential probability ratio test (SPRT) with the same prescribed levels of significance, and consequently, having the same limiting OC functions.

The SROT is motivated and proposed in the next section, followed by its termination proof in Section 3. The asymptotic OC and ASN expressions are derived respectively in Sections 4 and 5, while the asymptotic relative efficiency (ARE) results of the proposed SROT's as compared with other sequential tests are given in Section 6.

2. THE PROPOSED SROT

Consider the usual simple regression model

\[ x_i = \beta_0 + \beta c_i + \epsilon_i, \ i \geq 1, \ \bar{\epsilon} = (\bar{\epsilon}_0, \bar{\epsilon})', \]  

(2.1)
where the $c_i$ are known regression constants, $\beta$ is an unknown parameter (vector) and the $\varepsilon_i$ are independent and identically distributed random variables (i.i.d.r.v.) with an (unknown) distribution function (df) $F(x)$ absolutely continuous (with respect to Lebesgue measure) having pdf $f(x)$, $x \in E_1$, the Euclidean space. We are interested in testing sequentially

$$H_0: \beta = 0 \text{ Vs. } H_1: \beta = \Delta (>0), \text{ where } \Delta \text{ is known.} \quad (2.2)$$

[In (2.2), we could have framed $H_0: \beta = \beta^0 \text{ vs. } H_1: \beta = \beta^0 + \Delta$, for some known $\beta^0$.]

However, working with $X_i = X_i - \beta^0 c_i$, $i \geq 1$, one can always reduce the hypothesis to (2.2.)] The usual rank order statistic (used for the corresponding non-sequential testing problem) based on a sample of size $n (>1)$ is given by

$$T_n = \sum_{i=1}^{\infty} c_i \mathcal{J}_{\ni} ((n+1) \mathcal{R}_{\ni}), \quad c_i = \frac{(c_i - \bar{c})}{c_\bar{c}}, \quad i = 1, \ldots, n, \quad (2.3)$$

where $\bar{c}_n = n^{-1} \sum_{i=1}^{n} c_i$, $c_{\bar{c}} = \frac{1}{n} \sum_{i=1}^{n} (c_i - \bar{c})^2$, $\mathcal{R}_{\ni} = \frac{1}{n} \sum_{i=1}^{n} u(x_i - \bar{x}_i)$ ($u(t)$ being 1 or 0 according as $t \geq 0$ or $t < 0$) is the rank of $x_i$ among $X_1, \ldots, X_n$ ($1 \leq i \leq n$) and

$J_n (i/(n+1)) = EJ(u_{\ni}), \quad 1 \leq i \leq n$ where $J(u), 0 < u < 1$, is a suitable score function and

$U_{1 \leq i \leq n}$ are the order statistics in a random sample of size $n$ from a rectangular $(0,1)$ df. [The definition of $J_n (u)$ is extended over $u \in (0,1)$ by letting $J_n (u) = J_n (i/(n+1))$ for $i/(n+1) < u < i/n$, $1 \leq i \leq n$.] Further, on replacing

$X_n = (X_1, \ldots, X_n)$ by $X_{n-bc_i} = (X_1 - bc_1, \ldots, X_n - bc_n)$ in (2.3) [i.e., using the rank $r_{\ni}(b)$ of $X_i - bc_i$ among $X_1 - bc_1, \ldots, X_n - bc_n$], we define the corresponding rank order statistic as

$$T_n (b) = T(x_{n-bc_i}^*) = \sum_{i=1}^{n} c_i \mathcal{J}_{\ni} ((n+1) \mathcal{R}_{\ni}(b)), \quad -\infty \leq b < \infty, \quad (2.4)$$

Note that $T_n (b)$ is $+$ in $b$ [see, for example, Sen (1969)]. The point estimator

$\hat{\beta}_n$ of $\beta$ based on $T_n (b)$ may now be introduced [along the lines of Adichie (1967)] as follows:
\[ \hat{\beta}_n = \frac{1}{2}(\hat{\beta}_{n,1} + \hat{\beta}_{n,2}), \quad n \geq 1, \]  

(2.5)

where \( \hat{\beta}_{n,1} = \sup\{b: T_n(b) > 0\} \) and \( \hat{\beta}_{n,2} = \inf\{b: T_n(b) < 0\} \).

Suppose now that \( \hat{\beta}_n^* \) is the maximum likelihood estimator (m.l.e.) of \( \beta \) when \( F \) is specified, and let

\[ I(f) = \int_{-\infty}^{\infty} [(f'(x)/f(x))^2] dF(x) \]  

(2.6)

be the Fisher information, which we assume to be finite. Then the Bartlett-Cox SLRT rests on a test statistic which at the \( n \)th stage reduces to

\[ \Delta c^2 I(f)[\hat{\beta}_n^* - \frac{1}{2} \Delta]. \]  

(2.7)

Their procedure essentially uses three observations:

(i) the likelihood ratio process is asymptotically linear in the parametric variation in the locality of the true parameter point,

(ii) the likelihood ratio sequence is attracted by a Wiener process, and

(iii) the variance of the asymptotic distribution of \( C_n[\hat{\beta}_n - \beta] \) is \([I(f)]^{-1}\).

If we define

\[ v^2 = \int J^2(u) du - \left( \int J(u) du \right)^2, \]  

(2.8)

and we consider the sequence

\[ \Delta c^2 D^2(F)[\hat{\beta}_n - \frac{1}{2} \Delta] / v^2, \quad n \geq 1, \]  

(2.9)

where \( D(F) \) is defined by

\[ D(F) = \int_{-\infty}^{\infty} \{(d/dx)J(F(x))\} dF(x) \]  

(> 0),

(2.10)

then by virtue of the recent results of Ghosh and Sen (1972) and Sen and Ghosh (1972), it follows that \( \{T_n(b)\} \) also satisfies (i) and (ii) for \( b \) "close to" 0 and the variance of the asymptotic distribution of \( C_n[\hat{\beta}_n - \beta] \) is \( v^2/D^2(F) \). As such,
one is naturally inclined to the replacement of (2.7) [via (2.9)] by the following sequence:

\[ \Delta_n D(F) T_n (\frac{1}{2} \Delta) / \nu^2, \quad n \geq 1, \]  

(2.11)

where by definition of \( \hat{\beta}_n \) in (2.5) and by Theorem 3.2 of Ghosh and Sen (1972), we replace [in (2.9)] \( C_n D(F) [\hat{\beta}_n - \frac{1}{2} \Delta] \) by \( T_n (\Delta / 2) \). Since the df \( F \), and hence, the functional \( D(F) \) is unknown, we proceed as in Ghosh and Sen (1972) to provide a strongly consistent estimator \( D_n \) of \( D(F) \). For this, we define

\[ \hat{\beta}_{L,n} = \sup \{ b : T_n (b) > T_n^{(1)} \} \quad \text{and} \quad \hat{\beta}_{U,n} = \inf \{ b : T_n (b) < - T_n^{(1)} \}, \]  

(2.12)

where \( T_n^{(1)} \) satisfies \( P \{ |T_n| \leq T_n^{(1)} | \beta = 0 \} = 1 - \alpha_n \) (\( \alpha_n \) specified and \( \Rightarrow \alpha \), some specified \( 0 < \alpha < 1 \), as \( n \to \infty \)). Then, by Lemma 4.2 of Sen and Ghosh (1972), we have

\[ D_n = \frac{2T_n^{(1)}}{(\hat{\beta}_{U,n} - \hat{\beta}_{L,n})} \xrightarrow{a.s.} D(F), \]  

(2.13)

as \( n \to \infty \).

Our proposed SROT can now be formulated as follows:

For prescribed type I and type II errors \( \alpha_1, \alpha_2 \) (\( 0 < \alpha_1, \alpha_2 < 1 \)), consider two numbers \( A(1 - \alpha_2) / \alpha_1 \) and \( B(\alpha_2 / (1 - \alpha_1)) \), so that \( 0 < B < 1 < A < \infty \). Then, we start with an initial sample of size \( n_o (\Delta) \) (moderately large for small \( \Delta \)), and define a stopping variable \( N(= N(\Delta)) \) as the smallest positive integer \( n \geq n_o (\Delta) \) for which the following inequality is violated:

\[ b \nu^2 < \Delta_n D_n T_n (\frac{1}{2} \Delta) < a \nu^2, \quad (b = \log B, \ a = \log A). \]  

(2.14)

If \( N(\Delta) \leq n \) and \( \Delta_n D_n T_n (\frac{1}{2} \Delta) \) is < \( b \nu^2 \) (or \( \geq a \nu^2 \)), we accept \( H_0 \) (or \( H_1 \)). If the process does not terminate, we let \( N = \infty \).
3. TERMINATION OF SROT WITH PROBABILITY ONE

For the main result of this section, we assume that

(i) $J(u)$ is strictly $\uparrow$ in $u \in (0,1)$, $\int_0^1 |J(u)|^{2+\delta} du < \infty$ for some $\delta > 0$, \hspace{1cm} (3.1)

(ii) $\int_0^1 [J_n(u) - J(u)]^2 du + o(n^{-\frac{1}{2}})$ as $n \to \infty$, \hspace{1cm} (3.2)

(iii) $\max_{1 \leq i \leq n} |c_{ni}^*| = o(n^{-\frac{1}{2}})$, \hspace{1cm} (3.3)

(iv) $\lim_{n \to \infty} n^{-1} c_n^2 \geq c_0^2$, $0 < c_0^2 < \infty$, \hspace{1cm} (3.4)

(v) $\lim_{h \to 0} \sup_{-\infty < x < \infty} \{ \int_0^\infty J'(F(x+h_i)) f(x+h_i) dF(x) = D(F) > 0, \} \hspace{1cm} (3.5)$

where $D(F)$ is defined by (2.10).

Theorem 3.1. Under (3.1) through (3.5), for every (fixed) $\beta$, the SROT terminates with probability one, that is,

$P_\beta \{ N(\Delta) > n \} \to 0$ as $n \to \infty$. \hspace{1cm} (3.6)

Proof. Note that by (2.14), for every fixed $\beta$,

$P_\beta \{ N(\Delta) > n \} \leq P_\beta \{ bv^2 D_n^{-1} < \Delta^2 n^{-1} < av^2 D_n^{-1} \}$. \hspace{1cm} (3.7)

When $\beta = \frac{1}{2} \lambda$, the right hand side of (3.7) can be written as $P_0 \{ bv^2 D_n^{-1} < \Delta^2 n^{-1} < av^2 D_n^{-1} \}$. Since (3.1)-(3.3) imply (i) the square integrability of $J$ and (ii) $\max_{1 \leq i \leq n} |c_{ni}^*| = o(n^{-\frac{1}{2}})$ as $n \to \infty$, by Theorem a (on page 160) of Hájek and Šidák (1967), $T_n(0)$ is asymptotically normal with mean $0$ and variance $v^2$. Also, from the results of Jurečková (1969), it follows that under conditions less restrictive than (3.1)-(3.4), $D_n \sim D(F)$, in probability as $n \to \infty$. Finally, by (3.4), for every fixed $\Delta > 0$, $\lim_{n \to \infty} (\Delta^2 n^{-1}) = \infty$. Consequently, for $\beta = \frac{1}{2} \lambda$, the right hand side of (3.7) converges to $0$ as $n \to \infty$, i.e., (3.6) holds. For $\beta \neq \frac{1}{2} \lambda$, we make use of the
following representation of (3.7) where we let $T_n^*(b) = n^{-1} \nu_n^{-1} T_n(b)$ and
\[ \nu_n = n^{-1} \sum_{i=1}^{n} i/(n+1) = [n^{-1} \sum_{i=1}^{n} i/(n+1)]^2. \]
Thus,
\[ P(B/(\nu_n D_n n^{-1} \Delta C_n) < T_n^*(\beta \Delta) < (\nu_n D_n n^{-1} \Delta C_n)) ] 
\[ = P_0((b v^2)/(\nu_n D_n n^{-1} \Delta C_n) < T_n^*(\delta \Delta - \beta) < (b v^2)/(\nu_n D_n n^{-1} \Delta C_n)) \] (3.8)

Since $D_n \rightarrow D(F) > 0$, in probability, as $n \rightarrow \infty$ and $\nu_n \rightarrow \nu$, as $n \rightarrow \infty$, by (3.4), for every $\Delta > 0$,
\[ \lim_{n \rightarrow \infty} [(b v^2)/(\nu_n D_n n^{-1} \Delta C_n)] = 0, \quad \lim_{n \rightarrow \infty} [(a v^2)/(\nu_n D_n n^{-1} \Delta C_n)] = 0, \] (3.9)

and hence, to prove (3.6) for $\beta \neq \Delta$, it suffices to show that there exists a sequence $\{\tau_n^*(b)\}$ of normalizing constants, such that for every $b < 0$ (or $> 0$),
\[ |T_n^*(b) - \tau_n^*(b)| \rightarrow 0, \text{ in probability, and } \lim \inf_n \tau_n^*(b) > 0 \text{ (or } \lim \sup_n \tau_n^*(b) < 0). \]
Towards this end, we consider first two lemmas. Following Sen and Ghosh (1973) we let
\[ \tau_n^*(b) = n^{-1} \nu_n^{-1} \sum_{i=1}^{n} c_i \int_{-\infty}^{\infty} J(F_{n,1}^{-1}(x))dF_{n,1}^{-1}(x), \] (3.10)
where $F_{n,1}^{-1}(x) = F(x - \alpha + \epsilon c_1)$, $1 \leq i \leq n$ and $F_{n,1}^{-1} = n^{-1} \sum_{i=1}^{n} i c_i$. Also, note that (3.1) and (3.2) imply that

(i) $J(u)$ is of bounded variation in closed subintervals of $(0,1)$ and
\[ \int_0^1 |J(u)|du < \infty, \text{ and } \] (3.11)

(ii) $\int_0^1 |J_n(u) - J(u)|du \rightarrow 0$ as $n \rightarrow \infty$. (3.12)

Lemma 3.2. Under (3.3), (3.11) and (3.12), for every (fixed) $b$,
\[ T_n^*(b) - \tau_n^*(b) \rightarrow 0 \text{ a.s., as } n \rightarrow \infty. \] (3.13)
Proof. One can write $T^*_n(b) = \int_{-\infty}^{\infty} J_n(S_{n,b}(x))dS^*_n,b(x)$, where $S_{n,b}(x) = \sum_{i=1}^{n-1} \frac{1}{n} (x+bc_i - x_i)$ and $S^*_n,b(x) = \sum_{i=1}^{n} \frac{1}{n} c^*_i u(x+bc_i - x_i)$, $n \geq 1$, $-\infty < b < \infty$.

Since $\int_0^1 J(u)du = \int_0^1 J_n(u)du$, by (3.12), we claim that for every $\delta > 0$, there exists a $K_\delta(0 < K_\delta < \infty)$, such that on defining

$$J^*_n\left(\frac{i}{n+1}\right) = \begin{cases} J_n\left(\frac{i}{n+1}\right), & \text{if } |J_n\left(\frac{i}{n+1}\right)| \leq K_\delta, \\ 0, & \text{otherwise,} \end{cases}$$

we have

$$\int_0^1 |J^*(u) - J(u)| du < \delta,$$

and

$$n^{-1} \sum_{i=1}^{n} |J^*_n(i/(n+1)) - J_n(i/(n+1))| < \delta.$$  \hspace{1cm} (3.17)

Consequently, to prove the lemma, it suffices to consider the case where $J$ has bounded variation over all $(0,1)$. Note then,

$$T^*_n(b) - \tau^*_n(b) = \int_{-\infty}^{\infty} J_n(S_{n,b}(x))dS^*_n,b(x) - \int_{-\infty}^{\infty} J_n(\tilde{F}_{n,b}(x))d\tilde{F}^*_n,b(x)$$

$$= \int_0^1 J_n(u)dG^*_n,b(u) - \int_0^1 J_n(u)dH^*_n,b(u)$$

$$= \int_0^1 J_n(u)d[G^*_n,b(u) - H^*_n,b(u)] + \int_0^1 [J_n(u) - J(u)]dH^*_n,b(u),$$

where $G^*_n,b(u) = S^*_n,b(S^{-1}_{n,b}(u))$, $H^*_n,b(u) = F^*_n,b(F^{-1}_{n,b}(u))$, $0 < u < 1$, and $F^*_n,b(x) = \sum_{i=1}^{n-1} \frac{1}{n} c^*_i F_{i,b}(x)$. Thus, by definition of $H^*_n,b(u)$,
\[ \begin{align*}
&\frac{1}{n} \sum_{i=1}^{n} |c_{n1}^*| \int_{0}^{1} |J_n(u) - J(u)| \, F_{n1, b}(u) \, dF_{n1, b}(u) \\
&\leq \nu_n^{-1} \max_{1 \leq i \leq n} |c_{n1}^*| \int_{0}^{1} |J_n(u) - J(u)| \, dF_{n1, b}(u) \\
&= \nu_n^{-1} \max_{1 \leq i \leq n} |c_{n1}^*| \int_{0}^{1} |J_n(u) - J(u)| \, du \\
&\to 0, \text{ as } n \to \infty, \text{ by (3.3) and (3.12).}
\end{align*} \quad (3.19)\]

For the first term on the right hand side of (3.18), we note that for every \(\eta \in (0, \eta_n \leq 1)\),
\[|\int_{0}^{\eta} J_n(u) \, dG_{n1, b}(u)| = |\int_{0}^{\eta} G_{n1, b}(u) \, dJ_n(u)| \leq \nu_n^{-1} \max_{1 \leq i \leq n} |c_{n1}^*| \cdot \]
\[|\int_{1-\eta}^{1} J_n(u) \, dG_{n1, b}(u)| \leq 2\nu_n^{-1} \eta \text{ for } n \geq n_0, \text{ where } V \text{ stands for the variation of } J_n \text{ over } (0,1). \]
Similarly, \[|\int_{0}^{\eta} J_n(u) \, dH_{n1, b}(u)| \leq 2\nu_n^{-1} \eta \text{ for } n \geq n_0.
\]
Similar bounds also hold for \[|\int_{0}^{\eta} J_n(u) \, dH_{n1, b}(u)| \text{ and } |\int_{1-\eta}^{1} J_n(u) \, dH_{n1, b}(u)|.
\]
Further, by standard steps,
\[|\int_{0}^{1-\eta} J_n(u) \, d[G_{n1, b}(u) - H_{n1, b}(u)]| \leq V \sup_{n \leq u \leq 1-\eta} |G_{n1, b}(u) - H_{n1, b}(u)|. \quad (3.20)\]

Now, for \(\eta \in (\eta_n, 1-\eta)\) (i.e., being bounded away from 0 and 1), as \(n \to \infty\),
\[|S_{n, b}(u) - F_{n, b}(u)| \to 0 \text{ a.s.}, \text{ and hence, to show that the right hand side of (3.20) converges a.s. to 0 as } n \to \infty, \text{ it suffices to show that}
\]
\[\sup_{x} |S_{n, b}(x) - F_{n, b}(x)| \to 0, \text{ a.s., as } n \to \infty. \quad (3.21)\]

For this, we note that
\[S_{n, b}(x) - F_{n, b}(x) = \nu_n^{-1} \sum_{i=1}^{n} c_{n1}^* [u(x+bc_1 - \chi_i) - F_{n1, b}(x)] \]
\[= \nu_n^{-1} \sum_{i=1}^{n} [u(x+bc_1 - \chi_i) - F(x - \alpha + bc_1)] + Q_{n1}(x) + Q_{n2}(x), \quad (3.22)\]
where $S_{n1} = \{ i: c_{ni}^* \geq 0, 1 \leq i \leq n \}$ and $S_{n2} = \{ i: c_{ni}^* < 0, 1 \leq i \leq n \}$. Let us now choose $\{ \eta_{i,n}, i=0, \ldots, n \}$ by letting $\bar{F}_{n,b}(\eta_{i,n}) = i/n, 1 \leq i \leq n-1, \eta_{0,n} = -\infty$ and $\eta_{n,n} = +\infty$. Then, for every $x \in [\eta_{j-1,n}, \eta_{j,n}]$, by (3.22),

$$
\sum_{n1} c_{ni}^* [u(\eta_{j-1,n} + bc_i - x)] - F(\eta_{j,n} - \alpha + bc_i) \leq \sum_{n1} c_{ni}^* [u(\eta_{j,n} + bc_i - x)] - F(\eta_{j-1,n} - \alpha + bc_i),
$$

so that by a few standard steps we get that

$$
\sup_{-\infty < x < \infty} |Q_{n1}(x)| \leq \max_{1 \leq j \leq n} |Q_{n1}(\eta_{j,n})| + \frac{1}{n} \sum_{n1} [F(\eta_{j,n} - \alpha + bc_i) - F(\eta_{j-1,n} - \alpha + bc_i)] c_{ni}^*
$$

$$
\leq \left( \max_{1 \leq j \leq n} |Q_{n1}(\eta_{j,n})| \right) + \left( \max_{1 \leq i \leq n} |c_{ni}^*| \right) \frac{1}{n} \left( \sum_{n1} [F_{n,b}(\eta_{j,n}) - F_{n,b}(\eta_{j-1,n})] \right)
$$

$$
= \left( \max_{1 \leq j \leq n} |Q_{n1}(\eta_{j,n})| \right) + O(n^{-1}), \quad (3.23)
$$

by (3.3) and the choice of the $\{ \eta_{j,n} \}$. Finally, by (3.3), $n^k |c_{ni}^*|, i=1, \ldots, n$ are all bounded, so that $Q_{n1}(\eta_{j,n})$ involves a sum of independent bounded random variables (with zero expectations) on which using the exponential bounds of Hoeffding (1963), we have

$$
P\{ |Q_{n1}(\eta_{j,n})| > \varepsilon \text{ for at least one } j: 1 \leq j \leq n \}
$$

$$
\leq \sum_{j=1}^n P\{ |Q_{n1}(\eta_{j,n})| > \varepsilon \}
$$

$$
\leq n \exp\{-n\varepsilon^2/4K^2\}, \quad \varepsilon > 0, \quad (3.24)
$$

where $K = \sup\{ n^k \max_{1 \leq i \leq n} |c_{ni}^*| \} (\ll \infty)$. Using the Borel-Cantelli lemma and the fact that $\sum_{n=1}^\infty n \exp\{-cn\} < \infty, \forall c > 0$, we conclude from (3.23) and (3.24) that

$$
\sup_x |Q_{n1}(x)| \to 0 \text{ a.s., as } n \to \infty. \quad \text{A similar treatment holds for } \sup_x |Q_{n2}(x)|. \quad \text{Hence the proof of the lemma is complete.}
$$

Remark: For $J(u)$ not necessarily of bounded variation on $(0,1)$ but admitting
moments up to the order $2+\delta$, $\delta>0$, (3.13) has been proved in Theorem 1 of Sen and Ghosh (1972); the present lemma is a direct extension of a two-sample result of Hájek (1974) to the general regression problem. Since, in this paper, we assume $J(u)$ to be monotonic, the present lemma has its own flavor and relevance to our study.

Lemma 3.3. $\tau^*_n(b) \overset{d}{=} b(\cdots < b(\cdots)\cdots)$; $\tau^*_n(0) = 0$.

Proof. By definition in (3.10), $\tau^*_n(0) = n^{-1/2} \sum_{i=1}^n c_i \int_0^1 J(u) du = 0$. We prove that for any $b_1 < b_2 < 0$, $\tau^*_n(b_1) \geq \tau^*_n(b_2) > 0$ a similar proof holds for $0 \leq b_2 < b_1$. By definition in (3.10), it suffices to show that for $b_1 < b_2 < 0$,

$$\sum_{i=1}^n c_i \int_{-\infty}^\infty J(F_n, b_1(x)) dF_i(x) \geq \sum_{i=1}^n c_i \int_{-\infty}^\infty J(F_n, b_2(x)) dF_i(x), \quad (3.25)$$

where, without any loss of generality, we may assume that $c_1 \leq \ldots \leq c_n$. If we let $L_n, i(b) = \int_{-\infty}^\infty J(F_n, b(x)) dF_i(x) = \int_{-\infty}^\infty J(n^{-1/2} \sum_{j=1}^n F(x-\theta_j b(c_j-c_i))) dF(x)$, then we have $\tau^*_n(b) = \sum_{i=1}^n (c_i - \bar{c}_n) L_n, i(b)$. Consider now $b_1 < b_2 < 0$ and $k < \ell$. Then $[L_n, k(b_1) < L_n, \ell(b_1)] \Rightarrow [c_k > c_\ell] \Rightarrow [L_n, k(b_2) < L_n, \ell(b_2)]$, in view of the strict monotonicity of $J$ and $F$. This means that the permutation $[L_n, k(b_1), \ldots, L_n, n(b_n)]$ is better ordered than the permutation $[L_n, 1(b_1), \ldots, L_n, n(b_n)]$ in the sense of Lehmann (1966). Therefore, the lemma follows by using Corollary 2 to Theorem 5 of Lehmann (1966).

Returning now to the proof of Theorem 3.1, we note that for every (fixed) $\beta < 2\Delta$, and for every $n > 0$, under (3.4), there exist an $n_0$ and a $K(\infty)$, such that

$$b = \frac{1}{2} \Delta - \beta > n K(n^2 C_n^{-1})$$

for $n \geq n_0$. \quad (3.26)
Therefore, by Lemma 3.3, for \( \beta < \Delta, \ n > n_0 \),

\[
\tau^*_n(b) \leq \tau^*_n(\frac{\eta \ln^{\frac{1}{2}} c^{-1}_n}{n}) \cdot \frac{1}{\ln^{\frac{1}{2}} c^{-1}_n} \cdot \sum_{i=1}^{\infty} c^{*}_{ni} \cdot \int_{-\infty}^{\infty} J(n^{-1} \sum_{i=1}^{n} F(x-\eta \ln^{\frac{1}{2}} c^{-1}_n(c_i-c_j))dF(x),}
\]

whereby (3.3) and (3.5), the right hand side of (3.27) converges to

\[
-K\eta^{-1} \sum_{i=1}^{\infty} (c^{*}_{ni})^2 \int_{-\infty}^{\infty} J'(F(x))F(x)dF(x) = -K\eta^{-1} D(F) < 0.
\]

Hence, for every \( \beta < \Delta \), \( \limsup_{n \to \infty} \tau^*_n(\beta^\Delta - \beta) < 0 \). A similar proof holds for \( \beta > \Delta \).

Hence the proof of the theorem is complete.

4. THE OC FUNCTION OF THE SROT

For the sake of theoretical justification, in the sequel, we let \( \Delta \to 0 \); the asymptotic results should work out well when, in practice, \( \Delta \) is small. Also, for \( \Delta \to 0 \), our range of interest on \( \beta \) contracts to 0. Specifically, we assume that for some \( K > 1 \),

\[
\beta \in \mathcal{I}(\Delta) = \{ \beta = \phi \Delta : |\phi| \leq K \}.
\]

[It may be added that for every fixed \( \beta < 0 \) (or \( > 0 \)), the OC approaches 0 (or 1) as \( \Delta \to 0 \).] Also, we assume that the initial sample size \( n_0 = n_0(\Delta) \) satisfies the following:

\[
\lim_{\Delta \to 0} \frac{c^2}{n_0(\Delta)} = \infty \text{ but } \lim_{\Delta \to 0} \frac{\Delta^2 c^2}{n_0(\Delta)} = 0.
\]

Further, we denote by \( J_F(\phi, \Delta) \) the OC function of the SROT based on \( N(\Delta) \) of Section 2 when \( \beta = \phi \Delta \), \( J \) is the score function and \( F \) is the underlying df. Finally, for the OC, we assume that the score function \( J \) satisfies the following conditions:
J(u) = \Psi^{-1}(u), \ 0 < u < 1, \text{ where } \Psi(x) \text{ is an absolutely continuous df with a}
pdf } \psi \text{ for which}

\lim_{u \to 0, 1} \frac{\psi(\Psi^{-1}(u))}{u(1-u)} \geq K \text{ for some } 0 < K < \infty. \tag{4.3}

The above condition implies that J(u) is \dagger in u(0 < u < 1) and

\left| J(u) \right| \leq K_0 \{-\log u(1-u)\}, \quad \left| J'(u) \right| \leq K_0 [u(1-u)]^{-1}, \quad 0 < u < 1, \tag{4.4}

for some } 0 < K_0 < \infty, \text{ and there exists a } t_0(>0), \text{ such that}

M(t) = \int_{-\infty}^{\infty} \exp(tx) d\psi(x) < \infty, \quad \forall \left| t \right| \leq t_0. \tag{4.5}

**Theorem 4.1.** Under (3.2)-(3.5), (4.1)-(4.3) and the assumption that

\limsup_{n \to \infty} n^{-h} C_n^2 \leq C* < \infty \text{ for some } h(>1), \tag{4.6}

we have

\lim_{\Delta \to 0} \Pi_{J,F}(\phi, \Delta) = \begin{cases} (A_1^{-1} - 2\phi_1 - 1)/(A_1^{-1} - 2\phi_1 - B_1^{-1} - 2\phi_1), & \phi \neq \phi_1, \\ a/(a-b), & \phi = \phi_1. \end{cases} \tag{4.7}

**Remark.** (4.7) shows that SROT has the same asymptotic OC function as that of
the SPRT. Also, this is asymptotically true for all J satisfying (4.3) and
absolutely continuous F satisfying (3.5). In particular for \phi = 0, 1, (4.7)
reduces to } \alpha_1, \alpha_2, \text{ so that the SROT is "asymptotically consistent."

**Proof of the Theorem.** Note that, by definition,

\Pi_{J,F}(\phi, \Delta) = P_{\phi} \left\{ \Delta C_{m \Delta} (3 \Delta) \leq b v^2 / D_m \text{ for some } m \geq n_0(\Delta) \right\} \text{ before }

\Delta C_{m \Delta} (3 \Delta) \geq a v^2 / D_m \text{ for some } m \geq n_0(\Delta) \right\}

= P_{\phi} \{ \text{as above and } |D(F)/D_m - 1| \leq \eta \text{ for every } m \geq n_0(\Delta) \} \tag{4.8}

+ P_{\phi} \{ \text{as above and } |D(F)/D_m - 1| > \eta \text{ for some } m \geq n_0(\Delta) \},
where \( \eta(>0) \) is arbitrary. By Lemma 4.2 of Ghosh and Sen (1972), for every \( s>0 \), there exist positive constants \( K_S^{(1)}, K_S^{(2)} \) and a sample size \( n_s \), such that for \( n \geq n_s \):

\[
P(|D(F)/D_m - 1| > K_S^{(1)} \eta^{-\delta (\log n)^2}) \leq K_S^{(2)} n^{-s}, \delta > 0.
\]

By (4.9) and some standard steps, we claim that the last term on the right hand side of (4.8) converges to 0 as \( \Delta \to 0 \); (we let here \( s>1 \)). Let then, for every \( \eta > 0 \),

\[
a = \log A, \ b = \log B
\]

and on replacing in (2.14) \( a, b \) and \( D_n \) by \( a, b, \eta, i \), \( b, \eta, j \) and \( D(F) \), respectively, we define the stopping variable \( N^{(1, j)}(\eta) \), for \( i, j = 1, 2 \); let the corresponding OC functions be denoted by \( \Pi_{J,F}^{(i,j)}(\phi, \Delta) \), \( i, j = 1, 2 \). Then, by (4.8), (4.9) and the above discussion, it follows that for every \( \varepsilon > 0 \), there exists a \( \Delta \varepsilon > 0 \), such that

\[
\Pi_{J,F}^{(2, 1)}(\phi, \Delta - \varepsilon) \leq \Pi_{J,F}^{(i, j)}(\phi, \Delta) \leq \Pi_{J,F}^{(1, 2)}(\phi, \Delta) + \varepsilon, \forall \Delta < \Delta \varepsilon, \phi \in I.
\]

Next we show that for each \( i, j = 1, 2 \),

\[
\lim_{\Delta \to 0} \Pi_{J,F}^{(i, j)}(\phi, \Delta) = \begin{cases} 
(1-2\phi)a_{\eta, i-1}/(1-2\phi)a_{\eta, i-1} + (1-2\phi)b_{\eta, j} \\
(1-2\phi)a_{\eta, i}/(1-2\phi)b_{\eta, j}, \phi \neq \frac{1}{2} \\
\eta_i/\phi \eta_j, \phi = \frac{1}{2}.
\end{cases}
\]

The desired result then follows from the continuity of the OC function by letting \( \eta \to 0 \). For simplicity, we consider only the case of \( i = j = 0 \).

For a given \( \varepsilon > 0 \), let \( K_\varepsilon = (a-b)\varepsilon/\varepsilon \), and let

\[
n^* = n^*(\Delta, \varepsilon) + 1 = \inf\{n: \Delta^2 C^2_n > K_\varepsilon\}.
\]

Note that for a fixed \( \varepsilon(>0) \), as \( \Delta \to 0 \), \( n^*(\Delta, \varepsilon) \to \infty \), and \( [T_{n^*}(\varepsilon \Delta) - E_{\phi}(T_{n^*}(\varepsilon \Delta))] \) is asymptotically \( N(0, \nu^2) \), \( \forall \phi \in I \). Consequently, on proceeding as in (3.7) and (3.8),
with $D_n$ replaced by $D(F)$, it follows that $P_\phi \{ N_0(\Delta) > n^* \} < \varepsilon'$ (as $\Delta \to 0$), where $N_0(\Delta) = N(0,0)(\Delta)$ and $\varepsilon' \to 0$ as $\varepsilon \to 0$. On the other hand, for $n_0(\Delta) \leq n \leq n^*(\Delta, \varepsilon)$, by Theorem 3.2 of Ghosh and Sen (1972), as $\Delta \to 0$,

\[
\max_{n_0 < n < n^*} \left| T_n(\varepsilon^2 \Delta) - T_n(\phi \Delta) - (\phi^{-2})\Delta C_n D(F) \right| \to 0 \ a.s. \quad (4.14)
\]

Consequently, we claim that as $\Delta \to 0$,

\[
T_{N_0}(\Delta)^{(\varepsilon^2 \Delta)} - T_{N_0}(\Delta)^{(\phi \Delta)} - (\phi^{-2})\Delta C_{N_0}(\Delta) D(F) \to 0, \text{ in probability.} \quad (4.15)
\]

Further, under $P_\phi$, $T_n(\phi \Delta)$ has the same distribution as that of $T_n(0)$ under $P_0$.

Finally, from Theorem 1.2 of Sen and Ghosh (1972), we assert that under $P_0$,

\[
v^{-1}T_n(0) = W(n) + o(n^{\frac{1}{2}}) \ a.s., \text{ as } n \to \infty, \quad (4.16)
\]

where $W = \{ W(t), \ 0 \leq t < \infty \}$ is a standard Brownian motion on $[0, \infty)$. [Actually, in view of the stronger assumption (4.3), in (1.17) of Sen and Ghosh (1972), we can replace $o(t \log \log t)^{\frac{1}{2}}$ by $o(t^{\frac{1}{2}})$.] If $P(\phi, \gamma)$ is the probability that $\{ W_t, t > 0 \}$ first goes below the line by $\gamma^{-1} + (\varepsilon^2 - \phi)\gamma t$ for a $t$ smaller than any other $t$ for which it goes above $a\gamma^{-1} + (\varepsilon^2 - \phi)\gamma t$, then from the above we conclude that

\[
\lim_{\Delta \to 0} \Pi_{J,F}^{(0,0)}(\phi, \Delta) = \lim_{\gamma \to 0} P(\phi, \gamma) = P(\phi), \quad (4.17)
\]

where

\[
P(\phi) = \begin{cases} 
\frac{(\Lambda^{1-2\phi} - 1)}{(\Lambda^{1-2\phi} - 1 - 2\phi)}, & \phi \neq 0, \\
\frac{a}{(a-b)}, & \phi = 0
\end{cases} \quad (4.18)
\]

Hence, on replacing $a$ and $b$ in (4.18) by $\eta_{i,1}$ and $\eta_{i,j}$, respectively, for $i, j = 1, 2$, (4.12) follows by letting $\eta(>0)$ be arbitrarily small. Q.E.D.
5. THE ASN OF THE SROT

Unlike the one or two-sample situation [viz., Sen and Ghosh (1974)], here the performance of $T_n$ depends on $n$ through $C^2_n$, where by (3.4), $n^{-1}C^2_n$ is bounded from below (for large $n$) by a positive constant, but $n^{-1}C^2_n$ need not tend to a limit as $n \to \infty$. For this reason, it is possible to find an asymptotic value for $\Delta^2E_{\phi}C^2_{N(\Delta)}$ as $\Delta \to 0$, under the conditions of Section 4, but we need additional conditions to conclude on $\Delta^2E_{\phi}N(\Delta)$. As a result, we shall first consider the following theorem and then enlist additional results based on more stringent regularity conditions.

**Theorem 5.1.** Under (3.2)-(3.5), (4.1)-(4.3) and (4.6), for every $\phi \in I(\Delta)$,

$$\lim_{\Delta \to 0} \{\Delta^2E_{\phi}[C^2_{N(\Delta)}]\} = \psi(\phi, \tau), \quad (5.1)$$

where $E_{\phi}$ stands for the expectation under $\beta = \phi \Delta$,

$$\psi(\phi, \tau) = \begin{cases} \{a(1-P(\phi))+bP(\phi)\}[(\tau^2)/(\phi-\lambda_2)], & \phi \neq \lambda_2, \\ -\tau^2P'(\lambda_2)[(a-b)], & \phi = \lambda_2, \end{cases} \quad (5.2)$$

and $\tau^2 = \nu^2/p^2(\phi)$, $P(\phi)$ is defined by (4.18) and $P'(\lambda_2) = [(\partial/\partial \phi)P(\phi)]_{\lambda_2}$.

**Proof.** First, we prove (5.1) for $\phi \neq \lambda_2$. For some arbitrarily small (fixed) $\epsilon > 0$ and $k(>2)$, we let

$$n_1 = n_1(\epsilon, \lambda) = \inf\{n: C_n > \epsilon/\Delta\}, \quad n_2 = n_2(\Delta) = \inf\{n: C_n \geq \Delta^{-1}(-\log \Delta)^k\}. \quad (5.3)$$

Whenever there is no confusion, we omit the subscripts $J, F$ etc. in $N_{J, F}(\Delta)$, $\Pi_{J, F}(\psi, \lambda)$ etc. Then, we have

$$\Delta^2E_{\phi}[C^2_{N(\Delta)}] = \Delta^2\left[ \sum_{n < n_1} + \sum_{n_1 < n < n_2} + \sum_{n > n_2} C^2_{n, \phi}N(\Delta) = n \right], \quad (5.4)$$
where by (5.3), the first sum on the right hand side of (5.4) is $\leq \varepsilon$. Also,

$$
\sum_{n>n_2} C_n^2 P_{\phi}(N(\Delta)=n) = C_{n_2}^2 + \sum_{n>n_2} [C_n^2 - C_{n+1}^2] P_{\phi}(N(\Delta)>n).
$$

(5.5)

For $n \geq n_2$, by (4.9) and (5.3), for every $\eta>0$, $\phi\not\equiv 2$, we have

$$
P_{\phi}(N(\Delta)>n) \leq P_{\phi}(b\nu^2 < \Delta C_n T_n (\zeta_{2\Delta}) < a\nu^2)

\leq P_{\phi}(b\nu^2 (1+\eta)/D(F) < \Delta C_n T_n (\zeta_{2\Delta}) < a\nu^2 (1+\eta)/D(F)) + O(n^{-s})

(5.6)

= \Pr \left\{ \frac{b\nu^2 (1+\eta)}{D(F)\Delta n^2 C_n \nu} < \tau^*_{\phi}(\zeta_{2\Delta}) < \left[ \frac{\tau^*_{\phi}(\zeta_{2\Delta}) - \tau^*_{\phi}(\zeta_{\phi}\Delta)}{\Delta} \right] \right\} + O(n^{-s}),

$$

where we choose $s > h$, $h$ being defined by (4.6). Note that by (5.3) and the fact that $C_n/(\log C_n)^k$ is $\uparrow$ in $n$, we have for $n \geq n_2(\Delta)$, $\Delta$ sufficiently small,

$$
\Delta C_n \geq C_{\uparrow}(\log n)^k, \text{ where } 0 < C_{\uparrow} < \infty;
$$

(5.7)

$$
\left| \tau^*_{\phi}(\zeta_{2\Delta}) \right| \geq C_\phi \left| \phi_{-2}\right| (\log n)^k, \text{ } 0 < C_\phi < \infty.
$$

(5.8)

Since $n^{1/x} \rightarrow \infty$ as $n \rightarrow \infty$, from (5.6), (5.7) and (5.8), it follows that to show that the right hand side of (5.6) is $O(n^{-s})$, it is enough to prove the following.

Lemma 5.2. For every $s > 0$, there exist positive constants $K_{\uparrow}^{(1)}$, $K_{\downarrow}^{(2)}$ and an $n_0 = n_0(s)$, such that for $n \geq n_0$ and a given $b(-\infty, b) < \infty$,

$$
P\left( \left| \tau^*_{\uparrow}(b) - \tau^*_{\downarrow}(b) \right| > K_{\uparrow}^{(1)} n^{-s} (\log n)^2 \right) \leq K_{\downarrow}^{(2)} n^{-s},
$$

(5.9)

whenever (3.2) - (3.5), (4.3) and (4.6) hold.

Proof. As in the first line of (3.18),
\[ T^*(b) - \tau^*(b) = \int_{-\infty}^{\infty} [J_n(S_n, b(x)) - J(S_n, b(x))] dS^*_n, b(x) + \int_{-\infty}^{\infty} [J(S_n, b(x)) - J(\bar{F}_n, b(x))] dS^*_n, b(x) + \int_{-\infty}^{\infty} J(\bar{F}_n, b(x)) d[S^*_n, b(x) - F^*_n, b(x)] \]

\[ = I_1 + I_2 + I_3, \text{ say}, \quad (5.10) \]

where by (3.3) and the proof of Theorem 3.6.6 of Puri and Sen (1971, p. 409),

\[ |I_1| \leq \left\{ n^{-1/2} \max_{1 \leq i \leq n} \left| c_i^* \right| \right\} L_{i=1}^{\infty} J_n(i/(n+1)) - J(i/(n+1)) = o(n^{-1/2}). \]

Further, note that

\[ |dS^*_n, b(x)| \leq K_1 dS_n, b(x), K_1 < \infty, \text{ by (3.3)}, \quad (5.11) \]

and the same inequality holds for \((\bar{F}_n, b, F^*_n, b)\). Let us define then \(a_n\) and \(a_n^0\) by

\[ \bar{F}_n, b(a_n) = 1 - \bar{F}_n, b(a_n^0) = n^{-1/2} (\log n). \]

Then, by (4.3), \[ \int_{-\infty}^{a_n} J(\bar{F}_n, b(x)) dF^*_n, b(x) \leq K_1 \int_{-\infty}^{a_n} J(\bar{F}_n, b(x)) d\bar{F}_n, b(x) = 0(n^{-1/2} (\log n)^2). \]

Also, note that \[ \int_{-\infty}^{a_n} J(\bar{F}_n, b(x)) dS^*_n, b(x) \leq K_1 \int_{-\infty}^{a_n} J(\bar{F}_n, b(x)) dS_n, b(x) = Kn^{-1} \sum_{i=1}^{\infty} e_{ni} \text{ where } e_{ni} \]

\[ |J(\bar{F}_n, b(x_i-bc_i))|I(a_n-a_n^0, b, bc_i), \quad 1 \leq i \leq n. \]

As, such by a few standard steps, we obtain that \[ E[n^{-1} \sum_{i=1}^{\infty} e_{ni}] = \int_{-\infty}^{a_n} J(\bar{F}_n, b(x)) d\bar{F}_n, b(x) = 0(n^{-1/2} (\log n)^2) \]

and for \(k > 1\), \[ E[n^{-1} \sum_{i=1}^{\infty} e_{ni}]^2 k = 0(n^{-2k} [n^{-1/2} (\log n)^4]^k) = 0(n^{-3k/2} (\log n)^{4k}), \]

so that by the Markov inequality,

\[ P\{ |\int_{-\infty}^{a_n} J(\bar{F}_n, b(x)) dS^*_n, b(x) - \int_{-\infty}^{a_n} J(\bar{F}_n, b(x)) dF^*_n, b(x)| > n^{-1/2} (\log n)^2 \} \]

\[ \leq n^{-k} (\log n)^{-4k} 0(n^{-3k/2} (\log n)^{4k}) \]

\[ = o(n^{-k/2}) = o(n^{-s}), \text{ by letting } s \leq k/2. \quad (5.12) \]

A similar treatment holds for \[ \int_{-\infty}^{a_n} J(\bar{S}_n, b(x)) dS^*_n, b(x), \]
and also for the upper tail \((a_n^0, \infty)\), in each case. Thus, for both \(I_2\) and \(I_3\), we may consider the truncated range \((a_n, a_n^0)\), and neglect the tails with a probability \(> 1 - \text{const}(n^{-s})\).

Proceeding as in the proof of Lemma 3.2, it can be shown that
\[
\Pr(\sup_x |S_{n,b}(x) - F_{n,b}(x)| > K_S^{(1)} n^{-s}(\log n)) \leq K_S^{(2)} n^{-s},
\]
(5.13)

so that for \(a_n < x < a_n^o\), one has with a probability \(\geq 1 - O(n^{-s})\),
\[
|\int_{a_n}^{a_n^o} [J(S_{n,b}(x)) - J(F_{n,b}(x))] dS_{n,b}(x) - 1| \leq K_S^{(1)}(\log n).
\]
(5.14)

Consequently, by (5.13), (5.14) and (4.3), (5.11),
\[
\int_{a_n}^{a_n^o} \int_{a_n}^{a_n^o} [J(S_{n,b}(x)) - J(F_{n,b}(x))] dS_{n,b}(x) dS_{n,b}(y)
\leq \left\{ \sup_{a_n < x < a_n^o} |S_{n,b}(x) - F_{n,b}(x)| \right\} \int_{a_n}^{a_n^o} \int_{a_n}^{a_n^o} J'(\theta S_{n,b}(x) + (1-\theta) F_{n,b}(x)) dS_{n,b}(x) dS_{n,b}(y)
\leq 0(n^{-\frac{1}{2}} \log n) \cdot O(\log n)
\]
(5.15)

\[= 0(n^{-\frac{1}{2}}(\log n)^2), \text{ with probability } \geq 1 - \text{const.}(n^{-s}).\]

Finally, by partial integration,
\[
\int_{a_n}^{a_n^o} \int_{a_n}^{a_n^o} J(F_{n,b}(x)) d[S_{n,b}(x) - F_{n,b}(x)]
\leq \left\{ \sup_{a_n < x < a_n^o} |S_{n,b}(x) - F_{n,b}(x)| \right\} \int_{a_n}^{a_n^o} \int_{a_n}^{a_n^o} J(F_{n,b}(a_n)) - J(F_{n,b}(a_n^{-o})) dS_{n,b}(x) dS_{n,b}(y)
\]
(5.16)

where by the same technique as in the proof of Lemma 3.2, it follows that (5.13)
also holds for \(\{S_{n,b}(x) - F_{n,b}(x)\}\). Hence, the proof of the lemma is complete.

Returning now to the proof of Theorem 5.1, we note that by (3.3) and (4.6),
\[
(C_{n+1}^2 - C_n^2) = O(n^{-1}) \cdot C_n^2 = O(n^{-1}), \text{ so that in (5.6)-(5.9), on letting } s > h, \text{ we}
\]
note that \(\Lambda^2 \Sigma_{n>h} P_\phi \{N(\Delta > n) \in C_{n+1}^2 - C_n^2\} = \Lambda^2 O(1) \rightarrow 0 \text{ as } \Delta \rightarrow 0. \) Hence, it remains
to show that
\[ \lim_\Delta \Delta^2 \sum_{n_1 \leq n \leq n_2} \binom{\Delta^2}{n_1} \binom{\Delta^2}{n_2} \phi \{ N(\Delta) = n \} + \psi(\phi, \tau), \] (5.17)

and for this, it suffices to show that

\[ \lim_\Delta \Delta^2 \sum_{n_1 \leq n \leq n_2} [\binom{\Delta^2}{n_1+1} - \binom{\Delta^2}{n_2} \phi \{ N(\Delta) > n \} + \psi(\phi, \tau). \] (5.18)

Now, for \( n_1 \leq n \leq n_2 \) (i.e., \( \varepsilon \leq \Delta \leq (\log \frac{C_n}{n})^k = O(\log n)^k \)), Lemma 4.2 of Ghosh and Sen (1972) holds, so that \( P\{|D(F)/D_n -1| > \eta \text{ for some } n_1 \leq n \leq n_2 \} = \sum_{n=n_1}^{n_2} O(n^{-s}) = O([n_1(\Delta)]^{-s+1}) \to 0 \text{ as } \Delta \to 0. \) Also, \( \Delta^2 \sum_{n_1 \leq n \leq n_2} \binom{\Delta^2}{n_1+1} \binom{\Delta^2}{n_2} \phi \{ |D(F)/D_n -1| > \eta \text{ for some } n_1(\Delta) \leq n \leq n_2(\Delta) \} = [0(-\log \Delta)^k] \cdot O([n_1(\Delta)]^{-s+1}) \to 0 \text{ as } \Delta \to 0. \) Thus, we may write for some \( \Delta \to 0 \) and all \( 0 < \Delta \leq \Delta_0, \)

\[ \Delta^2 \sum_{n_1(\Delta)}^{n_2(\Delta)} \binom{\Delta^2}{n_1+1} \binom{\Delta^2}{n_2} \phi \{ \tilde{N}^{(1,1)}(\Delta) > n \} - \eta_\Delta \leq \Delta^2 \sum_{n_1(\Delta)}^{n_2(\Delta)} \binom{\Delta^2}{n_1+1} \binom{\Delta^2}{n_2} \phi \{ N(\Delta) > n \} \]

\[ \leq \Delta^2 \sum_{n_1(\Delta)}^{n_2(\Delta)} \binom{\Delta^2}{n_1+1} \binom{\Delta^2}{n_2} \phi \{ \tilde{N}^{(2,2)}(\Delta) > n \} + \eta_\Delta, \] (5.19)

where \( \eta_\Delta \to 0 \text{ as } \Delta \to 0. \) Using now Theorem 3.1 of Ghosh and Sen (1972), one gets again that

\[ \Delta^2 \sum_{n_1(\Delta)}^{n_2(\Delta)} \binom{\Delta^2}{n_1+1} \binom{\Delta^2}{n_2} \phi \{ \tilde{N}^{(1,1)}(\Delta) > n \} \]

\[ = \Delta^2 \sum_{n_1(\Delta)}^{n_2(\Delta)} \binom{\Delta^2}{n_1+1} \binom{\Delta^2}{n_2} \phi \{ \tilde{N}^{(1,1)}(\Delta) > n \} + O(1), \] (5.20)

where \( \tilde{N}^{(1,1)}(\Delta) \) is defined in the same way as \( N^{(1,1)}(\Delta) \) but with \( T_n(\Delta/2) \)

replaced by \( \tilde{Z}_n(\Delta) = T_n(\phi \Delta) + (\psi - \phi) A_C - D(F). \) Now, by steps similar to those in (5.6)-(5.8), but with added simplicity (because of the linear drift), it can be shown that

\[ P_{\phi} \{ \tilde{N}^{(1,1)}(\Delta) > n \} = O(n^{-s}), \text{ i.e. } n_1, n_2 \geq n_2(\Delta). \] (5.21)
Hence, it suffices to show that for every \( n > 0 \),

\[
\lim_{\Delta \to 0} \left| \Delta^2 \sum_{n=1}^{\infty} (C_{n+1} - C_n) P_\phi(\bar{N}(i,i)(\Delta) > n) - \psi(\phi, \tau) \right| < \epsilon, \tag{5.22}
\]

for \( i = 1, 2 \). To prove (5.22), it suffices to show that

\[
\lim_{\Delta \to 0} \left\{ \left| \Delta^2 \sum_{n=1}^{\infty} C_n^2 \phi(\bar{N}(i,i)(\Delta) = n) - \psi(\phi, \tau) \right| \right\} < \epsilon. \tag{5.23}
\]

Let now \( B_n^*(\beta) \) be the \( \sigma \)-field generated by \( R_{n1}(\beta), \ldots, R_{nn}(\beta), n \geq 1 \). From Lemma 2.1 of Sen and Ghosh (1972), it follows that for \( \beta = \phi \Delta, \{ C_n T_n(\phi \Delta), B_n^*(\phi \Delta); n \geq 1 \} \) is a martingale. Also, by definition, \( |C_n T_n(\phi \Delta)| \leq C_n \cdot n^\kappa \nu \leq C^\kappa n(1+\theta)/2 \nu \), for large \( n \). Hence, using power bounds for \( P(\bar{N}(i,i)(\Delta) > n) \) (as in earlier) and

choosing \( s \) appropriately, one gets \( \lim \inf \int_{[n > N]} |C_n T_n(\phi \Delta)| dP_\phi = 0 \). Thus, using Wald's lemma for margingales [viz., Chow, Robbins and Teicher (1965)], one gets

\[
E_\phi \{ C_{\bar{N}(i,i)}(\Delta) T_{\bar{N}(i,i)}(\Delta) (\phi \Delta) \} = E_0 \{ C_{\bar{N}(i,i)}(\Delta) T_{\bar{N}(i,i)}(\Delta) (0) \} = 0. \tag{5.24}
\]

The above equation implies that

\[
E_\phi \{ C_{\bar{N}(i,i)}(\Delta) Z_{\bar{N}(i,i)}(\Delta) \} = (\phi - \Delta) E_\phi \{ C_{\bar{N}(i,i)}(\Delta) D(F) \}. \tag{5.25}
\]

Since for small \( \Delta \), the excess over the boundaries can be neglected,

\[
C_{\bar{N}(i,i)}(\Delta) \quad \text{and} \quad Z_{\bar{N}(i,i)}(\Delta)
\]

can assume the two values \( bv^2(1+(-1)^\epsilon) / \Delta D(F) \) and \( av^2(1+(-1)^\epsilon) / \Delta D(F) \) with respective probabilities \( P^*_1(\phi \Delta) \) and \( P^*_2(\phi \Delta) \), say,

where \( \lim_{\Delta \to 0} P^*_1(\phi \Delta) = P(\phi) = \lim_{\Delta \to 0} [1-P^*_2(\phi \Delta)] \) and \( P(\phi) \) is defined by (4.18). Hence, \( \Delta \to 0 \)

\[
\lim_{\Delta \to 0} \Delta^2 E_\phi \{ C_{\bar{N}(i,i)}(\Delta) \} = \frac{v^2[b(1+(-1)^\epsilon)P(\phi) + a(1+(-1)^\epsilon)(1-P(\phi))]}{(\phi - \Delta) D^2(F)}. \tag{5.26}
\]

Since \( \epsilon(>0) \) is arbitrary, the proof follows from (5.23) and (5.26).
The above proof fails when \( \phi=\frac{1}{2} \). In that case, if we work with a sequence of \( \phi \) values, say, \( \frac{1}{2} \pm \varepsilon \), \( \varepsilon \rightarrow 0 \) as \( r \rightarrow \infty \), then by using the L'Hospital's rule, one gets from the above that

\[
\lim_{\Delta \to 0} \Delta^2 \mathbb{E}_\phi \left[ c^2 N(\Delta) \right] = -\tau^2 \mathbb{P} \left( \Theta \right) ((a-b)) \quad \text{Q.E.D.} \tag{5.27}
\]

In many practical problems, the \( \{c_i\} \) belong to a bounded interval and one may further assume that

\[
n^{-1}c_n \to C^2: \quad 0 < C < \infty, \quad \text{as } n \to \infty. \tag{5.28}
\]

Under (5.28) and the conditions of Theorem 5.1, it follows that

\[
\lim_{\Delta \to 0} \Delta^2 \mathbb{E}_\phi \left[ N(\Delta) \right] = C^{-2} \psi(\phi, \tau). \tag{5.29}
\]

As in the case of equispaced regression constants [viz., \( c_i = c_0 + hi, \ i \geq 1, \ h > 0 \)], in some other cases, though (5.28) may not hold, we may assume that

\[
C_n^2 = Q(n), \text{ where } Q \text{ is a convex function}. \tag{5.30}
\]

As such, \( EQ(N) \geq Q(EN) \), and hence,

\[
\lim_{\Delta \to 0} \Delta^2 Q(E_\phi N(\Delta)) \leq \psi(\phi, \tau); \tag{5.31}
\]

the equality sign in (5.31) is difficult to be achieved, as \( [\Delta^2 N(\Delta)] \) is not a degenerate (as \( \Delta \to 0 \)) random variable.

6. ARE RESULTS

The least squares (or normal theory ML) estimator of \( \beta \) is

\[
\beta_n^* = c_n^{-2} \sum_{i=1}^{n} (c_i - \bar{c}) X_i, \quad n \geq 1. \tag{6.1}
\]

As has been indicated after (2.6), if we let \( s_n^2 = \frac{1}{n-2} \sum_{i=1}^{n} (X_i - \bar{X}_n - \beta_n^*(c_i - \bar{c}))^2 \), \( n > 2 \), then on replacing in (2.7) \( I(\delta) \) by \( S_n^2 \), the corresponding procedure relates
to the normal theory SLRT. This terminates with probability one and as \( \Delta \to 0 \), it has the same OC function as in our Theorem 4.1. For this procedure, we have parallel to Theorem 5.1

\[
\lim_{\Delta \to 0} \Delta^2 E_N(\Delta) = \psi(\phi, \sigma), \quad (6.2)
\]

where \( \sigma^2 \) is the true variance of the \( X_i \). As a result, by (5.1), (5.2) and (6.2), we claim that under (5.28), the ARE of the SROT with respect to the SLRT, as judged by their ASN, is given by

\[
e_{J,L} = \psi(\phi, \sigma)/\psi(\phi, \tau) = \sigma^2/\tau^2 = D^2(F)\sigma^2/\nu^2. \quad (6.3)
\]

Similarly, if the actual form of the df of \( F \) were known and one would have used the Wald SPRT, under (5.28), the ARE of the SROT with respect to the SPRT is

\[
e_{J,W} = \psi(\phi, \nu^{-1}(f))/\psi(\phi, \tau) = D^2(F)/\nu^2 \nu(F). \quad (6.4)
\]

Since (6.3) and (6.4) have been discussed in detail in Sen and Ghosh (1974), we omit the details here. We may conclude, however, that for using the normal scores i.e., for \( J(u) = \Phi^{-1}(u) \), the inverse of the standard normal distribution, (6.3) is always \( \geq 1 \), where the equality sign holds only when \( F \) is normal. This indicates the desirable use of the normal scores SROT over the normal theory SLRT.
References


