AN ASYMPTOTICALLY OPTIMAL NONPARAMETRIC TEST FOR GROUPED DATA

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Summary. Generalizing the results of Sen [10], a class of nonparametric tests for the hypothesis of no regression in the multiple regression model is obtained here. The asymptotic power properties of the proposed tests are studied and the optimality of the tests is established under the conditions of Wald [11]. Applications of the results are also considered. In particular, for the several sample location problem, this enables us to study the optimality of the tests by Basu [1] and Gastwirth [2].

1. Introduction. Consider a sequence of random vectors \( X_{\nu} = (X_{\nu 1}, \ldots, X_{\nu N_{\nu}})' \) consisting of \( N_{\nu} \) independent random variables where \( X_{\nu i} \) has a continuous distribution function (d.f.) \( F_{\nu i}(x) \) given by

\[
F_{\nu i}(x) = F(\sigma^{-1}[x - \alpha - \beta'c_{\nu i}], \ i = 1, \ldots, N_{\nu},
\]

where \( \beta' = (\beta_1, \ldots, \beta_p) \), \( \alpha \) and \( \sigma (> 0) \) are real parameters and \( c_{\nu i} = (c_{\nu i 1}, \ldots, c_{\nu i p})' \), \( i = 1, \ldots, N_{\nu} \) are vectors of known constants satisfying (without any loss of generality)

\[
\sum_{i=1}^{N_{\nu}} c_{k i} = 0, \ (k = 1, \ldots, p).
\]

Define

\[
c_{k i}^* = c_{k i} / C_{k}, \ k = 1, \ldots, p, \ \text{where} \ C_{k}^2 = \sum_{i=1}^{N_{\nu}} c_{k i}^2, \ k = 1, \ldots, p.
\]

We assume

\[
\max_{1 \leq i \leq N_{\nu}} |c_{k i}^*| = o(1), \ \text{for all} \ k = 1, \ldots, p
\]

Let

\[
\Lambda_{\nu} = ((\sum_{i=1}^{N_{\nu}} c_{k i}^* c_{k' i}^*)) \rightarrow \Lambda = ((\lambda_{k k'})) \ \text{as} \ \nu \rightarrow \infty
\]

where \( \Lambda_{\nu} \) and \( \Lambda \) are assumed to be positive definite. Further, we assume that \( f(x) = F'(x) \) exists and

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\begin{align}
  (1.6) \quad \int_{-\infty}^{\infty} \left[ \frac{f'(x)}{f(x)} \right]^2 f(x) dx < \infty.
\end{align}

The $\mathbf{X}_\nu$ are not observable. We have a finite or countable set of class intervals
\begin{align}
  (1.7) \quad I_j: a_j < x \leq a_{j+1}, \quad j = 0, 1, \ldots, \infty,
\end{align}
where $-\infty = a_0 < a_1 < a_2 < \cdots < \infty$ is any (finite or countable) set of ordered points on the real line $(-\infty, \infty]$. The observable stochastic vector is $\mathbf{X}_\nu^* = (X_{\nu 1}, \ldots, X_{\nu N})$ where
\begin{align}
  (1.8) \quad X_{\nu i} = \sum_{j=0}^{\infty} I_j \cdot Z_{ij} \quad (i = 1, 2, \ldots, N) \nu
\end{align}
and
\begin{align}
  (1.9) \quad Z_{ij} = \begin{cases} 1 \text{ if } X_{\nu i} \in I_j \\ 0 \text{ otherwise} \end{cases}
\end{align}
for all $i = 1, 2, \ldots, N$, $1 \leq \nu < \infty$, $j = 0, \ldots, \infty$. Having observed $\mathbf{X}_\nu^*$, we want to test the null hypothesis
\begin{align}
  (1.10) \quad H_0: \beta = 0 \quad \text{(i.e. no regression)} \quad \text{against } \beta \neq 0.
\end{align}

The present investigation concerns some permutationally distribution-free tests for the above problem and shows that under a set of not too stringent regularity conditions, these are asymptotically optimum in a certain sense.

Let rank $\mathbf{A}_\nu = r$. We may assume without any loss of generality that $r = p$. Otherwise a reparametrisation in (1.1) will lead to a lower order $\mathbf{A}_\nu$ which will be of full rank.

2. Asymptotically Optimal Parametric Test. We define
\begin{align}
  (2.1) \quad F_j = F((a_j - \alpha)/\sigma), \quad P_j = F_{j+1} - F_j \quad (j = 0, 1, \ldots, \infty)
\end{align}
\begin{align}
  (2.2) \quad \Delta_j = \int_{F_j}^{F_{j+1}} \phi(u) du / \int_{F_j}^{F_{j+1}} du, \quad \text{when } P_j \neq 0
\end{align}
\begin{align}
  = 0 \quad \text{otherwise},
\end{align}
where
\begin{align}
  (2.3) \quad \phi(u) = -f'(F^{-1}(u))/f(F^{-1}(u)), \quad 0 < u < 1.
\end{align}
We can rewrite $\Delta_j$ as

$$\Delta_j = [f(F^{-1}(F_j)) - f(F^{-1}(F_{j+1}))]/P_j \quad (j = 0, 1, \ldots, \infty), \text{ when } P_j > 0. \tag{2.4}$$

Let

$$A^2(F, \{I_j\}) = \sum_{j=0}^{\infty} \Delta_j^2 P_j = \sum_{j \in J} \Delta_j^2 P_j, \text{ where } J = \{j; P_j > 0\}. \tag{2.5}$$

Hence,

$$0 < A^2(F, \{I_j\}) = \sum_{j \in J} \int_{F_j}^{F_{j+1}} \left[ \int_{\phi(u)du}^{F_{j+1}} \phi(u)du \right]^2 \int_{F_j}^{F_{j+1}} du \leq A^2(F) < \infty. \tag{2.6}$$

Assume

$$\sup_{j = 0, 1, \ldots, \infty} P_j < 1. \tag{2.7}$$

Consider the sequence of alternatives $\beta_k = \beta_k^\nu = \tau_k/C_k^\nu$. Let

$$\gamma_k = \tau_k/\sigma, \text{ for all } k = 1, \ldots, p, \text{ and } h_{v_i} = \sigma^{-1} \sum_{k=1}^{p} c_{kvi}^\nu \tau_k, \quad (i = 1, 2, \ldots, N^\nu). \tag{2.8}$$

Then,

$$F_{v_i}(a_{j+1}) - F_{v_i}(a_j) = P(X_{v_i} \in I_j) = F([a_{j+1} - \alpha]/\sigma - h_{v_i}) - F([a_j - \alpha]/\sigma - h_{v_i}). \tag{2.9}$$

Using Taylor expansion, (2.1) and (2.4), we get now,

$$F_{v_i}(a_{j+1}) - F_{v_i}(a_j) = P_j[l + h_{v_i} \Delta_j + h_{v_i} \theta_j g_j(h_{v_i})], \text{ for all } j \in J, \tag{2.10}$$

$$g_j(h_{v_i}) = P_j \int_{0}^{h_{v_i}} [f'((a_{j+1} - \alpha)/\sigma - \theta_j y) - f'((a_j - \alpha)/\sigma - \theta_j y)]dy \quad (\text{where } 0 < \theta_j < 1) \tag{2.11}$$

$$\approx o(1), \text{ uniformly in } j \in J, \quad i = 1, \ldots, N^\nu.$$

Let $\chi = (\gamma_1', \ldots, \gamma_k')'$.

Hence the likelihood function of $X^*_{v_1}, \ldots, X^*_{v_{N^\nu}}$ is given by

$$L(X^*_{v_i}|\chi) = \prod_{i=1}^{N^\nu} \left\{ \sum_{j=0}^{\infty} Z_{ij} P(X_{v_i} \in I_j) \right\} \tag{2.12}$$

$$= \prod_{i=1}^{N^\nu} \left\{ \sum_{j=0}^{\infty} Z_{ij} P_j[1 + h_{v_i} \Delta_j + \theta_j h_{v_i} g_j(h_{v_i})] \right\}$$
Let,

\( T_{k\nu} = [(\partial/\partial \gamma_k) \log L(X^*_\nu \mid \gamma)]_{\gamma=0} \approx \)

\[ \approx \left[(\partial/\partial \gamma_k) \log \left( \sum_{j=0}^{\infty} Z_{ij} P_j + h_{\nu i} \sum_{j=0}^{\infty} Z_{ij} P_j^* \Delta_j + \theta_j h_{\nu i} \sum_{j=0}^{\infty} Z_{ij} P_j \sum_{j=0}^{\infty} Z_{ij} P_j^* (h_{\nu i}) \right) \right]_{\gamma=0} \]

\[ = \sum_{i=1}^{N_\nu} \sum_{k=1}^{p} c_{k\nu i} \left( \sum_{j=0}^{\infty} Z_{ij} P_j^* \Delta_j / \sum_{j=0}^{\infty} Z_{ij} P_j \right) = \sum_{i=1}^{N_\nu} \sum_{j=0}^{\infty} c_{k\nu i} \sum_{j=0}^{\infty} Z_{ij} \Delta_j, \quad k = 1, \ldots, p. \]

For some variable \( Y\) asymptotically normal \((a, b, \nu)>0\) we mean

\((Y - a)/b\) converges in law to normal \((0, 1)\) distribution as \( \nu \to \infty \).

**LEMMA 2.1.** Under \( H_0, T_{1\nu}, \ldots, T_{p\nu} \) are asymptotically multinormally distributed with mean vector \( \mathbf{0} \) and dispersion matrix \( \Delta A^2(F, \{I_j\}) \).

**PROOF.** Let \( T_{\nu} = (T_{1\nu}, \ldots, T_{p\nu})' \). It is sufficient to show that for any real and finite \( e_1, \ldots, e_p, \varepsilon_{T_{\nu}} \) is asymptotically normal \((0, (\varepsilon_{A\nu} A)^2(F, \{I_j\}))\), where \( \varepsilon = (e_1, \ldots, e_p)' \). We can write,

\[ e_{T_{\nu}} = \sum_{i=1}^{N_\nu} g_{\nu i} W_{\nu i}, \]

where

\[ g_{\nu i} = \sum_{k=1}^{p} e_k c_{k\nu i}, \quad W_{\nu i} = \sum_{j=0}^{\infty} \Delta_j Z_{ij}, \quad (i = 1, \ldots, N_\nu). \]

Then,

\[ E(W_{\nu i} \mid H_0) = \sum_{j=0}^{\infty} \Delta_j P_j = \int_0^1 \phi(u) du = 0, \quad (i = 1, \ldots, N_\nu), \]

\[ E(W_{\nu i}^2 \mid H_0) = \text{Var}(W_{\nu i} \mid H_0) = A^2(F, \{I_j\}), \quad (i = 1, \ldots, N_\nu), \]

where \( E(\cdot \mid H_0) \) denote expectations under \( H_0 \), \( \text{Var}(\cdot \mid H_0) \) denote variances under \( H_0 \). So

\[ E(\varepsilon_{T_{\nu}} \mid H_0) = 0, \quad E(\varepsilon_{T_{\nu}}^2 \mid H_0) = (\varepsilon_{A\nu} A)^2(F, \{I_j\}) \sim (\varepsilon_{A\nu} A)^2(F, \{I_j\}), \]

where the sign \( \sim \) means that ratio of both sides tends to 1 as \( \nu \to \infty \).

Since, under \( H_0 \), \( W_{\nu 1}, \ldots, W_{\nu N_\nu} \) are all independently and identically distributed random variables \((i.i.d.r.v.)\) with \( 0 < V(W_{\nu i}) < \infty \), for all \( i = 1, \ldots, N_\nu \) to prove the lemma, all we need to show is that the coefficients...
\( g_{vi} \) in (2.15) satisfy the Noether condition, namely,

\[
(2.18) \quad \max_{1 \leq i \leq N_v} \frac{g_{vi}^2}{\sum_{i=1}^{N_v} g_{vi}^2} = o(1).
\]

But

\[
(2.19) \quad |g_{vi}| \leq \left( \max_{1 \leq k \leq p} |c_{xvi}^*| \right) \left( \sum_{k=1}^{p} |e_k| \right) = o(1)
\]

and

\[
(2.20) \quad \sum_{i=1}^{N_v} e_{vi}^2 = e_{v}^* \Lambda_v e \sim e_{v}^* e \quad \text{as} \quad v \to \infty.
\]

Hence, (2.18) is satisfied.

**Lemma 2.2.** Under \( H_0 \), \( \log[L(X_v^*|\gamma)/L(X_v^*|\gamma = 0)] \) is asymptotically normal

\(-\frac{1}{2}(\gamma' \Lambda \gamma) A^2(F, \{I_j\}), (\gamma' \Lambda \gamma) A^2(F, \{I_j\}).\)

**Proof.** Let \( \lambda_v^* = L(X_v^*|\gamma)/L(X_v^*|\gamma = 0) \). We have

\[
(2.21) \quad L(X_v^*|\gamma = 0) = \prod_{i=1}^{N_v} \left( \Sigma_{j=0}^{\infty} Z_{ij} \mathbb{P}_j \right).
\]

Hence,

\[
\log \lambda_v^* = \Sigma_{i=1}^{N_v} \log[1 + h_{vi}] \Sigma_{j=0}^{\infty} Z_{ij} \Delta_j + h_{vi} \Sigma_{j=0}^{\infty} \theta_j Z_{ij} g_j(h_{vi}) \]

and after some simplifications, we can write,

\[
\log \lambda_v^* = \Sigma_{i=1}^{N_v} h_{vi} \Sigma_{j=0}^{\infty} Z_{ij} (\Delta_j + \theta_j g_j(h_{vi})) - \frac{1}{2} \Sigma_{i=1}^{N_v} h_{vi}^2 \Sigma_{j=0}^{\infty} Z_{ij} \Delta_j^2 + o(1).
\]

Using the results, \( E(\Sigma_{j=0}^{\infty} Z_{ij} \Delta_j | H_0) = 0, \ E(\Sigma_{j=0}^{\infty} Z_{ij} g_j(h_{vi}) | H_0) = 0 \) and

\[
E(\Sigma_{j=0}^{\infty} Z_{ij} \Delta_j^2 | H_0) = A^2(F, \{I_j\}), \]

we get,

\[
(2.22) \quad E[\log \lambda_v^* | H_0] = -\frac{1}{2}(\gamma' \Lambda \gamma) A^2(F, \{I_j\}) + o(1) - \frac{1}{2}(\gamma' \Lambda \gamma) A^2(F, \{I_j\}).
\]

Also, since, \( \Sigma_{i=1}^{N_v} h_{vi} \Sigma_{j=0}^{\infty} Z_{ij} \Delta_j = \Sigma_{k=1}^{p} \gamma_k T_{kv} \), we get,

\[
(2.23) \quad \log \lambda_v^* = \Sigma_{k=1}^{p} \gamma_k T_{kv} + \frac{1}{2} \Sigma_{i=1}^{N_v} h_{vi}^2 \Sigma_{j=0}^{\infty} \Delta_j P_{ij} - \frac{1}{2} \Sigma_{i=1}^{N_v} h_{vi}^2 \Sigma_{j=0}^{\infty} Z_{ij} \theta_j g_j(h_{vi}) + o(1),
\]
Since \( h^*_\nu = \max_{1 \leq i \leq \nu} |h_{\nu i}| = o(1) \), splitting the summation over \( j \) in the first term of (2.23) into two sets \( \{ j : P_j > \eta_\nu \} \) and \( \{ j : P_j < \eta_\nu \} \) where \( \eta_\nu = O(h^*_\nu l^{1+\delta}) \), \( 0 < \delta < 1 \), first of the two resulting expressions converges in mean square to zero, while the second converges in mean to zero, since \( \sum_{j=0}^{\infty} \Delta_j^2 P_j \) is uniformly bounded. The second term in (2.23) converges in mean square to zero. Hence, using Markov-type inequalities and noting that

\[
\sum_{i=1}^{N_\nu} h_{\nu i}^2 \sum_{j=0}^{\infty} \Delta_j^2 P_j = (\gamma' \tilde{\Delta}_\nu \gamma) A^2(F, \{ I_j \}),
\]

we get

(2.24) \( \log \lambda^*_\nu - \gamma' T_\nu + \frac{1}{2} (\gamma' \tilde{\Delta}_\nu \gamma) A^2(F, \{ I_j \}) \rightarrow 0 \) in \( P_\nu \)-probability,

where \( P_\nu \) denotes the probability distribution of \( \tilde{X}^*_\nu \) under \( H_0 \). Since, \( \gamma' \tilde{\Delta}_\nu \gamma \sim \gamma' A \gamma \) as \( \nu \rightarrow \infty \), using lemma (2.1) we get the result.

Using the above lemma and the corollary to the lemma 1 of LeCam, [See[6], p.204] we get the probability measure \( Q_\nu \) (the probability distribution of \( \tilde{X}^*_\nu \) under alternative) contiguous to the probability measure \( P_\nu \). We shall make use of this fact in proving the following Lemma:

**Lemma 2.3.** Under the alternatives, \( T_\nu \) is asymptotically \( p \)-variate normal \((\tilde{\Delta}_\nu) A^2(F, \{ I_j \}), \tilde{\Delta}_\nu) A^2(F, \{ I_j \}).\)

**Proof.** As in Lemma 2.1, we need only to show \( e'T_\nu \) is asymptotically normal \((e' \tilde{\Delta}_\nu) A^2(F, \{ I_j \}), \tilde{\Delta}_\nu) A^2(F, \{ I_j \}))\) for any \( e \neq 0 \) of finite real constants. Consider the joint asymptotic distribution of \((\log \lambda^*_\nu, e'T_\nu)\). Since, for any real constants \( a, b \), \((a \log \lambda^*_\nu + be'T_\nu) - (a(\gamma' T_\nu + \frac{1}{2}\gamma' \tilde{\Delta}_\nu \gamma) A^2(F, \{ I_j \}) + be'T_\nu)\)

\( \rightarrow 0 \) in \( P_\nu \)-probability, \( a \log \lambda^*_\nu + be'T_\nu \) has asymptotically the same distribution as \((a(\gamma' + \frac{1}{2}\gamma' \tilde{\Delta}_\nu \gamma) A^2(F, \{ I_j \}))\) under \( H_0 \). The latter, however, is asymptotically normal \((-\frac{a}{2}(\gamma' \tilde{\Delta}_\nu \gamma) A^2(F, \{ I_j \}), \{(a^2(\gamma' \tilde{\Delta}_\nu \gamma) + b^2(e' \tilde{\Delta}_\nu e) + 2ab(\gamma' \tilde{\Delta}_\nu \gamma) A^2(F, \{ I_j \}))\) under \( H_0 \). Hence, under \( H_0 \), \((\log \lambda^*_\nu, e'T_\nu)\) is
asymptotically bivariate normal \((- \frac{1}{2} (\gamma' \Lambda \gamma) A^2(F, \{I_j\}), 0, (\gamma' \Lambda \gamma) A^2(F, \{I_j\}), (e' \Lambda e) A^2(F, \{I_j\}), \rho)\) where \(\rho = (\gamma' \Lambda e) / ((\gamma' \Lambda \gamma) (e' \Lambda e))^{\frac{1}{2}}\). Using now LeCam's third lemma (See [6], p. 208), \(e' T^\nu\) is under the alternatives asymptotically normal \(((\gamma' \Lambda \gamma) A^2(F, \{I_j\}), (e' \Lambda e) A^2(F, \{I_j\}))\) for any \(e \neq 0\) of finite real constants. Q.E.D.

Define the statistic

\[
S^*_\nu = T^\nu (\sum_{\nu} A^2(F, \{I_j\}))^{-1} T^\nu
\]

\[
= A^{-2}(F, \{I_j\}) T^\nu L^{-1} \Lambda_{\nu} T^\nu.
\]

We propose the following parametric test procedure for testing \(H_0\) against the alternatives based on the critical function \(\psi_1 (X^*_\nu)\):

\[
\psi_1 (X^*_\nu) = \begin{cases} 
1 & \text{if } S^*_{\nu} > S_{\nu, \epsilon} \\
\delta_{1v\epsilon} & \text{if } S^*_{\nu} = S_{\nu, \epsilon} \\
0 & \text{if } S^*_{\nu} < S_{\nu, \epsilon}
\end{cases}
\]

where \(S_{\nu, \epsilon}\) and \(\delta_{1v\epsilon}\) are chosen in such a way that \(E[\psi_1 (X^*_\nu) | H_0] = \epsilon\), \(0 < \epsilon < 1\), \(\epsilon\) being the desired level of significance of the test. We note that the above test is a similar size \(\epsilon\) test. Further, under the model (1.1) and the assumptions (1.2)-(1.6), \(S^*_{\nu}\) is distributed asymptotically under the null hypothesis as a central chi-square with \(p\) degrees of freedom and under the alternatives a non-central chi-square with \(p\) degrees of freedom and non-centrality parameter

\[
\eta = (\gamma' \Lambda \gamma) A^2(F, \{I_j\})
\]

Noting also that \(\delta_{1v\epsilon} \to 0\) and \(S_{\nu, \epsilon} \to \chi^2_{p, \epsilon}\) where \(\chi^2_{p, \epsilon}\) is the upper 100\(\epsilon\)% point of a central chi-square distribution with \(p\) degrees of freedom, we can easily obtain the following theorem:
THEOREM 2.1. Under the model (1.1) and the assumptions (1.2)-(1.6), the asymptotic power of the test procedure described in (2.26) is given by
\[ P \left[ \chi_p^2(\eta) \geq \chi_{p, \epsilon}^2 \right] , \]
where \( \chi_p^2(\eta) \) has a non-central chi-square distribution with \( p \) degrees of freedom and non-centrality parameter \( \eta \) defined in (2.27).

Let \( \lambda_\nu \) be the likelihood ratio test criterion for the testing problem considered in (1.10) i.e. \( \lambda_\nu = (L(\hat{X}_\nu^*|\gamma))/ (L(\tilde{X}_\nu^*|\gamma)) \), where
\[ \hat{X}_\nu^* = (\gamma_{N_\nu, 1}^*, \ldots, \gamma_{N_\nu, p}^*) \] is the maximum likelihood estimator (m.l.e.) of \( \gamma \).

Our next objective is to show \( S_\nu + 2 \log_\epsilon \lambda_\nu \rightarrow 0 \) in \( P_\nu \)-probability, and hence also in \( Q_\nu \)-probability because of the contiguity of the \( Q_\nu \)-probability distribution to the \( P_\nu \)-probability distribution. We make the following assumptions on the likelihood function \( L(\tilde{X}_\nu^*|\gamma) \) as in Wald [11].

ASSUMPTION I. Denote by \( D_{N_\nu} \) the set of all sample points \( E_{N_\nu} \) for which the m.l.e. \( \hat{Y}_{N_\nu} \) exists and the second order partial derivatives \( (\partial^2/\partial \gamma_k \partial \gamma_{k'} ) L(\tilde{X}_\nu^*|\gamma) \) are continuous functions of \( \gamma_1, \ldots, \gamma_p \). It is assumed that
\[ (2.28) \lim_{\nu \rightarrow \infty} P(D_{N_\nu}|\gamma) = 1, \text{ uniformly in } \gamma. \]
If for a sample point \( E_{N_\nu} \), there exists several maximum likelihood estimators, we can select one of them by some given rule. Hence, we shall consider \( \hat{Y}_{N_\nu} \) as a single valued function of \( E_{N_\nu} \), defined for all points of \( D_{N_\nu} \). For brevity of notation, we shall, henceforth, omit the suffix \( N_\nu \), whenever possible, without ambiguity.

ASSUMPTION II. For any positive \( \epsilon \),
\[ (2.29) \lim_{\nu \rightarrow \infty} P[|\hat{\gamma} - \gamma| < \epsilon | \gamma] = 1 , \]
uniformly in \( \gamma \), where
\[
\left| \hat{\gamma} - \gamma \right| = \left( \frac{\beta}{N} \sum_{k=1}^{P} \left( \gamma_k - \gamma \right)^2 \right)^{1/2}.
\]
Write, \( L(X_{\nu}^* | \gamma) = \prod_{i=1}^{N_{\nu}} L(X_{v_i}^* | \gamma) \). For any \( x \), any \( \delta > 0 \) and any \( \gamma_0 = (\gamma_{01}, \ldots, \gamma_{0p})' \), let
\[
(2.30) \quad \psi_{kk',i}(x, \gamma_0, \delta) = gL.b. \left( \frac{\partial^2}{\partial \gamma_k \partial \gamma_{k'}} \log L(X_{v_i}^* | \gamma) \right),
\]
and
\[
(2.31) \quad \phi_{kk',i}(x, \gamma_0, \delta) = L.u.b. \left( \frac{\partial^2}{\partial \gamma_k \partial \gamma_{k'}} \log L(X_{v_i}^* | \gamma) \right),
\]
where \( |\gamma - \gamma_0| \leq \delta \).

ASSUMPTION III

(a) For any sequences \( \{\gamma_{1N_{\nu}}\}, \{\gamma_{2N_{\nu}}\} \) and \( \{\delta_{N_{\nu}}\} \) for which
\[
\lim_{\nu \to \infty} \gamma_{1N_{\nu}} = \gamma \quad \text{and} \quad \text{lim}_{\nu \to \infty} \delta_{N_{\nu}} = 0,
\]
we have,
\[
(2.32) \quad \lim_{\nu \to \infty} \frac{1}{\gamma_{1N_{\nu}}} \psi_{kk',i}(x, \gamma_{2N_{\nu}}, \delta_{N_{\nu}}) = \lim_{\nu \to \infty} \phi_{kk',i}(x, \gamma_{2N_{\nu}}, \delta_{N_{\nu}})
\]
\[
= E_{\gamma} \left( \frac{\partial^2}{\partial \gamma_k \partial \gamma_{k'}} \log L(X_{v_i}^* | \gamma) \right),
\]
uniformly in \( \gamma \), where \( E \) stands for expectation.

(b) There exists an \( \epsilon > 0 \) such that
\[
E_{\gamma} \left[ \psi_{kk',i}(x, \gamma_2, \delta) \right]^2
\]
\[
= E_{\gamma} \left[ \phi_{kk',i}(x, \gamma_2, \delta) \right]^2
\]
are bounded functions of \( \gamma_1, \gamma_2 \) and \( \delta \) in the domain \( D \) defined by the inequalities \( |\gamma_1 - \gamma_2| \leq \epsilon \) and \( |\delta| \leq \epsilon \).

(c) The greatest lower bound with respect to \( \gamma \) of the absolute value of the determinant of the matrix
\[
(2.33) \quad E_{\gamma} \left( \frac{\partial^2}{\partial \gamma_k \partial \gamma_{k'}} \log L(X_{v_i}^* | \gamma) \right)
\]
is positive.

ASSUMPTION IV

(2.34) \( E_{\gamma} \left( \frac{\partial}{\partial \gamma_k} \log L(X^*_{v_i} | \gamma) \right) = E_{\gamma} \left( \frac{\partial^2}{\partial \gamma_k \partial \gamma_{k'}} \log L(X^*_{v_i} | \gamma) \right) = 0 \),
for all \( k, k' = 1, \ldots, p; \ i = 1, \ldots, N_{\nu} \).

ASSUMPTION V

There exists an \( \eta > 0 \) such that
\[
(2.35) \quad E_{\gamma} \left( \frac{\partial}{\partial \gamma_k} \log L(X^*_{v_i} | \gamma) \right)^{2+\eta}
\]
are bounded functions of \( \gamma \),
for all \( k = 1, \ldots, p; \ i = 1, \ldots, N_{\nu} \).
Let the surface \( S_c(\gamma) \) be defined by

\[
(2.36) \quad \gamma'((E_{\gamma}(-\partial^2/\partial\gamma_k\partial\gamma_l)) \log L(X_{\gamma}^*|\gamma))\gamma = c
\]

and the weight function \( \eta(\gamma) \) be defined by

\[
(2.37) \quad \eta(\gamma) = \lim_{\rho \to 0} \frac{A(\omega'(\gamma, \rho))}{A(\omega(\gamma, \rho))},
\]

where for any \( \gamma \) and any \( \rho > 0 \),

\[
(2.38) \quad \omega(\gamma, \rho) = \{ \gamma : |\gamma - \gamma| \leq \rho \},
\]

\( \gamma \) lies on the same surface \( S_c(\gamma) \) as \( \gamma \), \( \omega'(\gamma, \rho) \) is the image of \( \omega(\gamma, \rho) \) by transformation \( \gamma^* = \mathcal{B}_{\gamma} \gamma \) and \( \mathcal{B}_{\gamma} \) is a non-singular matrix such that

\[
(2.39) \quad \mathcal{B}_{\gamma}^T \mathcal{B}_{\gamma} = ((E_{\gamma}(-\partial^2/\partial\gamma_k\partial\gamma_l)) \log L(X_{\gamma}^*|\gamma)),
\]

and for any set \( \omega \), \( A(\omega) \) denotes the \((p-1)\) dimensional area of \( \omega \).

**THEOREM 2.3.** (Wald). Let \( S_c(\gamma) \) and \( \eta(\gamma) \) be defined as in (2.36) and (2.37). Then for testing \( H_0 \) as defined in (1.10), under the Assumptions I - V, the likelihood ratio test

(a) has asymptotically the best average power with respect to the surfaces \( S_c(\gamma) \) and weight function \( \eta(\gamma) \);

(b) has asymptotically best constant power on the surfaces \( S_c(\gamma) \);

(c) is an asymptotically most stringent test.

We next prove the following theorem:

**THEOREM 2.4.** \( S_\nu + 2\log e_\nu = 0 \) in \( P_\nu \)-probability.

**PROOF.** We have

\[
(2.40) \quad -2\log e_\nu = -2[\log e L(X_{\gamma}^*|\gamma)_{\gamma=0} - (\log e L(X_{\gamma}^*|\gamma)_{\gamma=\gamma})^\wedge].
\]

Using Taylor expansion of \( [\log e L(X_{\gamma}^*|\gamma)]_{\gamma=0} \) about \( \gamma = \gamma^\wedge \), and after some simplifications, we get,

\[
(2.41) \quad -2\log e_\nu = (\gamma^\wedge)^T((-\partial^2/\partial\gamma_k\partial\gamma_l) \log e L(X_{\gamma}^*|\gamma))_{\gamma=\gamma^\wedge}^\wedge
\]

\[
= (\gamma^\wedge)^T((-1/\nu^2\partial^2/\partial\gamma_k\partial\gamma_l) \log e L(X_{\gamma}^*|\gamma))_{\gamma=\gamma^\wedge}(\gamma^\wedge)
\]

where \( \gamma^\wedge \) lies in the \( p \)-dimensional rectangle \((0, \gamma^\wedge)\). Since,
\[ N_v^{-1}(\log_e L(X^*_\gamma | \gamma))_{\gamma=\gamma^*} - E_{\gamma=0}[N_v^{-1}(\log_e L(X^*_\gamma | \gamma))] \]

converges under \( P_v \)-probability to zero for all \( k, k' = 1, \ldots, p \) and \( \sqrt{N_v} \) has a non-degenerate limit distribution

\[ (-2\log_e \lambda) - \sum_{\gamma} E_{\gamma=0}[(\log_e L(X^*_\gamma | \gamma))] \]

converges in \( P_v \)-probability to zero. But under the regularity assumptions of Wald,

\[ E_{\gamma=0}[(\log_e L(X^*_\gamma | \gamma))] = E_{\gamma=0}[(\log_e L(X^*_\gamma | \gamma))] \]

and after some simplifications, we get,

\[ E_{\gamma=0}[N_v^{-1}(\log_e L(X^*_\gamma | \gamma))] = (N_v^{-1} \sum_{i=1}^{N_v} c_{k,v_i} c_{k',v_i}) A^2(F, \{I_j\}). \]

Hence,

\[ E_{\gamma=0}[N_v^{-1}(\log_e L(X^*_\gamma | \gamma))]_{k, k' = 1, \ldots, p} = N_v^{-1} \Lambda_v \Lambda^2(F, \{I_j\}). \]

(2.42) and (2.45) give

\[ -2\log_e \lambda - (\gamma \Lambda_v \gamma) A^2(F, \{I_j\}) \]

converges to zero in \( P_v \)-probability.

Next we show \( S_v - (\gamma \Lambda_v \gamma) A^2(F, \{I_j\}) \) converges to zero in \( P_v \)-probability.

We have, \( 0 = (\log_e L(X^*_\gamma | \gamma))_{\gamma=\gamma^*} = [\log_e L(X^*_\gamma | \gamma)]_{\gamma=\gamma} \]

\[ + \sum_{k'=1}^{p} \sum_{k'\neq k} \gamma_{k', \gamma_k}(\log_e L(X^*_\gamma | \gamma))_{\gamma=\gamma^*}, \]

where \( \gamma^* \) lies in the rectangle \([0, \gamma]\). Thus,

\[ T_{k,v} = \sum_{k'} \gamma_{k', \gamma_k}(\log_e L(X^*_\gamma | \gamma))_{\gamma=\gamma^*} \]

i.e.

\[ T_{k,v} = \sum_{k'} (\log_e L(X^*_\gamma | \gamma))_{\gamma=\gamma^*}. \]

But \( N_v^{-1}(\log_e L(X^*_\gamma | \gamma))_{\gamma=\gamma^*} \)

converges to zero in \( P_v \)-probability. Using this, (2.25) and Slutsky's Theorem, we get,
\[(2.49) \quad S_\nu - (\sum \Lambda) A^2(F, \{I_j\}) \to 0 \quad \text{in } P_\nu\text{-probability.}\]

The theorem is an immediate consequence of (2.46) and (2.49). \textbf{Q.E.D.}

Note that, since $Q_\nu$ are contiguous to $P_\nu$, $S_\nu + 2\log e\lambda_\nu + 0$ in $Q_\nu$-probability.

The above theorem implies that $S_\nu$ and $-2\log e\lambda_\nu$ have asymptotically the same distribution under the null hypothesis and also under the alternatives. Thus if the assumptions (1.2) - (1.6) along with the regularity conditions of Wald are satisfied, then under the model (1.1) the test procedure based on $S_\nu$ possesses the asymptotic optimal properties of the likelihood ratio test as given in Theorem 2.3.

3. Asymptotically optimum nonparametric test. We define as in Sen [10],

\[(3.1) \quad \sum_{i=1}^{N_\nu} Z_{ij} = N_{\nu j} \quad \text{for } j = 0, 1, \ldots, \infty, \text{ so that } N_\nu = \sum_{j=0}^{\infty} N_{\nu j} ; \]

\[(3.2) \quad F_{N_\nu,0} = 0, \quad F_{N_\nu,j+1} = \sum_{e=0}^{j} N_{\nu e}/N_\nu \quad \text{for } j = 0, 1, \ldots, \infty. \]

Define

\[(3.3) \quad \hat{\Delta}_{\nu j} = \left\{ f[F^{-1}(F_{N_\nu,j})] - f[F^{-1}(F_{N_\nu,j+1})] \right\} / \left[ F_{N_\nu,j+1} - F_{N_\nu,j} \right] \]

\[= \int_{F_{N_\nu,j}}^{F_{N_\nu,j+1}} \int_{u}^{1} \phi(u) du \quad \text{for } N_{\nu j} > 0 \]

and \[\hat{\Delta}_{\nu j} = 0 \quad \text{if } N_{\nu j} = 0.\]

Let

\[(3.4) \quad U_\nu = (U_{1\nu}, \ldots, U_{p\nu})', \]

where

\[(3.5) \quad U_{k\nu} = \sum_{i=1}^{N_\nu} c_{k\nu i} * \sum_{j=0}^{\infty} \hat{\Delta}_{\nu j} Z_{ij}. \]

We propose the test statistic

\[(3.6) \quad M_\nu = (U_\nu' \hat{\Delta}_\nu^{-1} U_\nu) A^2(F_{N_\nu}, \{I_j\}), \]
where

\[
A^2(F_{N_{ij}}, \{I_j\}) = \sum_{j=0}^{\infty} N_{ij} \frac{\Delta_{ij}^2}{N_{ij}}.
\]

We investigate first the joint null distribution of \(U_{1ij}, \ldots, U_{p_{ij}}\). As in Sen [10], we use the permutation argument to get a distribution-free test. Under \(H_0\) in (1.10) \(X_{ij}^*\) are i.i.d.r.v. and hence the joint distribution of \(X_{ij}^*\) remains invariant under any permutation of its \(N_{ij}\) arguments. We define a permutational probability measure denoted by \(P_{ij}\) on the set of \(N_{ij}!\) equiprobable points (actually there are \(N_{ij}! / \prod_{j=0}^{\infty} N_{ij}!\) distinct equally likely permutations of \(X_{ij}^*\)) on the \(N_{ij}\)-dimensional real space \(\mathbb{R}^{N_{ij}}\). Under \(P_{ij}\), all the \(N_{ij}!\) equally likely permutations have the common probability \((N_{ij}!)^{-1}\). But \(N_{ij}\) in (3.1) and \(\Delta_{ij}\) in (3.3) remain invariant under any permutation of co-ordinates of \(X_{ij}^*\). It follows now that

\[
E_{P_{ij}}(Z_{ij}) = N_{ij}/N_{ij}, \quad (i = 1, \ldots, N_{ij}; \quad j = 0, 1, \ldots, \infty);
\]

\[
E_{P_{ij}}(Z_{ij}Z_{ij}) = 0, \quad (i = 1, \ldots, N_{ij}; \quad j,j' = 0, 1, \ldots, \infty; \quad j \neq j');
\]

\[
E_{P_{ij}}(Z_{ij}Z_{ij'}) = N_{ij} (N_{ij'} - \delta_{jj'})/N_{ij}(N_{ij'} - 1), \quad \text{for all} \quad i \neq i',
\]

\(i, i' = 1, \ldots, N_{ij}; \quad j, j' = 0, 1, \ldots, \infty\) and \(\delta_{jj'}\) is the usual Kronecker delta; \(E_{P_{ij}}\) denotes expectation under the permutational probability measure \(P_{ij}\). Using these we get,

\[
E_{P_{ij}}(U_{k_{ij}}) = \sum_{i=1}^{N_{ij}} c_{k_{ij}} \int_{0}^{1} \phi(u)du = 0, \quad \text{for all} \quad k = 1, \ldots, p,
\]

and

\[
E_{P_{ij}}(U_{k_{ij}}U_{k'_{ij}}) = (N_{ij}/(N_{ij} - 1))A^2(F_{N_{ij}}, \{I_j\}) \sum_{i=1}^{N_{ij}} c_{k_{ij}} c_{k'_{ij}},
\]

for all \(k, k' = 1, \ldots, p\). In matrix notations, we can write,

\[
E_{P_{ij}}(U_{ij}) = 0;
\]

\[
\text{Var}_{P_{ij}}(U_{ij}) = (N_{ij}/(N_{ij} - 1))A^2(F_{N_{ij}}, \{I_j\})A_{ij}.
\]

Let,

\[
M_{ij} = U_{ij}^\prime (A^2(F_{N_{ij}}, \{I_j\})A_{ij})^{-1} U_{ij}
\]

\[
= A^{-2}(F_{N_{ij}}, \{I_j\}) U_{ij}^\prime A_{ij}^{-1} U_{ij}.
\]
We propose the following nonparametric test procedure:

\[
\psi_2(X^*_\nu) = \begin{cases} 
1 & \text{if } M_\nu > M_{\nu,\epsilon} \\
\delta_{2\nu\epsilon} & \text{if } M_\nu = M_{\nu,\epsilon} \\
0 & \text{if } M_\nu < M_{\nu,\epsilon}
\end{cases}
\]

where \(M_\nu,\epsilon\) and \(\delta_{2\nu\epsilon}\) are chosen in such a way that \(E_P(\psi_2(X^*_\nu)) = \epsilon\), the level of significance. This implies \(E(\psi_2(X^*_\nu)|H_0) = \epsilon\) i.e. \(\psi_2(X^*_\nu)\) is a similar size \(\epsilon\) test.

We next state the following useful lemma due to Sen [10].

**LEMMA 3.1.** Under the model (1.1) and the assumptions (1.2) - (1.6),

\(A^2(F_{N_\nu}, \{I_j\})\) converges in probability to \(A^2(F, \{I_j\})\) uniformly in \(\{I_j\}\).

Define,

\[
W^*_\nu = \hat{\sum}_{j=0}^\infty \hat{\Delta}_{\nu j} Z_{ij} \quad \text{for } i = 1, 2, \ldots, N_\nu.
\]

Under \(P_\nu\), \(\hat{\Delta}_{\nu j}\) are all invariant while \(Z_{ij}\) are stochastic. \(N_{\nu j}\) of \(W^*_\nu\) are equal to \(\hat{\Delta}_{\nu j}\) (\(j = 0, 1, \ldots, \infty\)). Also, \(U^*_{k\nu} = \sum_{i=1}^{N_\nu} c^*_{kvi} W^*_vi\).

We prove the following theorem:

**THEOREM 3.1.** Under \(P_\nu\), \(U_\nu\) is asymptotically \(p\)-variate normal with mean vector \(0\) and dispersion matrix \(A^2(F, \{I_j\})\) in probability.

**PROOF.** It is sufficient to show that for all non-null \(d = (d_1, \ldots, d_p)'\) where \(d_1, \ldots, d_p\) are real, finite, not all zeroes, \(d'U_\nu\) is asymptotically normal \((0, (d'\Lambda d)A^2(F, \{I_j\})\) under \(P_\nu\) in probability. We can write

\[
d'U_\nu = \sum_{i=1}^{N_\nu} e_{vi} W^*_vi,
\]

where \(e_{vi} = \sum_{k=1}^{p} d_k c^*_{kvi}\).

Using (3.13) and (3.14), we get

\[
E_P(d'U_\nu) = 0, \quad E_P((d'U_\nu)^2) = (N_\nu(N_\nu - 1))A^2(F_{N_\nu}, \{I_j\})d'\Lambda d.
\]

To prove the theorem, we shall make use of the Wald-Wolfowitz-Noether-Hájek permutational central limit theorem. The following theorem of Hájek ([4],
Theorem 4.2, p. 514), which we state in a slightly more general form convenient for our purpose, gives a necessary and sufficient condition for the asymptotic normality of \( \tilde{W}_v = \sum_{i=1}^{N_v} e_{vi} \tilde{W}^*_{vi} \) in probability.

**SUB-THOREM 3.1.** If the double sequence \( \{e_{vi}, 1 \leq i \leq N_v, \; v \geq 1\} \) satisfies the Noether condition

\[
\lim_{v \to \infty} \left\{ \max_{1 \leq i \leq N_v} \frac{(e_{vi} - \overline{e}_v)^2}{\sum_{i=1}^{N_v} (e_{vi} - \overline{e}_v)^2} \right\} = 0 \;
\text{ where } \overline{e}_v = \frac{-1}{\sum_{i=1}^{N_v} e_{vi}}
\]

and the double sequence \( \{\tilde{W}^*_{vi}, 1 \leq i \leq N_v, \; v \geq 1\} \) also satisfies the same condition in probability, then a necessary and sufficient condition that

\[
(W_v - E_{(W_v)})/(\text{Var}_v(W_v))^{1/2}
\]

is asymptotically normal \((0,1)\) in probability is

\[
(3.20) \quad \lim_{v \to \infty} k_v/N_v = 0 \Rightarrow
\]

\[
\lim_{v \to \infty} \max_{1 \leq i \leq \ldots \leq k_v \leq N_v} \frac{k_v}{\sum_{i=1}^{N_v} (W^*_vi - \overline{W}_v)^2} = 0 \;
\text{ in probability,}
\]

\[
\overline{W}_v = \frac{-1}{\sum_{i=1}^{N_v} \tilde{W}^*_{vi}}
\]

where \( \tilde{W}^*_v = \sum_{i=1}^{N_v} \overline{W}^*_{vi} \).

Note first that here \( \overline{e}_v = 0 \) and since \( N_v \) of the \( \tilde{W}^*_v \) are \( \Delta_{vj} \) for \( j = 0, 1, 2, \ldots \), \( \tilde{W}^*_v = \sum_{j=0}^{v} N_j \Delta_{vj} = 0 \). Since,

\[
\max_{1 \leq i \leq N_v} |e_{vi}| \leq \left( \max_{1 \leq i \leq N_v} |c_{kvi}| \right)^{1/2} \|d\| = o(1) \quad \text{and} \quad \sum_{i=1}^{N_v} e_{vi}^2 = O(1).
\]

\(d'\Delta d\sim d'\Delta d\), (non-zero and finite) it follows that \( \max_{1 \leq i \leq N_v} e_{vi}^2 \)

\(\sum_{i=j}^{N_v} e_{vi}^2 = o(1). \) So the sequence \( \{e_{vi}\} \) satisfies the Noether condition.

Further, if \( W^*_v(v_1) \leq W^*_v(v_2) \leq \ldots \leq W^*_v(v_{N_v}) \) denote the order statistics corresponding to \( W^*_v \)'s (not necessarily all distinct) we can write

\[
\max_{1 \leq i \leq \ldots \leq k_v \leq N_v} \sum_{i=1}^{N_v} \sum_{i=N_v-k_v+1}^{N_v} W^*_v(v_1) \;
\text{ which is invariant under}
\]

\[
1 \leq i \leq \ldots \leq i \leq N_v \; a=1 \; v_1 \alpha = N_v - k_v + 1
\]
all $N_{\nu}$ possible permutations of $W_{vi}^2$'s since order statistics remain invariant under permutation of arguments. Now, if we define a step function $a_{N_{\nu}}(u),\; 0 < u < 1$ such that

$$a_{N_{\nu}}(u) = W_{(vi)}^* \text{ for } (i-1)/N_{\nu} < u \leq i/N_{\nu} \text{ for } i = 1, 2, \ldots, N_{\nu},$$

we can write,

$$\sum_{i=N_{\nu} - k_{\nu} + 1}^{N_{\nu}} W_{(vi)}^* = \int_{0}^{1} a_{N_{\nu}}(u)du. \quad (3.21)$$

But,

$$\int_{0}^{1} a_{N_{\nu}}(u)du = \sum_{i=1}^{N_{\nu}} W_{(vi)}^* \leq \sum_{i=1}^{N_{\nu}} = A^2(F_{N_{\nu}}, \{I_{j}\}) \leq A^2(F) < \infty.$$ 

So $A^2(F_{N_{\nu}}, \{I_{j}\})$ is bounded uniformly in $N_{\nu}$ and $\{I_{j}\}$ and it now follows from (3.21) that $\sum_{i=N_{\nu} - k_{\nu} + 1}^{N_{\nu}} W_{(vi)}^* \to 0$ as $\nu \to \infty$ if $k_{\nu}/N_{\nu} \to 0$ as $\nu \to \infty$.

Further, $A^2(F_{N_{\nu}}, \{I_{j}\}) \to A^2(F, \{I_{j}\})$ in probability.

So, (3.20) holds in probability. In particular, putting $k_{\nu} = 1$ we get

$$\max_{1 \leq i \leq N_{\nu}} \frac{W_{vi}^2}{\sum_{i=1}^{N_{\nu}} W_{vi}^2} \to 0 \text{ in probability as } \nu \to \infty \text{ i.e. } \{W_{vi}\} \text{ satisfies Noether condition in probability.}$$

Next we show that the test procedure based on $\psi_2(X_{\nu}^*)$ is asymptotically power-equivalent to the one based on $\psi_1(X_{\nu}^*)$. To show this, we need in addition to Lemma (3.1), Lemma (3.2).

**Lemma 3.2.** $T_{\nu} - U_{\nu}$ converges to $0$ in $P_{\nu}$-probability.

**Proof.** It is sufficient to show that $T_{kv} - U_{kv} \to 0$ in $P_{\nu}$-probability for all $k = 1, \ldots, p$. Since convergence in mean square implies convergence in probability, it is sufficient to show that $E(T_{kv} - U_{kv})^2 \to 0$ under $H_0$, for all $k = 1, \ldots, p$. Proceeding exactly as in Lemma 3.2. of [10], (p.1235)

for each such squared mean, we get the result.
The above lemma implies that under the null hypothesis (and because of contiguity, also under the alternatives) \( U_v \) has asymptotically the same distribution as \( T_v \).

**Theorem 3.2.** Under the model (1.1), the assumptions (1.2) - (1.6), and under the alternatives considered, \( M_v \) and \( S_v \) have asymptotically the same distribution under \( P_v \) or \( Q_v \).

**Proof.** Using Lemma 3.2. and Slutsky's theorem, we get \( M_v - S_v \) converges in \( P_v \) (or \( Q_v \)) probability to zero. Hence the theorem.

4. Asymptotic Relative Efficiency (ARE). Suppose the true distribution function is \( G(x) \) instead of \( F(x) \) and let \( g(x) = G'(x) \). Let then

\[
A^2(G) = \int_{-\infty}^{\infty} [g'(x)/g(x)]^2 g(x)dx < \infty.
\]

Further, we define,

\[
G_j = G((a_j-\alpha)/\alpha), \quad j = 0, 1, \ldots, \infty;
\]

\[
P^*_j = G_{j+1} - G_j, \quad j = 0, 1, \ldots, \infty;
\]

\[
\phi^*(u) = - g'(G^{-1}(u))/g(G^{-1}(u));
\]

\[
\Delta^*_j = \int_{G_j}^{G_{j+1}} \phi^*(u)du/P^*_j, \quad j = 0, 1, \ldots, \infty;
\]

\[
\Delta^{**}_j = \int_{G_j}^{G_{j+1}} \phi(u)du/P^*_j, \quad j = 0, 1, \ldots, \infty;
\]

\[
A^2(G, \{I_j\}) = \sum_{k=0}^{\infty} \Delta^*_j P^*_j;
\]

\[
B^2(F, \{I_j\}) = \sum_{j=0}^{\infty} \Delta^{**}_j P^*_j;
\]

\[
C(F, G, \{I_j\}) = \sum_{j=0}^{\infty} \Delta^*_j \Delta^{**}_j P^*_j;
\]

\[
\rho(F, G, \{I_j\}) = C(F, G, \{I_j\})/[A(G, \{I_j\}) B(F, \{I_j\})];
\]
\[(4.11) \quad k^*_j(h_{vi}) = p_j^{-1} \int_0^{h_{vi}} [g'(a_{j+1} - \theta_j y) - g'(a_j - \theta_j y)] dy\]

(where \(0 < \theta_j < 1, i = 1, \ldots, N_j\); we consider all \(j \in J^* = \{j : j \in [0, \infty)\} \) and \(p_j^* > 0\)). Then, under the model (1.1) with \(F\) replaced by \(G\) and under the assumptions (1.2) - (1.6), we have,

\[(4.12) \quad \log \lambda^*_v = \sum_{k=1}^{b} \gamma^*_k k^*_v - \frac{1}{2} \sum_{i=1}^{N_v} \sum_{v_i}^{\infty} Z_{ij} A_j^* + \sum_{i=1}^{N_v} Z_{ij} \theta_j k_j^*(h_{vi}) + o(1),\]

\(Z_{ij}\) being defined as before, and

\[(4.13) \quad T^*_v = \sum_{x=1}^{N_v} \sum_{i=0}^{\infty} Z_{ij} A_j^*, \quad (k = 1, \ldots, \cdot).\]

Denote

\[(4.14) \quad T^*_v = (T^*_{1v}, \ldots, T^*_{pv})'.\]

Then we have,

\[(4.15) \quad E(T^*_v|H_0) = \Theta, \quad \text{Var}(T^*_v|H_0) = A^2(G, \{I_j\}) A_v .\]

Let

\[(4.16) \quad S^*_v = A^{-2}(G, \{I_j\}) T^*_v A_v^{-1} T^*_v .\]

Further, let,

\[(4.17) \quad T^{**}_v = \sum_{x=1}^{N_v} \sum_{i=0}^{\infty} Z_{ij} A_j^{**} ;\]

\[(4.18) \quad S^{**}_v = (T^{**}_v A_v^{-1} T^{**}_v)^{-1} B^2(F, \{I_j\}) .\]

Then,

\[(4.19) \quad E(T^{**}_v|H_0) = \Theta, \quad \text{Var}(T^{**}_v|H_0) = A_v B^2(F, \{I_j\}) .\]

If we now use the permutation test procedure as described in Section 3, it is asymptotically power-equivalent to the test procedure based on the test criterion \(\psi_3(X^*_v)\) described as follows:

\[(4.20) \quad \psi_3(X^*_v) = \begin{cases} 1 & \text{if } S^{**}_v > S^{**}_{v,\epsilon} \\ \delta^{3v}_v & \text{if } S^{**}_v = S^{**}_{v,\epsilon} \\ 0 & \text{if } S^{**}_v < S^{**}_{v,\epsilon} \end{cases},\]

where \(S^{**}_{v,\epsilon}\) and \(\delta^{3v}_v\) are so chosen that \(E[\psi_3(X^*_v)|H_0] = \epsilon .\)
As in Section 2, we can show here that under $H_0$, the model (1.1) and the assumptions (1.2) - (1.6) with $F$ replaced by $G$, for any real and finite $\mathbf{e} = (e_1, \ldots, e_p)'$,

\[\begin{align*}
\log^\star_{\nu} & \quad (\gamma', \mathbf{e}')(\mathbf{T}_\nu^*) \\
\text{is under} & \quad P_\nu \quad \text{asymptotically bivariate normal}
\end{align*}\]

\[(- \frac{1}{2}(\gamma, \mathbf{A}_\nu, \gamma)A^2(G, \{I_j\}), 0, (\gamma, \mathbf{A}_\nu, \gamma)A^2(G, \{I_j\}),(e, \mathbf{A}_\nu, e)B^2(F, \{I_j\}), \rho(F, G, \{I_j\}))].\]

Hence, using LeCam's third lemma,

\[\begin{align*}
\mathbf{e}^T \mathbf{T}_\nu^* & \quad \text{is under} \quad Q_\nu \quad \text{asymptotically normal} \\
((\gamma, \mathbf{A}_\nu, \gamma)C(F, G, \{I_j\}), (\gamma, \mathbf{A}_\nu, e)B^2(F, \{I_j\})).
\end{align*}\]

Hence, under $Q_\nu$,

\[\begin{align*}
\mathbf{T}_\nu^* & \quad \text{is asymptotically p-variate normal} \\
((\gamma, \mathbf{A}_\nu, \gamma)C(F, G, \{I_j\}), (\gamma, \mathbf{A}_\nu, e)B^2(F, \{I_j\})); \quad \text{also} \quad \mathbf{A}_\nu \rightarrow \mathbf{A} \quad \text{as} \quad \nu \rightarrow \infty, \quad \text{and so} \quad S_\nu^* \quad \text{has under} \quad Q_\nu \quad \text{an asymptotic non-central chi-square distribution with} \quad p \quad \text{degrees of freedom and non-centrality parameter}
\end{align*}\]

\[\begin{align*}
\eta'_o & = (\gamma, \mathbf{A}_\nu, \gamma)C^2(F, G, \{I_j\})/B^2(F, \{I_j\}).
\end{align*}\]

Also, $\mathbf{t}_\nu^*$ is under $Q_\nu$ asymptotically p-variate normal $((\gamma, \mathbf{A}_\nu, \gamma)A^2(G, \{I_j\}), (\gamma, \mathbf{A}_\nu, e)A^2(G, \{I_j\})).$ Hence $S_\nu^*$ (which is now the asymptotically optimum test statistic in the sense described in Section 2) has asymptotically a non-central chi-square distribution with $p$ degrees of freedom and non-centrality parameter

\[\begin{align*}
\eta_o & = (\gamma, \mathbf{A}_\nu, \gamma)A^2(G, \{I_j\}).
\end{align*}\]

Using the definition of ARE as given by Hannan [7], we get the ARE of the permutation test procedure as given in (3.16) relative to the asymptotically optimum parametric test procedure as given in (2.26) with $G$ replacing $F$ in the definition of $\mathbf{T}_\nu$ and hence $S_\nu$, is given by

\[\begin{align*}
e_F, G, \{I_j\} & = \eta'_o/\eta_o = \rho^2(I_j) .
\end{align*}\]
In case of ungrouped data where the \( X \) 's are observable while the model (1.1) and the assumptions (1.2) – (1.6) remain the same, for testing the hypothesis \( H_0: \beta = 0 \) against the alternatives \( \beta = (\sum_{i=1}^{N_{\nu}} c_{kvi})^{-1/2} \sum_{i=1}^{N_{\nu}} \tau \) one proceeds on lines analogous to that of Hájek [See [8]]; and if \( S_{\nu 0} \) denotes the statistic for ungrouped data, \( S_{\nu 0} \) has asymptotically a non-central chi-square distribution with \( p \) degrees of freedom and non-centrality parameter \( \eta_{oo} = (\gamma' \Lambda \gamma) A^2(F) \). Hence, the loss of efficiency due to grouping assuming that the true distribution function is \( F \) is given by

\[
(n_{oo} - \eta)/n_{oo} = \int_{F}^{J+1} \frac{1}{J} \int_{0}^{1} [\phi(u) - \Delta]^2 du / \int_{0}^{1} \phi^2(u) du
\]

\[= 1 - (A^2(F, \{I_j\}))/A^2(F),\]

which can be made arbitrarily small by making the width of the class intervals arbitrarily small.

Again in situations when the assumed distribution function is \( F \) while the true distribution function is \( G \), the loss of efficiency is

\[
[(\gamma' \Lambda \gamma)_{G}^2 A^2(G) - (\gamma' \Lambda \gamma)_{F}^2 (\{I_j\} A^2(G, \{I_j\}))/ (\gamma' \Lambda \gamma)_{G}^2 A^2(G)]
\]

\[= 1 - \left[ \frac{\rho(\{I_j\})}{\rho} \right]^2 A^2(G, \{I_j\})/A^2(G),\]

where \( \rho = \int_{0}^{1} \phi(u) \phi^*(u) du / (\int_{0}^{1} \phi^2(u) du) \int_{0}^{1} \phi^2(u) du )^{1/2}. \)

The expression (4.28) may be greater than, equal to or less than (4.27) with \( G \) replacing \( F \) depending on \( \rho(\{I_j\}) \) and \( \rho \). We might also remark that the grouped normal scores statistic, the grouped Wilcoxon's statistic and the grouped sign statistic belong to the class of tests considered here according as \( f(x) \) is normal, logistic or double exponential.

5. Applications. (1) \( p \)-sample problem: Consider the situation when the \( X \) 's are from \( p \) populations \( \Pi_1, \ldots, \Pi_p \) and out of the \( N \) observations \( n_{k,\nu} \) are from population \( \Pi_k \) \( (k = 1, \ldots, p) \). We also assume
(5.1) \[ \lim_{\nu \to \infty} \frac{n_{k\nu}}{N_{\nu}} = \pi_k \quad (0 < \pi_k < 1 \text{ for all } k = 1, \ldots, p; \sum_{k=1}^{p} \pi_k = 1). \]

In this case, the regression constants \( c_{k\nu i} \) are given by

(5.2) \[ c_{k\nu i} = \begin{cases} 1 - \frac{n_{k\nu}}{N_{\nu}} & \text{if the } i\text{-th observation is from } k\text{-th sample} \\ -\frac{n_{k\nu}}{N_{\nu}} & \text{otherwise} \end{cases} \]

\((i = 1, \ldots, N_{\nu}; k = 1, \ldots, p).\) Then,

\[ \sum_{i=1}^{N_{\nu}} c_{k\nu i} = 0, \quad \sum_{i=1}^{N_{\nu}} c_{k\nu i}^2 = \frac{N_{\nu}}{N_{\nu}}(N_{\nu} - n_{k\nu})/N_{\nu}, \quad \sum_{i=1}^{N_{\nu}} c_{k\nu i} c_{k'\nu i} = -\frac{n_{k\nu}}{N_{\nu}} \frac{n_{k'\nu}}{N_{\nu}} \]

for all \( k, k' = 1, \ldots, p. \) Hence,

(5.3) \[ \Lambda_{\nu} = \left( -1 + 2\delta_{kk'} \right) \left( \frac{\delta_{kk}^\nu N_{\nu} - n_{k\nu}}{\delta_{kk'}^\nu N_{\nu} - n_{k'\nu}} \right) \left( N_{\nu} - n_{k\nu} \right)\left( N_{\nu} - n_{k'\nu} \right)^{1/2} \]

\( k, k' = 1, \ldots, p. \)

Under (5.1), \( \Lambda_{\nu} \to \Lambda \), where,

\[ \Lambda = \left( -1 + 2\delta_{kk'} \right) \left( \frac{\delta_{kk'} - \pi_k}{\pi_k} \right) \left( \frac{\delta_{kk'} - \pi_k}{\pi_k} \right)^{1/2} \]

\( k, k' = 1, \ldots, p \)

\( \Lambda_{\nu} \) is a \( p \times p \) matrix of rank \((p-1).\) Writing

\[ \Lambda_{\nu} = \begin{vmatrix} \Lambda_{11\nu} & \Lambda_{12\nu} \\ \Lambda_{21\nu} & \Lambda_{22\nu} \end{vmatrix} \]

\( (p-1) \times (p-1) \)

We notice that \( \Lambda_{11\nu} \) is of full rank. Let \( D = (I_{p-1} \quad 0) \) where \( I_{p-1} \)

is the identity matrix of order \((p-1), \) \( D_{UV} \) has asymptotically the same distribution as \( D_{TUV} \) (under the null hypothesis and also under the alternatives) and so under \( Q_{\nu} \)-probability is \((p-1)\)-variate normal \((\Lambda DA^2(F, I_j)), \)

\( (D_{AD'})A^2(F, (I_j)). \) But \( DAD' = \Lambda_{11} \) where

\[ \Lambda = \begin{vmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{vmatrix} \]

\( (p-1) \times (p-1) \)

and thus by reparametrization we get a non-singular dispersion matrix for variables under consideration. Since \( U_{\nu}^* \Lambda_{\nu} \Lambda_{\nu}^* U_{\nu} = (D_{UV})^{-1} (D_{UV}), \)

\( M_{\nu} \) remains invariant under reparametrization. Also, writing
\[ A_{\nu} = H_{\nu} - u_{\nu} u_{\nu}' \text{ where } H_{\nu} = \begin{bmatrix} \frac{N}{(N - n_{\nu})} & 0 \\ 0 & \frac{N}{(N_{\nu} - n_{p-i\nu})} \end{bmatrix} \]

and
\[ u_{\nu}' = \left( \frac{n_{1\nu}^{1/2}}{(N_{\nu} - n_{1\nu})^{1/2}}, \ldots, \frac{n_{p-I\nu}^{1/2}}{(N_{\nu} - n_{p-I\nu})^{1/2}} \right), \]

we can find \( A_{\nu}^{-1} \) by using a result given in Rao ([9], p. 29).

We might remark that in the p-sample problem, Basu [1] has considered the censored case where only \( N_{\nu}^* \) of the ordered variables in the combined sample are observable while \( N_{\nu}^*/N_{\nu} \rightarrow p \ (0 < p < 1) \) as \( \nu \rightarrow \infty \). A similar problem follows as a special case of ours when \( I_0 : x < x_0 \) while \( I_1, I_2, \ldots, I_\infty \) have sufficiently small Lebesgue measure. However, while in Basu's case \( N_{\nu}^* \) is given while the corresponding truncation point is random, in our case the truncation points are fixed while \( N_{\nu j} \) (\( j = 0, 1, \ldots, \infty \)) are random. In spite of these basic differences, the asymptotic relative efficiencies of the two tests will be the same and further we can study the optimality properties of Basu's test according to the criteria described earlier. As such for the logistic distribution, Basu's test is optimal. Similar comparison can be made in particular with the test proposed by Gastwirth for the two sample problem in the censored case by taking \( p=2 \) (See also [10]).

(ii) **Paired Comparison Problem:** Consider \( p \) treatments in an experiment which yields paired observations for each pair \((i, j)\) of treatments and assume that \( N_{ij} \) difference scores \( z_{k}^{(i,j)} = x_{i\ell} - x_{j\ell} \ (\ell = 1, 2, \ldots, N_{ij}) \) have a common continuous d.f. \( \Pi_{ij}(x) \). This is the situation for example in the analysis of incomplete block experiments with each block of size two.

Suppose in our context we have \( \nu \) experiments, number of independent comparisons of the \((i, j)\)-th pair of treatment in the \( \nu \)-th experiment being \( N_{\nu ij} \) (\( 1 \leq i \leq j \leq p; \ \nu \leq 1 \)) and let \( N_{\nu} = \sum_{1 \leq i < j \leq p} N_{\nu ij} \). The model
we have considered includes the paired comparison model with
\[ c_{k\nu\ell} = d_{k\nu\ell} - \overline{d}_{k\nu} \] where \( d_{k\nu\ell} = 1, -1, \) or \( 0 \) according as the observation
\( X_{\nu\ell} \) is from a block where the \( k \)-th treatment is compared with another treatment \( k' \) (\( k' = k+1, \ldots, p \)), a treatment \( k' \) (\( k' = 1, 2, \ldots, k-1 \)) or is not involved at all. Note that, we are ignoring the fact of symmetry of the distributions of \( Z_{\nu}^{(i,j)} \) about zero under the null situation, and a more natural approach is to use statistics analogous to signed-ranked statistics. But it can be shown that [See [3]] test procedures based on such statistics lead to the same ARE as the test procedures based on statistics considered here.

(iii) **Categorical Data:** The above analysis is also applicable when the observable random variables are grouped in several ordered categories, while their underlying parent distribution is continuous.

6. **Concluding Remarks.** The present paper includes, as a particular case (when \( p = 1 \)), Sen's model where the null hypothesis has been tested against one-sided alternatives. Unlike the case of one-sided alternatives, we do not get here an asymptotically most powerful test, but a test asymptotically optimal in Wald's sense. Also when the class intervals are sufficiently small, the efficiency of the \( M_{\nu} \) test reduces to that of a similar test (for ungrouped data) considered by Jogdeo [8]. We may also remark that unlike Hájek [5] and Sen [10] we need only impose the Noether condition on the \( c_{k\nu i} \)'s, but no other assumption like \( \sum_{i=1}^{\nu} c_{k\nu i}^2 = 0(1), \) for all \( k = 1, \ldots, p. \) This helps avoiding some artificiality in taking the regression constants \( c_{k\nu i} \) (\( k = 1, \ldots, p; i = 1, \ldots, \nu; \nu \geq 1 \)) as for example, we may consider the case \( c_{k\nu i} = i - \frac{N_{\nu} + 1}{2} \) (\( k = 1, \ldots, p; i = 1, \ldots, \nu; \nu \geq 1 \)). On the other hand, we have considered a sequence \( \beta_{\nu} \) of alternatives which are "local", while Hajek and Sen have regarded the regression coefficient \( \beta \)
as fixed, and still in effect have considered "local" alternatives by imposing
the condition \( \sum_{v_1}^{N_0} c_1^2 = O(1) \), along with the Noether condition.

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