

Linear operator inequalities for strongly stable weakly regular linear systems

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Abstract: We consider the existence of solutions to certain linear operator inequalities (Lur'e equations) for strongly stable, weakly regular linear systems with generating operators $A, B, C, 0$. These operator inequalities are related to the spectral factorization of an associated Popov function and to singular optimal control problems with a nonnegative definite quadratic cost functional. We split our problem into two subproblems: the existence of spectral factors of the nonnegative Popov function and the existence of a certain extended output map. Sufficient conditions for the first problem are known and for the case that A has compact resolvent and its eigenvectors form a Riesz basis for the state space, we give an explicit solution to the second problem in terms of A, B, C and the spectral factor. The applicability of these results is demonstrated by various heat equation examples satisfying a positive-real condition. Although delay equations do not satisfy the above criterion, if the closed span of the eigenvectors equals the state space, a general approach is proposed that shows promise for retarded systems. The above results have been used to design adaptive observers for positive-real infinite-dimensional systems.

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1 Introduction

A key to solving a number of problems in systems and control is the existence of a solution to a linear operator (in)equality which is closely related to a spectral factorization problem and to a linear quadratic optimal control problem. More concretely, consider the stable finite-dimensional linear system

$$\dot{x}(t) = Ax(t) + Bu(t), x(0) = x_0; y(t) = Cx(t) \quad (1.1)$$

and the cost functional

$$J(x_0, u(\cdot)) = \int_0^\infty \left\langle \begin{pmatrix} Q & N^* \\ N & R \end{pmatrix} \begin{pmatrix} y(t) \\ u(t) \end{pmatrix}, \begin{pmatrix} y(t) \\ u(t) \end{pmatrix} \right\rangle dt, \quad (1.2)$$

where A, B, C, R, N, Q are matrices of appropriate dimensions, $R = R^*, Q = Q^*$, but Q, R are not assumed to be nonnegative definite. The *optimal control problem* is to find the input function $u^{opt} \in \mathbf{L}_2(0, \infty; \mathbb{C}^m)$ that minimizes $J(x_0, u(\cdot))$, where x, y are given by (1.1). The corresponding *Popov function* is given by

$$\Pi(i\omega) = R + N\mathbf{G}(i\omega) + \mathbf{G}(i\omega)^*N^* + \mathbf{G}(i\omega)^*Q\mathbf{G}(i\omega), \quad (1.3)$$

where $\mathbf{G}(i\omega) = C(i\omega I - A)^{-1}B$ and $\omega \in \mathbb{R}$. We consider the case corresponding to a positive Popov function: $\Pi(i\omega) \geq 0$ for $\omega \in \mathbb{R}$. The associated *linear operator inequality* is

$$\begin{pmatrix} A^*X + XA + C^*QC & (B^*X + NC)^* \\ B^*X + NC & R \end{pmatrix} \geq 0. \quad (1.4)$$

For each matrix solution X of (1.4) we obtain a factorization of Π

$$\Pi(i\omega) = \Xi(i\omega)^*\Xi(i\omega), \quad (1.5)$$

where $\Xi(s) = W + L(sI - A)^{-1}B$ and W, L arise from any factorization of the nonnegative definite matrix (1.4)

$$\begin{pmatrix} A^*X + XA + C^*QC & (B^*X + NC)^* \\ B^*X + NC & R \end{pmatrix} = \begin{pmatrix} L^* \\ W^* \end{pmatrix} \begin{pmatrix} L & W \end{pmatrix} \quad (1.6)$$

We shall also refer to (1.6) as the *Lur'e equations*. Of special interest are the solutions X for which W is square. For the case of a nonsingular R , this corresponds to solutions to the *algebraic Riccati equation*

$$A^*X + XA + C^*QC - (XB + C^*N^*)R^{-1}(B^*X + NC) = 0 \quad (1.7)$$

where $L = R^{-\frac{1}{2}}(B^*X + NC)$ and $W = R^{\frac{1}{2}}$. Other solutions to (1.4) correspond to solutions to the algebraic Riccati inequality

$$XA + A^*X - (XB + C^*N^*)R^{-1}(B^*X + NC) \geq 0. \quad (1.8)$$

Over the past years, many papers have addressed these problems for finite-dimensional linear systems; Willems [36], Clements and Glover [3], Willems, Kitapci and Silverman [14], Gohberg, Lancaster and Rodman [13], Wimmer [37], Molinari [20], Scherer [26], Geerts [10], Clements, Anderson, Laub and Matson [8] and the references therein are representative of the finite-dimensional literature. The case that is the most challenging is when R is singular and Π has imaginary axis zeros. A special case that has received particular attention is the so-called *positive-real lemma* where $Q = 0$, $N = I$ and $R \geq 0$ (see Anderson and Vongpanitlerd [1]). The solution $X = X^*$ to the Lur'e equations

$$\begin{aligned} A^*X + XA &= L^*L \\ B^*X + C &= W^*L \\ R &= W^*W \end{aligned} \quad (1.9)$$

provides a Lyapunov functional which is used to prove the stability of a nonlinear perturbation of (1.1) and in this connection it is called the *Lur'e problem*. If $R = 0$, the positive-real lemma has applications to adaptive control problems.

In this paper, we investigate some extensions of the above problems to the class of infinite-dimensional systems known as *strongly stable, weakly regular linear systems*. This represents a large class of systems, including those described by partial differential equations with boundary control and point observations as well as delay equations with delayed observations and control action (see Weiss [32], [31], [30], [33], [34]).

Staffans [27] [29] and Weiss and Weiss [35] solved the optimal control problem (1.1)–(1.2) for this class under the coercivity assumption $\Pi(i\omega) \geq \varepsilon I$ for some $\varepsilon > 0$ and all $\omega \in \mathbb{R}$. The solution is in terms

of a spectral factorization and solutions to an algebraic Riccati equation resembling (1.7). The main aim of this paper is to extend this to the singular case $\Pi(i\omega) \geq 0$, which is equivalent to obtaining sufficient conditions for the existence of solutions to the linear operator inequality (1.4); i.e., the following Lur'e equations

$$\begin{aligned} A^*X + XA + C^*QC &= L^*L \\ B^*X + NC &= W^*L \\ R &= W^*W. \end{aligned} \tag{1.10}$$

Our solutions will correspond to a singular version of the algebraic equation (1.7) insofar as L maps from the state space to the input space (i.e., W is “square”, $W \in \mathcal{L}(\mathcal{U})$).

The first papers to consider infinite-dimensional linear operator inequalities treated the positive-real lemma were by Yakubovich [38], [39], [40], but these made the restrictive assumption that A was bounded. Later papers with Likharnikov [15], [16] allowed for an unbounded A , but assumed exact controllability and other technical assumptions. Balakrishnan [2] treated the particular case of the positive-real lemma with $Q = 0$, $N = I$, $R = 0$ and assumed that A had compact resolvent, its eigenvectors formed a Riesz basis and B, C were very smooth rank one operators. He also assumed approximate controllability, although this was not used explicitly in the proof that was based on a spectral factorization approach. The next two papers used a regularization approach and showed that the solutions of the regularized Riccati equations converged under an exact controllability assumption. In Curtain [4], [7], B and C were finite-rank, bounded operators, whereas in Pandolfi [22], B could be unbounded. In the latter case, the exact controllability assumption is quite reasonable, as hyperbolic systems with boundary control often have this property. In a more recent paper, [21] with a bounded C , but a possibly unbounded control operator, Pandolfi drops the controllability assumption, but assumes instead that A is a Riesz spectral operator which generates an exponentially stable holomorphic semigroup. Furthermore, he considers the scalar case and assumes that there exists an $\alpha < 1$ such that the Popov function satisfies

$$\Pi(i\omega) \geq \frac{M}{|\omega|^\alpha} \text{ for sufficiently large } |\omega|. \tag{1.11}$$

The proof uses a spectral factorization approach and regularization.

Our approach is closest to that in Weiss and Weiss [35] which uses a spectral factorization assumption as its starting point. This necessitates introducing the known theory of weakly regular linear systems in Section 2. The advantage is that in Section 3, we can reduce our problem to two distinct subproblems

- the existence of a spectral factor Ξ satisfying (1.5)
- the existence of an extended output map.

Known sufficient conditions for the first problem are then summarized in Section 4 and sufficient conditions for the spectral factor to be regular are derived. The second problem is essentially showing the existence of a well-posed L operator in (1.9); this is the difficult infinite-dimensional part that is treated in Section 5. Our main assumption is that A has compact resolvent and the closed span of its eigenvectors equals the state space, which is satisfied for many parabolic, hyperbolic and retarded systems. Under this assumption, we give an explicit expression for L in terms of the spectral factor Ξ and the system parameters, A, B, C, Q, R . For the case that the eigenvectors form a Riesz basis, we can apply known sufficient conditions from Hansen and Weiss [12] to test whether L is admissible or not. This provides sufficient conditions for large classes of parabolic and hyperbolic systems. In Section 6, we illustrate the applicability of these results to several scalar parabolic systems satisfying a positive-real condition. We consider both unbounded B and C operators and we prove the existence of a spectral factor satisfying (1.5) and solutions to 1.9 under the assumption that there exist positive constants M and α such that

$$\xi(i\omega) \sim \frac{M}{|\omega|^\alpha} \text{ for sufficiently large } |\omega|. \quad (1.12)$$

These results are an improvement on those in Pandolfi [21], as his assumption that $\alpha < 1$ turns out to exclude interesting cases. We also give sufficient conditions for the operator L in (1.9) to be bounded and these are a considerable improvement on those in Balakrishnan [2]. We remark that the results in Section 6 have been used in Curtain, Demetriou and Ito [23] to design adaptive observers for parabolic systems.

Since the eigenvectors of retarded systems will not form a Riesz basis, the previous approach is not suitable for this class. However, in Section 7, we propose another, more adhoc, approach that can lead to a

successful solution to our problem and we illustrate it with a scalar and a multi-input multi-output example.

2 Preliminaries

Since our results are based on the theory of *well-posed linear systems* and *weakly regular linear systems*, we review the relevant theory and key new results from Weiss [34]. We begin with some notation.

Definition 2.1 *Let $\mathcal{Z}_1, \mathcal{Z}_2$ be Hilbert spaces, \mathcal{B} a Banach space and $\Omega \subset \mathbb{R}$.*

- $\mathbf{L}_2(\Omega, \mathcal{Z}_1)$ *is the class of Lebesgue-measurable, square integrable, \mathcal{Z}_1 -valued functions on Ω .*
- $\mathbf{L}_2^{loc}(0, \infty; \mathcal{Z}_1)$ *is the class of Lebesgue measurable functions from $[0, \infty)$ to \mathcal{Z}_1 , which are square integrable on $[0, \tau]$ for every $\tau > 0$, with the topology determined by the seminorms $\|f\|_{\mathbf{L}_2[0, \tau]}$.*
- $\mathbf{H}_2(\mathcal{Z}_1)$ *is the class of holomorphic, square-integrable, \mathcal{Z}_1 -valued functions on the open right half-plane.*
- $\mathbf{H}_\infty(\mathcal{B})$ *is the class of bounded, holomorphic, \mathcal{B} -valued functions on the open right half-plane.*
- $\mathbf{L}_\infty(\mathcal{L}(\mathcal{Z}_1, \mathcal{Z}_2))$ *is the class of essentially bounded, weakly Lebesgue measurable, $\mathcal{L}(\mathcal{Z}_1, \mathcal{Z}_2)$ -valued functions on the imaginary axis.*

Scalar-valued function spaces will be denoted $\mathbf{H}_2, \mathbf{H}_\infty, \mathbf{L}_\infty$ etc. For simplicity, we suppose that all Hilbert spaces are separable. Let \mathcal{W} be any such Hilbert space. We denote the right shift by τ on $\mathbf{L}_2^{loc}(0, \infty; \mathcal{W})$ by \mathbf{S}_τ , i.e.,

$$(\mathbf{S}_\tau w)(t) = \begin{cases} 0 & 0 \leq t < \tau \\ w(t - \tau) & t \geq \tau. \end{cases}$$

An operator \mathbb{F} on $\mathbf{L}_2(0, \infty; \mathcal{W})$ is called *shift-invariant* if $\mathbb{F}\mathbf{S}_\tau = \mathbf{S}_\tau\mathbb{F}$. \mathbf{P}_τ denotes the projection of $\mathbf{L}_2^{loc}(0, \infty; \mathcal{W})$ onto $\mathbf{L}_2(0, \tau; \mathcal{W})$ by truncation, defined for $w \in \mathbf{L}_2^{loc}(0, \infty; \mathcal{W})$ by

$$(\mathbf{P}_\tau w)(t) = \begin{cases} w(t) & 0 \leq t < \tau \\ 0 & t \geq \tau. \end{cases}$$

For $w_1, w_2 \in \mathbf{L}_2^{loc}(0, \infty; \mathcal{W})$ and $\tau \geq 0$, the τ -concatenation of w_1 and w_2 , denoted $w_1 \diamond_\tau w_2$ is defined by

$$(w_1 \diamond_\tau w_2)(t) = \mathbf{P}_\tau w_1 + \mathbf{S}_\tau w_2.$$

We now define well-posed linear systems for the Hilbert spaces $\mathcal{X}, \mathcal{U}, \mathcal{Y}$.

Definition 2.2 A well-posed linear system on \mathcal{U} , \mathcal{X} and \mathcal{Y} is a quadruple $\Sigma = (\mathbb{T}, \Phi, \Psi, \mathbb{F})$, where

- (i) $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0}$ is a strongly continuous semigroup of bounded linear operators on \mathcal{X} ;
- (ii) $\Phi = (\Phi_t)_{t \geq 0}$ is a family of bounded linear operators from $\mathbf{L}_2(0, \infty; \mathcal{U})$ to \mathcal{X} such that

$$\Phi_{\tau+t}(u \diamond_{\tau} v) = \mathbb{T}_t \Phi_{\tau} u + \Phi_t v,$$

for any $u, v \in \mathbf{L}_2(0, \infty; \mathcal{U})$ and any $\tau, t \geq 0$;

- (iii) Ψ is a continuous linear operator from \mathcal{X} to $\mathbf{L}_2^{loc}(0, \infty; \mathcal{Y})$ such that for any $x \in \mathcal{X}$ and $\tau > 0$,

$$\Psi x = \Psi x \diamond_{\tau} \Psi \mathbb{T}_{\tau} x;$$

- (iv) \mathbb{F} is a continuous linear operator from $\mathbf{L}_2(0, \infty; \mathcal{U})$ to $\mathbf{L}_2^{loc}(0, \infty; \mathcal{Y})$ such that for any $u, v \in \mathbf{L}_2(0, \infty; \mathcal{U})$,

$$\mathbb{F}(u \diamond_{\tau} v) = \mathbb{F}u \diamond_{\tau} (\Psi \Phi_{\tau} u + \mathbb{F}v).$$

\mathcal{U} is the input space, \mathcal{X} is the state space and \mathcal{Y} is the output space.

If A is the generator of the strongly continuous semigroup \mathbb{T} on \mathcal{X} , we denote by \mathcal{X}_1 the space $\mathcal{D}(A)$ with the norm $\|z\|_1 = \|(\beta I - A)z\|$, where $\beta \in \rho(A)$, and \mathcal{X}_{-1} is the completion of \mathcal{X} with respect to the norm $\|z\|_{-1} = \|(\beta I - A)^{-1}z\|$. The choice of β is unimportant, since different choices produce equivalent norms. Consequences of the definition are the existence of certain operators A, B, C . Assumption (i) implies the existence of the infinitesimal generator $A \in \mathcal{L}(\mathcal{X}_1, \mathcal{X}_{-1})$ of \mathbb{T} . Assumptions (i) and (ii) above imply the existence of a unique $B \in \mathcal{L}(\mathcal{U}, \mathcal{X}_{-1})$, called the *control operator* of Σ , such that for all $t \geq 0$,

$$\Phi_t u = \int_0^t \mathbb{T}_{t-\sigma} B u(\sigma) d\sigma. \quad (2.1)$$

The fact that $\Phi_t u \in \mathcal{X}$ means that B is an *admissible control operator* for \mathbb{T} (i.e., $\Phi_t \in \mathcal{L}(\mathbf{L}_2(0, t; \mathcal{U}), \mathcal{X})$). From (2.1) we see that $\Phi_t u$ depends only on $\mathbf{P}_t u$, and so Φ_t has a natural extension to $\mathbf{L}_2^{loc}(0, \infty; \mathcal{U})$. B is called *infinite-time admissible* for \mathbb{T} if for all $u \in \mathcal{L}(\mathbf{L}_2(0, \infty; \mathcal{U}))$

$$\sup_{0 \leq t < \infty} \left\| \int_0^t \mathbb{T}_{\sigma} B u(\sigma) d\sigma \right\|_{\mathcal{X}} < \infty. \quad (2.2)$$

In this case, we can define the *extended input map* $\tilde{\Phi} \in \mathcal{L}(\mathbf{L}_2(0, \infty; \mathcal{U}), \mathcal{X})$ by

$$\tilde{\Phi}v = \lim_{T \rightarrow \infty} \int_0^T \mathbb{T}_\sigma Bv(\sigma) \, d\sigma \quad (2.3)$$

If $x_0 \in \mathcal{X}$ is the initial state of Σ and $u \in \mathbf{L}_2^{loc}(0, \infty; \mathcal{U})$ is its input function, then the *state trajectory* of Σ , $x : [0, \infty) \rightarrow \mathcal{X}$ is defined by

$$x(t) = \mathbb{T}_t x_0 + \Phi_t u, \quad (2.4)$$

for all $t \geq 0$. The function x is continuous and it satisfies the differential equation

$$\dot{x}(t) = Ax(t) + Bu(t), \quad (2.5)$$

in the strong sense in \mathcal{X}_{-1} . The function x is the unique solution of (2.5) satisfying the initial condition $x(0) = x_0$. If u has the Laplace transform \hat{u} , and $x_0 = 0$, then x has the Laplace transform $\hat{x}(s) = (sI - A)^{-1}B\hat{u}(s)$, for all s with $\operatorname{Re}(s)$ sufficiently large. The operator Ψ in Definition 2.2 is called the *extended output map* of Σ . More generally, any operator Ψ which satisfies assumption (iii) in Definition 2.2 is called an *extended output map* for \mathbb{T} . For every such Ψ there exists a unique $C \in \mathcal{L}(\mathcal{X}_1, \mathcal{Y})$ called the *observation operator* of Ψ , such that

$$(\Psi x_0)(t) = C\mathbb{T}_t x_0, \quad (2.6)$$

for every $x_0 \in \mathcal{X}_1$ and every $t \geq 0$. This C determines Ψ , since \mathcal{X}_1 is dense in \mathcal{X} . The function $y_0 = \Psi x_0$ has a Laplace transform \hat{y}_0 and we have $\hat{y}_0(s) = C(sI - A)^{-1}x_0$, for all $x_0 \in \mathcal{X}$ and for $\operatorname{Re}(s)$ sufficiently large. If Ψ is *bounded*, i.e., $\Psi \in \mathcal{L}(\mathcal{X}, \mathbf{L}_2(0, \infty; \mathcal{Y}))$, then we say that C is *infinite-time admissible*. The Λ -*extension* of C is defined by

$$C_\Lambda z = \lim_{\lambda \rightarrow \infty} C\lambda(\lambda I - A)^{-1}z. \quad (2.7)$$

The domain $\mathcal{D}(C_\Lambda)$ consists of those $z \in \mathcal{X}$ for which the above limit exists. If we replace C by C_Λ in (2.6), then it holds for all $x_0 \in \mathcal{X}$ and almost every $t \geq 0$. The operator $C_{\Lambda w}$, the *weak Λ -extension* of C , is defined by

$$C_{\Lambda w} z = \text{weak} \lim_{\lambda \rightarrow \infty} C\lambda(\lambda I - A)^{-1}z. \quad (2.8)$$

The domain of C_{Λ_w} consists of those $z \in \mathcal{X}$ for which the above limit exists. C_{Λ_w} is an extension of C_Λ and they are equal if \mathcal{Y} is finite-dimensional.

The operator \mathbb{F} in Definition 2.1 is called the *extended input-output map* of Σ . \mathbb{F} is shift invariant, $\mathbb{F}\mathbf{S}_\tau = \mathbf{S}_\tau\mathbb{F}$, which implies that \mathbb{F} is *causal*:

$$\mathbf{P}_\tau\mathbb{F} = \mathbf{P}_\tau\mathbb{F}\mathbf{P}_\tau \quad \text{for all } \tau \geq 0. \quad (2.9)$$

Using (2.9) we can extend \mathbb{F} continuously to $\mathbf{L}_2^{loc}(0, \infty; \mathcal{U})$.

If \mathbb{T}_t is exponentially stable, then $\mathbb{F} \in \mathcal{L}(\mathbf{L}_2(0, \infty; \mathcal{U}), \mathbf{L}_2(0, \infty; \mathcal{Y}))$ and B and C are infinite-time admissible. A more general concept of stability is the following.

Definition 2.3 *The well-posed system $\Sigma = (\mathbb{T}, \Phi, \Psi, \mathbb{F})$ is strongly stable if \mathbb{T}_t is strongly stable ($\mathbb{T}_t x \rightarrow 0$ as $t \rightarrow \infty$), $\tilde{\Phi} \in \mathcal{L}(\mathbf{L}_2(0, \infty; \mathcal{U}), \mathcal{X})$, $\Psi \in \mathcal{L}(\mathcal{X}, \mathbf{L}_2(0, \infty; \mathcal{Y}))$ and $\mathbb{F} \in \mathcal{L}(\mathbf{L}_2(0, \infty; \mathcal{U}), \mathbf{L}_2(0, \infty; \mathcal{Y}))$.*

We can represent \mathbb{F} via the *transfer function* \mathbf{G} of Σ , which is a bounded analytic $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ -valued function on some right half-plane in \mathbb{C} . We do not distinguish between two transfer functions defined on different right half-planes, if one is a restriction of the other. The connection between \mathbb{F} and \mathbf{G} is as follows: if $u \in \mathbf{L}_2(0, \infty; \mathcal{U})$, then $y = \mathbb{F}u$ has a Laplace transform \hat{y} and, for $\text{Re}(s)$ sufficiently large,

$$\hat{y}(s) = \mathbf{G}(s)\hat{u}(s), \quad (2.10)$$

and $\mathbb{F} \in \mathcal{L}(\mathbf{L}_2(0, \infty; \mathcal{U}), \mathbf{L}_2(0, \infty; \mathcal{Y}))$ if and only if $\mathbf{G} \in \mathbf{H}_\infty(\mathcal{L}(\mathcal{U}, \mathcal{Y}))$. If \mathbf{G} is a bounded, analytic, $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ -valued function defined on some right half-plane in \mathbb{C} , then a *realization* of \mathbf{G} is a well-posed linear system Σ whose transfer function is \mathbf{G} . We state a result for the strongly stable case (see [35]).

Theorem 2.4 *Every $\mathbf{G} \in \mathbf{H}_\infty(\mathcal{L}(\mathcal{U}, \mathcal{Y}))$ has realizations $\Sigma = (\mathbb{T}, \Phi, \Psi, \mathbb{F})$. For any such Σ , $\mathbb{F} \in \mathcal{L}(\mathbf{L}_2(0, \infty; \mathcal{U}), \mathbf{L}_2(0, \infty; \mathcal{Y}))$ and we can choose a Σ such that $\Psi \in \mathcal{L}(\mathcal{X}, \mathbf{L}_2(0, \infty; \mathcal{Y}))$.*

In order to obtain nice state-space formulas we need to assume a regularity condition.

Definition 2.5 *The system Σ (or its transfer function \mathbf{G}) is called weakly regular if the following limit exists in \mathcal{Y} , for all $v \in \mathcal{U}$*

$$\text{weak } \lim_{\lambda \rightarrow \infty} \mathbf{G}(\lambda)v = Dv. \quad (2.11)$$

Σ (or \mathbf{G}) is called *regular* if the limit (2.11) exists in the norm topology of \mathcal{Y} . The operator $D \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ is called the *feedthrough operator* of Σ (or of \mathbf{G}). If \mathcal{Y} is finite-dimensional, then weak regularity equals regularity. Weakly regular systems have the following nice representations.

Theorem 2.6 *If Σ is weakly regular, then the following holds.*

(i) $\mathbf{G}(s) = C_{\Lambda_w}(sI - A)^{-1}B + D;$

(ii) $\mathbb{F} : \mathbf{L}_2^{loc}(0, \infty; \mathcal{U}) \rightarrow \mathbf{L}_2^{loc}(0, \infty; \mathcal{Y})$ is given by

$$(\mathbb{F}u)(t) = C_{\Lambda_w} \int_0^t \mathbb{T}_{t-\sigma} B u(\sigma) d\sigma + Du(t), \quad (2.12)$$

for almost all $t \geq 0$;

(iii) *If x is the state trajectory of Σ corresponding to the initial state $x_0 \in \mathcal{X}$ and the input function $u \in \mathbf{L}_2^{loc}(0, \infty; \mathcal{U})$, then the output function of Σ , $y = \Psi x_0 + \mathbb{F}u$ satisfies*

$$y(t) = C_{\Lambda_w} x(t) + Du(t), \quad \text{for almost all } t \geq 0. \quad (2.13)$$

If Σ is regular, then C_{Λ} may replace C_{Λ_w} in the above. A, B, C, D are called the generating operators of Σ .

One advantage of considering (weakly) regular linear systems is that the formulas for \mathbf{G} and \mathbb{F} are the same as the finite-dimensional ones with C_{Λ_w} replacing C (for well-posed systems they are rather complicated). Moreover, the regularity property is retained under feedback. The reason for introducing the new concept of weak regularity is to obtain a nice duality theory (see Section 6 of Weiss and Weiss [35], Staffans [27] and Mikkola [19]).

Theorem 2.7 *Let Σ be a well-posed linear system with semigroup generator A , control operator B , observation operator C , and transfer function \mathbf{G} . Denote by \mathcal{Z}_1 the Hilbert space $\mathcal{D}(A^*)$ with the norm $\|z\|_1^d = \|(\beta I - A^*)z\|$, where $\beta \in \rho(A^*)$, and by \mathcal{Z}_{-1} the completion of \mathcal{X} with respect to the norm $\|z\|_{-1}^d = \|(\beta I - A^*)^{-1}z\|$. Then there exists a unique well-posed linear system Σ^d , called the dual system of Σ , such that:*

- (i) the semigroup generator of Σ^d is $A^* \in \mathcal{L}(\mathcal{Z}_1, \mathcal{X})$,
- (ii) the control operator of Σ^d is $C^* \in \mathcal{L}(\mathcal{Y}, \mathcal{Z}_{-1})$,
- (iii) the observation operator of Σ^d is $B^* \in \mathcal{L}(\mathcal{Z}_1, \mathcal{U})$,
- (iv) the transfer function of Σ^d is $\mathbf{G}^d(s) = \mathbf{G}(\bar{s})^*$.

Σ^d is weakly regular if and only if Σ is and in this case the feedthrough operator of Σ^d is D^* , where D is the feedthrough operator of Σ .

We also need a representation of the operator \mathbb{F}^* from Theorem 6.3 in Weiss and Weiss [35].

Lemma 2.8 *Suppose that Σ is a weakly regular system with generating operators A, B, C, D . If $\Psi \in \mathcal{L}(\mathcal{X}, \mathbf{L}_2(0, \infty; \mathcal{Y}))$ and $\mathbb{F} \in \mathcal{L}(\mathbf{L}_2(0, \infty; \mathcal{U}), \mathbf{L}_2(0, \infty; \mathcal{Y}))$, then for any $w \in \mathbf{L}_2(0, \infty; \mathcal{Y})$ and almost all $t \geq 0$*

$$\mathbb{F}^* w(t) = B_{\Lambda_w}^* \lim_{T \rightarrow \infty} \int_t^T \mathbb{T}_{\sigma-t}^* C^* w(\sigma) d\sigma + D^* w(t), \quad (2.14)$$

where

$$B_{\Lambda_w}^* z = \text{weak} \lim_{\lambda \rightarrow \infty} B^* \lambda (\lambda I - A^*)^{-1} z. \quad (2.15)$$

Moreover, if $u(t) = \mathbb{F}^* w(t)$, then its Laplace transform \hat{u} is defined via its boundary function which is given by

$$\hat{u} = P_{\mathbf{H}_2} \{ \mathbf{G}^* \hat{w} \}, \quad (2.16)$$

where $P_{\mathbf{H}_2}$ is the orthogonal projection from $\mathbf{L}_2(i\mathbb{R}; \mathcal{U})$ onto $\mathbf{H}_2(\mathcal{U})$.

The interpretation of (2.16) is as follows: the function $\mathbf{G}(i\omega)^* \hat{w}(i\omega)$ is in $\mathbf{L}_2(i\mathbb{R}; \mathcal{U})$ and the orthogonal projection of this function onto $\mathbf{H}_2(\mathcal{U})$ (regarded as a subspace of $\mathbf{L}_2(i\mathbb{R}; \mathcal{U})$) equals $\hat{u}(i\omega)$, the boundary function of the Laplace transform $\hat{u}(s) \in \mathbf{H}_2(\mathcal{U})$.

The following technical theorem plays a crucial role in proving our main result.

Theorem 2.9 *Let $\Sigma = (\mathbb{T}, \Phi, \Psi, \mathbb{F})$ be a well-posed linear system on $\mathcal{U}, \mathcal{X}, \mathcal{Y}$, such that $\Psi \in \mathcal{L}(\mathcal{X}, \mathbf{L}_2(0, \infty; \mathcal{Y}))$ and $\mathbb{F} \in \mathcal{L}(\mathbf{L}_2(0, \infty; \mathcal{U}), \mathbf{L}_2(0, \infty; \mathcal{Y}))$.*

- (i) *If $\tilde{\mathbb{F}} \in \mathcal{L}(\mathbf{L}_2(0, \infty; \mathcal{U}), \mathbf{L}_2(0, \infty; \mathcal{Y}))$ is shift invariant, then $\Psi^{new} = \tilde{\mathbb{F}}^* \Psi$ is an extended output map for \mathbb{T} ;*

(ii) Suppose that $\tilde{\mathbb{F}}$ has a weakly regular transfer function $\tilde{\mathbf{G}} \in \mathbf{H}_\infty(\mathcal{L}(\mathcal{U}, \mathcal{Y}))$ and suppose that $\tilde{\Sigma} = (\tilde{\mathbb{T}}, \tilde{\Phi}, \tilde{\Psi}, \tilde{\mathbb{F}})$ is a realization such that $\tilde{\Psi}$ is bounded from the state space of $\tilde{\Sigma}$, $\tilde{\mathcal{X}}$, to $\mathbf{L}_2(0, \infty; \mathcal{Y})$. If the generating operators of $\tilde{\Sigma}$ are $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$, then the observation operator of Ψ^{new} is given by

$$C^{new} x_0 = \left(\tilde{B}_{\Lambda_w}^* L + \tilde{D}^* C \right) x_0, \quad \text{for all } x_0 \in \mathcal{D}(A), \quad (2.17)$$

where $L = \tilde{\Psi}^* \Psi \in \mathcal{L}(\mathcal{X}, \tilde{\mathcal{X}})$ maps $\mathcal{D}(A)$ into $\mathcal{D}(B_{\Lambda_w}^*)$.

Moreover, L is the solution of the Sylvester equation

$$\tilde{A}^* L + L A = -\tilde{C}^* C, \quad (2.18)$$

where all terms are in $\mathcal{L}(\mathcal{X}_1, \tilde{\mathcal{Z}}_1)$.

Proof. See Weiss and Weiss [35], Theorem 11.1 and Proposition 11.2.

■

3 The singular linear quadratic optimal control problem

In this section, we consider the weakly regular linear system $\Sigma = (\mathbb{T}, \Phi, \Psi, \mathbb{F})$ under the following assumptions:

- (a) Σ has feedthrough operator 0.
- (b) $\mathbb{F} \in \mathcal{L}(\mathbf{L}_2(0, \infty; \mathcal{U}), \mathbf{L}_2(0, \infty; \mathcal{Y}))$.
- (c) $\Psi \in \mathcal{L}(\mathcal{X}, \mathbf{L}_2(0, \infty; \mathcal{Y}))$.

We denote the generating operators of Σ by $A, B, C, 0$. Assumption (b) is equivalent to the transfer function $\mathbf{G} \in \mathbf{H}_\infty(\mathcal{L}(\mathcal{U}, \mathcal{Y}))$ and (c) is equivalent to $C(sI - A)^{-1}x \in \mathbf{H}_2(\mathcal{Y})$ for all $x \in \mathcal{X}$.

Let us consider the following cost functional associated with Σ .

$$\begin{aligned} & J(x_0, u(\cdot)) \\ &= \int_0^\infty \left\langle \begin{pmatrix} Q & N^* \\ N & R \end{pmatrix} \begin{pmatrix} y(t) \\ u(t) \end{pmatrix}, \begin{pmatrix} y(t) \\ u(t) \end{pmatrix} \right\rangle_{\mathcal{Y} \times \mathcal{U}} dt \end{aligned} \quad (3.1)$$

where $R = R^* \in \mathcal{L}(\mathcal{U})$, $Q = Q^* \in \mathcal{L}(\mathcal{Y})$ and $N \in \mathcal{L}(\mathcal{Y}, \mathcal{U})$. Note that no positivity assumptions are made on R or on Q . For each initial state $x_0 \in \mathcal{X}$, u and y are related as in Theorem 2.6. The *optimal control problem* is to find the input function $u^{opt} \in \mathbf{L}_2(0, \infty; \mathcal{U})$ that minimizes $J(x_0, u(\cdot))$. We substitute $y = \Psi x_0 + \mathbb{F}u$ into (3.1) to obtain

$$\begin{aligned} J(x_0, u(\cdot)) &= \left\langle \begin{pmatrix} \Psi^* Q \Psi & \Psi^* (Q\mathbb{F} + N^*) \\ (\mathbb{F}^* Q + N) \Psi & \mathcal{R} \end{pmatrix} \begin{pmatrix} x_0 \\ u \end{pmatrix}, \begin{pmatrix} x_0 \\ u \end{pmatrix} \right\rangle_{\mathcal{X} \times \mathbf{L}_2(0, \infty; \mathcal{U})} \end{aligned} \quad (3.2)$$

where

$$\mathcal{R} = R + N\mathbb{F} + \mathbb{F}^* N^* + \mathbb{F}^* Q\mathbb{F}. \quad (3.3)$$

Under our assumptions (b), (c), we see that $J(x_0, u(\cdot))$ is finite for all $x_0 \in \mathcal{X}$ and $u \in \mathbf{L}_2(0, \infty; \mathcal{U})$. \mathcal{R} is a Toeplitz operator whose symbol is the *Popov function* associated with the above control problem.

Definition 3.1 *The Popov function $\Pi : i\mathbb{R} \rightarrow \mathcal{L}(\mathcal{U})$ associated with the weakly regular linear system Σ and the cost function $J(x_0, u)$ in (3.1) is defined by*

$$\Pi(i\omega) = R + N\mathbf{G}(i\omega) + \mathbf{G}(i\omega)^* N^* + \mathbf{G}(i\omega)^* Q\mathbf{G}(i\omega), \quad (3.4)$$

where $\mathbf{G}(i\omega) = C_{\Lambda_\omega} (i\omega I - A)^{-1} B$ for almost every $\omega \in \mathbb{R}$.

We remark that a sufficient condition for the Popov function Π to be well-defined is that \mathbb{F} be a bounded operator from $\mathbf{L}_2(0, \infty; \mathcal{U})$ to $\mathbf{L}_2(0, \infty; \mathcal{Y})$. For in this case, $\mathbf{G} \in \mathbf{H}_\infty(\mathcal{L}(\mathcal{U}, \mathcal{Y}))$ and it has an extension to $s = i\omega$ in the sense that $\lim_{\sigma \rightarrow 0} \mathbf{G}(\sigma + i\omega)u$ exists for all $u \in \mathcal{U}$ and for almost all $\omega \in \mathbb{R}$ (see Theorem 4.5 in Rosenblum and Rovnyak [24]). Moreover, $\Pi \in \mathbf{L}_\infty(\mathcal{L}(\mathcal{U}))$.

In Staffans [28] and Weiss and Weiss [35] the linear quadratic control problem for stable regular linear systems was solved under a coercivity condition on the Popov function: $\Pi(i\omega) \geq \varepsilon I$ for some positive ε . In both papers, the starting point was to establish the existence of a spectral factorization, and this is the approach we take here too. In this section, we consider the singular case ($\varepsilon = 0$) following the style and notation from [35]. The optimal control problem will not have a solution, in general, so we consider the existence of a solution to the constrained Lyapunov equation (or linear operator inequality,

c.f.(1.4),(1.5)) instead. In this section, we separate the problem into two distinct parts: the existence of a spectral factorization and the existence of a certain extended output map.

Theorem 3.2 *Let $\Sigma = (\mathbb{T}, \Phi, \Psi, \mathbb{F})$ be a weakly regular linear system satisfying assumptions (a)–(c) and consider the associated cost function (3.1). Suppose that the corresponding Popov function $\Pi \in \mathbf{L}_\infty(\mathcal{L}(\mathcal{U}))$ has a spectral factorization: for almost all $\omega \in \mathbb{R}$,*

$$\Pi(i\omega) = \Xi(i\omega)^* \Xi(i\omega), \quad (3.5)$$

where the spectral factor Ξ is in $\mathbf{H}_\infty(\mathcal{L}(\mathcal{U}))$ and its range as a multiplication operator on $\mathbf{H}_2(\mathcal{U})$ is dense in $\mathbf{H}_2(\mathcal{U})$. Denote by \mathbb{F}_Ξ the shift-invariant operator corresponding to the transfer function Ξ and define its adjoint as in (2.16) with \mathbf{G} replaced by Ξ .

(i) *If there exists an extended output map Ψ_Ξ for Ξ that satisfies*

$$\mathbb{F}_\Xi^* \Psi_\Xi = (\mathbb{F}^* Q + N) \Psi, \quad (3.6)$$

then Ψ_Ξ is the unique solution to (3.6) and $\Sigma_\Xi = (\mathbb{T}, \Phi, \Psi_\Xi, \mathbb{F}_\Xi)$ is a well-posed linear system.

(ii) *If, moreover, $\Psi_\Xi \in \mathcal{L}(\mathcal{X}, \mathbf{L}_2(0, \infty; \mathcal{U}))$, then $X = \Psi^* Q \Psi - \Psi_\Xi^* \Psi_\Xi$ ($X = X^* \in \mathcal{L}(\mathcal{X})$) satisfies*

$$A^* X x + X A x = C_\Xi^* C_\Xi x - C^* Q C x, \quad (3.7)$$

for each $x \in \mathcal{D}(A)$ and

$$J(x_0, u(\cdot)) = \langle X x_0, x_0 \rangle + \|\Psi_\Xi x_0 + \mathbb{F}_\Xi u\|^2. \quad (3.8)$$

(iii) *If, moreover, Σ_Ξ is weakly regular, with generating operators A, B, C_Ξ, D_Ξ , then for each $x \in \mathcal{D}(A)$*

$$B_{\Lambda_w}^* X x + N C x = D_\Xi^* C_\Xi x. \quad (3.9)$$

Furthermore, the cost function satisfies

$$\begin{aligned} J(x_0, u(\cdot)) &= \langle x_0, (\Psi^* Q \Psi - \Psi_\Xi^* \Psi_\Xi) x_0 \rangle \\ &+ \langle \Psi_\Xi x_0 + \mathbb{F}_\Xi u, \Psi_\Xi x_0 + \mathbb{F}_\Xi u \rangle \\ &\geq \langle x_0, X x_0 \rangle. \end{aligned} \quad (3.10)$$

We remark that equations (3.7), (3.9) correspond to equation (1.6) in the introduction, identifying D_{Ξ} with W , C_{Ξ} with L and $B_{\Lambda_w}^*$ with B^* .

Proof.

- (i) To establish the well-posedness of Σ_{Ξ} we need to verify that \mathbb{F}_{Ξ} satisfies the functional equation in part (iv) of Definition 2.1:

$$\mathbb{F}_{\Xi}(u \diamond_{\tau} v)(t) = \mathbb{F}_{\Xi} u \diamond_{\tau} (\Psi_{\Xi} \Phi_{\tau} u + \mathbb{F}_{\Xi} v)(t).$$

We can split the equation above in two equations, one for $0 \leq t < \tau$ and one for $t > \tau$. The first one is a trivial equation, the second one is equivalent to showing that

$$\mathbf{S}_{\tau}^* \mathbb{F}_{\Xi}(u \diamond_{\tau} v) = \Psi_{\Xi} \Phi_{\tau} u + \mathbb{F}_{\Xi} v. \quad (3.11)$$

As in Weiss and Weiss [35] Section 11, (3.5) implies that

$$R + N\mathbb{F} + \mathbb{F}^* N^* + \mathbb{F}^* Q\mathbb{F} = \mathbb{F}_{\Xi}^* \mathbb{F}_{\Xi}. \quad (3.12)$$

Consider now

$$\begin{aligned} & \mathbf{S}_{\tau}^*(\mathbb{F}_{\Xi}^* \mathbb{F}_{\Xi}(u \diamond_{\tau} v)) \\ &= \mathbf{S}_{\tau}^*(R + N\mathbb{F} + \mathbb{F}^* N^* + \mathbb{F}^* Q\mathbb{F})(u \diamond_{\tau} v) \quad \text{from (3.12)} \\ &= Rv + \mathbb{F}^* N^* \mathbf{S}_{\tau}^*(u \diamond_{\tau} v) + (N + \mathbb{F}^* Q)(\Psi \Phi_t u + \mathbb{F}v) \\ & \quad \text{since } \mathbb{F}^* \text{ commutes with } \mathbf{S}_{\tau}^* \text{ and } \Sigma \text{ is well-posed} \\ &= (R + N\mathbb{F} + \mathbb{F}^* N^* + \mathbb{F}^* Q\mathbb{F})v + (N + \mathbb{F}^* Q)\Psi \Phi_t u \\ &= \mathbb{F}_{\Xi}^* \mathbb{F}_{\Xi} v + \mathbb{F}_{\Xi}^* \Psi_{\Xi} \Phi_t u \quad \text{from (3.12) and (3.6)} \\ &= \mathbb{F}_{\Xi}^*(\mathbb{F}_{\Xi} v + \Psi_{\Xi} \Phi_t u). \end{aligned}$$

Thus

$$\mathbb{F}_{\Xi}^*(\mathbf{S}_{\tau}^* \mathbb{F}_{\Xi}(u \diamond_{\tau} v) - \mathbb{F}_{\Xi} v - \Psi_{\Xi} \Phi_t u) = 0,$$

and (3.11) will be established if we can show that $\ker \mathbb{F}_{\Xi}^* = 0$, or equivalently, that the range of \mathbb{F}_{Ξ} is dense in $\mathbf{L}_2(0, \infty; \mathcal{U})$. But the latter is equivalent to the range of Ξ being dense in $\mathbf{H}_2(\mathcal{U})$ which was assumed to hold. The same condition implies that the solution of (3.6) is unique.

(ii) We first apply Theorem 2.9 to \mathbb{F}_Ξ, Ψ_Ξ to obtain the extended output operator $\Psi^{new} = \mathbb{F}_\Xi^* \Psi_\Xi$ with the observation operator

$$C^{new}x = (B_{\Lambda_w}^* L_1 + D_\Xi^* C_\Xi)x, \quad \text{for all } x \in \mathcal{D}(A), \quad (3.13)$$

where $L_1 = \Psi_\Xi^* \Psi_\Xi \in \mathcal{L}(\mathcal{X})$ is the observability Gramian of Σ'_Ξ and it satisfies the Lyapunov equation

$$A^* L_1 + L_1 A = -C_\Xi^* C_\Xi. \quad (3.14)$$

Next we apply the same theorem to $\mathbb{F}, Q\Psi$ to obtain the extended output operator $\Psi^{new2} = \mathbb{F}^* Q\Psi$ with the observation operator

$$C^{new2}x = B_{\Lambda_w}^* L_2 x, \quad \text{for all } x \in \mathcal{D}(A), \quad (3.15)$$

where $L_2 = (Q\Psi)^* \Psi \in \mathcal{L}(\mathcal{X})$ satisfies the Sylvester equation

$$A^* L_2 + L_2 A = -(QC)^* C. \quad (3.16)$$

Finally, (3.6) yields

$$C^{new}x = C^{new2}x + NC,$$

which together with (3.13)–(3.16) establishes (3.7)–(3.9) with $X = L_2 - L_1$.

(iii) Now we compute the cost

$$\begin{aligned} J(x_0, u) &= \langle x_0, \Psi^* Q \Psi x_0 \rangle + \langle (R + N\mathbb{F} + \mathbb{F}^* N^* + \mathbb{F}^* Q\mathbb{F})u, u \rangle \\ &\quad + \langle \Psi^* (Q\mathbb{F} + N^*)u, x_0 \rangle + \langle (\mathbb{F}^* Q + N)\Psi x_0, u \rangle \\ &= \langle x_0, \Psi^* Q \Psi x_0 \rangle + \langle \Psi_\Xi^* \mathbb{F}_\Xi u, x_0 \rangle + \langle \mathbb{F}_\Xi^* \Psi_\Xi x_0, u \rangle \\ &\quad + \langle \mathbb{F}_\Xi^* \mathbb{F}_\Xi u, u \rangle \quad \text{using (3.6) and (3.12)} \\ &= \langle x_0, \Psi^* Q \Psi x_0 \rangle - \langle x_0, \Psi_\Xi^* \Psi_\Xi x_0 \rangle \\ &\quad + \langle \Psi_\Xi x_0 + \mathbb{F}_\Xi u, \Psi_\Xi x_0 + \mathbb{F}_\Xi u \rangle. \\ &\geq \langle x_0, (L_2 - L_1)x_0 \rangle \\ &= \langle x_0, Xx_0 \rangle \quad \text{from (ii)}. \end{aligned}$$

■

It follows from the above proof that, if there exists a solution $u \in \mathbf{L}_2(0, \infty; \mathcal{U})$ to

$$\mathbb{F}_{\Xi} u = -\Psi_{\Xi} x_0, \quad (3.17)$$

then this control is optimal and the minimum cost is $\langle x_0, X x_0 \rangle$. From (3.11) with $v = 0$ we can deduce the following important corollary.

Corollary 3.3 *Let $\Sigma = (\mathbb{T}, \Psi, \Phi, \mathbb{F})$ be a weakly regular linear system satisfying assumptions (a)–(c) and consider the associated cost functional (3.1). If (A, B) is exactly controllable (i.e., the range of Φ_{τ} is \mathcal{X} from some $\tau > 0$), and the factorization (3.5) holds, then there exists a $\Psi_{\Xi} \in \mathcal{L}(\mathcal{X}, \mathbf{L}_2(0, \infty; \mathcal{U}))$ such that $\Sigma_{\Xi} = (\mathbb{T}, \Phi, \Psi_{\Xi}, \mathbb{F}_{\Xi})$ is a well-posed system.*

This corollary is useful for hyperbolic systems with boundary control which are often exactly controllable. If C is bounded and \mathcal{Y} has finite rank, then Σ_{Ξ} is also regular (see Proposition 12.10 in Weiss and Weiss [35]).

Theorem 3.2 reveals that there are two separate problems we need to solve in order to obtain the existence of solutions to the Lyapunov equations:

- (i) the existence of a weakly regular spectral factor $\Xi \in \mathbf{H}_{\infty}(\mathcal{L}(\mathcal{U}))$ satisfying (3.5);
- (ii) the existence of an extended output map Ψ_{Ξ} for \mathbb{T} satisfying (3.6) and $\Psi_{\Xi} \in \mathcal{L}(\mathcal{X}, \mathbf{L}_2(0, \infty; \mathcal{U}))$.

We remark that if \mathbb{T} is exponentially stable, and the extended output operator Ψ_{Ξ} exists, then it is automatically in $\mathcal{L}(\mathcal{X}, \mathbf{L}_2(0, \infty; \mathcal{U}))$. These problems are discussed in Sections 4 and 5, respectively.

4 The spectral factorization problem

In this section, we present known sufficient conditions for the existence of a spectral factor Ξ as required in Theorem 3.2. A detailed account on the factorization problem for nonnegative operator-valued functions can be found in Chapter 6 of Rosenblum and Rovnyak [24]. For our applications, the following generalization of Szegő's theorem from [24], Theorem 6.14, is relevant.

Theorem 4.1 *Let Π be a weakly measurable, nonnegative, $\mathcal{L}(\mathcal{U})$ -valued function which has invertible values almost everywhere on the imaginary axis. If*

$$\int_{-\infty}^{\infty} \frac{\log^+ \|\Pi(i\omega)\|_{\mathcal{L}(\mathcal{U})}}{1 + \omega^2} d\omega < \infty, \quad (4.1)$$

and

$$\int_{-\infty}^{\infty} \frac{\log^+ \|\Pi^{-1}(i\omega)\|_{\mathcal{L}(\mathcal{U})}}{1 + \omega^2} d\omega < \infty, \quad (4.2)$$

then there exists an analytic function Ξ defined on \mathbb{C}^+ , with values in $\mathcal{L}(\mathcal{U})$ and such that

$$\Pi(i\omega) = \Xi(i\omega)^* \Xi(i\omega) \quad \text{for almost all } \omega \in \mathbb{R} \quad (4.3)$$

If $\Pi \in \mathbf{L}_{\infty}(\mathcal{L}(\mathcal{U}))$, then $\Xi \in \mathbf{H}_{\infty}(\mathcal{L}(\mathcal{U}))$ is outer, and it is unique up to multiplication by a partial isometry with initial space containing $\text{Ran } \Xi(i\omega)$ (and any final space).

In the above theorem, we mean measurability as defined in [24] on p.81; the class of Popov functions defined by (3.4) is weakly measurable in this sense. We recall from Rosenberg and Rovnyak [24], p.94, that a function $\Xi \in \mathbf{H}_{\infty}(\mathcal{L}(\mathcal{U}, \mathcal{Y}))$ is called *outer* if the closed linear span of Ξf over all $f \in \mathbf{H}_2(\mathcal{U})$ equals $\mathbf{H}_2(\mathcal{Y})$.

More specialized results for matrix-valued functions can be found in Masani and Wiener [17] and Matveev [18]. We are particularly interested in the case that Π has scalar values which is usually called Szegő's theorem (see [24], p.110).

Corollary 4.2 *Suppose that $\Pi(i\cdot)$ is a measurable function defined on \mathbb{R} and taking values in $\text{Re}(s) \geq 0$. For the existence of a function $\xi(i\cdot) \in \mathbf{H}_2$ having no zeros in \mathbb{C}^+ and such that*

$$\Pi(i\omega) = \xi(i\omega)^* \xi(i\omega) \quad \text{for almost all } \omega \in \mathbb{R} \quad (4.4)$$

it is necessary and sufficient that

$$\left| \int_{-\infty}^{\infty} \frac{\log \Pi(i\omega)}{1 + \omega^2} d\omega \right| < \infty. \quad (4.5)$$

If $\Pi \in \mathbf{L}_{\infty}$, then ξ is outer.

The following lemma will be useful in our parabolic examples in Section 6 for which G is analytic in $\operatorname{Re}(s) \geq -\varepsilon$ for some positive ε . In such cases, the Popov function has an analytic extension in the strip $|\operatorname{Re}(s)| \leq \varepsilon$ to

$$\Pi(s) = R + N\mathbf{G}(s) + \mathbf{G}(-\bar{s})^*N^* + \mathbf{G}(-\bar{s})^*Q\mathbf{G}(s). \quad (4.6)$$

Lemma 4.3 *Suppose that Π defined by (4.6) is scalar-valued, analytic in a vertical strip around the imaginary axis and and positive almost everywhere. If there exist positive constants $\alpha, \eta, M > 1$, such that*

$$\frac{\gamma}{|\omega|^\mu} \leq \Pi(i\omega) \leq \frac{M}{|\omega|^\alpha} \quad \text{for } |\omega| > \eta, \quad (4.7)$$

then (4.1) holds and Π has a spectral factorization (4.3).

Proof. First we recall some integral inequalities

$$\int_{\eta}^{\infty} \frac{\log |\omega|}{1 + \omega^2} d\omega \leq q \int_{-\infty}^{\infty} \frac{\sqrt{\omega}}{1 + \omega^2} d\omega = \frac{q\pi}{8}$$

and that $\int_{-\infty}^{\infty} \frac{1}{1 + \omega^2} d\omega = \pi$.

(a) Suppose first that Π has no zeros on the imaginary axis and that

$$M_2 \leq \log \Pi(i\omega) \leq M_1 \quad \text{for } |\omega| < \eta.$$

Then

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\log \Pi(i\omega)}{1 + \omega^2} d\omega &= \left(\int_{-\eta}^{\eta} + \int_{\eta}^{\infty} + \int_{-\infty}^{-\eta} \right) \frac{\log \Pi(i\omega)}{1 + \omega^2} d\omega \\ &\leq \int_{-\eta}^{\eta} \frac{M_1}{1 + \omega^2} d\omega + \left(\int_{\eta}^{\infty} + \int_{-\infty}^{-\eta} \right) \frac{\log M}{1 + \omega^2} d\omega \\ &\quad - \alpha \left(\int_{\eta}^{\infty} + \int_{-\infty}^{-\eta} \right) \frac{\log |\omega|}{1 + \omega^2} d\omega \\ &\leq \pi(M_1 + \log M) - 2\alpha \left(\int_{\eta}^{\infty} \frac{\log |\omega|}{1 + \omega^2} d\omega \right) \\ &< \infty. \end{aligned}$$

Similarly,

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\log \Pi(i\omega)}{1 + \omega^2} d\omega &\geq \int_{-\eta}^{\eta} \frac{M_2}{1 + \omega^2} d\omega + \left(\int_{\eta}^{\infty} + \int_{-\infty}^{-\eta} \right) \frac{\log \gamma}{1 + \omega^2} d\omega \\ &\quad - \mu \left(\int_{\eta}^{\infty} + \int_{-\infty}^{-\eta} \right) \frac{\log |\omega|}{1 + \omega^2} d\omega \\ &> -\infty, \end{aligned}$$

since all integrals are finite.

Since Π is analytic in a strip around the imaginary axis, it has at most finitely many zeros of finite multiplicity in $|\omega| < \eta$. Since $\Pi(i\omega) = \Pi(-i\omega)$, the multiplicity is even. Suppose that it has a zero at $i\omega_0$ of multiplicity k . Then in a neighbourhood of ω_0 ,

$$\Pi(i\omega) = C_1(\omega - \omega_0)^{2k} + C_2(\omega - \omega_0)^{2k+1} + \dots,$$

and $\log \Pi(i\omega) \sim \log C_1 + 2k \log |\omega - \omega_0|$. Consider the integral

$$\begin{aligned} I &= \int_{\omega_0 - \varepsilon}^{\omega_0 + \varepsilon} \frac{\log |\omega - \omega_0|}{1 + \omega^2} d\omega \\ &= \int_{-\varepsilon}^{\varepsilon} \frac{\log |y|}{1 + (y + \omega_0)^2} dy \end{aligned}$$

Now

$$M_4 = \frac{1}{1 + (1 + |\omega_0|)^2} < \frac{1}{1 + (y + \omega_0)^2} < 1$$

for $|y| \leq \varepsilon < 1$. So we have

$$2M_4 \int_0^{\varepsilon} \log |y| dy \leq \int_{-\varepsilon}^{\varepsilon} \frac{\log |y|}{1 + (y + \omega_0)^2} dy \leq 2 \int_0^{\varepsilon} \log |y| dy.$$

Finally, since $\varepsilon \log \varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$, we have $\int_0^{\varepsilon} \log |y| dy = \varepsilon \log \varepsilon - \varepsilon$ and this shows that the contribution of the integral (4.5) around a zero is bounded. ■

Theorem 4.1 provides testable sufficient conditions for the existence of the spectral factor required in part (i) of Theorem 3.2. Part (ii), however, requires a regular spectral factor and as discussed in [35], this is difficult to test a priori. The following new results give sufficient conditions which are easy to test. The first covers the case that the Popov function has zero limit at infinity.

Lemma 4.4 *Let $\Pi \in \mathbf{L}_\infty(\mathcal{L}(\mathcal{U}))$ satisfy*

$$\Pi(i\omega) \geq 0 \text{ for almost all } \omega \in \mathbb{R}. \quad (4.8)$$

and suppose that there exists a spectral factor $\Xi \in \mathbf{H}_\infty(\mathcal{L}(\mathcal{U}))$ satisfying (4.3). Then Ξ is regular with the zero feedthrough operator if for some integer $k > 0$

$$\int_0^\infty \frac{\lambda \|\Pi(i\omega)\|_{\mathcal{L}(\mathcal{U})}^k}{\lambda^2 + \omega^2} d\omega \rightarrow 0 \text{ as } \lambda \rightarrow \infty. \quad (4.9)$$

Proof. From Theorem A on p.90 in [24] we have the following representation for $\Xi^{2k} \in \mathbf{H}_\infty(\mathcal{L}(\mathcal{U}))$

$$\Xi^{2k}(x + iy) = \frac{1}{\pi} \int_{-\infty}^\infty \frac{x \Xi(i\omega)^{2k}}{x^2 + (y - \omega)^2} d\omega.$$

Thus for real $\lambda > 0$, we obtain

$$\begin{aligned} \|\Xi(\lambda)^{2k}\| &\leq \frac{1}{\pi} \int_{-\infty}^\infty \frac{\lambda \|\Xi(i\omega)\|^{2k}}{\lambda^2 + \omega^2} d\omega \\ &= \frac{2}{\pi} \int_0^\infty \frac{\lambda \|\Pi(i\omega)\|^k}{\lambda^2 + \omega^2} d\omega \\ &\rightarrow 0 \text{ as } \lambda \rightarrow \infty \end{aligned}$$

since (4.9) holds.

■

We remark that a sufficient condition for (4.9) to hold is (4.7), and so the spectral factor in Lemma 4.3 is necessarily regular.

The last result covers the case in which the Popov function $\Pi(i\omega)$ is scalar-valued and has a limit as $|\omega| \rightarrow \infty$. This is the case, for example, if Σ in Theorem 3.2 is weakly regular with feedthrough operator D and $G(s) - D$ is the Laplace transform of a function in $\mathbf{L}_1(0, \infty)$.

Lemma 4.5 *Let $\Pi \in \mathbf{L}_\infty$ satisfy (4.8) and suppose that there exists a spectral factor $\xi \in \mathbf{H}_\infty$ satisfying (4.3). If $\Pi(i\omega)$ has a limit d as $\omega \rightarrow \infty$, then there exists a regular spectral factor with feedthrough operator \sqrt{d} .*

Proof. We recall the definition of a scalar outer function defined on the unit disk from Duren [9], formula (6) on p.24. This is translated to outer functions defined on the upper half-plane in the proof of

Theorem 11.6 on p.193 of the same book. It is a simple matter to translate this to an outer function ξ defined on the right half-plane, namely,

$$\xi(s) = \exp\left\{\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1 - i\omega s}{s - i\omega} \frac{\log \Pi(i\omega)}{1 + \omega^2} d\omega\right\} e^{i\gamma}, \quad (4.10)$$

where $\Pi(i\omega) = \xi^*(i\omega)\xi(i\omega)$.

Note that

$$\begin{aligned} \xi(s) &= \exp\left\{\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{x(1 + \omega^2) + i(*)}{x^2 + (\omega - y)^2} \frac{\log \Pi(i\omega)}{1 + \omega^2} d\omega\right\} e^{i\gamma} \\ &= \exp\left\{\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{x}{x^2 + (\omega - y)^2} \log \Pi(i\omega) d\omega\right\} e^{i\bar{\gamma}}. \end{aligned} \quad (4.11)$$

So for real positive λ a spectral factor is

$$\xi(\lambda) = \exp\left\{\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\lambda \log \Pi(i\omega)}{\lambda^2 + \omega^2} d\omega\right\},$$

and applying the Lebesgue dominated convergence theorem completes the proof.

■

5 Existence of an extended output operator

The hardest part of the problem lies in establishing the existence of an extended output operator Ψ_{Ξ} satisfying (3.2). First we give fairly general necessary conditions.

Theorem 5.1 *Let $\Sigma = (\mathbb{T}, \Phi, \Psi, \mathbb{F})$ be a strongly stable, weakly regular linear system with generating operators $A, B, C, 0$. and a cost function given by (3.1). Suppose that the associated Popov function $\Pi \in \mathbf{L}_{\infty}(\mathcal{L}(\mathcal{U}))$ has a weakly regular spectral factor $\Xi \in \mathbf{H}_{\infty}(\mathcal{L}(\mathcal{U}))$, which is outer. If there exists an extended output map Ψ_{Ξ} satisfying (3.6), then $\Sigma_{\Xi} = (\mathbb{T}, \Phi, \Psi_{\Xi}, \mathbb{F}_{\Xi})$ is a weakly regular linear system with generating operators A, B, C_{Ξ}, D_{Ξ} . Moreover, for any isolated eigenvalue λ_n of A with corresponding eigenfunction e_n (3.6) is equivalent to*

$$\Xi(-\bar{\lambda}_n)^* C_{\Xi} e_n = \mathbf{G}(-\bar{\lambda}_n)^* Q C e_n + N C e_n. \quad (5.1)$$

Proof. Σ satisfies assumptions (a)–(c) and so Theorem 3.2 shows that Σ_{Ξ} is well-posed. It is weakly regular since Ξ is. We apply Lemma 2.7 to (3.6) and (2.6) to obtain

$$\begin{aligned}\mathbb{F}_{\Xi}^* \Psi_{\Xi} e_n(t) &= B_{\Lambda_w}^* \lim_{T \rightarrow \infty} \int_0^T \mathbb{T}_{\sigma}^* C_{\Xi}^* C_{\Xi} \mathbb{T}_{\sigma} \mathbb{T}_t e_n \, d\sigma \\ &= B_{\Lambda_w}^* \lim_{T \rightarrow \infty} \int_0^T \mathbb{T}_{\sigma}^* C_{\Xi}^* C_{\Xi} e^{\lambda_n(\sigma+t)} e_n \, d\sigma \\ &= B_{\Lambda_w}^* (-\lambda_n I - A^*)^{-1} C_{\Xi}^* C_{\Xi} e^{\lambda_n t} e_n \\ &= \Xi(-\bar{\lambda}_n)^* C_{\Xi} e^{\lambda_n t} e_n.\end{aligned}$$

Similarly, using Lemma 2.7 and (2.6), we obtain

$$\begin{aligned}\mathbb{F}^* Q \Psi e_n(t) &= B_{\Lambda_w}^* \lim_{T \rightarrow \infty} \int_0^T \mathbb{T}_{\sigma}^* C^* Q C e^{\lambda_n(\sigma+t)} e_n \, d\sigma \\ &= \mathbf{G}(-\bar{\lambda}_n)^* Q C e^{\lambda_n t} e_n,\end{aligned}$$

and setting $t = 0$ proves (5.1).

■

Notice that if $\Xi(-\bar{\lambda}_n)$ has an inverse in $\mathcal{L}(\mathcal{U})$, (5.1) gives a formula for a candidate C_{Ξ} . As one would expect, for sufficiency we need that C_{Ξ} be admissible and, surprisingly, we do not need a controllability assumption.

Lemma 5.2 *Consider the strongly stable, weakly regular linear system $\Sigma = (\mathbb{T}, \Phi, \Psi, \mathbb{F})$ with generating operators $A, B, C, 0$, the cost function (3.1) and the associated Popov function (3.4). Suppose that the following conditions hold:*

- (i) *the Popov function $\Pi \in \mathbf{L}_{\infty}(\mathcal{L}(\mathcal{U}))$ has a weakly regular spectral factor $\Xi \in \mathbf{H}_{\infty}(\mathcal{L}(\mathcal{U}))$;*
- (ii) *A has compact resolvent with eigenvalues $\lambda_n, n \in \mathbb{N}$ and its eigenvectors, e_n , are such that $\text{span} \{e_n, n \in \mathbb{N}\}$ is dense in \mathcal{X} ;*
- (iii) *$\Xi(-\bar{\lambda}_n)$ is invertible in $\mathcal{L}(\mathcal{U})$ for all $n \in \mathbb{N}$;*
- (iv) *C_{Ξ} is an infinite-time admissible observation operator for \mathbb{T} , where C_{Ξ} is defined by*

$$C_{\Xi} e_n = (\Xi(-\bar{\lambda}_n))^{-*} (\mathbf{G}(-\bar{\lambda}_n)^* Q C e_n + N C e_n). \quad (5.2)$$

Then the extended output operator Ψ_{Ξ} associated with C_{Ξ} is the unique solution to (3.6) and $\Psi_{\Xi} \in \mathcal{L}(\mathcal{X}, \mathbf{L}_2(0, \infty; \mathcal{U}))$.

Proof. (iii), (iv) imply the existence of a bounded extended output map Ψ_{Ξ} . Under assumption (ii), to show that (3.6) holds, it suffices to prove it for all eigenvectors e_n . Under (iii) (5.2) is equivalent to (5.1) and in Theorem 5.1 we showed that this is exactly

$$\mathbb{F}_{\Xi}^* \Psi_{\Xi} e_n = \mathbb{F}^* Q \Psi e_n + N \Psi e_n$$

for $n \in \mathbb{N}$.

■

In Hansen and Weiss [12] the following sufficient conditions are given for a control operator to be infinite-time admissible with respect to a diagonal semigroup \mathbb{T} on $\mathcal{X} = l^2$. Diagonal semigroups are generated by Riesz spectral operators, i.e., A has compact resolvent and its eigenvalues form a Riesz basis for the state space. We give the dual formulation for an observation operator.

Lemma 5.3 *Suppose that A , the generator of \mathbb{T} , is a diagonal matrix on $\mathcal{X} = l^2$ with eigenvalue eigenvector pairs $\lambda_n, e_n, n \in \mathbb{N}$ satisfying the following conditions:*

- (i) $\operatorname{Re} \lambda_n < 0$ for $n \in \mathbb{N}$;
- (ii) either \mathbb{T} is analytic or there exist numbers $0 < a \leq b$ and $\alpha \geq 0$ such that

$$a |\operatorname{Im} \lambda_n|^\alpha \leq -\operatorname{Re} \lambda_n \leq b |\operatorname{Im} \lambda_n|^\alpha. \quad (5.3)$$

Then $C \in \mathcal{L}(\mathcal{X}_1, l^2)$ is an infinite-time admissible observation operator for \mathbb{T} if and only if there exists $M \geq 0$ such that

$$\left\| \sum_{-\lambda_n \in R(h, \omega)} C e_n (C e_n)^* \right\|_{\mathcal{L}(l^2)} \leq M h, \quad (5.4)$$

where $R(h, \omega) = \{z \in \mathbb{C} | 0 < \operatorname{Re} z \leq h, \omega - h \leq \operatorname{Im} z < \omega + h\}$.

Note that an exponentially stable, invertible diagonal semigroup satisfies (5.3) with $\alpha = 0$. (5.2) specifies a candidate for C_{Ξ} if $\Xi(-\bar{\lambda}_n)$ is boundedly invertible, and if we have some lower bound on the norm of $\Xi(-\bar{\lambda}_n)^{-1}$, Lemma 5.3 can be used to test whether the candidate is admissible. While this approach is suitable for a large class of parabolic and hyperbolic exponentially stable systems, finding a suitable lower bound is not easy in general; the challenge is to do this using only the given data on $\Pi(i\omega)$. In the following lemma, we give some useful bounds on the spectral factors of Popov functions satisfying (4.7) in Lemma 4.3. They are obtained using the characterization of a scalar outer function (4.10).

Lemma 5.4 *Suppose that the scalar-valued function Π satisfies the assumptions of Lemma 4.3. Then Π has a spectral factorization (4.3), and the spectral factor ξ can be bounded below as follows.*

(i) *For sufficiently large $|\omega|$ there holds*

$$|\xi(i\omega)| \geq \frac{\sqrt{\gamma}}{|\omega|^{\mu/2}}; \quad (5.5)$$

(ii) *For $s \in \mathbb{S}_1 = \{s = x + iy \mid x \geq 0, |y| \leq \beta|x|, \beta \geq 0\}$ there exists a positive constant γ_1 such that*

$$|\xi(s)| \geq \frac{\gamma_1}{|s|^{\mu/2}} \text{ for sufficiently large } x \geq 0. \quad (5.6)$$

(iii) *For $s \in \mathbb{S}_2 = \{s = x + iy \mid 0 \leq a \leq x \leq b\}$ there exists a positive constant γ_2 such that*

$$|\xi(s)| \geq \frac{\gamma_2}{|s|^{\mu/2}} \text{ for sufficiently large } |y|. \quad (5.7)$$

Proof.

(a) (i) follows from (4.3).

(b) Let us suppose for the time being that Π has no zeros on the imaginary axis. First we make some simplifications in the analysis by some elementary observations. We note that, since $\Pi(i\omega)$ is continuous in ω , we can always choose γ and η so that

$$\Pi(i\omega) \geq \begin{cases} \gamma & |\omega| \leq \eta \\ \frac{\gamma}{|\omega|^\mu} & |\omega| \geq \eta. \end{cases} \quad (5.8)$$

Since ξ is outer, we deduce the following from (4.11)

$$\log |\xi(s)|^2 = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x \log \Pi(i\omega)}{x^2 + (\omega - y)^2} d\omega, \quad (5.9)$$

and subtracting $\log \gamma$ from both sides gives

$$\log \frac{|\xi(s)|^2}{\gamma} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x \log \{\Pi(i\omega)/\gamma\}}{x^2 + (\omega - y)^2} d\omega.$$

This shows that, without loss of generality, in proving (5.5) - (5.7) we may take $\gamma = 1$ in (5.8).

(c) We now prove (ii) under the assumption that Π has no zeros on the imaginary axis. From (5.9) we obtain

$$\begin{aligned} & \pi \log |s|^\mu |\xi(s)|^2 \\ &= \int_{-\infty}^{\infty} \frac{x \log \{|s|^\mu \Pi(i\omega)\}}{x^2 + (\omega - y)^2} d\omega \\ &= \int_{-\infty}^{\infty} \log \left\{ (x^2 + y^2)^{\mu/2} \Pi(i(y - tx)) \right\} \frac{dt}{1 + t^2} \quad (5.10) \\ &= \int_{-\infty}^{\infty} \log \left\{ \frac{(x^2 + y^2)^{\mu/2}}{|y - tx|^\mu} \right\} \frac{dt}{1 + t^2} \\ &\quad + \int_{-\infty}^{\infty} \log \{|y - tx|^\mu \Pi(i(y - tx))\} \frac{dt}{1 + t^2} \\ &\geq \frac{\pi\mu}{2} \log \left(1 + \left(\frac{y}{x} \right)^2 \right) - \int_{-\infty}^{\infty} \log \left| \frac{y}{x} - t \right|^\mu \frac{dt}{1 + t^2} \\ &\quad + \log \gamma \int_{|y-tx|>\eta} \frac{dt}{1 + t^2} + \gamma \int_{|y-tx|<\eta} \log |y - tx|^\mu \frac{dt}{1 + t^2} \\ &= \frac{\pi\mu}{2} \log \left(1 + \left(\frac{y}{x} \right)^2 \right) - \mu \int_{-\infty}^{\infty} \log \left| \frac{y}{x} - t \right| \frac{dt}{1 + t^2} \\ &\quad + \log \gamma \int_{|y-tx|>\eta} \frac{dt}{1 + t^2} \\ &\quad + \gamma\mu \log x \int_{|y-tx|<\eta} \frac{dt}{1 + t^2} + \gamma\mu \int_{|y-tx|<\eta} \log \left| \frac{y}{x} - t \right| \frac{dt}{1 + t^2} \\ &= \frac{\pi\mu}{2} \log \left(1 + \left(\frac{y}{x} \right)^2 \right) + \mu \log x \int_{|y-tx|<\eta} \frac{dt}{1 + t^2} \\ &\quad + \mu \int_{|y-tx|<\eta} \log \left| \frac{y}{x} - t \right| \frac{dt}{1 + t^2} - \mu \int_{-\infty}^{\infty} \log \left| \frac{y}{x} - t \right| \frac{dt}{1 + t^2}, \end{aligned}$$

where we in the above we have used (4.7) and $\gamma = 1$. Now for $s \in \mathbb{S}_1$, the first two terms are bounded below for sufficiently large x , and the last two sum to

$$\begin{aligned} & -\mu \int_{|y-tx|>\eta} \log \left| \frac{y}{x} - t \right| \frac{dt}{1+t^2} \\ \geq & -\mu \int_{|y-tx|>\eta} \log ||t| + \beta| \frac{dt}{1+t^2} \quad \text{for } s \in \mathbb{S}_1 \end{aligned}$$

But $\int_{-\infty}^{\infty} \log |t + \beta| \frac{dt}{1+t^2} < \infty$, using similar arguments as in the proof of Lemma 4.3 and this proves (5.6).

- (d) We now prove (iii) under the assumption that Π has no zeros on the imaginary axis. From (5.10) we have

$$\begin{aligned} \pi \log (|s|^\mu \xi(s)^2) = & \\ & \int_{|y-tx|<\eta} + \int_{|y-tx|>\eta} \log \left\{ (x^2 + y^2)^{\frac{\mu}{2}} \Pi(i(y-tx)) \right\} \frac{dt}{1+t^2}. \end{aligned}$$

We estimate the first integral

$$\begin{aligned} I_1 = & \\ & \frac{\mu}{2} \log(x^2 + y^2) \int_{|y-tx|<\eta} \frac{dt}{1+t^2} + \int_{|y-tx|<\eta} \log \Pi(iy-tx) \frac{dt}{1+t^2} \\ & \geq \frac{\mu}{2} \log(x^2 + y^2) \int_{|y-tx|<\eta} \frac{dt}{1+t^2} + \log \gamma \int_{|y-tx|<\eta} \frac{dt}{1+t^2} \\ & \geq \frac{\mu}{2} \log(a^2 + y^2) \int_{|y-tx|<\eta} \frac{dt}{1+t^2} + 0 \\ & \geq 0 \quad \text{for sufficiently large } |y|, \end{aligned}$$

where we have taken $\gamma = 1$ used $s \in \mathbb{S}_2$.

For the second integral we have

$$\begin{aligned} I_2 = & \int_{|y-tx|>\eta} \log \left\{ \frac{(x^2 + y^2)^{\mu/2}}{|y-tx|^\mu} \right\} \frac{dt}{1+t^2} + \\ & \int_{|y-tx|>\eta} \log \{ |y-tx|^\mu \Pi(i(y-tx)) \} \frac{dt}{1+t^2} \\ \geq & \frac{\mu}{2} \log \left(1 + \left(\frac{x}{y} \right)^2 \right) \int_{|y-tx|>\eta} \frac{dt}{1+t^2} \end{aligned}$$

$$\begin{aligned}
& - \mu \int_{|y-tx|>\eta} \log \left| 1 - \frac{tx}{y} \right| \frac{dt}{1+t^2} + \log \gamma \int_{|y-tx|>\eta} \frac{dt}{1+t^2} \\
& \geq -\mu \int_{|y-tx|>\eta} \log \left| 1 - \frac{tx}{y} \right| \frac{dt}{1+t^2} \\
& \geq -\mu \int_{|y-tx|>\eta} \log \left(1 + \left| \frac{at}{y} \right| \right) \frac{dt}{1+t^2} \\
& \geq -\mu \int_{|y-tx|>\eta} c \left(1 + \left| \frac{at}{y} \right| \right)^{\frac{1}{2}} \frac{dt}{1+t^2} \\
& \geq -\mu \int_{-\infty}^{\infty} c \left(1 + |c_1 t| \right)^{\frac{1}{2}} \frac{dt}{1+t^2} \\
& \geq 0 \quad \text{for sufficiently large } |y|,
\end{aligned}$$

where we have used $s \in \mathbb{S}_2$ and taken $\gamma = 1$. Combining the estimates for I_1 and I_2 proves (iii).

- (e) Next we extend (ii) and (iii) to allow for zeros on the imaginary axis. As in (c) of the proof of Lemma 4.3, there will be at most finitely many zeros on the imaginary axis and they will have even multiplicity. Let $i\omega_0$ be a zero of Π with multiplicity $2k$. Note that

$$\Pi_0(i\omega) = \Pi(i\omega) \left| \frac{1 + i(\omega - \omega_0)}{i\omega - i\omega_0} \right|^{2k}$$

also satisfies (5.8) and it has no zero at $i\omega_0$. On examining the proofs in (c) and (d), we see that we only need to show that the following integral is bounded below

$$\begin{aligned}
I_3 &= \int_{-\infty}^{\infty} \frac{x}{x^2 + (\omega - y)^2} \log \left| \frac{i\omega - i\omega_0}{1 + i(\omega - \omega_0)} \right|^{2k} d\omega \\
&= 2k \int_{-\infty}^{\infty} \frac{x}{x^2 + (\omega - y)^2} \log \left| \frac{i\omega - i\omega_0}{1 + i(\omega - \omega_0)} \right| d\omega.
\end{aligned}$$

Now the function $f(s) = \frac{s - i\omega_0}{1 + s - i\omega_0}$ is outer, since it is in \mathbf{H}_∞ and it has no zeros in the right half-plane. So using (4.11) we have

$$\begin{aligned}
I_3 &= 2k\pi \log |f(s)| \quad \text{for } x \geq 0 \\
&= k\pi \log \left[\frac{x^2 + (y - \omega_0)^2}{(x + 1)^2 + (y - \omega_0)^2} \right]
\end{aligned}$$

Consider first the case $s \in \mathbb{S}_1$:

$$\begin{aligned} I_3 &\geq k\pi \log \left[\frac{1}{\left(1 + \frac{1}{x}\right)^2 + \left(\left|\frac{y}{x}\right| + \left|\frac{\omega_0}{x}\right|\right)^2} \right] \\ &> k\pi \log \left[\frac{1}{\left(1 + \frac{1}{2|\omega_0|}\right)^2 + \left(\beta + \frac{1}{2}\right)^2} \right] \quad \text{for } x > 2|\omega_0|, \end{aligned}$$

and so I_3 is bounded below for $s \in \mathbb{S}_1$ and sufficiently large x . Consider now the case $s \in \mathbb{S}_2$:

$$I_3 \geq k\pi \log \left[\frac{\left(1 - \left|\frac{\omega_0}{y}\right|\right)^2}{\left(\frac{b+1}{y}\right)^2 + \left(1 + \left|\frac{\omega_0}{y}\right|\right)^2} \right]$$

which is clearly bounded below for $s \in \mathbb{S}_2$ and sufficiently large $|y|$.

■

We note that a result like part (ii) was proved using a similar proof in Pandolfi [21]. The \mathbb{S}_1 bound corresponds to exponentially stable analytic semigroups (e.g., parabolic) for which the spectrum of A is contained in a wedge in the left half-plane, while \mathbb{S}_2 corresponds to exponentially stable hyperbolic semigroups for which A has its spectrum in a vertical strip in the left half-plane. A significant class not covered by the lemmas in this section is those systems described by delay equations. Although the span of the eigenfunctions may be dense in the state space (and so Lemma 5.2 holds), they will not form a Riesz basis and so the sufficient conditions of Lemma 5.3 do not apply. We suggest an alternative approach in Section 7. Finally, we give a result concerning the existence of an optimal control.

Lemma 5.5 *Suppose that the conditions of Lemma 5.2 are satisfied and, in addition, that $\Xi(s)$ has an inverse in $\mathcal{L}(\mathcal{U})$ for $s \in \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. Then there exists an optimal control $u \in \mathbf{L}_2(0, \infty; \mathcal{U})$ satisfying (3.17) if and only if for all $x_0 \in \mathcal{X}$*

$$\Xi(s)^{-1} C_{\Xi}(sI - A)^{-1} x_0 \in \mathbf{H}_2(\mathcal{U}). \quad (5.11)$$

Proof. (3.17) has the frequency-domain equivalent

$$\Xi(s)\hat{u}(s) = -C_{\Xi}(sI - A)^{-1} x_0 \quad (5.12)$$

and $u \in \mathbf{L}_2(0, \infty; \mathcal{U})$ if and only if $\hat{u} \in \mathbf{H}_2(\mathcal{U})$.

■

We remark that we would not expect that condition (5.11) would be satisfied, in general. Consider the scalar case where the outer property of Ξ implies that $\Xi(s)^{-1}$ is holomorphic in \mathbb{C}^+ , but not that it is bounded in norm there. Unfortunately, it is not sufficient to check (5.11) on the imaginary axis, so even in the scalar case (5.11) is hard to verify. Since most applications do not require the existence of an optimal control, but just a solution to the constrained Lyapunov equation (3.7)–(3.9), we shall not pursue the existence of solutions to (3.11) further.

6 Scalar Riesz Spectral Systems

Theorems 3.2, 4.1, and Lemmas 4.7, 5.2, 5.3 and 5.4 can be combined to show the existence of a solution to the Lur'e equations (3.7), (3.9) for systems with a scalar transfer function and a generating operator A that is Riesz spectral and its eigenvalues are contained either in a wedge or in a vertical strip in the left half-plane. In this section, we illustrate this approach by considering examples of scalar parabolic systems that satisfy a positive-real condition. In this case the Popov function satisfies a bound (4.7) and we can use the useful estimates in Lemmas 4.3 and 5.4. The singular positive-real lemma for parabolic systems was first considered in Balakrishnan [2] and it is interesting to compare our results with his and those in Pandolfi [21]. Moreover, our results have immediate applicability to the theory for adaptive observers and compensators in Curtain, Demetriou and Ito [23]. We consider the class of systems described as follows.

$$\begin{aligned} \frac{\partial z}{\partial t} &= \frac{\partial^2 z}{\partial x^2} + b(x)u_1(t), \quad z(0, t) = 0, \quad z(\pi, t) = u_2(t), \\ y_1(t) &= \int_0^1 c(x)z(x, t)dx, \quad y_2(t) = z(x_2, t), \quad y_3(t) = \frac{\partial z}{\partial x}(x_3, t), \end{aligned}$$

where $b, c \in \mathbf{L}_2(0, \pi) = \mathcal{X}$, the state space.

We let

$$\mathcal{D}(A) = \left\{ \begin{array}{l} h \in \mathbf{L}_2(0, \pi) : h, \frac{dh}{dx} \text{ are absolutely continuous,} \\ \frac{d^2h}{dx^2} \in \mathbf{L}_2(0, \pi) \text{ and } h(0) = 0 = h(\pi) \end{array} \right\}$$

and define

$$Ah = \frac{d^2 h}{dx^2} \quad \text{for } h \in \mathcal{D}(A).$$

Then A has compact resolvent, eigenvalues $\lambda_n = -n^2$, $n \in \mathbb{N}$ and eigenvectors $e_n = \sqrt{2} \sin(nx)$, $n \in \mathbb{N}$, which form an orthonormal basis for $\mathbf{L}_2(0, \pi)$. A is self-adjoint and it generates an exponentially stable contraction semigroup. We shall consider the following scalar systems:

Example 6.1 Distributed control and smooth observation: $u_2 = 0$, $y = y_1$, and

$$b(x) = c(x) = \begin{cases} \frac{1}{2\varepsilon} & x_0 - \varepsilon \leq x \leq x_0 + \varepsilon \\ 0 & \text{elsewhere} \end{cases};$$

Example 6.2 Boundary control and smooth observation: $u_1 = 0$, $y = y_1$, $c(x) = 1$;

Example 6.3 Distributed control and point observation: $u_2 = 0$, $b(x) = 1$, $y = y_3$, $x_3 = 0$.

All examples can be formulated as regular linear systems with generating operators A, B, C with respect to a suitable state space, where

$$\begin{aligned} Bu &= bu, & b &= \sum_{n=1}^{\infty} b_n e_n \\ Cx &= \langle x, c \rangle, & c &= \sum_{n=1}^{\infty} c_n e_n. \end{aligned}$$

Example 6.1 has bounded B and C operators, but the others have an unbounded B or C and to ensure admissibility it is necessary to choose the state space \mathcal{Z} carefully (see Curtain and Weiss [5]). Lemma 5.3 shows that B will be infinite-time admissible if

$$\sum_{k=1}^{k=n} b_k^2 \leq M(n^2 + 1), \quad (6.1)$$

for some positive constant M ; hence it suffices to show that

$$|b_n| \leq M_1 \sqrt{n}, \quad |c_n| \leq M_1 \sqrt{n}, \quad (6.2)$$

for some positive constant M_1 .

Example 6.1 $\mathcal{Z} = \mathcal{X} = \mathbf{L}_2(0, \pi)$, $b_n = c_n = \frac{\sqrt{2}}{n\varepsilon}(\sin nx_0 \sin n\varepsilon)^2$;

Example 6.2 $\mathcal{Z} = \mathcal{X}_{-\frac{1}{2}}$, where the latter is the completion of \mathcal{X} with respect to the norm $\|x\|_{-\frac{1}{2}} = \|(-A)^{-\frac{1}{2}}x\|$.

$$b_n = \sqrt{2n}(-1)^n, c_n = \sqrt{\frac{2}{n}}(1 - (-1)^n);$$

Example 6.3 $\mathcal{Z} = \mathcal{D}((-A)^{\frac{1}{2}})$, $b_n = \sqrt{2n}(1 - (-1)^n)$, $c_n = \sqrt{\frac{2}{n}}(-1)^n$, or, alternatively, $\mathcal{Z}_1 = \mathcal{X}_1$, $b_n = \sqrt{2}(1 - (-1)^n)$, $c_n = \sqrt{2}(-1)^n$.

The transfer functions are of the form

$$\mathbf{g}^j(s) = \sum_{n=1}^{\infty} \frac{a_n^j}{s + n^2}, \quad (6.3)$$

where j refers to Example 6. j , $j = 1, 2, 3$, and $a_n^j = c_n b_n$ are

$$a_n^1 = \frac{2}{(n\varepsilon)^2}(\sin nx_0 \sin n\varepsilon)^2;$$

$$a_n^2 = a_n^3 = \begin{cases} 4 & \text{for odd } n \\ 0 & \text{for even } n \end{cases}$$

Example 6.1 is both approximately controllable and observable, provided that x_0 is not a rational multiple of π . Example 6.2 is approximately controllable, but not approximately observable, while Example 6.3 is approximately observable, but not approximately controllable. Choosing $Q = 0$, $N = 1$, $R = 0$ we obtain the Popov function

$$\Pi^j(i\omega) = \mathbf{g}^j(i\omega)^* + \mathbf{g}^j(i\omega) = \sum_{n=1}^{\infty} \frac{2n^2 a_n^j}{\omega^2 + n^4}, \quad (6.4)$$

and Π^j is positive and continuous for $j = 1, 2, 3$. In order to use Lemma 5.4(ii), we need some estimates on $\Pi^j(i\omega)$. Using the comparison of

$$\sum_{n=1}^{\infty} \frac{1}{n^p(\omega^2 + n^4)} \quad \text{with} \quad \int_1^{\infty} \frac{dx}{x^p(\omega^2 + x^4)}$$

for integers p and interpolation for fractions, we obtained asymptotic estimates with positive μ and γ of the form

$$\Pi^j(i\omega) \sim \frac{\gamma}{|\omega|^\mu} \quad \text{for sufficiently large } \omega, \quad (6.5)$$

k	$\geq 3 + \delta$	2	3/2	1	1/2	0	$-1 + \delta$
μ	2	3/2	5/4	1	3/4	1/2	$\delta/2$

Table 1: $\delta > 0$.

where μ depends on the asymptotic behaviour of the coefficients a_n^j :

$$a_n^j \sim \frac{1}{n^k} \text{ for sufficiently large } n \quad (6.6)$$

as displayed in Table 1.

$$\text{(Note that } \int_1^\infty \frac{x^2 dx}{x^k(x^4 + \omega^2)} \sim \frac{1}{|\omega|^\mu} \text{).}$$

If k in (6.6) satisfies $k \geq -1 + \delta$ for some positive δ , then Π^j will be continuous and it will satisfy an estimate (6.5) for some $\mu > 0$. So by Lemmas 4.3 and 4.4, there exists a regular spectral factor with feedthrough operator 0 for any Π^j of the form (6.4) where a_n^j satisfies (6.6) for a $k \geq -1 + \delta$. From Lemma 5.2, the candidate for the observation operator of ξ is given by

$$C_\xi e_n = \xi(n^2)^{-1} C e_n = \xi(n^2)^{-1} c_n. \quad (6.7)$$

From (6.2) we see that for the admissibility of C_ξ we require that

$$|C_\xi e_n| \leq M_1 \sqrt{n}. \quad (6.8)$$

We recall from Lemma 5.4 that

$$\xi(n^2) \geq \frac{\gamma_1}{n^\mu} \text{ for sufficiently large } n, \quad (6.9)$$

and so combining (6.7), (6.9), we obtain the estimate

$$|C_\xi e_n| \leq \frac{n^\mu}{\gamma_1} |c_n|, \quad (6.10)$$

and so we require that

$$n^\mu |c_n| \leq M_2 \sqrt{n}, \quad (6.11)$$

for some positive constant M_2 .

We now apply this analysis to our examples.

Example 6.1: $k = 1$, $\mu = 1$, $c_n \sim \frac{1}{n}$. So $|C_\xi e_n| \sim \text{constant}$ and C_ξ is admissible, but unbounded. In this example, although B and C are bounded operators, the C_ξ operator is unbounded.

Example 6.2: $k = 0$, $\mu = \frac{1}{2}$, $c_n \sim \frac{1}{\sqrt{n}}$. So $|C_\xi e_n| \leq M_3$ and C_ξ is admissible, but unbounded, with respect to the new state space \mathcal{Z} . In this example, C was bounded, but B unbounded (both with respect to \mathcal{X}).

Example 6.3: $k = 0$, $\mu = \frac{1}{2}$. With the state space $\mathcal{X}_{\frac{1}{2}}$, $c_n \sim \sqrt{n}$ and so $|C_\xi e_n| \leq M_3 n$ and C_ξ is not admissible. However, if we choose the state space to be \mathcal{X}_1 , $c_n \sim 1$ and so $|C_\xi e_n| \leq M_4 \sqrt{n}$ and C_ξ is admissible with respect to \mathcal{X}_1 . In this example, we had a bounded B , but unbounded C with respect to \mathcal{X}_1 .

So in all the above examples we obtain a solution $X \in \mathcal{L}(\mathcal{Z})$ to the Lur'e equations

$$A^* X z + X A z = C_\xi^* C_\xi z; (B^* X + C) z = 0, \quad (6.12)$$

where $z \in \mathcal{D}(A) = \mathcal{Z}_1$, provided that we choose the state space \mathcal{Z} as given above. We remark that the above analysis is valid for any spectral system with a self-adjoint operator A with eigenvalues of the order of n^2 , and a transfer function of the form (6.3). In fact, even for the worst case where $|c_n b_n| \sim n^{1-\delta}$, $k = -1 + \delta$ for a positive δ . Then from Table 1 we obtain $\mu = \frac{\delta}{2}$ and $|\xi(n^2)| \geq \text{const.} n^{\frac{\delta}{2}}$. Allowing for the possibility of shifting the state space, we can take $b_n \sim \sqrt{n}$ and we obtain an (just) admissible C_ξ . In the analysis of Pandolfi [21], he assumes that the C operator is bounded, B may be unbounded, and $\mu \leq 1$. We do not need these restrictions; our Example 6.1 would be excluded from his analysis, despite it having bounded B and C operators. Example 6.2 is excluded because C is unbounded. Note, however, that we do need to shift the state space in Example 6.3.

It is interesting to ask what conditions we must place on the coefficients a_n^j to obtain a bounded observation operator C_ξ . This depends on both c_n and b_n in a rather complicated way. We require that there exist positive constants M_5 , ε , such that

$$c_n \leq \frac{M_5}{n^{\frac{1}{2} + \mu + \varepsilon}}, \quad (6.13)$$

where μ is related to the k in the estimate (6.6) for a_n^j . We show that (6.10) will be satisfied for the following choice of an unbounded B and

a bounded C .

Example 6.4: We choose \mathcal{X} as the state space and $c_n = \frac{1}{n^2}$ and $b_n = \sqrt{n}$. C is a bounded observation operator and B is an unbounded, but admissible control operator. We have $k = \frac{3}{2}$, $\mu = \frac{5}{4}$ and $C_\xi e_n \sim \frac{1}{n^{\frac{3}{4}}}$ which is a bounded operator.

In the above example, the analysis is helped by having an unbounded B , but a smooth C operator; $c \in \mathcal{D}(A^{\frac{1}{2}})$. In Balakrishnan [2], to obtain a bounded C_ξ he required much stronger conditions; $b, c \in \mathcal{D}(A)$. Our analysis shows that this leads to $k = 5 + \delta$, $\mu = 2$, and $C_\xi e_n \sim \frac{1}{n^{\frac{1}{2} + \varepsilon}}$, where $\varepsilon > 0$. So, despite the extra smoothness of B and C , C_ξ is only just bounded.

We remark that for Examples 6.2 and 6.3 it is possible to obtain a closed form expression for the transfer function using the methods in chapter 4 in Curtain and Zwart [6]. We illustrate this approach with a final parabolic example with both B and C unbounded.

Example 6.4:

$$\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2}, \quad z(0, t) = 0, \quad z(\pi, t) = u(t), \quad y(t) = z(1, t).$$

This has the following transfer function

$$g(s) = \frac{\sinh \pi \sqrt{s}}{\pi \sqrt{s} \cosh \pi \sqrt{s}}$$

and we obtain

$$\operatorname{Re} g(i\omega) = \frac{1}{\pi \sqrt{2\omega}} \frac{\sinh \pi \sqrt{2\omega} + \sin \pi \sqrt{2\omega}}{\cosh \pi \sqrt{2\omega} + \cos \pi \sqrt{2\omega}}.$$

So Π is positive and $\Pi(i\omega) \sim \frac{1}{\sqrt{\omega}}$. The operator A for this example is as before except that its domain it has the boundary conditions $h(0) = 0$, $h'(1) = 0$. It has the eigenvalues $\lambda_n = -(n + \frac{1}{2})^2$ and eigenvectors $e_n(x) = \sqrt{2} \sin(n + \frac{1}{2})x$, $n = 0, 1, 2, \dots$. Using the same notation as before, we obtain $c_n = b_n = \sqrt{2}$, $a_n \sim 1$, $k = 0$, $\mu = \frac{1}{2}$. So

$$\xi(-\lambda_n) \sim \frac{1}{\sqrt{n}}, \quad |C_\xi| \leq \sqrt{n}.$$

So we obtain an admissible observation operator C_ξ .

7 Retarded systems

We motivate an approach suitable for delay systems using a positive-real example.

Example 7.1 Consider the delay system

$$\dot{x}(t) = -ax(t) - bx(t-1) + u(t); \quad a, b > 0 \quad (7.1)$$

$$y(t) = x(t) \quad (7.2)$$

with the transfer function

$$g(s) = \frac{1}{s + a + be^{-s}}. \quad (7.3)$$

Clearly, $g(s) \in \mathbf{H}_\infty$ if $a - |b| \geq \mu > 0$. Now consider the problem for the special case $Q = 0, N = I, R = 0$ and calculate the Popov function.

$$\begin{aligned} \Pi(j\omega) &= g(j\omega) + g(j\omega)^* \\ &= \frac{2(a + b \cos \omega)}{(a + b \cos \omega)^2 + (\omega - b \sin \omega)^2} \\ &\geq 0 \quad \text{if } a \geq |b|. \end{aligned}$$

So the system is positive-real if $a \geq |b|$. In this case, it is easy to find the spectral factor

$$\xi(s) = \frac{\alpha + \beta e^{-s}}{s + a + be^{-s}}, \quad \text{where } \alpha^2 + \beta^2 = 2a, \quad \alpha\beta = b. \quad (7.4)$$

$\xi \in \mathbf{H}_\infty$ and the candidate for C_ξ is

$$(C_\xi x)(t) = \alpha x(t) + \beta x(t-1). \quad (7.5)$$

The delay system (7.1), (7.2) can be formulated on the state-space $\mathcal{X} = \mathbb{C} \oplus \mathbf{L}_2(-1, 0)$ with generating operators defined by

$$Bu = \begin{pmatrix} u \\ 0 \end{pmatrix}, \quad C \begin{pmatrix} r \\ f(\cdot) \end{pmatrix} = r, \quad (7.6)$$

$$\mathcal{D}(A) = \left\{ \begin{pmatrix} r \\ f(\cdot) \end{pmatrix} \in \mathcal{X} \mid f \text{ is absolutely continuous,} \right. \\ \left. \frac{df}{d\theta}(\cdot) \in \mathbf{L}_2(-1, 0) \text{ and } f(0) = r \right\},$$

$$A \begin{pmatrix} r \\ f(\cdot) \end{pmatrix} = \begin{pmatrix} -ar - bf(-1) \\ \frac{df}{d\theta} \end{pmatrix} \quad (7.7)$$

(see Curtain and Zwart [6], Chapter 2.4). It is interesting to note that while C and B are bounded operators, C_ξ is not. However, it is known that it is admissible (Salamon, [25]), and that the span of the eigenvectors of A is dense in \mathcal{X} (see Curtain and Zwart [6] Theorem 2.5.10). So appealing to Lemma 5.2, it suffices to verify (5.2) for each eigenvector

$$e_n = \begin{pmatrix} 1 \\ e^{\lambda_n} \end{pmatrix} \quad (7.8)$$

corresponding to the eigenvalue λ_n :

$$\Delta(\lambda_n) = \left(\lambda_n + a + be^{-\lambda_n} \right) = 0 \quad (7.9)$$

(see Curtain and Zwart [6], Theorem 2.4.6). Now with $Q = 0, N = I$, we have

$$Ce_n = 1, \quad C_\xi e_n = \alpha + \beta e^{-\lambda_n},$$

and

$$\begin{aligned} \xi(-\bar{\lambda}_n)^* C_\xi e_n &= \frac{\alpha + \beta e^{\lambda_n}}{-\lambda_n + a + be^{\lambda_n}} \cdot (\alpha + \beta e^{-\lambda_n}) \\ &= \frac{\alpha^2 + \beta^2 + \alpha\beta(e^{\lambda_n} + e^{-\lambda_n})}{-\lambda_n + a + be^{\lambda_n}} \\ &= \frac{2a + b(e^{\lambda_n} + e^{-\lambda_n})}{2a + b(e^{\lambda_n} + e^{-\lambda_n})} \\ &= \text{from (7.4) and (7.9)} \\ &= Ce_n \quad \text{as desired.} \end{aligned}$$

In fact, it is easily verified that the solution to the equations (3.7), (3.9) is

$$Q = \begin{pmatrix} I & 0 \\ 0 & \beta^2 I \end{pmatrix}.$$

Note that the candidate solution to (3.17) for the optimal control is $\hat{u}(s) = x_0$ which is not on \mathbf{H}_2 .

From the above example, we learn that for retarded systems even with bounded B and C operators we can expect an unbounded C_{Ξ} operator. Since we have explicit representations for admissible observation operators for retarded systems, we can obtain more explicit conditions for the existence of solutions to our problem by generalizing the approach in the above example to allow for several inputs and outputs and finitely many discrete delays. We consider the following class of retarded systems.

$$\dot{x}(t) = A_0 x(t) + \sum_{j=1}^p A_j x(t - h_j) + B_0 u(t); \quad (7.10)$$

$$x(0) = r, \quad x(\theta) = f(\theta), \quad -h_p \leq \theta < 0; \quad (7.11)$$

$$y(t) = C_0 x(t), \quad (7.12)$$

where $0 < h_1 < \dots < h_p$ represent the point delays, $x(t) \in \mathbb{C}^n$, $A_j \in \mathcal{L}(\mathbb{C}^n)$, $j = 0, 1, \dots, p$, $B_0 \in \mathcal{L}(\mathbb{C}^m, \mathbb{C}^n)$, $C_0 \in \mathcal{L}(\mathbb{C}^n, \mathbb{C}^q)$, $r \in \mathbb{C}^n$, and $f \in \mathbf{L}_2(-h_p, 0; \mathbb{C}^n)$. The delay system (7.10)–(7.12) can be formulated on the state space $\mathcal{X} = \mathbb{C}^n \oplus \mathbf{L}_2(-h_p, 0; \mathbb{C}^n)$ with generating operators defined by

$$Bu = \begin{pmatrix} B_0 u \\ 0 \end{pmatrix}, \quad C \begin{pmatrix} r \\ f(\cdot) \end{pmatrix} = C_0 r, \quad (7.13)$$

$$\mathcal{D}(A) = \left\{ \begin{pmatrix} r \\ f(\cdot) \end{pmatrix} \in \mathcal{X} \mid f \text{ is absolutely continuous,} \right. \\ \left. \frac{df}{d\theta}(\cdot) \in \mathbf{L}_2(-h_p, 0; \mathbb{C}^n) \text{ and } f(0) = r \right\},$$

$$A \begin{pmatrix} r \\ f(\cdot) \end{pmatrix} = \begin{pmatrix} A_0 r + \sum_{j=1}^p A_j f(-h_j) \\ \frac{df}{d\theta} \end{pmatrix}. \quad (7.14)$$

A generates a C_0 semigroup \mathbb{T} on \mathcal{X} and B, C are bounded operators (see Curtain and Zwart [6], Chapter 2.4 and Theorems 4.2.6, 4.2.10). The spectrum of A , $\sigma(A)$ consists of eigenvalues which are the solutions of $\det(\Delta(\lambda)) = 0$, where for $\lambda \in \mathbb{C}$

$$\Delta(\lambda) = \lambda I - A_0 - \sum_{j=1}^p A_j e^{-\lambda h_j}. \quad (7.15)$$

\mathbb{T} is exponentially stable if

$$\sigma(A) \subset \{s | \operatorname{Re} s < -\mu\} \quad \text{for some } \mu > 0. \quad (7.16)$$

The eigenvectors of A are given by

$$e_n = \begin{pmatrix} r_n \\ e^{\lambda_n \cdot} r_n \end{pmatrix}, \quad \text{where } \Delta(\lambda_n)r_n = 0, r_n \neq 0. \quad (7.17)$$

The span of the generalized eigenvectors of A are dense in \mathcal{X} if ([6], Theorem 2.5.10)

$$\det(A_p) \neq 0. \quad (7.18)$$

We recall that

$$(sI - A)^{-1}e_n = \frac{1}{s - \lambda_n}e_n. \quad (7.19)$$

(see [6], Chapter 2.4). We also need to consider a more general type of observation operator

$$C_{\Xi} \begin{pmatrix} r \\ f(\cdot) \end{pmatrix} = C_{-1}r + \sum_{j=1}^p C_j f(-h_j) + \int_{-h_p}^0 C_{00}(\theta) f(\theta) d\theta \quad (7.20)$$

where $C_j \in \mathcal{L}(\mathbb{C}^n, \mathbb{C}^q)$, $j = -1, 1, \dots, p$ and $C_{00}(\cdot) \in \mathbf{L}_2(-h_p, 0; \mathcal{L}(\mathbb{C}^n, \mathbb{C}^q))$. $C_{\Xi} \in \mathcal{L}(\mathcal{X}_1, \mathbb{C}^q)$ is also an infinite-time admissible observation operator for \mathbb{T} if (7.16) holds (see Salamon [25]). The transfer function \mathbf{G} of the system with generating operators $A, B, C, 0$ is given by

$$\mathbf{G}(s) = C_0 \Delta(s)^{-1} B_0, \quad (7.21)$$

and A, B, C_{Ξ}, D_{Ξ} are generating operators of a regular linear system with transfer function Ξ given by

$$\Xi(s) = C_h(s) \Delta(s)^{-1} B_0 + D_{\Xi}, \quad (7.22)$$

where

$$C_h(s) = C_1 + \sum_{j=1}^p C_j e^{-sh_j} + \int_{-h_p}^0 C_{00}(\theta) e^{s\theta} d\theta. \quad (7.23)$$

The above facts together with Lemma 5.2 provide an approach for the class of retarded systems defined by (7.10)–(7.12).

Lemma 7.2 Consider the regular linear system Σ with generating operators $A, B, C, 0$ given by (7.13)–(7.14) and a Popov function given by (3.4). Suppose that the following assumptions are satisfied:

- (i) \mathbb{T} is exponentially stable ((7.16) holds);
- (ii) the span of the eigenvectors of A is dense in \mathcal{X} ((7.18) holds and there are no generalized eigenvectors);
- (iii) there exists a spectral factor Ξ for Π of the form (7.22) for a certain operator C_h given by (7.23);
- (iv) C_h satisfies

$$\begin{aligned} \Xi(-\bar{\lambda}_n)^* C_h(-\lambda_n) r_n = \\ B_0^* \Delta(-\bar{\lambda}_n)^{-*} C_0^* Q C_0 r_n + N C_0 r_n, \end{aligned} \quad (7.24)$$

for all $n \in \mathbb{N}$, where r_n, Δ are given by (7.17), (7.15), respectively.

Then there exists an $X \in \mathcal{L}(\mathcal{X})$ such that (3.7), (3.9) hold.

Once we have a candidate for Ξ , we can try to match it with an admissible \tilde{C} from the known terms on the right-hand side of (7.20). Of course, there is no guarantee that we will succeed, but it is a starting point. We use the above approach in two examples, the first a positive-real multi-input multi-output system.

Example 7.3 Consider the delay system (7.10)–(7.12) with $p = 1, h_1 = h, A_1 = \alpha I$ and the Popov function parameters $Q = 0, N = 1, R = 0$ under the following assumptions:

- (A_0, B_0, C_0) is controllable and observable;
- $A_0 + \alpha I$ is Hurwitz ;
- $\mathbf{G}_0(s) = C_0(sI - A_0)^{-1} B_0$ satisfies

$$G_0(i\omega)^* + G_0(i\omega) \geq 0 \quad \text{for } \omega \in \mathbb{R}.$$

Then by the finite-dimensional positive-real lemma ([1]) there exist matrices $L_0, P = P^* \geq 0, Q = Q^* > 0$ that satisfy the Lur'e equations

$$A_0^*P + PA_0 = -L_0^*L_0 \quad (7.25)$$

$$B_0^*P = C_0 \quad (7.26)$$

$$(A_0^* + \alpha I)Q + Q(A_0 + \alpha I) = -I \quad (7.27)$$

Suppose further that the following two conditions hold:

$$L_0^*L_0 - 2|\alpha|P \geq 0, \quad (7.28)$$

$$h < \frac{\lambda_{\min}(Q)}{\lambda_{\max}^2(Q)|\alpha|(\|A_0\| + |\alpha|)}. \quad (7.29)$$

An easy calculation gives

$$\begin{aligned} \Pi(i\omega) &= \mathbf{G}(i\omega)^* + \mathbf{G}(i\omega) \\ &= B_0^*(-i\omega I - A_0^* - \alpha e^{i\omega h})^{-1}(L_0^*L_0 - 2\alpha P \cos \omega h)(i\omega I - A_0 - \alpha e^{-i\omega h})^{-1}B_0 \\ &\geq B_0^*(-i\omega I - A_0^* - \alpha e^{i\omega h})^{-1}(L_0^*L_0 - 2|\alpha|P)(i\omega I - A_0 - \alpha e^{-i\omega h})^{-1}B_0 \\ &\geq 0, \end{aligned}$$

where we have used (7.25), (7.26) and (7.28). (7.27) and (7.29) imply that A generates an exponentially stable semigroup (see Halanay [11], p.377). We now find a suitable C_{Ξ} operator of the form:

$$C_{\Xi} \begin{pmatrix} r \\ f(\cdot) \end{pmatrix} = C_1 r + C_2 f(-h).$$

Ξ given by (7.22) is a spectral factor if we can find square matrices C_1, C_2 that satisfy

$$C_1^*C_1 + C_2^*C_2 = -L_0^*L_0; \quad C_1^*C_2 = C_2^*C_1 = -\alpha P.$$

Without loss of generality we can take C_1, C_2 to be symmetric and α to be positive. We choose a square matrix F such that

$$(C_1 + C_2)^*(C_1 + C_2) = L_0^*L_0 - 2\alpha P = F^*F.$$

Then, substituting $C_2 = F - C_1$ we obtain the Riccati equation for C_1 :

$$C_1F + F^*C_1 - 2C_1C_1 + 4\alpha P = 0.$$

This has a solution which we take to be C_1 and $C_2 = F - C_1$. So we have satisfied all the assumptions of Lemma 7.2 and there exists a solution of the Lur'e equations.

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