AN INVENTORY MODEL WITH SPECIAL SALE

by

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Institute of Statistics
Mimeograph Series No. 583
May 1968
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CHAPTER 1: INTRODUCTION

1.1 Description of an Inventory Model

We consider an inventory process dealing with the procuring, storing, and selling of a single commodity. The inventory process is divided into "review periods" of equal length. Stock level is reviewed periodically and is known only at the beginning of a review period. We will refer to the set of points where the stock level is known as "review points." Procurement decisions are made at review points, and delivery of goods is instantaneous. By "instantaneous," we mean that delivery takes a negligible amount of time compared to the length of a period.

There are two modes of selling within each period. One is called the "regular sale" and the other the "special sale." The regular sale is more or less a regular type of selling as usually encountered in inventory literature. The regular sale period covers the whole review period except perhaps when the special sale is in effect. Sales are made in the regular sale period whenever the stock is adequate to meet the demand. The part of demand which exceeds stock is considered lost and not carried over to the next period. A suitably defined shortage cost is charged in this case.

The special sale can be given at the beginning of a period after the procurement and before the regular sale. Goods are sold in the special sale usually under a lower-than-regular price. The underlying assumption is that there exists demand which cannot be attracted to the regular sale, but can be attracted to the special sale through lower pricing. Some of the reasons why a special sale may be desirable are:
a. When large stock has been piled up and high holding cost is expected;

b. When goods are perishable and possible loss of goods in stock is expected;

c. When the special sale has promotional effect for the succeeding regular sale;

d. When the special sale itself is a profitable business.

To effectively relieve the situations stated in a and b, as well as not to interfere with the regular sale too long, it is required that the special sale be a quick sale - "quick" in the sense that the special sale period is almost instantaneous compared to the length of the regular sale period. This requirement can be realized if the special sale price is sufficiently low.

Besides the above timing requirement, we also require that the special sale and the regular sale be conducted under one roof so that goods allocated to the special sale but not sold there can be immediately transferred back to the regular sale with zero cost. This "under-one-roof" requirement does not preclude the possibility to sell the commodity to a second market at a different location. For instance, if the sale to the second market is through a local dealer who can be contacted by telephone, then we consider that the special sale is conducted where the telephone is.

Conceivably, the inventory situation can be such that a special sale is not desirable at all for some particular period; or if it is, then the best strategy is to allocate only a part of the physical stock to the special sale. These questions are subject to management's decision.
By "management," we mean the party who is responsible for the optimization of the total cost of the inventory process. The amount allocated to the special sale will be referred to as the special stock. During the special sale period, sales are made as long as the special stock is adequate to meet the demand. The part of demand which exceeds the special stock is considered lost but no shortage cost is charged. The hypothesis is that management has no obligation to satisfy demand in special sales, while it does have the obligation in regular sales.

Since inventory decisions are made periodically, we will be concerned only with the total demand in the special sale period and the total demand in the regular sale period. The former will hereafter be referred to as the special-sale demand and the latter, the regular demand. The special-sale demand and the regular demand of any period are random variables. As it is likely that the special sale will have some effect on the succeeding regular demand, we make the assumption that the regular demand depends on the amount of goods sold in the special sale. This amount will be defined as zero when there is no special sale in the period. The dependence on the special sale can, of course, go either way. When the special sale has promotional effect, then the larger the special sale, the larger will be the succeeding regular demand. In other cases, a large amount of special sale may hurt the succeeding regular demand. Demand in the $i^{th}$ time period is independent of demand in any other time period.

The relevant inventory costs include the usual procuring cost, holding cost, shortage cost, and sale revenue (the last item is interpreted as a negative cost). All costs are assumed proportional to the
quantity involved. Thus we can talk about cost on a unit goods basis. The objective is to optimize the expected total cost where the expectation is taken over all possible demand realizations. Decision variables subject to optimization are:

a. The amount to be procured in each period (including zero as a possibility).

b. The amount to be allocated to the special stock (again including zero as a possibility).

1.2 Some Realistic Aspects of the Model

Let us look at some of the real-life examples in which special sale is offered alongside with regular sale.

a. **Inventory-Type Special Sale.** The motivation for this type of special sale is mainly to relieve high inventory costs or to dispose of goods which may perish soon. The special sale itself may lose money but is still desirable as the lesser of two evils. The inventory-type special sale can be found in practice in all kinds of stores. When special sale is given to relieve inventory cost, certainly no procurement will be made for that particular period. That positive procurement and special sale do not occur together is a characteristic of the optimal policy for this type of special sale.

b. **Supplementary-Type Special Sale.** The motivation for this type of special sale is the same as that for the regular sale - profit. Basement sales in a department store, or special sales which are offered quite regularly, usually are
mere attempts to catch as many customers and make as much profit as possible and may not be necessitated by large inventory situation. Another example is the dairy store which delivers dairy products to subscribing customers, but also sells to customers who occasionally visit the store. The two sales differ in service and could as well differ in price. But it is likely that the dairy store makes money on both of them.

c. The Promotional-Type Special Sale. The motivation for this special sale is either to introduce a new product or to keep reminding customers of an old product. The sale itself is treated as some sort of advertising and thus is not required to make immediate profit by itself. It is still a profit-oriented sale as distinguished from the inventory-type special sale (which does not seek profit at all) in the sense that the promotional special sale is believed to be responsible for the profits made in the succeeding regular sale. Many soft-drink companies use the strategy of lowering the price of a beverage once in a while to lure customers back. In some extreme cases, companies send out free samples to potential customers, and thus incurring even a negative sale revenue.

Now we comment on the particular set-up of the special sale in our model. Let us arbitrarily say that a time period starts with a stock-review. Since the question "how many goods should be procured" can be answered only if we know how many goods are there in stock, it seems logical that we adhere to the usual assumption that the procuring point
follows the review point immediately. There are now two possible cases: either the special sale proceeds the regular sale, or vice versa. For both cases, it is in general desirable to have a decision variable controlling the amount of stock allocated to the first sale, since an optimal allocation is not necessarily the whole stock. However, it is not possible to place a similar control on the second sale simply because the stock level at the second sale is assumed unknown. In this study we will consider the case when special sale preceds regular sale. In those cases when starting inventory are high and an inventory-type special sale is called for, then it is desirable to have special sale preceding regular sale so that saving of holding cost and of loss of perished goods could be achieved during the time period.

1.3 Motivation and Methodology

In mathematical inventory literature, the historical interest has been overwhelmingly concerned with the procuring activity. "Selling" has rarely been considered as a decision variable except possibly in Karlin and Carr's article on pricing policy [15]. In this study, we treat selling as a decision variable from a different viewpoint. We still consider price as fixed in our model, but we allow two modes of selling which differ not only in price but also in other aspects, and work on the optimal combination of them. The problem is further complicated by the fact that the regular demand depends on the amount of goods sold in the preceding special sale.

By the introduction of a selling-decision variable in addition to the procuring-decision variable, we now have a two-dimensional optimization problem. Our approach is first to work on the special
case - the single-stage model, where the inventory process consists of a single period. Then we apply Bellman's "Principle of Optimality" and construct recurrence equations for the multistage model. This is the classical "functional equation method" so popular in inventory literature. Optimal policies are found to be of simple forms characterized by critical numbers. Analytic solutions for critical numbers are obtainable for the infinite-stage model when demand distributions do not vary over time. They are useful not only in providing approximate solutions of critical numbers for the finite-stage process, but also in revealing the structural features of optimal policies.

When demand distributions do vary over time, we extend Karlin's work from his one-decision-variable case to our two-decision-variable case. In this case we are able to obtain more general results due to the special features of our particular model.

1.4 Review of Literature

Arrow, Harris, and Maschak [1] popularized the functional equation method in dealing with mathematical inventory problems. Their attention, in their 1951 paper [1], was focused on a special-type policy. Dvoretzky et al. [7] examined the results of Arrow et al., with full generality. Bellman [4], Bellman et al. [5], Karlin [13], and Iglehart [11], [12] considered the existence and uniqueness of the solution to the functional equation in an infinite-stage model.

For the one-decision-variable optimization inventory problem, Bellman et al. [5] established conditions under which the optimal policy will have the simple form of a series of critical numbers
\( \bar{x}_1, \bar{x}_2, \ldots \), called a base stock policy. If the stock level at the beginning of period \( n \) is below \( \bar{x}_n \), a procurement in the amount of the difference is made; otherwise no procurement is made.

When demand distributions vary from period to period, it is in general difficult to obtain analytic solutions for the critical numbers. Karlin [14] showed that when demand distributions are increasing stochastically over time, then the optimal policy of the first period is identical to the first-period optimal policy of a problem in which demand distributions are identical to the first period. Veinott [18], [19] extended Karlin's results to the case when demand distributions are "stochastically increasing in translation." Namely, let \( D_i \) be the \( i \)th period demand variable and \( \phi_{D_i} \) its distribution function, then the inequality \( \phi_{D_i}(\varepsilon + a_i) \geq \phi_{D_{i+1}}(\varepsilon) \) holds for all \( \varepsilon \) and \( i = 1, 2, 3, \ldots \), where \( a_i > 0 \) is the minimum value \( D_i \) can take.

For the treatment of two-dimensional (two-decision-variables) optimization inventory problems, Fukuda [8] considered the optimal procuring and disposal policy in a multiechelon inventory system. He excluded explicitly the possibility that procuring and disposal may occur in the same period. Phelps [16] considered a similar problem for the single-echelon system, except that in his model there are two modes of procuring. The one with a cheaper procuring cost has a limited supply whose amount depends on previous policy. Barankin [3], Daniel [7], and Fukuda [9] considered regular and emergency procuring policies where a premium is paid for shorter delivery time.

For general reference on inventory control theory, see [2], [10], [17], and [20].
CHAPTER 2: THE SINGLE-STAGE PROCESS

2.1 Model

In the single-stage process the decision maker wants to determine his optimal procurement and sales policy when his time horizon is limited to a single period only. These policies will obviously depend on the nature of the market demand and supply functions.

From the supply side, it is assumed that the firm can procure instantaneously an unlimited quantity of the particular commodity at a fixed price c per unit quantity. Let x be the stock level at the beginning of the period and let y be the inventory level after procurement. Then Max(0, y-x) is the amount procured. Let Max(0, y-z) be the amount of goods allocated for special sale. Then the stock level available for regular sale is at least z.

In this study we shall assume that the special-sale demand D* is a random variable with a known probability distribution function \( P_{D*} \) at the special sale price \( r^* \) per unit quantity. The minimum value of \( D^* \) is a known number \( a^* \) where \( a^* \geq 0 \). If "I" is the amount sold in the special sale, then obviously \( I = \min(D^*, y-z) \).

The demand distribution at the regular sale will depend upon the amount sold in the special sale. If the two sales are competitive, then regular sale is likely to be reduced; whereas if the special sale is promotional in nature, then regular sale is likely to increase. Let the random variable \( D(I) \) denote the regular demand at price \( r \) per unit quantity where the quantity sold on special sale is known to be I. In this discussion, we shall assume \( D(I) = D - qI \) where the parameter q is known and \( D \) is a random variable following known probability
distribution function \( \Phi_D \) where \( \Phi_D \) is independent of \( \Phi_{D^*} \). Note that \( D(0) = D \), thus \( D \) can be thought of as what the regular demand will be if nothing is sold in the special sale. Given \( I \), then the regular demand distribution function is \( \Phi_{D-qI} \), i.e., \( P_r(DI/I=i) \leq \epsilon) = \Phi_{D-qI}(\epsilon) = \Phi_D(\epsilon+qi) \).

The parameter \( q \) in our model will be negative if the special sale is promotional in nature where as if it is competitive, then \( q \) will be positive. Demands in regular sale and special sale will obviously be independent if \( q = 0 \). In this study, we shall consider only those distributions which have continuous densities and the expectations of the associated random variables are finite.

One of the basic assumptions about the random variable \( D(I) \) is that it is nonnegative. This introduces a certain restriction on the maximum value "I" can take when \( q \) is positive. Let the minimum value of \( D \) be \( a > 0 \). Then \( D(I) \) will be a nonnegative random variable for every realization of \( I \) if we assume that the upper bound of \( D^* \) is \( a/q \) where \( q > 0 \). (It is not necessary to put any such restriction on the range of \( D^* \) if \( q \leq 0 \).) Thus there is an obvious limitation on our model when \( q > 0 \). But in many such situations, "a" is a relatively large number and "q" is a small positive fraction, so that the assumption \( \Psi_{D^*}(a/q) = 1 \) is not likely to be a severe restriction.

In order to introduce more flexibility in the model, we shall introduce a parameter \( p \) to take into account any kind of leakage within the system. If there is no leakage, then the left-over amount after special sale, e.g., \( y-I \), will be available for regular sale. But in our model we shall assume that \( (1-p)(y-I) \) amount of goods will not be available at the regular sale where \( p \) lies between 0 and 1. The
proportion 1-p takes into account any kind of leakage within the system during the regular sale period. Since the special sale is conducted for a short duration at the beginning of the time period under study, the loss due to leakage during this period will be assumed to be negligible. For a system without leakage, p, of course, will be one.

Besides the procurement cost, a second set of costs is associated with the stock of inventories on hand. Storage or holding costs may be incurred by the actual maintenance of stocks or the rent of storage space. Each unit of commodity sold in the special sale is charged a holding cost $h_1 > 0$. Each unit of goods not sold in the special sale is charged a holding cost $h_1 + h_2 > h_1$. We have assumed that storage or holding cost is proportional to the size of the stock of inventory. However, other cases may be considered. For example, if storage takes place in a warehouse, the unit storage cost may jump from a low value to a high value if the stock exceeds a certain quantity.

The holding cost arises when supply exceeds demand. And when demand exceeds supply (since it may be impossible or too costly to guarantee that demand will be met under all circumstances), another type of cost known as penalty cost or shortage cost arises. The failure to meet demand generates cost though in different ways under different situations. Often, it involves loss of good will on the part of the customer, thereby giving rise to the ticklish problem of determining the monetary value of goodwill. In this study we have assumed that the firm incurs a cost $s > 0$ when demand exceeds supply by unit amount and the total shortage cost is proportional to the total excess demand. Moreover shortage cost applies only to regular sale.
Finally, there is the problem of disposal of goods left at the end of the period. It is assumed that they can be disposed of with zero cost.

Within the framework outlined above, the firm wants to determine the procurement level $y$ and the amount to be allocated for special sale $y-z$ in such a way that the expected total cost is minimized. Naturally, for a policy to be admissible, $y$ should be at least as large as $x$ and $z$ should lie between 0 and $y$.

2.2 Mathematical Formulation

Let $f_x(y,z,\varepsilon,\varepsilon^*)$ denote the total cost of the single-period process given a policy $(y,z)$ and the initial stock level $x$ when the special-sale demand is $\varepsilon^*$ and the regular demand is $\varepsilon$. Then

$$f_x(y,z,\varepsilon,\varepsilon^*) = c(y-x) + h_1 y - r^*\varepsilon^* + h_2(y-\varepsilon^*) - r\varepsilon$$

for $\varepsilon^* \leq y-z$ and $\varepsilon \leq p(y-\varepsilon^*)$.

$$= c(y-x) + h_1 y - r^*\varepsilon^* + h_2(y-\varepsilon^*) - rp(y-\varepsilon^*) + s(\varepsilon-p\varepsilon^*)$$

for $\varepsilon \geq p(y-\varepsilon^*)$.

$$= c(y-x) + h_1 y - r^*(y-z) + h_2 z - rpz + s(\varepsilon-pz)$$

for $\varepsilon \leq pz$. (2.2.1)
Let

\[
L_x(y,z) = E[l_x(y,z,\varepsilon,\varepsilon*)],
\]

where the expectation is over the regular demand and the special demand variables.

Then

\[
L_x(y,z) = c(y-x) + h_1 y
\]

\[
+ \int_{y-z}^{y-z} \left\{ - r^* \varepsilon^* + h_2 (y-\varepsilon^*) + \int_{a-\varepsilon^*}^{p(y-\varepsilon)} - r \varepsilon \phi_D(\varepsilon+q\varepsilon^*)d\varepsilon \right\} d\varepsilon
\]

\[
+ \int_{p(y-\varepsilon^*)}^{\infty} \left\{ [ - r p(y-\varepsilon^*) + s(\varepsilon-py+q\varepsilon^*)] \phi_D(\varepsilon+q\varepsilon^*)d\varepsilon \right\} \psi_D(\varepsilon^*)d\varepsilon
\]

\[
+ \int_{y-z}^{\infty} \left\{ - r^* (y-z) + h_2 z + \int_{a-q(y-z)}^{p(z)} - r \varepsilon \phi_D(\varepsilon+qy-qz)d\varepsilon \right\} d\varepsilon
\]

\[
+ \int_{p(z)}^{\infty} \left\{ [ - r p z + s(\varepsilon-pz)] \phi_D(\varepsilon+qy-qz)d\varepsilon \right\} \psi_D(\varepsilon^*)d\varepsilon
\]

(2.2.2)

where \( \phi_D^* \) and \( \varepsilon_D \) denote the density function of the random variables \( D^* \) and \( D \) respectively. We also assume that \( D^* \) and \( D \) have finite mean values so that \( L_x(y,z) \) is finite iff \( y \) is finite. It is also evident that \( L_x(y,z) \) is a continuous function of \( y \) and \( z \).
Definition: If a policy \((y', z')\) is such that

\[
L_x(y', z') = \min_{x<y<\infty} \min_{0<z<y} L_x(y, z),
\]

then \((y', z')\) is an optimal policy. Let

\[
g(y, a) = cy + h_1 y
\]

\[
+ \left\{ \begin{array}{l}
y-z \\
- r^*x^* + h_2(y-x^*) + \int \frac{p(y-\varepsilon)}{a-\varepsilon^*} \int \phi_D(\varepsilon + q \varepsilon^*) d\varepsilon \\
\end{array} \right.
\]

\[
+ \left\{ \begin{array}{l}
\int \frac{[- r^p(y-x^*) + s(\varepsilon - py + p \varepsilon^*)] \phi_D(\varepsilon + q \varepsilon^*) d\varepsilon}{p(y-\varepsilon^*)} \\
\end{array} \right.
\]

\[
+ \left\{ \begin{array}{l}
y-z \\
- r^*(y-z) + h_2 z + \int \frac{p^z}{a-q(y-z)} \int \phi_D(\varepsilon + qy-\varepsilon qz) d\varepsilon \\
\end{array} \right.
\]

\[
+ \left\{ \begin{array}{l}
\int \frac{[- r^p z + s(\varepsilon - p^2)] \phi_D(\varepsilon + qy-\varepsilon qz) d\varepsilon}{p^z} \\
\end{array} \right. \psi_D(\varepsilon^*) d\varepsilon^*.
\]

Then

\[
L_x(y, z) = -cx + g(y, z)
\]

In order to obtain the optimum value of \(y\) and \(z\), we use the technique of stepwise minimization. Let \(z_y\) be the value of \(z\) at which \(g(y, z)\) attains its minimum when \(y\) is a finite known nonnegative
number. Since \( g(y,z) \) is a continuous function of \( z \) and the range of \( z \) is a closed interval, there exists a \( \bar{z}_y \) at which \( g(y,z) \) attains its minimum.

Let \( W(y) = g(y,\bar{z}_y) \). Now if \( W(y) \) attains its minimum at a finite value \( \bar{y} \), then \( (\bar{y},\bar{z}_y) \) will be our optimum policy. For if \( (y,z) \) be an arbitrary feasible policy, then

\[
g(y,z) \geq g(y,\bar{z}_y) = W(y) \geq W(\bar{y}) = g(\bar{y},\bar{z}_y).
\]

2.3 Optimal Special Sale Policy

Let \( \bar{z}_y \) denote an optimal special sale policy when the procurement level \( y \) is given. Then the structure of \( \bar{z}_y \) is given by Theorem 1.1 when \( p \geq q > 0 \). The structure of \( \bar{z}_y \) when \( p \geq q > 0 \) is not true will be discussed later in the chapter.

Theorem 1.1: If \( p \geq q \geq 0 \) and

(a) if \( r^* + h_2 + sq - rp - sp \geq 0 \), then \( \bar{z}_y = 0 \)

(b) if \( r^* + h_2 - rq \leq 0 \); then \( \bar{z}_y = y \)

(c) if \( r^* + h_2 + sq - rp - sp < 0 < r^* + h_2 - rq \);

then there exists a uniquely determined number \( Z \), \( a < Z < \infty \), such that

\[
\bar{z}_y = 0 \quad \text{for} \quad \frac{Z}{q} < y < \infty
\]

\[
= \frac{Z-qy}{p-q} \quad \text{for} \quad \frac{Z}{p} \leq y \leq \frac{Z}{q}
\]

\[
= y \quad \text{for} \quad 0 \leq y < \frac{Z}{p}
\]

Proof: Since \( L_x(y,z) = -cx + g(y,z) \), it is sufficient to minimize \( g(y,z) \) to obtain the minimum of \( L_x(y,z) \) for a given value of \( x \) and \( y \).

Differentiating \( g(y,z) \) with respect to \( z \), we obtain:
\[ D_2 g(y,z) = (r^* + h_2 + sq - rp - sp + \int_{a-q(y-z)}^{pz} (p-q)(r+s)\phi(\varepsilon + qy-qz) d\varepsilon) \]

\[ [1 - \psi_{D^*}(y-z)] \]

\[ = V(qy+pz-qz)[1 - \psi_{D^*}(y-z)] \text{ for } y-z \leq \frac{a}{q}, \quad (2.3.1) \]

where

\[ V(\theta) = r^* + h_2 + sq - rp - sp + (r+s)(p-q)\phi_D(\theta). \quad (2.3.2) \]

and

\[ D_2 g(y,z) = 0 \quad \text{for } y-z > \frac{a}{q} \quad (2.3.3) \]

since

\[ 1 - \psi_{D^*}(y-z) = 0 \quad \text{for } y-z > \frac{a}{q} \]

(a) Now

\[ r^* + h_2 + s q - rp - sp + (r+s)(p-q)\phi_D(qy+pz+qz) \]

\[ \geq r^* + h_2 + sq - rp - sp \]

since \( p \geq q \) and \( \phi_D \geq 0 \).

Moreover \( 1 - \psi_{D^*}(y-z) \geq 0 \).

\[ . \quad D_2 g(y,z) \geq 0 \text{ in (a).} \]

\[ . \quad g(y,z) \text{ is a monotone-nonincreasing continuous function of } z \text{ for a fixed } y. \text{ Now if } D_2 g(y,z) \text{ is strictly positive then } \overline{z}_y = \min_{0 \leq z \leq y} z = 0. \]

Otherwise there may be a closed interval of optimal values of \( z \).

Since in any case, 0 will be included in the closed interval, \( \overline{z}_y = 0 \) is an optimal policy in (a).
(b) \[ r^* + h_2 + sq - rp - sp + (r+s)(p-q)\phi_D(qy+pz-qz) \]
\[ \leq r^* + h_2 - rq \quad \text{as} \quad \phi_D(qy+pz-qz) \leq 1 \]

\[ \therefore \quad D_2g(y,z) \leq 0 \text{ in (b)} \]

\[ \therefore \quad g(y,z) \text{ is a monotone-nonincreasing function of } z \text{ for every finite value of } y. \]

\[ \therefore \quad \text{The minimum of } g(y,z) \text{ is attained either at } z = y \text{ or there may be a closed interval with } y \text{ as the right end point such that at any point of the interval, } g(y,z) \text{ attains minimum. In any case } z_y = y \text{ is an optimum policy.} \]

(c) If \[ r^* + h_2 + sq - rp - sp < 0 < r^* + h_2 - rq, \]
then because of the continuity of \( \phi_D \), there exists a unique number \( Z \) such that \( V(Z) = 0 \) and \( V(\theta) < 0 \) for all \( \theta < Z \). If \( y > \frac{Z}{q} \), then \( qy + (p-q)z > Z \) for every \( 0 \leq z \leq y \); and if \( y < \frac{Z}{p} \), then \( qy + (p-q)z < Z \) for every admissible \( z \). Since \( \phi_D \) is a monotone-nondecreasing function,
\[ D_2g(y,z) \geq 0 \text{ if } y > \frac{Z}{q}; \quad \text{and} \quad D_2g(y,z) \leq 0 \text{ if } y < \frac{Z}{p}. \]
Accordingly as in case (a) and case (b), an optimum solution of \( z \) is given by \( z_y = 0 \) if \( \frac{Z}{q} < y < \infty \), and \( z_y = y \) when \( 0 \leq y < \frac{Z}{p} \).

Now when \( \frac{Z}{p} < y < \frac{Z}{q} \):
\[ D_2g(y,z) \leq 0 \text{ when } qy + (p-q)z \leq Z \text{ or } z \leq \frac{Z-qy}{p-q}, \text{ and} \]
\[ D_2g(y,z) \geq 0 \text{ when } z > \frac{Z-qy}{p-q}. \]

\[ \therefore \quad g(y,z) \text{ will attain minimum when } Z = \frac{Z-qy}{p-q}. \]

In this case also, there may be a range of values of \( z \) for which \( g(y,z) \) will attain minimum, but \( \frac{Z-qy}{p-q} \) will be an optimal \( z \).
2.4 Optimal Procurement Policy

Substituting \( \bar{z} \) for \( z \) in \( g(y,z) \) and calling the new function \( W(y) \), we will show that \( W(y) \) is convex in \( y \).

For easier reference, we list the results of \( D_1 g(y,z) \), \( D_2 g(y,z) \), \( D_{11} g(y,z) \), \( D_{22} g(y,z) \) and \( D_{12} g(y,z) \) here. For their derivations, consult Appendix 7.1.

\[
D_1 g(y,z) = \frac{\partial g(y,z)}{\partial y} \\
= \int_{a^*}^{y-z} [c + h_1 + h_2 - rp - sp \psi_D(\epsilon^*) \psi_D(\epsilon^*) d\epsilon^* + (r+s)\psi_D(py - pe^* + qe^*)] \psi_D(\epsilon^*) d\epsilon^* \\
+ \int_{y-z}^{\infty} [c + h_1 - r^* - sq \psi_D(qy + pz - qz)] \psi_D(\epsilon^*) d\epsilon^* \tag{2.4.1}
\]

\[
D_2 g(y,z) = \frac{\partial g(y,z)}{\partial z} \\
= [r^* + h_2 + sq - rp - sp + \int_{a-q(y,z)}^{pz} (p-q)(r+s)\psi_D(\epsilon^*) d\epsilon] \\
[1 - \psi_D(\epsilon^*) (y-z)] \tag{2.4.2}
\]
\[
D_{11} g(y, z) = \frac{\partial^2 g(y, z)}{\partial y^2} \\
= \int y^2 (p\phi_D(py^2q\psi_D^\epsilon \psi_D^\epsilon)dy^2q\psi_D^\epsilon d\epsilon \\
+ [r^2+h^2+sq-rp-sp+\int a^2 (p-q)(r-s)\phi_D(qy^2q\psi_D^\epsilon \psi_D^\epsilon d\epsilon)dy^2q\psi_D^\epsilon d\epsilon [1 - \psi_D^\epsilon (y-z)] \\
(2.4.3)
\]

\[
D_{22} g(y, z) = \frac{\partial^2 g(y, z)}{\partial z^2} \\
= [r^2+h^2+sq-rp-sp+\int a^2 (p-q)(r+s)\phi_D(qy^2q\psi_D^\epsilon \psi_D^\epsilon d\epsilon)dy^2q\psi_D^\epsilon d\epsilon [1 - \psi_D^\epsilon (y-z)] \\
(2.4.4)
\]

\[
D_{12} g(y, z) = - [r^2+h^2+sq-rp-sp+\int a^2 (p-q)(r+s)\phi_D(qy^2q\psi_D^\epsilon \psi_D^\epsilon d\epsilon)dy^2q\psi_D^\epsilon d\epsilon [1 - \psi_D^\epsilon (y-z)] \\
+ (r+s)(p-q)q\phi_D(qy^2q\psi_D^\epsilon \psi_D^\epsilon d\epsilon)dy^2q\psi_D^\epsilon d\epsilon [1 - \psi_D^\epsilon (y-z)]. \\
(2.4.5)
\]

Theorem 1.2: \( W'(y) \) is a continuous monotone-nondecreasing function for \( 0 \leq y < \infty \).

Proof: Note that:

\[
W'(y) = D_{1g}g(y, y) + D_{2g}g(y, y) \frac{dy}{dx} (2.4.6)
\]
and

\[ W''(y) = D_{11}g(y, \bar{z}_y) + 2D_{12}g(y, \bar{z}_y) \frac{dz}{dy} + D_{22}g(y, \bar{z}_y) \left( \frac{dz}{dy} \right)^2 \]

\[ + D_{2}g(y, \bar{z}_y) \frac{d^2 z}{dy^2} \]

whenever \( \frac{dz}{dy} \) and \( \frac{d^2 z}{dy^2} \) exist.

Proof will be partitioned into three cases.

a. \( 0 \leq r^* + h_2 + sq - rp - sp \)

From Theorem 1.1, \( z_y = 0 \). \( \therefore \) \( \frac{dz}{dy} = 0 \) and \( \frac{d^2 z}{dy^2} = 0 \).

From (2.4.6),

\[ W'(y) = D_{1}g(y,0) \] (2.4.8)

From (2.4.7),

\[ W''(y) = D_{11}g(y,0). \] (2.4.9)

Substituting \( z = 0 \) in (2.4.3), then \( \psi_{D^*}(y-z) = 1 \) and \( \psi_{D^*}(y-z) = 0 \)

since \( y-z > Z/q > a/q \). Hence

\[ D_{11}g(y,0) = \int_a^y (r+s)p^2 \phi_D(py-p\epsilon^*+q\epsilon^*)\psi_{D^*}(\epsilon^*) d\epsilon^* \geq 0. \]

b. \( r^* + h_2 - rq \leq 0 \).

From Theorem 1.1, \( z_y = y \). \( \therefore \) \( \frac{dz}{dy} = 1 \) and \( \frac{d^2 z}{dy^2} = 0 \). From (2.4.6),

\[ W'(y) = d_1g(y,y) + D_{2}g(y,y) \] (2.4.10)
From (2.4.7),

\[ W''(y) = D_{11}g(y,y) + 2D_{12}g(y,y) + D_{22}g(y,y) \]

\[ = (r+s)q^2 \phi_D(qy) + 2(r+s)(p-q)q \phi_D(qy) + (r+s)(p-q)^2 \phi_D(qy) \]

\[ = (r+s)p^2 \phi_D(qy) \geq 0. \tag{2.4.11} \]

\[ c. \quad r^* + h_2 + sq - rp - sp < 0 < r^* + h_2 - rp \]

From Theorem 1.1

\[ \bar{z}_y = 0 \quad \text{for } \frac{Z}{q} < y \]

\[ = \frac{Z-qy}{p-q} \quad \frac{Z}{p} \leq y \leq \frac{Z}{q} \]

\[ = y \quad 0 \leq y \leq \frac{Z}{p} \]

\[ \frac{dz}{dy} \text{ exists except on the two points } \frac{Z}{p} \text{ and } \frac{Z}{q}. \]

For \( \frac{Z}{q} < y \), \( \frac{dz}{dy} = 0 \), and \( \frac{d^2z}{dy^2} = 0 \). Thus from (2.4.6) and (2.4.7):

\[ W'(y) = D_{11}g(y,0) \tag{2.4.12} \]

\[ W''(y) = D_{11}g(y,0) \tag{2.4.13} \]

Substitute \( z = 0 \) in (2.4.3). The second term is now zero since \( y-z = Z/q > a/q > a^* \) and \( \psi_D(y-z) = 0 \). Hence (2.4.9) is nonnegative.
For \( y < \frac{Z}{p^*} \), \( \frac{d^2 z_y}{dy^2} = 1 \) and \( \frac{d^2 z_y}{dy^2} = 0 \). From (2.4.6) and (2.4.7):

\[
W'(y) = D_{1}g(y,y) + D_{2}g(y,y) \tag{2.4.14}
\]

\[
W''(y) = D_{11}g(y,y) + 2D_{12}g(y,y) + D_{22}g(y,y) \geq 0 \tag{2.4.15}
\]

as shown in case b.

Finally for \( \frac{Z}{p} \leq y \leq \frac{Z}{q} \), note:

\[
V(qy+p \bar{z}_y - q \bar{z}_y) \left[ 1 - \psi_{D^*}(y-\bar{z}_y) \right]
\]

\[
= V(Z) \left[ 1 - \psi_{D^*}(y-\bar{z}_y) \right]
\]

\[
= 0
\]

Thus from (2.3.1) and (2.3.3), \( D_2 g(y, \frac{Z-qy}{p-q}) = 0 \).

Hence

\[
W'(y) = D_{1}g(y, \frac{Z-qy}{p-q}) \tag{2.4.16}
\]

\[
W''(y) = D_{11}g(y, \frac{Z-qy}{p-q}) - \frac{q}{p-q} D_{12}g(y, \frac{Z-qy}{p-q}) \\
= \int_{a^*}^{(py-Z)/(p-q)} (r+s)p^2 \phi_{D}(py-p\varepsilon^*+q\varepsilon^*) \psi_{D^*}(\varepsilon^*)d\varepsilon^*
\]
\[+ (r+s)q^2 \phi_D(Z) \left[ 1 - \psi_D^*(\frac{py-Z}{p-q}) \right] - (r+s)q^2 \phi_D(Z) \left[ 1 - \psi_D^*(\frac{py-Z}{p-q}) \right] = \int_{a^*} \frac{\psi_D(p) - q\epsilon + q\epsilon \star \psi_D^*(\epsilon \star \epsilon) \psi_D^*(\epsilon \star \epsilon) d\epsilon \star \star \geq \geq 0 \quad (2.4.17)\]

From (2.4.12) and (2.4.16), \(W'(y)\) is continuous at \(\frac{Z}{q}\) since \(D_1 g(y,z)\) is continuous in both \(y\) and \(z\), and \(\frac{Z-qy}{p-q} = 0\) at \(y = \frac{Z}{q}\). From (2.4.14) and (2.4.16), \(W'(y)\) is also continuous at \(\frac{Z}{p}\) since \(\frac{Z-qy}{p-q} = \frac{Z}{p} = y\) at \(y = \frac{Z}{p}\) and \(D_2 g(y,y) \rightarrow D_2 g(\frac{Z}{p}, \frac{Z}{p}) = 0\).

Thus \(W'(y)\) is a continuous function. Furthermore from (2.4.13), (2.4.15), (2.4.17), \(W'(y)\) is monotone-nondecreasing. Theorem 1.2 is proved.

**Corollary:** \(W(y)\) is a convex function for \(0 \leq y < \infty\).

**Theorem 1.3:** When \(0 \leq q \leq p\), then:

(a) \(\operatorname{Min}(c+h_1+h_2-rp-sp, c+h_1-r^*sp) \leq W'(y) \leq c + h_1 + h_2\) for all \(0 \leq y < \infty\).

(b) \(0 \leq \operatorname{Min}(c+h_1+h_2-rp-sp, c+h_1-r^*sp) \Rightarrow \bar{y} = x\)

(c) \(\operatorname{Min}(c+h_1+h_2-rp-sp, c+h_1-r^*sq) < 0\)

\(\Rightarrow\) there exists a uniquely determined number \(Y, 0 < Y < \infty\), such that: \(W'(Y) = 0\) and \(W'(y) < 0\) for all \(0 \leq y < Y\)

\(\Rightarrow \bar{y} = \operatorname{Max}(x,Y).\)
Proof:

(a) When \(0 \leq r^* + h_2 + sq - rp - sp\), from (2.4.8)

\[
W'(y) = D_1 g(y, 0).
\]

\[
\lim_{M \to 0} W'(M) = \lim_{M \to 0} D_1 g(M, 0) = c + h_1 + h_2
\]

\[
W'(0) = D_1 g(0, 0) = c + h_1 - r^* - sq
\]

When \(r^* + h_2 - rq \leq 0\), from (2.4.10)

\[
W'(y) = D_1 g(y, y) + D_2 g(y, y)
\]

\[
\lim_{M \to \infty} W'(M) = \lim_{M \to \infty} D_1 g(M, M) + \lim_{M \to \infty} D_2 g(M, M)
\]

\[
= (c + h_1 - r^* + rq) + (r^* + h_2 - rq)
\]

\[
= c + h_1 + h_2
\]

\[
W'(0) = D_1 g(0, 0) + D_2 g(0, 0)
\]

\[
= (c + h_1 - r^* - sq) + (r^* + h_2 + sq - rp - sp)
\]

\[
= c + h_1 + h_2 - rp - sp.
\]
When \( r^* + h_2 + sq - rp - sp < 0 < r^* + h_2 - rp \)

\[
1_\infty W'(M) = 1_\infty D_1 g(M,0) = c + h_1 + h_2
\]

\[
W'(0) = D_1 g(0,0) + D_2 g(0,0)
\]

\[= c + h_1 + h_2 - rp - sp.\]

Thus (a) is proved.

(b) \( 0 \leq \text{Min}(c+h_1+h_2-rp-sp, c+h_1-r^*-sq) \)

\[\Rightarrow 0 \leq W'(y) \text{ for all } 0 < y < \infty \]

\[\Rightarrow W(y) \text{ is monotone-nondecreasing for } 0 < y < \infty \]

\[\Rightarrow \bar{y} = \text{Min}_{x < y < \infty} y = z.\]

(c) Statement is obviously true by the monotone-nondecreasingness of \( W'(y) \).

2.5 A Verbal Interpretation of the Dependence of Optimal Policy on Cost Structure

Given a policy \((y,z)\), now consider the case when an additional \( \Delta \) amount of goods is procured where \( \Delta \) is an infinitesimal small positive number, and we want to determine whether this \( \Delta \) should be allocated to the special sale or not.

First of all, when the special-sale demand is not greater than \( y-z \), then whether this \( \Delta \) amount of goods is allocated to special
sale or not is of no consequence. So let us assume that the special-
sale demand is greater than \(y - z\). In other words, this \(\Delta\) amount of
goods will be sold in the special sale if it is made available.

Two cases now have to be considered. When the regular demand
is greater than \(z\), then the cost of \(\Delta\) if allocated to the special
sale is:

\[
(c + h_1 - r^* - sq) \cdot \Delta.
\]  

(2.5.1)

Because by selling an additional \(\Delta\) amount of goods in special
sale, the regular demand is lowered down by the amount \(q \cdot \Delta\), and a
saving of shortage cost in the amount of \(s \cdot q \Delta\) is realized.

On the other hand, the cost of \(\Delta\) if withheld from the special
sale is

\[
(c + h_1 + h_2 - rp - sp) \Delta
\]  

(2.5.2)

Due to the leakage factor, the amount \(p \Delta\) only is available to
the regular sale. A revenue of \(r \cdot p \cdot \Delta\) and a saving of shortage cost
\(sp \Delta\) are thus realized.

Now if \(c + h_1 - r^* - sq \leq c + h_1 + h_2 - rp - rq\), or equivalently,
\(r^* + h_2 + sq - rp - sp \geq 0\), it says that even if this \(\Delta\) amount of
goods can be sold in the regular sale, it is still better to sell it
in the special sale. This being the case, then all goods will be
allocated to the special sale. This is the conclusion in Theorem
case a.

When the regular demand is not greater than \(z\), then the cost of
\(\Delta\) if allocated to the special sale is:

\[
(c + h_1 - r^* + rq) \Delta
\]  

(2.5.3)
This is because by selling an additional $\Delta$ amount of goods in the special-sale, the regular demand is lowered by the amount $q\Delta$, and thus a revenue of $rq\Delta$ is lost.

On the other hand, the cost of $\Delta$ if withheld from the special sale is simply:

$$ (c + h_1 + h_2)\Delta $$

(2.5.4)

Now if $c + h_1 + h_2 \leq c + h_1 - r^* + rq$, or equivalently $r^* + h_2 - rq \leq 0$, it says that even if the $\Delta$ amount of goods will not be sold in the regular sale, it is still preferable to withhold it from the special sale. This being the case, then no goods will be allocated to special sale. This is the conclusion in Theorem 1.1, case b.

Finally, when $r^* + h_2 + sq - rp - sp < 0 < r^* + h_2 - rq$, it says: having an $\Delta$ amount of goods sold in the regular sale is preferable to having it sold in the special sale; while having it sold in the special sale is preferable to reserving it for the regular sale but not being able to sell it there. This cost structure is thus seen to be a somewhat normal situation. In this normal case, there will exist some equilibrium points at which the cost of allocating an $\Delta$ amount of goods to the special sale is indifferent to the cost of reserving it for the regular sale. And since both costs are monotone nondecreasing in marginal cost, the equilibrium points will be optimum points if they are admissible.

In Theorem 1.1, case c, the equilibrium point is solved and its admissibility checked for all ranges of $y$.

We have discussed the problem of how to allocate a $\Delta$ amount of goods procured for the inventory system. Now we turn our attention
to the problem whether this $\Delta$ amount of goods should be procured at all. Recall that $(c+h_1-r^*-sq)\Delta$ is the cost of this $\Delta$ amount of goods when it is sold in the special sale while it could be sold in regular sale; and $(c+h_1+h_2-rp-rq)\Delta$ is the cost of the $\Delta$ amount of goods when it is sold in the regular sale. If $\operatorname{Min}(c+h_1-r^*-sq, c+h_1+h_2-rp-sp) \geq 0$, then procuring cannot be profitable. So the optimal procuring policy is to procure nothing. If $\operatorname{Min}(c+h_1-r^*-sq, c+h_1+h_2-rp-sp) < 0$, then there exists a level $Y$ such that procuring up to that level will be profitable. When this $Y$ is less than the initial stock level $x$, then again no procurement is made. Theorem 1.3 is thus verbally explained. Let

$$f(x) = \operatorname{Min}_{x < y < \infty} \operatorname{Min}_{0 < z < y} L_x(y, z)$$  \hspace{1cm} (2.5.5)

The minimum exists since for all finite $x$, $L_x(y, z)$ is finite iff $y$ is finite.

Then

$$f(x) = -cx + W(x) \text{ for } Y < x$$

$$= -cx + W(Y) \text{ for } x \leq Y$$  \hspace{1cm} (2.5.6)

\therefore $f(x)$ is also a convex function.

Taking derivative:

$$f'(x) = -c + W'(x) \text{ for } Y < x$$

$$= -c \leq Y$$  \hspace{1cm} (2.5.7)

From Theorem 1.3, it follows trivially:

**Theorem 1.4:** When $0 \leq q \leq p$, then:
(a) $f'(x)$ is a continuous monotone nondecreasing function.

(Implying that $f(x)$ is convex and $f''(x)$ exists a.e.)

(b) \[ \lim_{M \to \infty} f'(M) = h_1 + h_2 \]

(c) $f'(x) = -c$ for all $x < Y$. In particular $f'(0) = -c$.

2.6 Some Further Results of Optimal Policy

Summarizing Theorem 1.1 and Theorem 1.3, we obtain the following table.

The last case, i.e., $c + h_1 + h_2 \geq c + h_1 - r^* + rq$, $c + h_1 - r^* - sq \geq c + h_1 + h_2 - rp - sp$, and $\min(c + h_1 - r^* - sq, c + h_1 + h_2 - rp - sp) \leq 0$, is more or less a normal cost structure. We are able to obtain more specific results in this case by further breaking down the cost structure.

Substituting $y + Z/p$ and $\bar{z}_y = Z-\text{eq}/p-q = y$ in $W'(y)$ and recalling from Section 2.3, $S'(y) = g(y, \bar{z}_y)$ for $Z/p \leq y \leq Z/q$, we obtain:

\[ W'(\frac{Z}{p}) = D_{1g}(\frac{Z}{p}, \frac{Z}{p}) \]

\[ = c + h_1 - r^* - sq + (r+s)q \Phi_D(Z) \]

\[ = c + h_1 - r^* - sq - \frac{q}{p-q} (r^* + h_2 + sq - rp - sp) \]

(since $r^* + h_2 + sq - rp - sp + (r+s)(p-q) \Phi_D(Z) = V(Z) = 0$)

\[ = c + h_1 - \frac{r^* p + h_2 q - rpq}{p-q} \] ; \hspace{1cm} (2.6.1)
Table 1. Table of optimal policy corresponding to various cost structure

<table>
<thead>
<tr>
<th>$c + h_1 + h_2$</th>
<th>$c + h_1 - r^* - sq$</th>
<th>$\min \left( \frac{c + h_1 - r^* - sq}{c + h_2 - rp - sp} \right) \leq 0$</th>
<th>x</th>
<th>$\bar{y} =$</th>
<th>$\bar{z} =$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\geq c + h_1 - r^* + rq$</td>
<td>$\geq c + h_1 + h_2 - rp - sp$</td>
<td>for all x</td>
<td>x</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>yes</td>
<td>no</td>
<td>yes</td>
<td>$Y &lt; x$</td>
<td>x</td>
<td>0</td>
</tr>
<tr>
<td>no</td>
<td>yes</td>
<td>no</td>
<td>for all x</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>no</td>
<td>yes</td>
<td>yes</td>
<td>$Y &lt; x$</td>
<td>x</td>
<td>Y</td>
</tr>
<tr>
<td>yes</td>
<td>yes</td>
<td>no</td>
<td>$\frac{Z}{q} &lt; x$</td>
<td>x</td>
<td>0</td>
</tr>
<tr>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>$\frac{Z}{p} &lt; x &lt; \frac{Z}{q}$</td>
<td>x</td>
<td>$\frac{Z - qx}{p-q}$</td>
</tr>
<tr>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>$\max(x, Y) &lt; \frac{Z}{q}$</td>
<td>$\max(x, Y)$</td>
<td>0</td>
</tr>
<tr>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>$\frac{Z}{p} \leq \max(x, Y) \leq \frac{Z}{q}$</td>
<td>$\max(x, Y)$</td>
<td>$\frac{Z - q \max(x, Y)}{p-q}$</td>
</tr>
<tr>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>$\max(x, Y) &lt; \frac{Z}{p}$</td>
<td>$\max(x, Y)$</td>
<td>$\max(x, Y)$</td>
</tr>
</tbody>
</table>
and

\[ W'(\frac{Z}{q}) = D_n g(\frac{Z}{q}, 0) \]

\[ = \int_{a^*}^{Z/q} \left[ c + h_1 + h_2 - rp - sp + (r+s)\phi_D(\frac{PZ}{q} - pe^* + qe^*) \right] \psi_{D^*}(e^*)de^* \]

\[ + \int_{\frac{Z}{q}}^{\infty} (c+h_1-r^*-sq)\psi_{D^*}(e^*)de^* \]

\[ = \int_{a^*}^{Z/q} \left[ r^* + h_2 + sq - rp - sp + (r+s)\phi_D(\frac{PZ}{q} - pe^* + qe^*) \right] \psi_{D^*}(e^*)de^* \]

\[ + (c+h_1-r^*-sq) \]  \hspace{1cm} (2.6.2)

Since \( W'(y) \) is monotone-non-decreasing in \( y \) and \( Z/q \geq Z/p \), clearly

\[ (2.6.2) \geq (2.6.1) \]

Furthermore, from \( W'(Y) = 0 \), we will know the ordering of \( \frac{Z}{q} \), \( Y \),

and \( \frac{Z}{p} \) by knowing the signs of (2.6.1) and (2.6.2). Hence the last

case in Table 1 can be stated as:

Theorem 1.5: When \( 0 \leq q \leq p \), and

(a) If

\[ \int_{a}^{\frac{Z}{q}} \left[ r^* + h_2 + sq - rp - sp + (r+s)\phi_D(\frac{PZ}{q} - pe^* - qe^*) \right] \psi_{D^*}(e^*)de^* \]

\[ + (c+h_1-r^*-sq) \geq 0 \]
then
\[ \bar{y} = x, \quad \bar{z} = 0 \quad \text{for } Y \leq x \]
\[ \bar{y} = Y, \quad \bar{z} = 0 \quad \text{for } x \leq Y \]

(b) If
\[ c + h_1 - \frac{r^* p + h_2 q - rpq}{p-q} \leq 0 \]
\[ \frac{Z}{q} \leq \left\{ \begin{array}{l}
    c + h_2 + sq - rp - sp + (r+s)p_D \left( \frac{Z}{q} - p\epsilon^* - q\epsilon^* \right) \psi_D^*(\epsilon^*)d\epsilon^*, \\
    \end{array} \right. \]
\[ + (c+h_1-r^*-sq) \]

then
\[ \bar{y} = x, \quad \bar{z} = 0 \quad \text{for } \frac{Z}{q} < x \]
\[ \bar{y} = x, \quad \bar{z} = \frac{Z-qx}{p-q} \quad \text{for } Y < x < \frac{Z}{q} \]
\[ \bar{y} = Y, \quad \bar{z} = \frac{Z-qY}{p-q} \quad \text{for } x \leq Y. \]

(b) If
\[ 0 < c + h_1 - \frac{r^* p + h_2 q - rpq}{p-q} \]
then
\[ \bar{y} = x, \quad \bar{z} = 0 \quad \text{for } \frac{Z}{q} < x \]
\[ \bar{y} = x, \quad \bar{z} = \frac{Z-qx}{p-q} \quad \text{for } \frac{Z}{p} < x < \frac{Z}{q} \]
\[ \bar{y} = x, \quad \bar{z} = x \quad \text{for } Y < x < \frac{Z}{p} \]
\[ \bar{z} = Y, \quad \bar{z} = Y \quad \text{for } x \leq Y. \]

2.7 The Case $0 \leq p < q$ and the Case $q < 0 \leq p$

Results in the previous sections correspond to the case $0 \leq q \leq p$.

However, analysis in the other two cases $0 \leq p < q$ and $q < 0 \leq p$ is
essentially identical, and results are similar except perhaps differing on the boundary conditions of the $V(\theta)$ function and the $W(y)$ function. Let us see how the ordering of $0$, $q$, and $p$ affects the results.

From (2.3.3):

$$V(\theta) = r^* + h_2 + sq - rp - sp + (r+s)(p-q)\Phi_D(\theta).$$

When $0 \leq p < q$, then $V(\theta)$ is monotone-nonincreasing in $\theta$ and:

$$r^* + h_2 - rq \leq V(\theta) \leq r^* + h_2 + sq - rp - sp$$

Corresponding to Theorem 1.1, we now have: Theorem 1.6. If $0 \leq p < q$ and

(a) if $r^* + h_2 - rq \geq 0$, then $\bar{z}_y = 0$

(b) if $r^* + h_2 + sq - rp - sp \leq 0$, then $\bar{z}_y = y$

(c) if $r^* + h_2 - rq < 0 < r^* + h_2 + sq - rp - sp$, then there exists a real number $z$, $a < z < \infty$, such that

$$\bar{z}_y = y \quad \text{for } \frac{z}{p} < y < \infty$$

$$= \frac{z-qy}{p-q} \quad \text{for } \frac{z}{q} \leq y < \frac{z}{p}$$

$$= 0 \quad \text{for } 0 \leq y < \frac{z}{q}$$

(2.7.1)

**Proof:** Follows proof of Theorem 1.1 immediately.

When $q \leq 0 < p$, then $V(\theta)$ is still monotone-nondecreasing and

$$r^* + h_2 + sq - rp - sp \leq V(\theta) \leq r^* + h_2 - rq$$
Hence we have:

Theorem 1.7: When \( q \leq 0 \leq p \) and

(a) if \( r^* + h_2 + sq - rp - sp \geq 0 \), then \( \overline{z}_y = 0 \)
(b) if \( r^* + h_2 - rq \leq 0 \), then \( \overline{z}_y = y \)
(c) if \( r^* + h_2 + sq - (r+s)p < 0 < r^* + h_2 - rq \), then there exists a real number \( Z \), \( a < Z < \infty \), such that

\[
\overline{z}_y = y \quad \text{for } y < \frac{Z}{p}
\]

\[
= \frac{Z-qy}{p-q} \quad \text{for } \frac{Z}{p} \leq y < \infty \quad (2.7.2)
\]

Proof: Follows proof of Theorem 1.1 immediately.

Theorem 1.2, Theorem 1.3, and Theorem 1.4 apply also to the case \( 0 \leq q < p \) and \( q \leq 0 < p \). Proofs are identical except perhaps the partitioning into different ranges of \( y \) should follow (2.7.1) and (2.7.2), respectively. Theorem 1.5 applies to the case \( q \leq 0 < p \) with the part \( W(Z/q) \) omitted, and will apply to the case \( 0 \leq p < q \) if all inequalities stated in (a), (b) and (c) are reversed.
CHAPTER 3: THE MULTISTAGE PROCESS

3.1 The General Setup

Without loss of generality, we assume that the multistage process is composed of N periods. In each period we have a single-stage process as described in the last chapter, except that goods left at the end of period i are carried over to period i + 1, for i = 1, 2, 3, \ldots, N - 1. Goods left at the end of period N are disposed of with zero cost.

The following is also assumed:

a. Each period is of identical length, and the cost structure is stationary over time.

b. The leakage factor "p" is stationary over time.

c. Demands from period to period are independent but not necessarily identical random variables. Typically the i\(^{th}\) period special-sale demand \(D_{i}^{*}\) has a probability distribution \(\psi_{i}\), and the i\(^{th}\) period regular demand \(D_{i}^{*}\), given that there is no special sale in period i, has a probability distribution \(\phi_{i}\), \(D_{i}^{*}\), \(D_{i}\) have minimums \(a_{i}^{*}\), \(a_{i}\), respectively. (Within each period, the dependence of the regular demand on the amount sold in the special sale is the same as what we proposed for the single-period process. The parameter "q" is stationary over time.)

d. A discount factor \(\alpha\) is charged on future cost. Specifically, cost incurred in period i is discounted by the factor \(\alpha^{i-1}\) to period one, \(i = 1, 2, 3, \ldots, N\).

Since our inventory system is such that at each stage the system is characterized by the stock level, and the purpose of the process is to minimize some function of the stock level. And furthermore, the past
history of the system, given the present stock level, is of no importance in determining future actions, Bellman's "Principle of Optimality" applies. It says "An optimal policy has the property that whatever the initial decisions are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision."

(P. 83 of [5].)

On this basis, we construct the following recurrence equation for the N-stage model.

**Definition:**

$f_{ij}(x_i)$ is the expected total cost from period $i$ to period $j$ under an optimal policy where $x_i$ is the starting stock level at period $i$.

Then we can write:

$$f_{1N}(x_1) = \inf_{x_1 \leq y \leq x} \inf_{0 \leq z \leq y} \left[ L(y, z) + \alpha \left( \int_{0}^{\infty} f_{2N}(x_2) dH_{yz}(x_2) \right) \right]$$

(3.1.1)

where $H_{yz}(x_2)$ denote the cumulative distribution function of $x_2$ given the first period policy $(y, z)$.

Again let $\varepsilon^*$ and $\varepsilon$ denote the special demand outcome and the regular demand outcome of the first period, then

- for $\varepsilon \leq y - z$ and $\varepsilon \leq p(y - \varepsilon^*)$, we have $x_2 = p(y - \varepsilon^*) - \varepsilon$;
- for $\varepsilon \leq y - z$ and $\varepsilon > p(y - \varepsilon^*)$, we have $x_2 = 0$
- for $\varepsilon > y - z$ and $\varepsilon \leq pz$, we have $x_2 = pz - \varepsilon$;
- for $\varepsilon > y - z$ and $\varepsilon > pz$, we have $x_2 = 0$. 
Let:

\[ g_{1N}(y, z) = cy + h_1y \]

\[ + \int_{a^*}^{y-z} \left[ -r_1^*x^* + h_2(y-x^*) \right] \phi_1(x+q_1^*x^*)d\epsilon \]

\[ + \int_{a_1-q^*}^{p(y-x^*)} \left[ -r_1^*(p(y-x^*)) + s[\epsilon - p(y-x^*)] + \alpha f_{2N}(0) \right] \phi_1(x+q_1^*x^*)d\epsilon \]

\[ + \int_{p(y-x^*)}^{\infty} \phi_1(x+q_1^*x^*)d\epsilon \psi_1(x^*)d\epsilon^* + \]

\[ + \int_{y-z}^{\infty} \left[ -r_1^*(y-z) + h_2z \right] \phi_1(x+q_1^*x^*)d\epsilon \]

\[ + \int_{a-q(y-z)}^{p(z)} \left[ -r_1^*(p(z)-\epsilon) + \alpha f_{2N}(p(z)-\epsilon) \right] \phi_1(x+q(y-z))d\epsilon \]

\[ + \int_{p(z)}^{\infty} \left[ -r p z + s(x-pz) + \alpha f_{2N}(0) \phi_1(x+q(y-z))d\epsilon \right] \psi_1(x^*)d\epsilon^*. \]

**Definition:**

A policy for the N-stage process is a sequence of N pairs of numbers \( (y_1, z_1, y_2, z_2, \ldots, y_n, z_n) \) such that at the \( i^{th} \) period, \( \max\{0, y_i - x_i\} \) is to be procured, and \( \max\{0, y_i - z_i\} \) is to be allocated to the special sale. The \( i^{th} \) period component of a policy is called a \( i^{th} \) period policy.
3.2 An Optimal Policy

We have discussed in the single-state process the various results corresponding to different parametric structures. Since our goal in this chapter is mainly to see the link between the single-stage process and the multistage process, we will restrict our attention only to a particular parametric structure.

**Theorem 3.1:** For the \(N\)-stage model, assuming the following:

\[
0 \leq r + s - \alpha c,
\]

\[
0 \leq q \leq p,
\]

\[
r^* + h_2 + sq - rp - sp \leq 0 \leq r^* + h_2 - rp - (p-q)\alpha c
\]

\[
c + h_1 - r^* - sq < 0 < (1-p)c + h_1 + h_2
\]

(Note that \(c + h_1 + h_2 - rp - sp < 0\) is implied.)

then:

(a) There exists a uniquely determined critical number \(Z_{1N}'\),

\[
a_1 \leq Z_{1N}' < \infty,
\]

such that a first-period optimal allocation policy given \(y\) is:

\[
\tilde{z}_y = 0 \quad \text{for} \quad \frac{Z_{1N}}{q} \leq y < \infty
\]

\[
= \frac{Z_{1N} - qy}{p-q} \quad \text{for} \quad \frac{Z_{1N}}{p} \leq y \leq \frac{Z_{1N}}{q}
\]

\[
= y \quad \text{for} \quad 0 < y < \frac{Z_{1N}}{p}
\]

(b) There exists a uniquely determined critical number \(Y_{1N}'\),

\[
0 < Y_{1N}' < \infty,
\]

such that a first period optimal procurement policy is \(y + \max(x,Y_{1N}')\).
(c) \( f'_{1N}(x) \) is a continuous monotone-nondecreasing function,

(implying that:

\( f_{1N}(x) \) is convex

\( f''_{1N}(x) \) exist and \( f''_{1N}(x) \geq 0 \) almost everywhere.)

\[
f'_{1N}(M) \xrightarrow{M \to \infty} (h_1 + h_2) \sum_{j=0}^{N-1} (ap)^j,
\]

\( f'_{1N}(x) = -c \) for \( x \leq Y_{1N} \).

(In particular \( f'_{1N}(0) = -c \), and \( f'_{1N}(x) \geq -c \) for all \( x \geq 0 \).)

**Proof:** The parameteric structure superimposed on Theorem 3.1 should be recognizable from our discussion in the single-stage process. A fraction term of \( c \) is sometimes added to correspond to the fact that goods remaining in the stock at the end of period one are not disposed of but carried over to period two.

Proof is by induction. We have seen that Theorem 3.1 holds for the single-stage process. Let us make the inductive assumption that Theorem 3.1 holds for \( n = 2, 3, \ldots, N-1 \). We proceed to show that it also holds for \( n = N \).

Note that the inductive assumptions:

\( f'_{1,N-1}(x) \) is continuous monotone-nondecreasing,

\( f''_{1,N-1}(x) \) exists and \( \geq 0 \) a.e.,

\( f'_{1,N-1}(0) = -c \) and \( f'_{1,N-1}(x) \geq -c \) for all \( x \geq 0 \) is equivalent to the assumptions:

\( f''_{2N}(x) \) is continuous monotone-nondecreasing,

\( f''_{2N}(x) \) exists and \( \geq 0 \) a.e.,

\( f'_{2N}(0) = -c \) and \( f'_{2N}(x) \geq -c \) for all \( x \geq 0 \).
Now the proof follows closely our procedure in the single-stage process.

Taking derivative of $g_{1N}(y, z)$ with respect to $z$, we obtain:

$$
D_2 g_{1N}(y, z) = \left\{ \begin{array}{c}
\frac{\partial}{\partial z} \left( r^* + h_2 + sq - rp - sp \right) \\
\frac{p}{z} \left[ (p-q) \left[ r + s + \alpha f_{2N}^{\prime} (p \cdot z) \right] - \phi_{1}(z) \right] \right\} [1 - \psi_1(y-z)]
\end{array} \right.
$$

$$
= \left\{ \begin{array}{c}
\frac{\partial}{\partial z} \left( r^* + h_2 + sq - rp - sp \right) \\
\frac{q}{y} \left[ (p-q) \left[ r + s + \alpha f_{2N}^{\prime} (q \cdot y \cdot z) \right] - \phi_{k}(z) \right] \right\} [1 - \psi_1(y-z)]
\end{array} \right.
$$

$$
= V_{1N}(q+y+z) [1 - \psi_1(y-z)] \quad \text{for } y \cdot z \leq \frac{a}{q}
$$

$$
D_2 g_{1N}(y, z) = 0 \quad \text{for } y \cdot z > \frac{a}{q} \quad (3.2.1)
$$

where:

$$
V_{1N}(z) = r^* + h_2 + sq - rp - sp
$$

$$
+ \left\{ \begin{array}{c}
\frac{\partial}{\partial z} \left( (p-q) \left[ r + s + \alpha f_{2N}^{\prime} (z) \right] \right) - \phi_1(z) \right\} \right\} [1 - \psi_1(y-z)]
\end{array} \right.
$$

$$
= V_{1N}(q+y+z) [1 - \psi_1(y-z)] \quad \text{for } y \cdot z \leq \frac{a}{q}
$$

$$
D_2 g_{1N}(y, z) = 0 \quad \text{for } y \cdot z > \frac{a}{q} \quad (3.2.2)
$$
Since:
\[
V_{1N}(\Theta) = \int_{a_1}^\Theta (p-q)[r + s + af''_{2N}(\Theta-\varepsilon)]\phi_1(\varepsilon)d\varepsilon
\]
\[+ (p-q)(r+s-\alpha c)\phi_1(\Theta) \geq 0.
\]

\(V_{1N}(\Theta)\) is a continuous monotone-nondecreasing function

\(V_{1N}(z_1) = r^* + h_2 + sq - rp - sp \leq 0\)

\[
\lim_{M \to \infty} V_{1N}(M) = r^* + h_2 + sq - rp - sp
\]
\[
+ \lim_{M \to \infty} \int_{a_1}^M (p-q)[r + s + af'_{2N}(M-\varepsilon)]\phi_1(\varepsilon)d\varepsilon
\]
\[= r^* + h_2 + sq - rp - sp
\]
\[+ (p-q)[r + s + \alpha(h_1 + h_2) \sum_{j=0}^{N-2} (ap)^j] \geq r^* + h_2 - rq \geq 0.
\]

So there exists a \(Z_{1N}'\), \(a_1 \leq Z_{1N}' < \infty\), such that \(V_{1N}(Z_{1N}') = 0\), and \(V_{1N}(\Theta) < 0\) for all \(\Theta < Z_{1N}'\).

Now for \(y > \frac{Z_{1N}}{q_2}\), \(V_{1N}(qy+pz-qz) \geq V_{1N}(Z_{1N}) = 0\). Since \(1 - \psi_1(y,z) \geq 0\), \(g_{1N}(y,z)\) is thus nonnegative.

\[\therefore \quad \bar{z}_y = \min_{0 \leq z \leq y} z = 0\]
For \( \frac{z_{1N}}{p} \leq y \leq \frac{z_{1N}}{q} \), then \( V_{1N}(qy+pz-qz) = V_{1N}(z_{1N}) = 0 \) at \( z = \frac{z_{1N}-qy}{p-q} \).

Since \( D_{2g_{1N}}(y,z) \geq 0 \) for \( z \geq \frac{z_{1N}-qy}{p-q} \) and \( D_{2g_{1N}}(y,z) \leq 0 \) for \( z = \frac{z_{1N}-qy}{p-q} \),

\[
\bar{z}_y = \frac{z_{1N}-qy}{p-q}
\]
is a minimum point. Furthermore it is admissible since

\[
0 \leq \frac{z_{1N}-qy}{p-q} \leq y \quad \text{for} \quad \frac{z_{1N}}{p} \leq y \leq \frac{z_{1N}}{q}.
\]

Finally, for \( 0 \leq y < \frac{z_{1N}}{p} \), \( V_{1N}(qy+pz-qz) \leq V_{1N}(z_{1N}) = 0 \) for all \( z \). Therefore, \( D_{2g_{1N}}(y,z) \leq 0 \).

\[
\therefore \quad \bar{z}_y = \max_{0 \leq z \leq y} z = y
\]

Part (a) is proved.

(b) Substitute \( \bar{z}_y \) in \( g_{1N}(y,z) \) and define

\[
W_{1N}(y) = g_{1N}(y,\bar{z}_y) \quad (3.2.3)
\]

Note that due to the inductive assumptions on \( f''_{2N}(x) \) and \( f''_{2N}(x) \), the following \( D_{1g_{1N}}(y,z) \), \( D_{2g_{1N}}(y,z) \), \( D_{11g_{1N}}(y,z) \), \( D_{12g_{1N}}(y,z) \) and \( D_{22g_{1N}}(y,z) \) are all well defined. For derivations, consult Appendix 7.1.

\[
D_{1g_{1N}}(y,z) = \begin{cases} \frac{y-z}{a_{1z}} & \left\{ \begin{array}{l} c + h_1 + h_2 - rp - sp \\ + \int p(y-\varepsilon^*) \\ + \int_{-\infty}^{\infty} \frac{y-z}{a_{1z}} & \left\{ \begin{array}{l} c + h_1 - r* - sq \\ + \int p(z) \\ + \int q(z) \end{array} \right\} \psi_1(\varepsilon^*)d\varepsilon \right\} \psi_1(\varepsilon^*)d\varepsilon \end{cases} \end{cases}
\]

\[
(3.2.4)
\]
$$D_{22} g_{1N}(y,z) = \begin{cases} r^* h_2 + sq - rp - sp & \\
 & \int^{p_z} \left[ (p-q) \left[ \frac{r+s+af'_{2N}(pz-\epsilon)}{2N} \right] \phi_1(\epsilon+q_y-q_z) d\epsilon \right] a_1-q(y-z) \\
 & \left[ 1 - \Psi_1(y-z) \right] \\
 & \end{cases}$$

(3.2.5)

$$D_{11} g_{1N}(y,z) = \int^{y-z} \left[ p^2 (r+s-ac) \phi_1(p_y-p_\epsilon+q_\epsilon \ast) \right] a_1-q_\epsilon \ast \left[ p(y_\epsilon \ast) \right] + \int^{p_\epsilon \ast} p^2 a^2 f'_{2N}(p_y-p_\epsilon \ast-\epsilon) \phi_1(\epsilon+q_\epsilon \ast) d\epsilon \right] \Psi_1(\epsilon \ast) d\epsilon \ast$

$$+ \int^{\infty} \left[ q^2 (r+s-ac) \phi_1(q_y+p_z-q_z) \right] a_1-q(y-z)$$

$$+ \int^{p_z} q^2 a^2 f''_{2N}(p_z-\epsilon \ast) \phi_1(q_y+p_z-q_z) d\epsilon \right] \Psi_1(\epsilon \ast) d\epsilon \ast.$$

(3.2.6)

$$D_{22} g_{1N}(y,z) = \begin{cases} r^* h_2 + sq - rp - sp & \\
 & \int^{p_z} \left[ (p-q) \left[ \frac{r+s+af'_{2N}(pz-\epsilon)}{2N} \right] \phi_1(\epsilon+q_y-q_z) d\epsilon \right] \Psi_1(y-z) \\
 & \left[ q(p-q)(r+s-ac) \phi_1(q_y+p_z-q_z) \right] a_1-q(y-z) \\
 & \int^{\infty} \left[ (p-q)^2 a^2 f''_{2N}(p_z-\epsilon \ast) \phi_1(q_y+p_z-q_z) d\epsilon \right] \Psi_1(\epsilon \ast) d\epsilon \ast. \\
 & \end{cases}$$

(3.2.7)
\[ D_{12g1N}(y,z) = - \left\{ r^* + h_2 + sq - rp - sp \right. \]

\[ \left. + \int_{p^2} (p-q)[r+s+af_{2N}(pz-\varepsilon)]\phi_1(c+qy-qz)d\varepsilon \right\} \psi_1(y-z) \]

\[ + \int_{y-z}^{\infty} [q(p-q)(r+s-ac)\phi_1(qy+pz-qz)]d\varepsilon \]

\[ + \int_{y-z}^{\infty} q(p-q)f_{2N}''(pz-\varepsilon^*)\phi_1(qy+pz-qz)d\varepsilon \psi_1(\varepsilon^*)d\varepsilon^*. \]

(3.2.8)

From (3.2.3), we obtain:

\[ W_{1N}^1(y) = D_{1g1N}(y, z_y) + D_{2g1N}(y, z_y) \frac{d\bar{z}_Y}{dy} \]

(3.2.9)

and

\[ W_{1N}^2(y) = D_{11g1N}(y, z_y) + D_{12g1N}(y, z_y) \frac{d\bar{z}_Y}{dy} \]

\[ + D_{22g1N}(y, z_y) \frac{d\bar{z}_Y}{dy} + D_{2g1N}(y, z_y) \frac{d\bar{z}_Y}{dy} \]

(3.2.10)

whenever \( \frac{d\bar{z}_Y}{dy} \) and \( \frac{d^2\bar{z}_Y}{dy} \) exist. For \( \frac{z_{1N}}{q} < y \), then \( z_y = 0 \), \( \frac{d\bar{z}_Y}{dy} = 0 \), and \( \frac{d^2\bar{z}_Y}{dy} = 0 \).

\[ W_{1N}^1(y) = D_{1g1N}(y, 0) \]

(3.2.11)
\( W'_{1N}(y) = D_{11}g_{1N}(y,0) \)  

(3.2.12)

Substitute \( z = 0 \) in (3.2.6). Then \( \psi_1(y-z) = 1 \) and \( \psi_1(y-z) = 0 \) since \( y-z > \frac{Z_{1N}}{q} > a_1/q \). Hence

\[
D_{11}g_{1N}(y,z) = \int_a^y [p^2(r+s+ac)\psi_1(py-pe*+qe*)]_{a_1}^{y} \\
+ \int_{a_1-qe*}^{p(y-\epsilon*)} p^2\alpha f''_{2N}(py-pe*+\epsilon)\phi_1(\epsilon+qe*)d\epsilon]\phi_1(\epsilon*)d\epsilon* \geq 0
\]

For \( 0 \leq y < \frac{Z_{1N}}{p} \),

\[
W'_{1N}(y) = D_{11}g_{1N}(y,y) + D_{22}g_{2N}(y,y) \quad (3.2.13)
\]

\[
W''_{1N}(y) = D_{11}g_{1N}(y,y) + 2D_{12}g_{1N}(y,y) + D_{22}g_{1N}(y,y)
\]

\[
= \int_0^\infty [q^2(r+s+ac)\phi_1(py) \\
+ \int_{a_1}^{py} q^2\alpha f''_{2N}(pq-\epsilon)\phi_1(py)d\epsilon]\phi_1(\epsilon*)d\epsilon* \\
+ 2\int_0^\infty [q(p-q)(r+s+ac)\phi_1(py) \\
+ \int_{a_1}^{py} q(p-q)\alpha f''_{2N}(pq-\epsilon)\phi_1(py)d\epsilon]\phi_1(\epsilon*)d\epsilon*
\]
\begin{align*}
&+ \int_0^\infty [ (p-q)^2 (r+s-ac) \phi_1(py) \\
&+ \int_{a_1}^{py} (p-q)^2 a_1^{2N} (py-\varepsilon) \phi_1(py) d\varepsilon ] \psi_1(\varepsilon^*) d\varepsilon^* \\
&= \int_0^\infty [ p^2 (r+s-ac) \phi_1(py) \\
&+ \int_{a_1}^{py} p^2 a_1^{2N} (py-\varepsilon) \phi_1(py) d\varepsilon ] \psi_1(\varepsilon^*) d\varepsilon^* \geq 0 \tag{3.2.14}
\end{align*}

by the inductive assumption \( f_2^{(n)}(x) \geq 0 \) a.e.

For \( \frac{Z_{1N}}{p} \leq y \leq \frac{Z_{1N}}{q} \), \( z = \frac{Z_{1N} - qy}{p-q} \). Note:

\begin{align*}
V_{1N}(qy + p\bar{z}_y - q\bar{z}_y) & [ 1 - \psi_1(y-\bar{z}_y) ] \\
&= V_{1N}(Z_{1N}) [ 1 - \psi_1(y-\bar{z}_y) ] \\
&= 0 \tag{3.2.15}
\end{align*}

From (3.2.1), \( D_{2N}(y, z) = 0 \). Hence

\begin{align*}
W_{1N}'(y) = D_{1N}(y, \frac{Z_{1N} - qy}{p-q}) \tag{3.2.16}
\end{align*}
\[ W_{1N}^\prime(y) = D_{11} g_{1N}(y, \frac{Z_1 - qy}{p-q}) - \frac{q}{p-q} D_{12} g_{1N}(y, \frac{Z_1 - qy}{p-q}) \]

\[ = \int_{a_1}^{\infty} \frac{(py - Z_1)}{(p-q)} [p^2 (r+s-ac) \phi_1 (py - p\varepsilon + q\varepsilon)] \psi_1 (\varepsilon) d\varepsilon \]

\[ + \int_{a_1}^{\infty} \frac{p (y - \varepsilon)}{(p-q)} p^2 af_{2N}'' (py - p\varepsilon - \varepsilon) \phi_1 (\varepsilon + q\varepsilon)] \psi_1 (\varepsilon) d\varepsilon \]

\[ + \int_{a_1}^{\infty} \frac{p (Z_1 - qy)}{(p-q)} \frac{q^2 af_{2N}'' (\frac{pZ_1 - pqy}{p-q} - \varepsilon)}{a_1 - q(py - Z_1)}/(p-q) \phi_1 (Z_1) \psi_1 (\varepsilon) d\varepsilon \]

\[ - \int_{a_1}^{\infty} \frac{p (Z_1 - qy)}{(p-q)} \frac{q^2 af_{2N}'' (\frac{pZ_1 - pqy}{p-q} - \varepsilon)}{a_1 - q(py - Z_1)}/(p-q) \phi_1 (Z_1) \psi_1 (\varepsilon) d\varepsilon \]
\[
\begin{align*}
\int \frac{(py-z_{1N})}{(p-q)} \left[ p^2(r+s-\alpha c) \phi_1(py-p\varepsilon^*-q\varepsilon^*) \right]_{a_1^*} \text{d}\varepsilon^* \\
+ \int \left[ p^2 \alpha f''_{2N}(py-p\varepsilon^*-\varepsilon) \phi_1(\varepsilon+q\varepsilon^*) \right] \psi_1(\varepsilon^*) \text{d}\varepsilon^* \geq 0. (3.2.17)
\end{align*}
\]

From (3.2.11) and (3.2.16), \( W'_{1N}(y) \) is continuous at \( \frac{Z_{1N}}{q} \), since \( D_{1g_{1N}}(y,z) \) is continuous in both \( y \) and \( z \), and \( \frac{Z_{1N}-qy}{p-q} = 0 \) at \( y = \frac{Z}{q} \). From (3.2.13) and (3.2.16), \( W'_{1N}(y) \) is also continuous at \( \frac{Z_{1N}}{p} \), since \( \frac{Z_{1N}-qy}{p-q} = \frac{Z_{1N}}{p} = y \) at \( y = \frac{Z_{1N}}{p} \), and \( D_{2g_{1N}}(y,y) \) \( \rightarrow \) \( \frac{Z_{1N}}{p} \) \( \downarrow y \) \( \uparrow \frac{Z_{1N}}{p} \). \( D_{2g_{1N}} \left( \frac{Z_{1N}}{p}, \frac{Z_{1N}}{p} \right) = 0. \)

Thus \( W'_{1N}(y) \) is a continuous function. From (3.2.12), (3.2.14), and (3.2.17) \( W''_{1N}(y) \) is monotone nondecreasing. Furthermore:

\[
\lim_{M \to \infty} W'_{1N}(M) = \lim_{M \to \infty} D_{1g_{1N}}(M,0)
\]

\[
= \lim_{M \to \infty} \int_{a_1^*}^{M} \left\{ c + h_1 + h_2 - rp - sp \right\} \text{d}\varepsilon^* \\
+ \int_{a_1^*-q\varepsilon^*}^{p(M-\varepsilon^*)} \left[ p[r + s + \alpha f''_{2N}(pM-p\varepsilon^*-\varepsilon)] \right] \psi_1(\varepsilon^*) \text{d}\varepsilon^*
\]

\[
\cdot \phi_1(\varepsilon+q\varepsilon^*) \text{d}\varepsilon^* \psi_1(\varepsilon^*) \text{d}\varepsilon^*
\]
\[ \begin{align*}
= c + h_1 + h_2 - rp - sp + p [r + s + a(h_1 + h_2)] \\
N-2 \\
\cdot \sum_{j=0}^{\infty} (ap)^j \\
\end{align*} \]

\[ c + (h_1 + h_2) \sum_{j=0}^{N-1} (ap)^j > 0 \]  \hspace{1cm} (3.2.18)

and

\[ W_{1N}'(0) = D_1s_{1N}(0,0) + D_2s_{1N}(0,0) \]

\[ = \int_{0}^{\infty} (c+h_1-r*-sq)\psi_1(\varepsilon*)d\varepsilon* + (r*+h_2+sq-rp-sp) \]

\[ = c + h_1 + h_2 - rp - sp < 0 \]  \hspace{1cm} (3.2.19)

Thus there exists a unique \( Y_{1N} \), \( 0 < Y_{1N} < \infty \), such that \( W_{1N}'(Y_{1N}) = 0 \) and \( W_{1N}'(y) < 0 \) for all \( y < Y_{1N} \). Since \( W_{1N}'(y) \) is a monotone nondecreasing function, \( Y_{1N} \) is an optimal procuring level if admissible. Thus part (b) is proved.

(c) By definition of \( f_{1N}(x) \),

\[ f_{1N}(x) = -cx + W_{1N}(x) \hspace{1cm} \text{for} \ Y_{1N} \leq x \]

\[ = -cx + W_{1N}(Y_{1N}) \hspace{1cm} \text{for} \ x < Y_{1N} \]  \hspace{1cm} (3.2.20)
Taking derivative:

\[ f'_{1N}(x) = -c + w'_{1N}(x) \quad \text{for } y_{1N} \leq x \]

\[ = -c \quad \text{for } x < y_{1N} \quad (3.2.21) \]

Since \( w'_{1N}(x) \xrightarrow{x \to y_{1N}} w'_{1N}(y_{1N}) = 0 \), \( f'_{1N}(x) \) is continuous. It is monotone nondecreasing since \( w'_{1N}(x) \) is.

From (3.2.21) and (3.2.18):

\[ \lim_{N \to \infty} f'_{1N}(M) = -c + c + \left( h_1 + h_2 \right) \sum_{j=0}^{N-1} (ap)^j \]

\[ = \left( h_1 + h_2 \right) \sum_{j=0}^{N-1} (xp)^j \]

Part (c) is thus proved. Q.E.D.

Note that since \( N \) is arbitrary, Theorem 3.1 will obviously still be true if results are stated in \( Z_{1N}, y_{1N} \) and \( f_{1N}(x) \).

**Lemma 3.2:** Under the assumptions of Theorem 3.1, then

\[ c + h_1 - \frac{r^*p + h_2q - rpq}{p-q} \leq 0 \iff \frac{Z_{1N}}{p} \leq y_{1N} \quad \text{for all } i = 1, 2, \ldots, N. \]

**Proof:**

\[ w'_{1N}\left( \frac{Z_{1N}}{p} \right) = D_{1} g_{1N}\left( \frac{Z_{1N}}{p} \right), \quad \frac{Z_{1N}}{p} \]

\[ = c + h_1 - r^* - sq + \int_a^{Z_{1N}} q[r + s + \alpha f^*_{2N}(Z_{1N} - \epsilon) \phi_1(\epsilon) \text{d}\epsilon} \]
\[ \begin{align*}
&= c + h_1 - r^* - sq + \frac{q}{p-q} \left[ Y_{1N}^i(Z_{1N}^i) - (r^* + h_2 + sq - rp - sp) \right] \\
&= c + h_1 - \frac{r^* p + h_2 q - rpq}{p-q} \\
\leq 0 &= W_{1N}^i(Y_{1N}^i).
\end{align*} \]

Since \( W_{1N}^i(y) \) is monotone nondecreasing, Lemma 3.2 follows.

### 3.3 Some Ordering Properties of Critical Numbers

In this section, we shall derive some ordering properties of the critical numbers \( Y_{1N}, Z_{1N}, i = 1, 2, \ldots, N \), when demand distributions are identical from period to period. Let \( \{ \phi, \psi \} \) denote the common distributions, and \( \{ \phi, \psi \} \) the densities.

**Theorem 3.3:** Suppose \( c + h_1 - \frac{r^* p + h_2 q - rpq}{p-q} \leq 0 \) and \( \phi_1 = \phi, \psi_1 = \psi \).

Then under the assumptions of Theorem 3.1, \( Z_{1N} \) and \( Y_{1N} \) are monotone nondecreasing in \( i \).

**Proof:**

We will first prove \( Z_{1N} > Z_{2N}, Y_{1N} > Y_{2N} \).

From Lemma 3.2,

\[
\frac{Z_{1N}}{p} \leq Y_{1N} \tag{3.3.1}
\]
\[ V_{2N}(Z_{2N}) - V_{1N}(Z_{2N}) \]

\[ = \left\{ r + h_s + sq - (r+s)p + \int_a^{Z_{2N}} (p-q)[r + s + \alpha f_3^{(Z_{2N} - \varepsilon})] \phi(\varepsilon) d\varepsilon \right\} \]

\[ - \left\{ r^* + h_s + sq - (r+s)p + \int_a^{Z_{2N}} (p-q)[r + s + \alpha f_2^{(Z_{2N} - \varepsilon})] \phi(\varepsilon) d\varepsilon \right\} \]

\[ = \int_a^{Z_{2N}} (p-q)\alpha [f_3^{(Z_{2N} - \varepsilon}) f_2^{(Z_{2N} - \varepsilon})] \phi(\varepsilon) d\varepsilon \]

\[ \geq 0 \]

since from Theorem 3.1, \( f_3^{(Z_{2N} - \varepsilon}) \geq -c \) and \( f_2^{(Z_{2N} - \varepsilon}) = -c \) as \( Z_{2N} - \varepsilon \leq Y_{2N} \) for all \( \varepsilon \geq 0 \).

But \( V_{1N}(Z_{2N}) \leq V_{2N}(Z_{2N}) = 0 = V_{1N}(Z_{1N}) \) implies \( Z_{2N} \leq Z_{1N} \) since \( V_{1N}(\theta) \) is monotone-nondecreasing in \( \theta \).

The proof for \( Y \) part is similar. Again we show \( W_1^{(Y_{2N})} \)

\( \leq W_2^{(Y_{2N})} \). However, since \( W_1^{(Y)} \), \( W_2^{(Y)} \) have various functional forms at different ranges of \( y \), and we do not know in which range \( Y_{2N} \) is, we have to consider all possibilities.

From (3.3.1), \( \frac{Z_{2N}}{p} \leq Y_{2N} \); also we have proved \( Z_{2N} \leq Z_{1N} \). Thus \( Y_{2N} \) can lie in any interval of the following sequence:
\[ \frac{Z_{2N}}{p} \leq \min \left( \frac{Z_{1N}}{p}, \frac{Z_{2N}}{q} \right) \leq \max \left( \frac{Z_{1N}}{p}, \frac{Z_{2N}}{q} \right) \leq \frac{Z_{1N}}{p} < \infty. \]

First, let \( \frac{Z_{1N}}{q} < Y_{2N} < \infty \). Then \( \frac{Z_{2N}}{q} < Y_{2N} \). From (3.2.11),

\[ W_{1N}(Y_{2N}) = D_{1g_{1N}}(Y_{2N},0), \quad W_{2N}(Y_{2N}) = D_{1g_{2N}}(Y_{2N},0). \]

But:

\[ D_{1g_{2N}}(Y_{2N},0) - D_{1g_{1N}}(Y_{2N},0) \]

\[ = \int_{a^*}^{Y_{2N}} \left\{ c + h_1 + h_2 - rp - sp \right. \]

\[ + \int_{a-q \varepsilon^*}^{p(Y_{2N}-\varepsilon^*)} p[r + s + \alpha f'_{2N}(pY_{2N}-p\varepsilon^*-\varepsilon)] \phi(\varepsilon+q\varepsilon^*)d\varepsilon \right\} \psi(\varepsilon^*)d\varepsilon^* \]

\[ + \int_{Y_{2N}}^{\infty} (c+h_1-r^*-sq) \psi(\varepsilon^*)d\varepsilon^* \]

\[ - \int_{a^*}^{Y_{2N}} \left\{ c + h_1 + h_2 - rp - sp \right. \]

\[ + \int_{a-q \varepsilon^*}^{p(Y_{2N}-\varepsilon^*)} p[r + s + \alpha f'_{2N}(pY_{2N}-p\varepsilon^*-\varepsilon)] \phi(\varepsilon+q\varepsilon^*)d\varepsilon \right\} \psi(\varepsilon^*)d\varepsilon^* \]
\[ + \int_{Y_{2N}}^\infty (c+h_{1-r^*-sq}) \psi(\varepsilon^*) d\varepsilon^* \]

\[ = \int_{Y_{2N}}^{p(y_{2N}-\varepsilon^*)} \int_{a^*-q\varepsilon^*} p \alpha f'_{3N}(pY_{2N}-p\varepsilon^*-\varepsilon) - f'_{2N}(pY_{2N}-p\varepsilon^*+\varepsilon) \psi(\varepsilon^*) d\varepsilon^* \]

\[ \geq 0 \]

Since \( f'_{3N}(pY_{2N}-p\varepsilon^*-\varepsilon) \geq -c \) and \( f'_{2N}(pY_{2N}-p\varepsilon^*+\varepsilon) = -c \).

Next let \( \text{Max} \left( \frac{Z_{1N}}{p}, \frac{Z_{2N}}{q} \right) \leq Y_{2N} \leq \frac{Z_{1N}}{q} \), then from (3.2.16) and (3.2.11):

\[ W'_{1N}(Y_{2N}) = D_{1\delta_{1N}}(Y_{2N}, \frac{Z_{1N}-qY_{2N}}{p-q}) \quad \text{and} \quad W'_{2N}(Y_{2N}) = D_{1\delta_{2N}}(Y_{2N}, 0). \]

But \( D_{1\delta_{1N}}(y,z) \) is monotone-nondecreasing in \( z \) for \( z \leq \frac{Z_{1N}-qy}{p-q} \) since \( D_{1\delta_{1N}}(y,z) \) is nonnegative. (The first term in (3.2.7) is either 0 or \(-V_{1N}(qy+pz-qz) \geq -V_{1N}(Z_{1N}) = 0 \) for \( z \leq \frac{Z_{1N}-pq}{p-q} \).)

Hence

\[ W'_{1N}(Y_{2N}) - W'_{2N}(Y_{2N}) \geq D_{1\delta_{1N}}(Y_{2N}, 0) - D_{1\delta_{2N}}(Y_{2N}, 0) \]

That the last difference is nonnegative has been proved in the first case.

Third, let \( \frac{Z_{1N}}{p} \leq Y_{2N} < \frac{Z_{2N}}{q} \) by supposing

\[ \text{Max} \left( \frac{Z_{1N}}{p}, \frac{Z_{2N}}{q} \right) = \frac{Z_{2N}}{q}. \]
From (3.2.16):

\[
W^i_{1\bar{N}}(Y_{2\bar{N}}) = D_{1\bar{N}}^i(Y_{2\bar{N}}, \frac{Z_{1\bar{N}} - qY_{2\bar{N}}}{p-q})
\]

\[
W^i_{2\bar{N}}(Y_{2\bar{N}}) = D_{2\bar{N}}^i(Y_{2\bar{N}}, \frac{Z_{2\bar{N}} - qY_{2\bar{N}}}{p-q})
\]

Again by noting:

\[
f^i_{3\bar{N}}(pY_{2\bar{N}} - \epsilon) \geq -c
\]

\[
f^i_{2\bar{N}}(p \cdot \frac{Z_{2\bar{N}} - qY_{2\bar{N}}}{p-q} - \epsilon) \geq -c
\]

\[
f^i_{2\bar{N}}(pY_{2\bar{N}} - \epsilon) = -c,
\]

\[
f^i_{2\bar{N}}(p \cdot \frac{Z_{2\bar{N}} - qY_{2\bar{N}}}{p-q} - \epsilon) = -c,
\]

while the last equality is because from \(\frac{Z_{2\bar{N}}}{p} \leq Y_{2\bar{N}}\):

\[
p \cdot \frac{Z_{2\bar{N}} - qY_{2\bar{N}}}{p-q} \leq p \cdot \frac{Z_{2\bar{N}} - q}{p-q} = Z_{2\bar{N}} \leq Y_{2\bar{N}}
\]

But:

\[
D_{1\bar{N}}^i(Y_{2\bar{N}}, \frac{Z_{2\bar{N}} - qY_{2\bar{N}}}{p-q}) - D_{1\bar{N}}^i(Y_{2\bar{N}}, \frac{Z_{1\bar{N}} - qY_{2\bar{N}}}{p-q})
\]

\[
= \left\{ \begin{array}{l}
\frac{pY_{2\bar{N}} - Z_{2\bar{N}}}{p-q} \\
a* \{ c + h_1 + h_2 - rp - sp \\
\end{array} \right.
\]
\[
+ \int p(Y_{2N} - \epsilon^*) \left\{ \begin{array}{l}
p \left[ r + s + af'_{2N}(pY_{2N} - p\epsilon^* - \epsilon) \right] \phi(\epsilon + q\epsilon^*) d\epsilon \end{array} \right\} \psi(\epsilon^*) d\epsilon^*
\]

\[
+ \int_{pY_{2N} - Z_{2N}}^{\infty} \left\{ c + h_1 - r^* - sq \right\} \frac{Z_{2N} - qY_{2N}}{p-q} \left( p \cdot \frac{Z_{2N} - qY_{2N}}{p-q} - \epsilon \right)
\]

\[
\quad + \left\{ p \cdot \frac{Z_{2N} - qY_{2N}}{p-q} \right\} \frac{Z_{2N} - qY_{2N}}{p-q} \phi(\epsilon + q \cdot \frac{Z_{2N} - qY_{2N}}{p-q}) d\epsilon \right\} \psi(\epsilon^*) d\epsilon^*
\]

\[
- \int \frac{pY_{2N} - Z_{1N}}{p-q} \left\{ c + h_1 + h_2 - (r+s)p \right\} \frac{Z_{2N} - qY_{2N}}{p-q} \left( p \cdot \frac{Z_{2N} - qY_{2N}}{p-q} - \epsilon \right)
\]

\[
\quad + \left\{ p \cdot \frac{Z_{2N} - qY_{2N}}{p-q} \right\} \frac{Z_{2N} - qY_{2N}}{p-q} \phi(\epsilon + q \cdot \frac{pY_{2N} - Z_{1N}}{p-q}) d\epsilon \right\} \psi(\epsilon^*) d\epsilon^*
\]

\[
- \int_{pY_{2N} - Z_{1N}}^{\infty} \left\{ c + h_1 - r^* - sq \right\} \frac{Z_{2N} - qY_{2N}}{p-q} \left( p \cdot \frac{Z_{2N} - qY_{2N}}{p-q} - \epsilon \right)
\]

\[
\quad + \left\{ p \cdot \frac{Z_{2N} - qY_{2N}}{p-q} \right\} \frac{Z_{2N} - qY_{2N}}{p-q} \phi(\epsilon + q \cdot \frac{pY_{2N} - Z_{1N}}{p-q}) d\epsilon \right\} \psi(\epsilon^*) d\epsilon^*
\[
\left\{ \begin{array}{l}
\frac{p_{Y_{2N}} - Z_{2N}}{p-q} \\
\frac{p_{Y_{2N}} - Z_{1N}}{p-q}
\end{array} \right. \\
\left\{ \begin{array}{l}
c + h_1 + h_2 - (r+s)p \\
c + h_1 - r* - sq
\end{array} \right.
\]
\[+
\int_a^{p_{Y_{2N}} - (p-q)e^*} p[r+s + f'_{3N}(p_{Y_{2N}} - p e^* + q e^* - e)] \phi(e) de \right\} \psi(e^*) de^*
\]
\[+
\int_a^{Z_{2N}} q[r+s + \alpha f'_{3N}(Z_{2N} - e)] \phi(e) de \right\} \psi(e^*) de^*
\]
\[-
\int\left\{ \begin{array}{l}
[V_{1N}(Z_{1N})] \psi(e^*) de^*
\end{array} \right. \\
\left\{ \begin{array}{l}
\frac{p_{Y_{2N}} - Z_{2N}}{p-q} \\
\frac{p_{Y_{2N}} - Z_{1N}}{p-q}
\end{array} \right. \\
\left\{ \begin{array}{l}
Z_{2N} \\
p[r+s + \alpha f'_{3N}(Z_{2N} - e)] \phi(e) de \psi(e^*) de^*
\end{array} \right. \\
\int_a^{Z_{2N}} \\
a
\]
\[
\left\{ \begin{array}{l}
\frac{pY_{2N} - Z_{2N}}{p-q} \\
pY_{2N} - (p-q) e^* \\
pY_{2N} - Z_{1N} \end{array} \right\} \left\{ \begin{array}{l}
pY_{2N} c^* \\
p[ r + s + a \frac{1}{3N} (pY_{2N} - pe^* + qe^* - e)] \\
a \end{array} \right\}
\]

\[
\phi(e) d\varepsilon (e^*) d\varepsilon^* \geq 0
\]

since \( pY_{2N} (p-q)e^* \geq pY_{2N} (p-q) \frac{pY_{2N} - Z_{2N}}{p-q} = Z_{2N} \).

Finally for the other two cases:

\[
\frac{Z_{2N}}{q} \leq Y_{2N} < \frac{Z_{1N}}{p} \quad \text{when } \max\left(\frac{Z_{1N}}{p}, \frac{Z_{2N}}{p}\right) = \frac{Z_{1N}}{p}, \quad \text{and } \frac{Z_{2N}}{p} \leq Y_{2N} < \frac{Z_{1N}}{p},
\]

we have \( \frac{Z_{2N}}{p} < Y_{2N} < \frac{Z_{1N}}{p} \) in both cases.

Hence:

\[
W_{2N}^i (Y_{2N}) - W_{1N}^i (Y_{2N})
\]

\[
\geq W_{2N}^i \left(\frac{Z_{2N}}{p}\right) - W_{1N}^i \left(\frac{Z_{1N}}{p}\right)
\]

\[
= (c + h_1 - \frac{r^* p + h_2 q - r p q}{p-q}) - (c + h_1 - \frac{r^* p + h_2 q - r p q}{p-q})
\]

\[
= 0
\]

by using Lemma 3.2.
Thus we have proved $W'_{2N}(Y_{2N}) > W'_{1N}(Y_{2N})$ for $Y_{2N}$ lying in all possible ranges. But:

$$W'_{1N}(Y_{1N}) = 0 = W'_{2N}(Y_{2N}) > W'_{1N}(Y_{2N})$$

implies $Y_{1N} > Y_{2N}$ since $W'_{1N}(y)$ is monotone-nondecreasing.

We have succeeded in proving $Y_{1N} > Y_{2N}$, $Z_{1N} > Z_{2N}$ for arbitrary $N$. Since $Y_{1N} \geq Y_{i+1,N}$ and $Z_{1N} \geq Z_{i+1,N}$ are equivalent to $Y_{1,N-i+1} \geq Y_{2,N-i+1}$ and $Z_{1,N-i+1} \geq Z_{2,N-i+1}$ for all $i = 1, 2, \ldots, N-1$,

Theorem 3.4 is proved.

**Theorem 3.5:** Suppose $c + h_1 - \frac{r^p+h_2(q-r)p}{p-q} > 0$ and $\phi_i = \phi$, $\psi_i = \psi$ for all $i = 1, 2, \ldots, N$. Then $Y_{1N}$ are monotone nondecreasing in $N$.

**Proof:**

Since now we do not have the inequality $Z_1 \geq Z_2$, the essential ingredient in proving Part b of Result 3.2.1, we have to resort to a different approach.

Define

$$\Omega(y) = c + h_1 + h_2 - (r+s)p + \int_{a}^{P_y} p(r+s-\alpha c)\phi(\varepsilon)d\varepsilon; \quad (3.3.2)$$

then $\Omega(y)$ is clearly continuous and monotone-nondecreasing.

Also

$$\Omega(M) \xrightarrow{M \to \infty} c(1-\alpha p) + h_1 + h_2 \geq 0$$

and

$$\Omega(\frac{N}{p}) = c + h_1 + h_2 - (r+s)p < 0.$$
So there exists a unique $\Omega^{\Omega}$, $0 < y \leq \Omega < \infty$ such that $\Omega(y) = 0$, and $\Omega(y) < 0$ for all $y < Y^{\Omega}$.

Now $c + h_1 - \frac{r^*p + h_2q - rpq}{p-q} > 0$ implies $\frac{z_{1N}}{p} > Y_{2N}$ from Lemma 3.2.

By Theorem 3.3, $z_{Y_{2N}}^{\Omega} = Y_{2N}$.

From (3.2.13):

$$W_{2N}'(Y_{2N}) = D_{12N}(Y_{2N}, Y_{2N}) + D_{22N}(Y_{2N}, Y_{2N}).$$

By noting

$$f_{3N}(pY_{2N} - \varepsilon) \geq -c,$$

we obtain:

$$D_{12N}(Y_{2N}, Y_{2N}) + D_{22N}(Y_{2N}, Y_{2N})$$

$$= c + h_1 - r^* - sq + \int_a^{pY_{2N}} q[r + s + \alpha f_{3N}(pY_{2N} - \varepsilon)]\phi(\varepsilon)d\varepsilon + V_{2N}(pY_{2N})$$

$$= c + h_1 + h_2 - (r+s)p + \int_a^{pY_{2N}} p[r + s + \alpha f_{3N}(pY_{2N} - \varepsilon)]\phi(\varepsilon)d\varepsilon$$

$$\geq c + h_1 + h_2 - (r+s)p + \int_a^{pY_{2N}} p(r+s-\alpha)c\phi(\varepsilon)d\varepsilon$$

$$= \Omega(Y_{2N}).$$

From $\Omega(Y_{2N}) = 0 = W_{2N}'(Y_{2N}) \geq \Omega(Y_{2N})$ and the monotone-nondecreasingness of $\Omega(y)$, we conclude $Y^{\Omega} \geq Y_{2N}$. 


Now we will show $Y_{1N} \geq Y_{2N}$ by using a contradiction argument.

Supposing $Y_{1N} < Y_{2N}$, then $f'_2(Y_{1N}^- - \varepsilon) = -c$. Also since $Y_{1N} < \frac{Z_{1N}}{p}$,
\[ z_{Y_{1N}} = Y_{1N} \]
So from (3.2.13):

\[ W'_{1N}(Y_{1N}) = d_{1g_{1N}}(Y_{1N}, Y_{1N}) + d_{2g_{1N}}(Y_{1N}, Y_{1N}) \]

\[ = c + h_1 + h_2 - (r+s)p + \int_a^b \frac{pY_{1N}}{p[r + s + af_{2N}(pY_{1N}^- - \varepsilon)]\phi(\varepsilon)d\varepsilon} \]

\[ = c + h_1 + h_2 - (r+s)p + \int_a^b \frac{pY_{1N}}{p(r+s-\alpha c)\phi(\varepsilon)d\varepsilon} \]

\[ = \Omega(Y_{1N}) \]

From $\Omega(Y_{1N}^\Omega) = 0 = W'_{1N}(Y_{1N}) = \Omega(Y_{1N})$ and the way $Y_{1N}^\Omega$ is selected, we conclude $Y_{1N} \geq Y$.

But $Y_{1N} \geq Y_{1N}^\Omega \geq Y_{2N}$ is contradictory to $Y_{1N} < Y_{2N}$.

Again since $N$ is arbitrary and $Y_{1N} \geq Y_{i+1,N}$ is equivalent to $Y_{1,N-i+1} \geq Y_{2,N-i+1}$, Theorem 3.4 is proved.

**Theorem 3.5:** When $\phi_i = \phi$, $\psi_i = \psi$ for all $i = 1, 2, \ldots, N$, then under the assumptions of Theorem 3.1, $Y_{1N}$ are uniformly bounded for all $i$ and $N$, $i \leq N$.

**Proof:**

When $c + h_1 - \frac{r^p + h_2q - rq}{p-q} > 0$, then we have seen in the proof of Theorem 3.4, $Y_{2N} \leq Y_{1N}^\Omega$ when $Y_{1N}^\Omega$ is the number determined from
\( \Omega(Y) = 0, \Omega(y) \leq 0 \) for all \( y < Y \). Since \( N \) is arbitrary and \( Y_{1N} \) is equivalent to \( Y_{2,N-i+2} \), we conclude \( Y_{1N} \leq Y \) for all \( i \) and \( N, i \leq N \),

\[
\frac{r*p+h_2q-rpq}{p-q} > 0 \text{ holds.}
\]

When \( c + h_1 - \frac{r*p+h_2q-rpq}{p-q} \leq 0 \), from Lemma 3.2, \( \frac{Z_{1N}}{p} < Y_{1N} \).

First assume \( \frac{Z_{1N}}{q} < Y_{1N} \), then from (3.2.11) and (3.2.4):

\[
W_{1N}^{*}(Y_{1N}) = D_{1N}^{*}(Y_{1N}, 0)
\]

\[
= \int_{a^*}^{Y_{1N}} \left\{ c + h_1 + h_2 - rp - sp \right. \\
+ \int_{a-qe^*}^{p(Y_{1N}-e^*)} p[r + s + \alpha f_{2N}(pY_{1N} - pe^* - e)] \\
\cdot \phi(e + qe^*)de^* \\
\left. \right\} \psi(e^*)de^* \\
+ \int_{Y_{1N}}^{\infty} (c + h_1 - r* - sq)\psi(e^*)de^*
\]

(3.3.3)

Define:

\[
\Omega_1(y) = \int_{a^*}^{Y_{1N}} [c + h_1 + h_2 - rp - sp
\]
\[ + \int_{a-q\varepsilon^*}^{p(y-\varepsilon^*)} p(r+s-ac)\phi(\varepsilon+q\varepsilon^*)d\varepsilon \, \psi(\varepsilon^*)d\varepsilon^* \]

\[ + \int_{y}^{\infty} (c + h_1 - r^*-sq)\psi(\varepsilon^*)d\varepsilon^* \]  

(3.3.4)

Then since:

\[ \Omega'_1(y) = \int_{a^*}^{y} \int_{a-q\varepsilon^*}^{p(y-\varepsilon^*)} p^2(r+s-ac)\phi(\varepsilon+q\varepsilon^*)d\varepsilon \, \psi(\varepsilon^*)d\varepsilon^* \]

\[ + [c + h_1 + h_2 - rp - sp + p(r+s-ac)\phi_1(qy)]\psi(y)-(c+h_1 - r^*-sq)\psi(y) \]

\[ \geq \int_{a^*}^{y} \int_{a-q\varepsilon^*}^{p(y-\varepsilon^*)} p^2(r+s-ac)\phi(\varepsilon+q\varepsilon^*)d\varepsilon \, \psi(\varepsilon^*)d\varepsilon^* \]

\[ + V(qy)\psi(y) \]

\[ \geq 0 \quad \text{for} \quad \frac{Z_1}{q} \leq Y. \]
\( \Omega_1(y) \) is monotone-increasing. Furthermore:

\[
\lim_{M \to \infty} \Omega_1(M) \rightarrow c + h_1 + h_2 - rp - sp + p(r+s-ac) = (1-ap)c + h_1 + h_2
\]

\( > 0 \),

and

\[
\Omega_1 \left( \frac{Z_{1\mathbb{N}}}{q} \right) \leq W'_{1\mathbb{N}} \left( \frac{Z_{1\mathbb{N}}}{q} \right) < W'_{1\mathbb{N}}(\mathbb{Y}_{1\mathbb{N}}) = 0.
\]

there exists a uniquely determined number \( Y_{01} \), \( \frac{Z_{1\mathbb{N}}}{q} < Y_{01} < \omega \), such that \( \Omega_1(Y_{01}) = 0 \) and \( \Omega_1(y) > 0 \) for all \( y > Y_{01} \).

Compare (3.3.3) and (3.3.4). We find: \( W'_{1\mathbb{N}}(\mathbb{Y}_{1\mathbb{N}}') \geq \Omega_1(Y_{1\mathbb{N}}) \)

since \( f_2'(pY_{1\mathbb{N}} - pe^{\star - \epsilon}) \geq -c \). Also, \( \Omega_1(Y_{01}) = 0 = W'_{1\mathbb{N}}(\mathbb{Y}_{1\mathbb{N}}) \geq \Omega_1(Y_{1\mathbb{N}}) \).

\( Y_{01} \geq Y_{1\mathbb{N}} \) by the monotone nondecreasingness of \( \Omega \), and our selection of \( Y_{01} \).

Next assume \( \frac{Z_{1\mathbb{N}}}{p} \leq Y_{1\mathbb{N}} \leq \frac{Z_{1\mathbb{N}}}{q} \). We will show the uniform boundedness of \( Y_{1\mathbb{N}} \) by showing that \( Z_{1\mathbb{N}} \) are uniformly bounded.

Define

\[
\Omega_2(\Theta) = r^* + h_2 + sq - rp - sp + (p-q)(r+s-ac)\phi(\Theta) \quad (3.3.6)
\]

Then \( \Omega_2(\Theta) \) is monotone-nondecreasing in \( \Theta \) and

\[
\lim_{M \to \infty} \Omega_2(M) \rightarrow r^* + h_2 - rq - a(p-q)c \geq 0
\]

\[
\Omega_2(0) = r^* + h_2 + sq - rp - sp \leq 0.
\]
There exists a $Z_{Ω^2}$, $0 ≤ Z_{Ω^2} ≤ M$, such that $Ω_2(Z_{Ω^2}) = 0$ and $Ω_2(θ) > 0$ for all $θ > Z_{Ω^2}$.

Compare (3.2.2) with (3.3.6), we note $Y_{1N}(Z_{1N}) ≥ Ω_2(Z_{1N})$ since $f_{2N}(x) ≥ -c$ for all $x$.

But $Ω_2(Z_{Ω^2}) = 0 = V_{1N}(Z_{1N}) ≥ Ω_2(Z_{1N})$. Since $Ω_2$ is monotone-nondecreasing and from the way we select $Z_{Ω^2}$, we conclude $Z_{Ω^2} ≥ Z_{1N}$.

Thus we have proved that when $c + h_1 - \frac{r*p+h_2*q-rpq}{p-q} ≤ 0$,

$Y_{1N} ≤ \text{Max} \{Y_{1N}^{Ω1}, Z_{Ω^2}/q\}$. Again since $N$ is arbitrary and $Y_{1N}$ is equivalent to $Y_{1,N-1+1}$, we conclude $Y_{1N}$ are uniformly bounded for all $i$ and $N$, $i ≤ N$. 
CHAPTER 4: THE INFINITE-STAGE PROCESS

4.1 Infinite Period Model When Demand Distributions in Different Time Periods are Independent and Identical

Theorem 4.1: Suppose

(a) \( r + s - ac > 0 \)
(b) \( p \geq q > 0 \)
(c) \( r_k + h_2 + sq - rp - sp < 0 \leq r_k + h_2 - rq - (p-q)ac \)
(d) \( c + h_1 - r_k sq < 0 < (1-ap)c + h_1 + h_2 \)
(e) demand distributions \( \{\phi_i, \psi_i\} \) are identical for every time period and the demand on special sale or regular sale in the \( i^{th} \) time period is independent of the demand in the special sale and regular sale in any other time period.

Then \( f_{iN}(x) \) converges uniformly for every finite \( i \) to a unique function \( f_1(x) \) for every finite close interval \( 0 \leq x \leq b \). Moreover, \( f_1(x) \) is a bounded continuous convex function in the interval \( 0 \leq x \leq b \) for every finite \( b \). And \( f_1(x) \) is the unique solution among the class of continuous and uniformly bounded solution of the infinite-stage functional equation.

Proof:

Under the assumptions stated, we know from Theorem 3.1 there exists for every finite \( i \) and \( N (i \leq N) \) critical numbers \( 0 < y_{iN} < \infty \).

From Theorem 3.6, \( y_{iN} \) are uniformly bounded for every \( i \) and \( N \).

Let \( (\overline{y}_{iN}, \overline{z}_{iN}) \) be the \( i^{th} \) period component of the \( N \)-stage optimal policy. Then:
\[ f_{iN}(x_i) = L_{x_i}(\tilde{y}_{iN}, \tilde{z}_{iN}) + \alpha \int_{0}^{\infty} f_{i-1,N}(x_{i+1}) dH_{\tilde{y}_{iN}, \tilde{z}_{iN}}(x_{i+1}) \]

\[ \leq L_{x_i}(\tilde{y}_{i,N-1}, \tilde{z}_{i,N-1}) + \alpha \int_{0}^{\infty} f_{i+1,N}(x_{i+1}) dH_{\tilde{y}_{i,N-1}, \tilde{z}_{i,N-1}}(x_{i+1}) \]

by definition of \((\tilde{y}_{iN}, \tilde{z}_{iN})\). Similarly:

\[ f_{i,N-1}(x_i) = L_{x_i}(\tilde{y}_{i,N-1}, \tilde{z}_{i,N-1}) \]

\[ + \alpha \int_{0}^{\infty} f_{i+1,N-1}(x_{i+1}) dH_{\tilde{y}_{i,N-1}, \tilde{z}_{i,N-1}}(x_{i+1}) \]

\[ \leq L_{x_i}(\tilde{y}_{i,N}, \tilde{z}_{i,N}) + \alpha \int_{0}^{\infty} f_{i+1,N-1}(x_{i+1}) \]

\[ \cdot dH_{\tilde{y}_{i,N}, \tilde{z}_{i,N}}(x_{i+1}) \]

Hence:

\[ |f_{iN}(x_i) - f_{i,N-1}(x_i)| \]

\[ \leq \text{Max} \left\{ \alpha \int_{0}^{\infty} |f_{i+1,N}(x_{i+1}) - f_{i+1,N-1}(x_{i+1})| dH_{\tilde{y}_{iN-1}, \tilde{z}_{iN-1}}(x_{i+1}) \right\} \]
\[ a \int_{0}^{\infty} |f_{i+1,N}(x_{i+1}) - f_{i+1,N-1}(x_{i+1})| \, dH_{y_{i+1},z_{i+1}} \]

\[ \leq a \sup |f_{i+1,N}(x_{i+1}) - f_{i+1,N-1}(x_{i+1})| \quad (4.1.1) \]

where the supreme is taken over all possible \( x_{i+1} \).

Now \( x_{i+1} \leq y_{i} = \max(x_{i},y_{i,N}) \) and in general,

\[ x_{i+1} \leq \max(x_{i},y_{i,N},y_{i+1,N},\ldots,y_{i+j-1,N}) \leq B < \infty \]

That such a "\( B \)" exists is because of \( 0 \leq x_{i} \leq b \), and the uniform boundedness of \( y_{i,N} \).

Let

\[ U_{i} = \sup_{0 \leq x_{i+1} \leq B} |f_{i+1,N}(x_{i+1}) - f_{i+1,N-1}(x_{i+1})| \quad (4.1.2) \]

Since (4.1.1) is true for all \( 0 \leq x_{i} \leq B \), in particular:

\[ U_{i,N} \leq a U_{i+1,N} \quad (4.1.3) \]

In general, we will have:

\[ U_{i,N} \leq a U_{i+1,N} \leq \ldots \leq a^{N-i-1} U_{NN} \]

\[ = \alpha^{N-i} \sup_{0 \leq x \leq B} |f_{NN}(x)| \]

\[ = \alpha^{N-i} \sup_{0 \leq x \leq B} \left| L_{x} (\tilde{y}_{NN},\tilde{z}_{NN}) \right| \quad (4.1.4) \]
Now \( L_x(\overline{y}_{NN}, \overline{z}_{NN}) \), the single-stage minimum expected cost is continuous in \( x \) and is finite for every \( 0 \leq x \leq B \). Therefore there exists a number \( m < \infty \) such that \( \sup_0 \leq x \leq B L_x(\overline{y}_{NN}, \overline{z}_{NN}) \leq m \), where \( m \) depends on the demand distribution only.

\[
\therefore U_{iN} \leq \alpha^{-i} \frac{m}{\alpha^{1-i}} \tag{4.1.5}
\]

Since \( \sum_{N=1}^{\infty} U_{iN} \leq \sum_{N=1}^{\infty} \alpha^{-i} \frac{m}{\alpha^{1-i}} \), \( f_{iN}(x) \) converges uniformly to a function \( f_(x) \) for all \( 0 \leq x \leq B \). Since the convergence is uniform, \( f_(x) \) is a continuous function. Since \( f_{iN}(x) \) is convex for each \( N \), \( f_(x) \) is convex. Since \( i \) is arbitrary and \( f_{iN}(x) \equiv f_{1,N-i+1}(x) \), \( f_(x) \equiv f_1(x) \). Let us denote this common limiting function by \( f_1(x) \). That \( f_1(x) \) is a solution of the infinite-stage functional equation:

\[
f(x) = L_x(\overline{y}, \overline{z}) + \alpha \int_0^\infty f(x) dH_{\overline{y}, \overline{z}}(x) \tag{4.1.6}
\]

is due to the uniform convergence of \( f_{iN} \) to \( f_1 \). (see p. 15 of [5]).

To prove the uniqueness of \( f_1 \) as solution of (4.1.6):

Let \( F_1(x_1) \) be any other solution such that \( F_1(x_1) \) is continuous and uniformly bounded for all \( 0 \leq x_1 \leq B \). Then:

\[
F_1(x_1) = L_x(\overline{y}_1, \overline{z}) + \alpha \int_0^\infty F_1(x_2) dH_{\overline{y}, \overline{z}}(x_2) \tag{4.1.7}
\]

An optimal policy \((\overline{y}, \overline{z})\) exists since the right-hand side of (4.1.7) is continuous and finite for finite \( \overline{y} \). Since \( f_1(x) \) is also a solution:
\[ f_\perp(x_1) = \mathcal{L}_{x_1}(y, z) + \alpha \int_0^\infty f_\perp(x_2) d\mathcal{H}_\perp(y, z, x_2) \quad (4.1.8) \]

By the definition of \((\bar{y}, \bar{z})\) and \((\bar{y}, \bar{z})\)

\[ F_\perp(x_1) = \mathcal{L}_{x_1}(\bar{y}, \bar{z}) + \alpha \int_0^\infty F_\perp(x_2) d\mathcal{H}_\perp(y, z, x_2) \quad (4.1.9) \]

and

\[ f_\perp(x_1) \leq \mathcal{L}_{x_1}(\bar{y}, \bar{z}) + \alpha \int_0^\infty f_\perp(x_2) d\mathcal{H}_\perp(y, z, x_2) \quad (4.1.10) \]

\[ \therefore \quad |F_\perp(x_1) - f_\perp(x_1)| \]

\[ \leq \text{Max} \left\{ \alpha \int_0^\infty |F_\perp(x_2) - f_\perp(x_2)| d\mathcal{H}_\perp(y, z, x_2), \right. \]

\[ \left. \alpha \int_0^\infty F_\perp(x_2) - f_\perp(x_2) d\mathcal{H}_\perp(y, z, x_2) \right\} \]

\[ \leq \alpha \text{Max}_{0 \leq x_2 \leq B} |F_\perp(x_2) - f_\perp(x_2)| \quad (4.1.11) \]

The maximum exists since \(F_\perp\) and \(f_\perp\) are both continuous and finite in the interval \(0 \leq x_2 \leq B\) where \(B = \text{Max}(x_1, \bar{y}, \bar{z})\). Let \(u = \text{Max}_{0 \leq x \leq B} |F_\perp(x) - f_\perp(x)|\). Then since (4.1.11) is true for all \(0 \leq x_1 \leq B\), we obtain:

\[ u \leq \alpha u \quad (4.1.12) \]
For $0 < \alpha < 1$, $u = 0$. Or equivalently, $F_I(x) = f_I(x)$ for all $x$ lying in a finite interval.

**Theorem 4.2:** Under the assumptions of Theorem 4.1:

(a) There exists a uniquely determined number $Z_I$, $a < Z_I < \infty$, such that an optimal allocation policy given $y$ is:

$$\bar{z} = 0, \frac{Z_I}{q} < y < \infty$$

$$= \frac{Z_I - qy}{p - q}, \frac{Z_I}{p} < y < \frac{Z_I}{q}$$

$$= y, 0 < y < \frac{Z_I}{p}.$$  

(b) There exists a uniquely determined number $Y_I$, $0 < Y_I < \infty$, such that an optimal procurement policy is $y = \text{Max}(x, Y_I)$.

(c) $f_I(x)$ is a continuous monotone-nondecreasing function,

$$f_I(M) \xrightarrow{M \to \infty} (h_1 + h_2)/(1-\alpha p)$$

$$f_I(x) = -c \text{ for } x \leq Y_I.$$

**Proof:**

Let

$$g_I(y,z) = cy + h_1y + \int_{a^*}^{y-z} \left\{-r^*e^* + h_2(y-e^*)\right\}d\epsilon$$

$$+ \int_{a-qe^*}^{p(y-e^*)} \left[-r^* + uf_I(py-pe^* - \epsilon)\right]\phi(\epsilon + qe^*)d\epsilon$$
\[
+ \int \frac{[- \, rp(y-\varepsilon*) + s(\varepsilon-py+pe*) + af_1(0)]}{p(y-\varepsilon*)} \psi(\varepsilon*) \, d\varepsilon
\]

\[
\{ \phi(\varepsilon+q\varepsilon*) \psi(\varepsilon*) \, d\varepsilon \}
\]

\[
+ \int_0^\infty \left\{ - r*(y-z) + h_2 z + \int_{a-q(y-z)}^{p z} [- \, r e + af_1(pz-\varepsilon)] \psi(\varepsilon*) \, d\varepsilon \right\}
\]

\[
\phi(e+qy-qz) \, d\varepsilon
\]

\[
\int_0^\infty \left\{ - rpz + s(\varepsilon-pz) + af_1(0) \phi(\varepsilon+qy+qz) \psi(\varepsilon*) \, d\varepsilon \right\}
\]

\[
\psi(\varepsilon*) \, d\varepsilon
\]

Then:

\[
f_1(x) = - cx + \min_{x \leq y < \infty} \min_{0 < z < y} g_1(y, z) \quad (4.1.14)
\]

From (4.1.13):

\[
D_1 g_1(y, z) = \int_a^{y-z} \left\{ c + h_1 + h_2 - rp - sp \right\}
\]

\[
+ \int_{a-q\varepsilon*}^{p(y-\varepsilon*)} \left\{ p[r + s + af_1(py-pe*-\varepsilon)] \phi(\varepsilon+q\varepsilon*) \psi(\varepsilon*) \, d\varepsilon \right\}
\]
\[ + \int_{y-z}^{\infty} \left\{ c + h_1 - r^* - sq \right\} \psi(\varepsilon) d\varepsilon^* \]

\[ + \int_{a-q(y-z)}^{p_z} q[r + s + a f_1^+(p z - \varepsilon)] \phi(\varepsilon + q y - q z) d\varepsilon \psi(\varepsilon) d\varepsilon^* \]

\[ D_2 g_1(y, z) = \left\{ r^* + h_2 + sq - rp - sp + \int_{z-q(y-z)}^{p_z} (p-q)[r + s + a f_1^+(p z - \varepsilon)] \right. \]
\[ \cdot \phi(\varepsilon + q y - q z) d\varepsilon \left\} [1 - \psi(y-z)] \]

\[ = \left\{ r^* + h_2 + sq - rp - sp + \int_{a}^{q y + (p-q)z} (p-q) \right. \]
\[ \cdot [r + s + a f_1^+(q y + p z - q z - \varepsilon)] \phi(\varepsilon) d\varepsilon \left\} [1 - \psi(y-z)] \]

\[ = V_1(p y + p z - q z)[1 - \psi(y-z)] \quad \text{for} \quad y-z \leq \frac{a}{q} \]

\[ D_2 g_1(y, z) = 0 \quad \text{for} \quad y-z < \frac{a}{q} \quad (4.1.16) \]
where

\[ V_1(\theta) = r^* + h_2 + sq - rp - sp \]

\[ + \int_{a}^{\theta} (p-q) [r + s + \alpha f_1(\theta - \epsilon)] \phi(\epsilon) d\epsilon \quad (4.1.17) \]

From Theorem 4.1, \( f_1(x) \) is a convex function defined on any finite close interval of \( x \). Therefore \( f_1(x) \) exists almost everywhere, and \( D_1 g_1(y, z) \), \( D_2 g_1(y, z) \) and \( V_1(\theta) \) are well defined continuous functions. Furthermore, \( f_1(x) \) is monotone-nondecreasing and bounded between \(-c\) and \((h_1 + h_2)/(1-\alpha p)\) since by having an additional \( \Delta \) amount of goods in the starting stock, the best could happen is that a procuring cost of \( c\Delta \) will be saved; and the worst could happen is that no procurement will be made and this \( \Delta \) amount of goods will not be sold hereafter, and thus incurring a holding cost \((h_1 + h_2) \sum_{j=0}^{\theta} (\alpha p)^j \Delta = (h_1 + h_2) \Delta/(1-\alpha p)\).

Now \( V_1(\theta) \) is monotone-nondecreasing since the integrand \((p-q)[r + s + \alpha f_1(\theta - \epsilon)]\) is monotone-nondecreasing and nonnegative.

Moreover:

\[ V_1(a) = r^* + h_2 + sp - rp - sp < 0 \]

\[ V_1(H) \xrightarrow{N \to \infty} r^* + h_2 - rq + (p-q)a(\Delta_1 + h_2)/1-\alpha p > 0 \]

Hence there exists a uniquely determined \( Z_1 \) such that \( V_1(Z_1) = 0 \) and \( V_1(\theta) < 0 \) for all \( \theta < Z_1 \).
Considering (4.1.16) and using arguments similar to the single-stage process or the N-stage process, we obtain part (a).

Let

\[ W_1(y) = g_1(y, z_y). \]

Again using arguments similar to the single-stage process or the N-stage process, we have:

\[ W_1^1(y) = D_1 g_1^1(y, 0) \quad \text{for } \frac{Z_1}{q} < y < \infty \]

\[ = D_1 g_1^1(y, \frac{Z_1 - qy}{p - q}) \quad \text{for } \frac{Z_1}{p} \leq y \leq \frac{Z_1}{q} \]

\[ = D_1 g_1^1(y, y) + D_2 g_1^1(y, y) \quad \text{for } 0 \leq y < \frac{Z_1}{p} \]

and \( W_1^1(y) \) is continuous.

To show the monotone nondecreasingness of \( W_1^1(y) \), we cannot take \( W_1^1(y) \) since we can say nothing about the existence of \( f_1^1(x) \) and thus \( D_{11} g_1^1(y, z), D_{12} g_1^1(y, z) \) and \( D_{22} g_1^1(y, z) \) are not well defined. However, we can take the difference of \( W_1^1(y) \) at any two points \( y_1 > y_2 \) and show that \( W_1^1(y_1) \geq W_1^1(y_2) \).

First let \( y \) be any point \( \geq \frac{Z_1}{q} \) and let \( \Delta \) be any positive number, then:
\[ W_1^+(y + \Delta) - W_1^+(y) \]

\[ = D_1 g_1^+(y + \Delta, 0) - D_1 g_1^+(y, 0) \]

\[ = \int_{\mathbb{R}^+}^{y + \Delta} \left\{ c + h_1 + h_2 - rp - sp \right\} \right. \]

\[ + \int_{a - q\varepsilon^*}^{p(y + \Delta - \varepsilon^*)} \left[ p[r + s + \alpha f_1^+(py + p\Delta - p\varepsilon^* - \varepsilon)] \psi(\varepsilon + q\varepsilon^*) \psi(\varepsilon^*) d\varepsilon \right] \]

\[ + \int_{y + \Delta}^{\infty} (c + h_1 - r\varepsilon^* - sq) \psi(\varepsilon^*) d\varepsilon^* - \int_{a - q\varepsilon^*}^{y} \left\{ c + h_1 + h_2 - rp - sp \right\} \]

\[ + \int_{a - q\varepsilon^*}^{p(y - \varepsilon^*)} \left[ p[r + s + \alpha f_1^+(py - p\varepsilon^* - \varepsilon)] \psi(\varepsilon + q\varepsilon^*) \psi(\varepsilon^*) d\varepsilon \right] \]

\[ - \int_{y}^{\infty} (c + h_1 - r\varepsilon^* - sq) \psi(\varepsilon^*) d\varepsilon^* \]

\[ \int_{y}^{y + \Delta} \left\{ r\varepsilon^* + h_2 + sq - rp - sp \right\} \right. \]

\[ + \int_{a - q\varepsilon^*}^{p(y + \Delta - \varepsilon^*)} \left[ p[r + s + \alpha f_1^+(py + p\Delta - p\varepsilon^* - \varepsilon)] \psi(\varepsilon + q\varepsilon^*) \psi(\varepsilon^*) d\varepsilon \right] \]
\[\int_{y}^{y+\Delta} \left\{ r + h_2 + sq - rp - sp \right\} \psi(\varepsilon) d\varepsilon + \int_{a}^{p(y+\Delta)-(p-q)\varepsilon} \phi(\varepsilon) d\varepsilon \]

\[= \int_{y}^{y+\Delta} V_{1}^{*} (pq+p\Delta-pq^{*}+qq^{*}) \psi(\varepsilon) d\varepsilon \]

\[\geq \int_{y}^{y+\Delta} V_{1}^{*} (qy+q\Delta) \psi(\varepsilon) d\varepsilon \]

\[\geq \int_{y}^{y+\Delta} V_{1}^{*} (Z_{1}) \psi(\varepsilon) d\varepsilon \]

\[= 0. \quad (4.1.18)\]

Next let \( y \) be any point lying in the interval \( \left[ \frac{Z_{1}}{p}, \frac{Z_{1}}{q} \right] \) and \( \Delta \) a positive number such that \( y + \Delta \leq \frac{Z_{1}}{q} \). Then:

\[W_{1}^{x}(y+\Delta) - W_{1}^{x}(y)\]

\[= D_{1}\mathcal{B}^{1}(y + \Delta, \frac{Z_{1}-qy-q\Delta}{p-q}) - D_{1}\mathcal{B}^{1}(y, \frac{Z_{1}-qy}{p-q})\]
\[
\begin{align*}
&\int_{y+\Delta - \frac{Z_{\frac{1}{2}}-qy-q\Delta}{p-q}}^y \left\{ c + h_1 + h_2 - rp - sp \right\} \, a^* \\
&+ \int_{a-q\epsilon^*}^p (y+\Delta-\epsilon^*) \, p[r + s + \alpha f_{\frac{1}{2}}(py+p\Delta-p\epsilon^*-\epsilon)] \phi(\epsilon+q\epsilon^*) \, d\epsilon \\
&+ \int_{a-q\epsilon^*}^\infty \left\{ c + h_1 - r^* - sq \right\} \, d\epsilon^* \\
&+ \int_{a-q(y+\Delta - \frac{Z_{\frac{1}{2}}-qy-q\Delta}{p-q})}^p q[r + s + \alpha f_{\frac{1}{2}}(p \cdot \frac{Z_{\frac{1}{2}}-qy-q\Delta}{p-q} - \epsilon)] \phi(\epsilon+qy+q\Delta-q \cdot \frac{Z_{\frac{1}{2}}-qy-q\Delta}{p-q}) \, d\epsilon \\\n&+ \int_{a^*}^{y-\frac{Z_{\frac{1}{2}}-qy}{p-q}} \left\{ c + h_1 + h_2 - rp - sp \right\} \, d\epsilon^* \\
&+ \int_{a-q\epsilon^*}^{p(y-\epsilon^*)} p[r + s + \alpha f_{\frac{1}{2}}(py-p\epsilon^*-\epsilon)] \phi(\epsilon+q\epsilon^*) \, d\epsilon \\
&\psi(\epsilon^*) \, d\epsilon^* 
\end{align*}
\]
\[
- \int \left\{ \left( c + h_1 - r^* - sq \right) \right. \\
y - \frac{Z_1-\text{qy}}{p-q} \left. \right\} \\
+ \int \left\{ \right. \\
p \cdot \frac{Z_1-\text{qy}}{p-q} \\
a-q(y - \frac{Z_1-\text{qy}}{p-q}) \\
q\left[ r + s + \alpha f_1^+(p \cdot \frac{Z_1-\text{qy}}{p-q} - \varepsilon) \right] \\
\left. \right\} \psi(\varepsilon^*)d\varepsilon^* \\
\left. \right\} \psi(\varepsilon^*)d\varepsilon^*
\]

\[
\int \left\{ \left( c + h_1 + h_2 - rp - sp \right) \right. \\
y + \Delta - \frac{Z_1-\text{qy-q}\Delta}{p-q} \left. \right\} \\
+ \int \left\{ \right. \\
p(y+\Delta)-(p-q)e^* \\
a \\
p[r + s + \alpha f_1^+(py+p\Delta-p\varepsilon^*+qe^*+e^*\varepsilon^*)] \psi(\varepsilon^*)d\varepsilon^* \\
\left. \right\} \psi(\varepsilon^*)d\varepsilon^*
\]

\[
\int \left\{ \left( c + h_1 - r^* - sq \right) \right. \\
y - \frac{Z_1-\text{qy}}{p-q} \left. \right\} \\
+ \int \left\{ \right. \\
p \cdot \frac{Z_1-\text{qy}}{p-q} \\
a-q(y - \frac{Z_1-\text{qy}}{p-q}) \\
q\left[ r + s + \alpha f_1^+(p \cdot \frac{Z_1-\text{qy}}{p-q} - \varepsilon) \right] \\
\left. \right\} \psi(\varepsilon^*)d\varepsilon^* \\
\left. \right\} \psi(\varepsilon^*)d\varepsilon^*
\]

\[
\int \left\{ \left( c + h_1 + h_2 - rp - sp \right) \right. \\
y + \Delta - \frac{Z_1-\text{qy-q}\Delta}{p-q} \left. \right\} \\
+ \int \left\{ \right. \\
p(y+\Delta)-(p-q)e^* \\
a \\
p[r + s + \alpha f_1^+(py+p\Delta-p\varepsilon^*+qe^*+e^*\varepsilon^*)] \psi(\varepsilon^*)d\varepsilon^* \\
\left. \right\} \psi(\varepsilon^*)d\varepsilon^*
\]
\[ + \int_{a}^{Z_{\frac{1}{1}} \left[ r + s + \alpha f_{1}^{1} (Z_{\frac{1}{1}} - \varepsilon) \phi (\varepsilon) \right] \psi (\varepsilon^*)} d\varepsilon^* \]

\[ y + \Delta - \frac{Z_{\frac{1}{1}} - qy - q\Delta}{p-q} \]

\[ \geq \int_{y - \frac{Z_{\frac{1}{1}} - qy}{p-q}}^{y + \Delta - \frac{Z_{\frac{1}{1}} - qy - q\Delta}{p-q}} \left\{ r^* + h_2 + sq - rp - sp \right\} d\varepsilon^* \]

\[ + \int_{a}^{Z_{\frac{1}{1}}} (p-q) \left[ r + s + \alpha f_{1}^{1} (Z_{\frac{1}{1}} - \varepsilon) \phi (\varepsilon) \right] \psi (\varepsilon^*) d\varepsilon^* \]

(Since \( p(y+\Delta) - (p-q)\varepsilon^* \geq p(y+\Delta) - (p-q)(y + \Delta - \frac{Z_{\frac{1}{1}} - qy - q\Delta}{p-q}) = Z_{\frac{1}{1}} \) and

\[ \int_{a}^{\theta} \left[ r + s + \alpha f_{1}^{1} (\theta - \varepsilon) \phi (\varepsilon) \right] d\varepsilon \text{ is monotone nondecreasing in } \theta. \]

\[ y + \Delta - \frac{Z_{\frac{1}{1}} - qy - q\Delta}{p-q} \]

\[ = \int_{y - \frac{Z_{\frac{1}{1}} - qy}{p-q}}^{V_{\frac{1}{1}}(Z_{\frac{1}{1}}) \psi (\varepsilon^*)} d\varepsilon^* \]

\[ = 0 \quad \text{(4.1.19)} \]

Finally let \( y \) lie in the interval \([0, \frac{Z_{\frac{1}{1}}}{p}]\) and \( \Delta \) be a positive number such that \( y + \Delta \leq \frac{Z_{\frac{1}{1}}}{p} \). Then:
\[ W_L^1(y+\Delta) - W_L^1(y) \]

\[ = D_1g_L^1(y+\Delta, y+\Delta) + D_2g_L^1(y+\Delta, y+\Delta) - D_1g_L^1(y, y) - D_2g_L^1(y, y) \]

\[ = \int_0^\infty \left\{ c + h_1 - r^* - sq + \int_a p(y+\Delta) \right\} \left\{ q[r + s + af_1^1(py+p\Delta-\epsilon)] \cdot \phi(\epsilon)d\epsilon \right\} \psi(\epsilon^*)d\epsilon^* \]

\[ + \left\{ r^* + h_2 + sq - rp - sp + \int_a p(y+\Delta) \right\} \left( p-q)[r + s + af_1^1(py+p\Delta-\epsilon)]\phi(\epsilon)d\epsilon \right\} \]

\[ - \int_0^\infty \left\{ c + h_1 - r^* - sq + \int_a py \right\} \left\{ q[r + s + af_1^1(py-\epsilon)]\phi(\epsilon)d\epsilon \right\} \psi(\epsilon^*)d\epsilon^* \]

\[ - r^* + h_2 + sq - rp - sp + \int_a py \left( p-q)[r + s + af_1^1(py-\epsilon)]\phi(\epsilon)d\epsilon \right\] \[ > 0 \quad (4.1.20) \]

From (4.1.18), (4.1.19), (4.1.20), and the continuity of \( W_L^1(y) \) at the points \( \frac{Z_1}{P} \) and \( \frac{Z_1}{Q} \), we conclude that \( W_L^1(y) \) is monotone nondecreasing. Furthermore:
\[ \lim_{M \to \infty} W_{1}(M) = \lim_{M \to \infty} D_{1}g_{1}(M,0) \]

\[ = c + h_{1} + h_{2} - rp - sp + p[r + s + a(h_{1} + h_{2})/(1-\alpha)] \]

\[ = c + (h_{1} + h_{2})/(1-\alpha) > 0 \]

and

\[ W_{1}(0) = D_{1}g_{1}(0,0) + D_{1}g_{2}(0,0) \]

\[ = (c+h_{1} - r*s - q) + (r*h_{2} + s*r - r*p - s) \]

\[ = c + h_{1} + h_{2} - rp - sp < 0. \]

.'.' There exists a uniquely determined number \( Y_{1} \), \( 0 < Y_{1} < \infty \), such that \( W_{1}(Y_{1}) = 0 \) and \( W_{1}(y) < 0 \) for all \( y < Y_{1} \). Part (b) is proved.

Part c trivially follows from the fact

\[ f_{1}(x) = -cx + W_{1}(x) \quad \text{for } Y \leq x \]

\[ = -cx + W_{1}(Y) \quad \text{for } x < Y \]
Moreover, since $f_1^1(x)$ is monotone-nondecreasing and exists everywhere, $f_1^2(x)$ now exists almost everywhere for any closed interval of $x$.

4.2 Critical Numbers of the Infinite-Period Process with Independent and Identical Demand Distributions

Since:

$$W_1^1\left(\frac{Z_1}{p}\right) = D_1 g_1^1\left(\frac{Z_1}{p}, \frac{Z_1}{p}\right)$$

$$= \int_0^\infty \left\{ c + h_1 - r^* - sq + \int_0^{Z_1} q[r + s + \alpha f_1^1(Z_1 - \epsilon)] \phi(\epsilon) d\epsilon \right\} \psi(\epsilon^*) d\epsilon^*$$

$$= c + h_1 - r^* - sq + \frac{q}{p-q} \left[ V_1(Z_1) - r^* + h_2 + sq - rp - sp \right]$$

$$= c + h_1 - \frac{r^*p + h_2q - rpq}{p-q} \quad (4.2.1)$$

We have:

Lemma 4.3:

$$c + h - \frac{r^*p + h_2q - rpq}{p-q} - 0 \iff \frac{Z_1}{p} - \frac{Y_1}{p}.$$
When \( c + h_1 + \frac{r^*p+h_2q-rpq}{p-q} \leq 0 \), from Lemma 4.3, \( Z_1 \leq \frac{Z_1}{p} \leq Y_1 \).

From (4.1.17) and Theorem 4.2

\[
V_1(Z_1) = r^* + h_2 + sq - rp - sp
\]

\[
+ \int_a^{Z_1} (p-q)[r + s + \alpha f_1(Z_1-\varepsilon)] \phi(\varepsilon) d\varepsilon
\]

\[
= r^* + h_2 + sq - rp - sp + (p-q)(r+s-ac)\phi(Z_1)
\]

\[
= 0
\]

\[
\therefore Z_1 = \phi^{-1}\left(\frac{rp+sp-r^*-h_2-sq}{(p-q)(r+s-ac)}\right)
\]

(4.2.2)

To solve \( Y_1 \), note that for \( Y_1 > \frac{Z_1}{q} \):

\[
W_1(Y_1) = D_1 S_1(Y_1, 0)
\]

\[
= \int_{Y_1}^{Y_1} \left\{ c + h_1 + h_2 - rp - sp
\right.
\]

\[
+ \int_{a-q\varepsilon^*}^{p(Y_1-\varepsilon^*)} p[r + s + \alpha f_1(pY_1-p\varepsilon^* - \varepsilon)]
\]

\[
\times \phi(\varepsilon+q\varepsilon^*) d\varepsilon
\]

\[
\{ \psi(\varepsilon^*) d\varepsilon^* \}
\]
\[ + \int_{Y_1^-}^{\infty} (c + h_1 - r^* - sq) \psi(\varepsilon^*) d\varepsilon^* \]

\[ = \int_{a^*}^{Y_1^-} \left\{ r^* + h_1 + h_2 - rp - sp \right\} \psi(\varepsilon^*) d\varepsilon^* + (c + h_1 - r^* - sq) \]

\[ + p(r+s-ac)\phi(pY_1^--p\varepsilon^*+q\varepsilon^*) \psi(\varepsilon^*) d\varepsilon^* \]

\[ = 0 \quad (4.2.3) \]

since \( f_1^*(pY_1^- - p\varepsilon^* - \varepsilon) = -c \) from Theorem 4.2.

If the solution of \( Y_1^- \) from (4.2.3) is not greater than \( \frac{Z_1^-}{q} \), then clearly \( Y_1^- < \frac{Z_1^-}{q} \). Now

\[ W_1^t(Y_1^-) = D_1 g_1(Y_1^-, \frac{Z_1^- - qY_1^-}{p-q}) \]

\[ = \int_{a^*}^{pY_1^- - Z_1^- \over p-q} \left\{ c + h_1 + h_2 - rp - sp \right\} \psi(\varepsilon^*) d\varepsilon^* \]

\[ + \int_{a-q\varepsilon^*}^{p(Y_1^- - \varepsilon^*)} p[r + s + f_1^*(pY_1^- - p\varepsilon^* - \varepsilon)] \phi(\varepsilon + q\varepsilon^*) d\varepsilon \psi(\varepsilon^*) d\varepsilon^* \]
\[ + \int_{pY_1-Z_1 \over p-q}^{\infty} \left\{ c + h_1 - r^* - sq \right\} \frac{pY_1-Z_1}{p-q} \]

\[ + \int_{p \cdot \frac{Z_1-qY_1}{p-q}} \left\{ q[r + s + f_1(p \cdot \frac{Z_1-qY_1}{p-q} - \varepsilon)]\phi(\varepsilon + q \cdot \frac{pY_1-Z_1}{p-q}) d\varepsilon \right\} \]

\[ \cdot \psi(\varepsilon^*) d\varepsilon^* \]

\[ = \int_{pY_1-Z_1 \over p-q}^{a^*} \left[ c + h_1 + h_2 - rp - sp + p(r+s-ac)\phi(pY_1-pe^*-q\varepsilon^*) \right] \]

\[ \cdot \psi(\varepsilon^*) d\varepsilon^* \]

\[ + \int_{pY_1-Z_1 \over p-q}^{\infty} [c + h_1 - r^* - sq + q(r+s-ac)\phi(Z_1)]\psi(\varepsilon^*) d\varepsilon^* \]

\[ = c + h_1 - \frac{r^*p+q-rpq}{p-q} + \int_{pY_1-Z_1 \over p-q}^{a^*} \int_{Z_1}^{pY_1-(p-q)\varepsilon^*} p(r+s-ac)\phi(\varepsilon) d\varepsilon \psi(\varepsilon^*) d\varepsilon^* \]

\[ = 0 \quad (4.2.4) \]
Since $W_1^2(y)$ is a monotone nondecreasing function, (4.2.3) and (4.2.4) can be solved without great difficulty. When $c + h_1 - \frac{r^* p + h_2 q - r p q}{p - q} > 0$, from Lemma 4.3, $\frac{Z_1}{p} > Y_1$. Now there is no easy way to solve $Z_1$ explicitly since the term $f_1^2(Z_1 - \varepsilon)$ in $V_1(Z_1)$ cannot be made equal to "-c." However, once the stock level is down to $Y_1$, then for all succeeding periods, $\bar{y}_1 = \max(x_1, y_1) = Y_1$, $\frac{Z_1}{p}$. But from Theorem 4.2, $\bar{y}_1 < \frac{Z_1}{p}$ implies $\bar{x}_1 = \bar{y}_1 = Y_1$. So the solving of $Z_1$ is only of academic interest.

$Y_1$ can still be solved from:

$W_1^2(Y_1) = D_1 g_1(Y_1, Y_1) + D_2 g_2(Y_1, Y_1)$

$$= \int_0^\infty \left\{ c + h_1 - r^* - s q + \int_a^{p Y_1} q[r + s + \alpha f_1^1(p Y_1 - \varepsilon)] \phi(\varepsilon) d\varepsilon \right\} \psi(\varepsilon^*) d\varepsilon^*$$

$$+ \left\{ r^* + h_2 + s q - r p - s p + \int_a^{p Y_1} (p - q) \right\} [r + s + \alpha f_1^1(p Y_1 - \varepsilon)] \phi(\varepsilon) d\varepsilon$$
\[ = c + h_1 + h_2 - rp - sp + p(r + s - \alpha c)\phi(pY^1) \]
\[ = 0 \quad (4.2.5) \]

since \( f_1^1(pY^1 - \varepsilon) = -c \). From (4.2.5), we solve:

\[ Y^1 = \frac{1}{p} \phi^{-1}\left(\frac{rp + sp - c - h_1 - h_2}{p(r + s - \alpha c)}\right) \quad (4.2.6) \]

Lemma 4.4: If the following conditions hold

(a) \( F_1, F_2, F_3, F_4 \) are probability distributions with densities \( f_1, f_2, f_3, f_4 \).

(b) \( L'_1(\varepsilon) = -\ell_1, L'_2(\varepsilon) = \ell_2, L'_3(\varepsilon) = -\ell_3, L'_4(\varepsilon) = \ell_4 \), where \( dL'_1(\varepsilon)/d\varepsilon \) and \( \ell_1, \ell_2, \ell_3, \ell_4 \) are nonnegative constants.

(c) \( F_1[L_1(\varepsilon)] \geq F_3[L_3(\varepsilon)], F_2[L_2(\varepsilon)] \geq F_4[L_4(\varepsilon)] \) for all \( \varepsilon \).

(d) \( A + BF_3[L_3(b)] \geq 0, F_2[L_2(a)] = 0 \) where \( Z, B, a, b \) are constants; \( B \) is nonnegative, \( a \leq b \).

(Note that \( F_4[L_4(a)] \leq F_2[L_2(a)] = 0 \) from (b) and (d).) Then,

\[ \int_a^b [A + BF_1[L_1(\varepsilon)]\]f_2[L_2(\varepsilon)]d\varepsilon \geq \int_a^b \frac{\ell_4}{\ell_2} [A + BF_3[L_3(\varepsilon)]\]f_4[L_4(\varepsilon)]d\varepsilon.\]

Proof:

Using integration by parts:

\[ \int_a^b [A + BF_1[L_1(\varepsilon)]\]f_2[L_2(\varepsilon)]d\varepsilon \]
\[ \frac{1}{L_2} \left[ (A + BF_L)(e) \right] f_2(L_2(e)) + \int_a^b \frac{L_3}{L_2} f_3(L_3(e)) \cdot BF_2(L_2(e)) \, de \]

\[ \geq \frac{1}{L_2} \left[ (A + BF_L)(e) \right] f_4(L_4(e)) + \int_a^b \frac{L_3}{L_2} f_3(L_3(e)) \cdot BF_4(L_4(e)) \, de \]

\[ \geq \frac{1}{L_2} \left[ (A + BF_L)(e) \right] f_4(L_4(e)) - \frac{1}{L_2} f_3(L_3(b)) \cdot BF_4(L_4(b)) \]

\[ + \int_a^b \frac{L_4}{L_2} \cdot BF_3(L_3(e)) f_4(L_4(e)) \, de \]

\[ = \int_a^b \frac{L_4}{L_2} \left[ (A + BF_L)(e) \right] f_4(L_4(e)) \, de. \]

Theorem 4.5: Let \( Y_A, Z_A \) be the critical numbers when demand distributions are identically \( \phi_A \) and \( \psi_A \) at each period. Let \( Y_B, Z_B \) be the critical numbers when demand distributions are identically \( \phi_B \) and \( \psi_B \). Moreover let \( \phi_A(e) \geq \phi_B(e) \) for all \( e \), and \( \psi_A(e^*) \geq \psi_B(e^*) \) for all \( e^* \), then under assumptions (a), (b), (c) and (d) of Theorem 4.1, \( Y_A \leq Y_B \).

Proof: First consider the case \( c + h_1 - \frac{r^* p + h_2 q - r p q}{p - q} > 0 \). From (4.2.6):
\[ \phi_A(pY_A) = \frac{rp+sp-c_1-h_1-h_2}{p(r+s-ac)} = \phi_B(pY_B) \leq \phi_A(pY_B) \]

\[ \therefore Y_A \leq Y_B. \]

Next consider the case \( c + h_1 - \frac{r^*p+h_2q-rpq}{p-q} \leq 0 \). From (4.2.2):

\[ \phi_A(Z_A) = \frac{rp+sp-r^*-h_2-sq}{(p-q)(r+s-ac)} = \phi_B(Z_B) \leq \phi_A(Z_B) \]

\[ \therefore Z_A \leq Z_B \] (4.2.7)

From Lemma 4.3, \( \frac{Z_B}{p} \leq Y_B \). We will consider the two possibilities

\[ \frac{Z_B}{p} \leq Y_B < \frac{Z_B}{q} \quad \text{and} \quad \frac{Z_B}{q} < Y_B. \]

Assuming \( \frac{Z_B}{q} < Y_B \), then from (4.2.7), \( \frac{Z_A}{q} < Y_B \). Hence \( W'_A(Y_B) = D_{1g_A}(Y_B,0) \) and \( W'_B(Y_B) = D_{1g_B}(Y_B,0) \).

Let \( a \) and \( a^* \) be the minimum values of the distributions \( \phi_A \) and \( \psi_A \) respectively; and \( b \) and \( b^* \) be the minimum values of the distributions \( \phi_B \) and \( \psi_B \) respectively. Then \( b \geq a, b^* \geq a^* \). By noting \( f'_A(x) \geq -c \) and using Lemma 4.4:

\[ D_{1g_A}(Y_B,0) = \int_{a^*}^{Y_B} \left( c + h_1 + h_2 - rp - sp \right) \]

\[ + \int \left[ p(Y_B-\varepsilon^*) - \alpha f'_A(pY_B-p\varepsilon^*+\varepsilon) \phi_A(\varepsilon+q\varepsilon^*)d\varepsilon \right] \psi_A(\varepsilon^*)d\varepsilon^* \]

\[ + \int_{Y_B}^{\infty} (c+h_1-r^*-sq)\psi_A(\varepsilon^*)d\varepsilon^* \]
\[ > \int_{a^*}^{Y_B} \left[ r^* + h_2 + sq - rp - sp + p(r+s-\alpha c) \phi_A(p_{Y_B} - p\varepsilon^* + q\varepsilon^*) \psi_A(\varepsilon^*)d\varepsilon^* + (c+h_1-r^*-sq) \right] \]

\[ > \int_{a^*}^{Y_B} \left[ r^* + h_2 + sq - rp - sp + p(r+s-\alpha c) \phi_B(p_{Y_B} - p\varepsilon^* + q\varepsilon^*) \psi_B(\varepsilon^*)d\varepsilon^* + (c+h_1-r^*-sq) \right] \]

\[ = \int_{b^*}^{Y_B} \left[ c + h_1 + h_2 - rp - sp + p(r+s-\alpha c) \phi_B(p_{Y_B} - p\varepsilon^* + q\varepsilon^*) \psi_B(\varepsilon^*)d\varepsilon^* \right] \]

\[ + \int_{Y_B}^{\infty} (c+h_1-r^*-sq) \psi_B(\varepsilon^*)d\varepsilon^* \]

and

\[ D_{1g_B}(Y_B, 0) = \int_{b^*}^{Y_B} \left\{ c + h_1 + h_2 - rp - sp + p(Y_B - \varepsilon^*) \right\} \psi_B(\varepsilon^*)d\varepsilon^* \]

\[ + \int_{b^*}^{p(Y_B - \varepsilon^*)} \left\{ p[r + s + af_{B}^I(p_{Y_B} - p\varepsilon^* - \varepsilon)] \phi_B(\varepsilon^* + q\varepsilon^*)d\varepsilon \right\} \psi_B(\varepsilon^*)d\varepsilon^* \]
\[ + \int_{Y_B}^{\infty} (c + h_1 - r^* - sq) \psi_B(\epsilon^*) \, d\epsilon^* \]

\[ = \int_{b^*}^{Y_B} [c + h_1 + h_2 - rp - sp] \psi_B(\epsilon^*) \, d\epsilon^* \]

\[ + p(r+s-\alpha c) \phi_B(pY_B-p\epsilon^*+q\epsilon^*)] \psi_B(\epsilon^*) \, d\epsilon^* \]

\[ + \int_{Y_B}^{\infty} (c + h_1 - r^* - sq) \psi_B(\epsilon^*) \, d\epsilon^* \]

since \( f_B'(pY_B-p\epsilon^*+\epsilon) = -c \).

\[ \therefore W_A'(y_B) = D_1 g_A(y_B, 0) - D_1 g_B(y_B, 0) = W_B'(y_B) \]

Next assuming \( \frac{Z_B}{p} \leq y_B \leq \frac{Z_B}{q} \), then \( W_B'(y_B) = D_1 g_B(y_B, \frac{Z_B - q^* y_B}{p-q}) \). From (4.2.7), \( \frac{Z_A}{p} \leq y_B \). If \( \frac{Z_A}{q} \leq y_B \), then \( W_A'(y_B) = D_1 g_A(y_B, \frac{Z_A - q^* y_B}{p-q}) \). If \( \frac{Z_A}{q} < y_B \) then \( W_A'(y_B) = D_1 g_A(y_B, 0) \). However, \( D_1 g_A(y, z) \) is independent of \( z \) for \( y - z \geq a^* \). Therefore \( D_1 g_A(y_B, 0) = D_1 g_A(y_B, \frac{Z_A - q^* y_B}{p-q}) \) for \( y_B > \frac{Z_A}{q} \geq \frac{a}{q} \geq a^* \). So in either case,

\[ W_A'(y_B) - W_B'(y_B) = D_1 g_A(y_B, \frac{Z_A - q^* y_B}{p-q}) - D_1 g_B(y_B, \frac{Z_A - q^* y_B}{p-q}) \]

Again by noting \( f_A'(x) \geq -c, Z_A \leq Z_B, \phi_A(Z_A) = \phi_B(Z_B) \) and Lemma 4.4:
\[ D_1 s_A (Y_B; \frac{Z_A - qY_B}{p-q}) \]

\[ = \int_{a^*}^{Y_B} \left\{ c + h_1 + h_2 - rp - sp \right\} \frac{Z_A - qY_B}{p-q} d\epsilon \]

\[ + \int_{a^*-q^*}^{p(Y_B - \epsilon^*)} p[r + s + \alpha f_A'(pY_B - p\epsilon^* - \epsilon)] \phi_A(\epsilon + q\epsilon^*) d\epsilon \]

\[ \psi_A(\epsilon^*) d\epsilon^* \]

\[ + \int_{a^*}^{y_B} \left\{ c + h_1 - r^* - sq \right\} \frac{Z_A - qY_B}{p-q} d\epsilon \]

\[ + \int_{a^* - q^*(Y_B - \frac{Z_A - qY_B}{p-q})}^{p \cdot \frac{Z_A - qY_B}{p-q}} q[r + s + \alpha f_A'(p \cdot \frac{Z_A - qY_B}{p-q} - \epsilon)] \]

\[ \phi_A(\epsilon + qY_B - q \cdot \frac{Z_A - qY_B}{p-q}) d\epsilon \]

\[ \psi_A(\epsilon^*) d\epsilon^* \]

\[ - \int_{a^*}^{Y_B - \frac{Z_A - qY_B}{p-q}} \left\{ c + h_1 + h_2 - rp - sp + p(r+s-\alpha) \phi_A(pY_B - p\epsilon^* + q\epsilon^*) \right\} \psi_A(\epsilon^*) d\epsilon^* \]

\[ + \int_{y_B - \frac{Z_A - qY_B}{p-q}}^{\infty} \left\{ c + h_1 - r^* - sq + q(r+s-\alpha) \phi_A(Z_A) \right\} \psi_A(\epsilon^*) d\epsilon^* \]
\[ \begin{align*}
\int_{a^*}^{Y_B} Y_B - \frac{Z_A - qY_B}{p-q} & \quad [r^* + h_2 + sq - rp - sp - q(r+s-ac) \phi_A(Z_A) \\
+ p(r+s-ac) \phi_A (pY_B - p\epsilon^* - q\epsilon^*)] \psi_A(\epsilon^*) d\epsilon^* \\
+ [c + h_1 - r^* - sq + q(r+s-ac) \phi_A(Z_A)] \\
\end{align*} \]

\[ \begin{align*}
\int_{a^*}^{Z_B - qY_B} Y_B - \frac{Z_A - qY_B}{p-q} & \quad [r^* + h_2 + sq - rp - sp - q(r+s-ac) \phi_A(Z_A) \\
+ p(r+s-ac) \phi_B (pY_B - p\epsilon^* + q\epsilon^*)] \psi_B(\epsilon^*) d\epsilon^* \\
+ [c + h_1 - r^* - sq + q(r+s-ac) \phi_A(Z_A)] \\
\end{align*} \]
\[
\mathcal{D}_{1, B}(Y_B, \frac{Z_B - q Y_B}{p - q})
\]

\[
\int_{Y_B - \frac{Z_B - q Y_B}{p - q}}^{Y_B} \left[ c + h_1 + h_2 - r p - s p + p (r + s - \alpha \epsilon) \Phi_B(p, Y_B, \frac{Z_B - q Y_B}{p - q} + \epsilon) \right] \psi_B(\epsilon) d\epsilon
\]

and

\[
\int_{0}^{\infty} \left[ c + h_1 - r^* - s q + q (r + s - \alpha \epsilon) \Phi_B(Z_B) \right] \psi_B(\epsilon) d\epsilon
\]
\[
\psi_B(\varepsilon^*) \left\{ \int_{Y_B}^{Y_B - \frac{Z_B - qY_B}{p-q}} \left[ c + h_1 + h_2 - rp - sp \right] + p(r+s-ac)\phi_B(pY_B - p\varepsilon^* + q\varepsilon^*) \right\} \psi_B(\varepsilon^*) d\varepsilon^*
\]

\[
+ \int_{Y_B}^{\infty} \left[ c + h_1 - r* - sq + q(r+s-ac)\phi_B(Z_B) \right] \psi_B(\varepsilon^*) d\varepsilon^*
\]

since \( f_B'(pY_B - p\varepsilon^* - \varepsilon) = -c \) and \( f_B'(p \cdot \frac{Z_B - qY_B}{p-q} - \varepsilon) = -c \). (The latter equality is due to:

\[
p \cdot \frac{Z_B - qY_B}{p-q} \leq p \cdot \frac{Z_B - q \cdot \frac{Z_B}{p-q}}{p-q} = Z_B \leq Y_A
\]

\[
W_A'(Y_B) = D_A(Y_B, \frac{Z_A - qY_B}{p-q}) \geq D_B(Y_B, \frac{Z_B - qY_B}{p-q}) = W_B'(Y_B)
\]

To summarize, we have proved that when \( c + h_1 - \frac{r*p + h_2 + q - rpq}{p-q} \leq 0 \),

\( W_A'(Y_B) \geq W_B'(Y_B) \) regardless of the value of \( Y_B \). But

\[ W_A'(Y_B) = W_A'(Y_B) = 0 = W_A'(Y_A) \]
implies $Y_B \geq Y_A$ since $W'_A(y)$ is a monotone nondecreasing function.

Theorem 4.5 is proved.

Theorem 4.6: Under assumptions (a), (b), (c) and (d) of Theorem 4.1, then $Y_{1N} \leq Y_{1}$ for all $N = 1, 2, 3, \ldots$

Where $Y_{1N}$ is the first-period critical number of a $N$-stage process with varying demand distributions over the time period.

Proof:

First assume $c + h_1 - \frac{r_2 p + h_2 q - r p q}{p - q} > 0$. Then from Lemma 3.2,

$Y_{1N} \leq \frac{Z_{1N}}{p}$. Hence

$W_{1N}'(Y_{1N}) = D_{1N} g_{1N}(Y_{1N}, Y_{1N}) = D_{2N} g_{2N}(Y_{1N}, Y_{1N})$

$$= \int_{0}^{\infty} \left\{c + h_1 - r^* - sq + \int_{a_1}^{pY_{1N}} q[r + s + \alpha f'_{2N}(pY_{1N} - \varepsilon)] \phi_1(\varepsilon) d\varepsilon \right\} \psi_1(\varepsilon^*) d\varepsilon^*$$

$$+ \left\{r^* + h_2 + sq - rp - sp + \int_{a_1}^{pY_{1N}} (p - q) \right\}$$

$$\cdot [r + s + \alpha f'_{2N}(pY_{1N} - \varepsilon)] \phi_1(\varepsilon) d\varepsilon$$

$$\geq \int_{0}^{\infty} \left[c + h_1 - r^* - sq + q(r + s - ac) \phi_1(pY_{1N}) \right] \psi_1(\varepsilon^*) d\varepsilon^* \varepsilon$$
\[ + \left[ r^* + h_2 + sq - rp - sp + (p-q)(r+s-ac)\phi_1(pY_1) \right] \]

\[ = c + h_1 + h_2 - rp - sp + p(r+s-ac)\phi_1(pY_1) \]

\[ = 0 \]

\[ = W'_1(Y_{1N}) \]

\[ \therefore \quad Y_{1N} = Y_{1N}^* \]

Next assume \( c + h_1 - \frac{r^*p+h_2q-rpq}{p-q} \leq 0 \).

\[ V_{1N}(Z_1^{-}) = r^* + h_2 + sq - rp - sp \]

\[ + \int_{a_1}^{Z_1^{-}} (p-q)[r + s + \alpha f''_{2N}(Z_1^{-})]\phi_1(\varepsilon)d\varepsilon \]

\[ \geq r^* + h_2 + sq - rp - sp + (p-q)(r+s-ac)\phi_1(Z_1^{-}) \]

\[ = 0 \]
\[ = V_{1N}(Z_{1N}) \]

\[ Z_{1N} \geq Z_{1N} \quad \text{(4.2.8)} \]

To prove \( W_{1N}^1(Y_{1-}) - W_{1N}^1(Y_{1-}) \geq 0 \), we need to consider all possible ranges of \( Y_{1-} \) so that the difference can be written out explicitly. However, the proof is exactly the same as the proof of the corresponding part for Theorem 3. We will find that those terms \( f_1^1(x) \) in \( W_{1N}^1(Y_{1-}) \) have arguments \( x \) less than \( Y_{1-} \) and hence \( f_1^1(x) = -c \) while those corresponding terms \( f_{1N}^1(x) \) in \( W_{1N}^1(Y_{1-}) \) is \( \geq -c \). This plus (4.2.8) provides the tools needed for the proof.

4.3 Infinite Period Model When Demand Distributions in Different Time Periods are Independent but Varying

Theorem 4.7: Suppose

(a) \( r + s - \alpha c \geq 0 \)

(b) \( p \geq q \geq 0 \)

(c) \( r^* + h_2 + sq - rp - sp \leq 0 \leq r^* + h_2 - rq - (p-q)dc \)

(d) \( c + h_1 - r^* - sq < 0 < (1-\alpha)p + h_1 + h_2 \)

(e) the \( i^{th} \) period demand distributions \( \{\phi_i, \psi_i\} \) are independent but not necessarily identical to the demand distributions of any other time period,

(f) there exists two distribution functions \( \{\phi, \psi\} \) with finite expectations such that \( \phi_i(\epsilon), \leq \phi(\epsilon) \) for all \( \epsilon \)

and \( \psi_i(\epsilon^*) \leq \psi(\epsilon^*) \) for all \( \epsilon^*, i = 1, 2, 3, \ldots \).

Then \( f_{1N}^1(x) \) converges uniformly for every finite \( i \) to a unique function \( f_1^1(x) \) for every finite close interval \( 0 \leq x \leq b \). Moreover,
\( f_i(x) \) is a bounded continuous convex function in the interval 
\( 0 \leq x \leq b \) for every finite \( b \).

**Proof:**

Theorem 4.3 is a restatement of Theorem 4.1 except that now demand distributions are allowed to vary over time. The proof of Theorem 4.1, aside from the portion dealing with the uniqueness of the solution to the functional equation, will still be true here if we can show \( y_i^N \) are still uniformly bounded for all \( i \) and \( N, i \leq N \).

Let \( (Y, Z) \) be the critical numbers corresponding to the infinite-stage process where demand distributions at each period are identically \( \{0, 1\} \). From Theorem 4.6, \( y_i^N \leq y_i^1 \) for all \( N = 1, i+1 \ldots \) where \( y_i^1 \) is the critical number when demand distributions are identically \( \{0, 1\} \) for all periods. From Theorem 4.5, \( y_i^1 \leq Y \) for all \( i = 1, 2, 3, \ldots \). From Theorem 4.2, \( Y < \infty \). Thus we see that \( y_i^N \) are uniformly bounded and Theorem 4.7 follows.

From Theorem 4.7, \( f_i(x) = \lim_{N \to \infty} f_i^N(x) \) exists and is a bounded continuous convex function over all finite close interval of \( x \). It follows that \( f_i'(x) \) exists almost everywhere. Therefore \( g_i(y,z) = \lim_{N \to \infty} g_i^N(y,z) \) exists and \( g_i^1(y,z), g_i^2(y,z) \) are well defined continuous functions. Also:

\[
y_i(0) = r^* + h_2 + sq - rp - sp
\]

\[
+ \int_0^{\infty} (p-q)[r + s + \alpha f_{i+1}^N(\theta-\varepsilon)]\psi(\varepsilon)d\varepsilon \quad (4.3.1)
\]
are well defined and

\[ D_2 g_1(y, z) = V_1(qy + pz - qz)[1 - \psi_1(y-z)] \quad \text{for } y-z \leq a_1/q \]

\[ = 0 \quad \text{for } y-z > a_1/q \quad (4.3.2) \]

With the boundary conditions checked, then there exists a uniquely determined number \( Z_1 \), \( a_1 < Z_1 < \infty \), such that \( V_1(Z_1) = 0 \) and \( V_1(\theta) < 0 \) for all \( \theta < Z_1 \).

From (4.3.2), we can construct the \( \bar{z}_y \) function as we have done many times now.

Let

\[ W_1(y) = g_1(y, \bar{z}_y) \quad (4.3.3) \]

We can show as before:

\[ W'_1(y) = D_1 g_1(y, 0) \quad \text{for } \frac{Z_1}{q} < y < \infty \]

\[ = D_1 g_1(y, \frac{Z_1 - qy}{p-q}) \quad \text{for } \frac{Z_1}{p} \leq y \leq \frac{Z_1}{q} \]

\[ = D_1 g_1(y, y) + D_2 g_1(y, y) \quad \text{for } 0 \leq y \leq \frac{Z_1}{p} \quad (4.3.4) \]

and that \( W'_1(y) \) is continuous monotone-nondecreasing. (See proof of Theorem 4.2.)

So there exists a uniquely determined number \( Y_1, 0 < Y_1 < \infty \), such that \( W'_1(Y_1) = 0 \) and \( W_1(y) < 0 \) for all \( y < Y_1 \). Now from:
\[ f_i(x) = -cx + W_i(x) \quad \text{for } Y_i \leq x \]

\[ = -cx + W_i(Y_i) \quad \text{for } x < Y_i \]

(4.3.5)

We obtain

\[ f'_i(x) = -c + W'_i(x) \quad \text{for } Y_i \leq x \]

\[ = -c \quad \text{for } x < Y_i \]

(4.3.6)

Since \( W'_i(x) \) is continuous and \( W'_i(Y) = 0 \), \( f'_i(x) \) is a continuous function.

In short, we have shown that:

**Theorem 4.8:** The results of Theorem 4.2 are still valid when the assumptions are replaced by those of Theorem 4.7.

From Theorem 4.8, we obtain the functional form of \( W'_i(y) \) for all values of \( y \). Now an analogy of Theorem 4.6 for the varying demand distributions case could be proved by following the proof of 4.6 step by step. Thus we have:

**Lemma 4.9:** Under assumptions (a), (b), (c), and (d) of Theorem 4.7, then \( Y_i < Y_i^- \).

**Corollary:**

When \( c + h_i - \frac{r^*p^+h_2q-rpq}{p-q} \leq 0 \), \( Z_i \leq Z_i^- \).

**Proof:**

\[ V_i(Z_i^-) = r^* + h_2 + sq - rp - sp \]

\[ + \int_a^{Z_i^-} (p-q)[r + s + \phi_i(Z_i^- - \epsilon)] \phi_i(\epsilon) \, d\epsilon \]
\[ \geq r^* + h_2 + sq - rp - sp + (p-q)(r+s-ac)\phi_i(Z_i) \]

\[ = 0 \]

\[ = v_i(Z_i) \]

:. \[ Z_i \geq Z_i \] as \[ v_i(\theta) \] is monotone-nondecreasing.

**Lemma 4.10:** Under the assumptions of Theorem 4.7, then

\[ c + h_1 - \frac{r^*p + h_2 q - rpq}{p-q} < 0 \iff \frac{Z_i}{p} < Y_i \]

**Proof:**

\[ W'_i(\frac{Z_i}{p}) = c + h_1 - \frac{r^*p + h_2 q - rpq}{p-q} \] (the derivation is omitted as it has been done many times under similar situations), \[ W'_i(Y_i) = 0 \] by definition of \[ Y_i \], and the monotone-decreasingness of \[ W'_i(y) \] assure Lemma 4.10.

**Theorem 4.11:** Under the assumptions of Theorem 4.7, then

\[ \{f_i(x) : i = 1, 2, 3, \ldots\} \] is the unique solution of the infinite-stage functional equation among the class of solution which is continuous, uniformly bounded over \( x \) for all \( x \) lying in a finite interval, and has uniformly bounded optimal procuring policy.

**Proof:**

Let \[ \{F_i(x) : i = 1, 2, 3, \ldots\} \] be any other solution which belongs to the class specified in the theorem. Then we can write
\[ F_i(x_i) = L_{x_i} (\overline{y}, \overline{z}) + \alpha \int_{0}^{\infty} F_{i+1}(x_{i+1}) \text{d}H=_{y,z}(x_{i+1}) \]  

(4.3.7)

where \((\overline{y}, \overline{z})\) is the \(i\)th period component of an optimal policy.

(The existence of which is guaranteed by the fact that the right-hand side is continuous and finite for finite \(\overline{y}\).) Since \(\{f_i(x_i): i = 1,2,3,...\}\) is also a solution of the infinite-stage functional equation:

\[ f_i(x_i) = L_{x_i} (\overline{y}, \overline{z}) + \alpha \int_{0}^{\infty} f_{i+1}(x_{i+1}) \text{d}H=_{y,z}(x_{i+1}) \]  

(4.3.8)

Now by the definition of an optimal policy:

\[ F_i(x_i) \leq L_{x_i} (\overline{y}, \overline{z}) + \alpha \int_{0}^{\infty} F_{i+1}(x_{i+1}) \text{d}H=_{y,z}(x_{i+1}) \]  

(4.3.9)

\[ f_i(x_i) \leq L_{x_i} (\overline{y}, \overline{z}) + \alpha \int_{0}^{\infty} f_{i+1}(x_{i+1}) \text{d}H=_{y,z}(x_{i+1}) \]  

(4.3.10)

From (4.3.7), (4.3.8), (4.3.9) and (4.3.10):

\[ |F_i(x_i) - f_i(x_i)| \leq \text{Max} \left\{ \alpha \int_{0}^{\infty} |F_{i+1}(x_{i+1}) - f_{i+1}(x_{i+1})| \text{d}H=_{y,z}(x_{i+1}), \right\} \]
\[ \leq \alpha \sup |F_{i+1}(x_{i+1}) - f_{i+1}| \]  
(4.3.11)

where the supreme is over all possible \( x_{i+1} \).

Now \( x_{i+1} \leq \max(y, \tilde{y}) \) and in general

\[ x_{i+j} \leq \max(\tilde{y}, y_{i+1}, \ldots, y_{i+j-1}, x_{i+1}, y_{i+1}, \ldots, y_{i+j-1}) \]

By our assumption \( y_k \) are uniformly bounded for all \( k \). From Lemma 4.9, \( Y_k \leq \frac{y}{k} < Y \). Thus for all \( x_i \) lying in a finite interval \( x_{i+j} \) is bounded by a number \( B < \infty \) uniformly over \( j \).

Let

\[ U_i = \sup_{0 < x < B} |F_i(x) - f_i(x)| \]  
(4.3.12)

Then since (4.3.11) is true for all \( 0 \leq x_i \leq B \), we obtain:

\[ U_i \leq \alpha U_{i+1} \]  
(4.3.13)

And in general:

\[ U_i \leq \alpha^n U_{i+n} \]  
(4.3.14)

But \( U_{i+n} = \sup_{0 < x < B} |F_{i+n}(x) - f_{i+n}(x)| \) is uniformly bounded over \( n \) since \( F_{i+n}(x) \) and \( f_{i+n}(x) \) are both uniformly bounded for all \( x \) lying in a finite interval.

Let \( n \to \infty \) in (4.3.14), we obtain \( U_i = 0 \). Since \( i \) is arbitrary, Theorem 4.11 is proved.
4.4 Some Theorems on Critical Numbers of Varying Demand Distributions

Case

When demand distributions are varying from period to period, it is in general difficult to obtain analytic solutions for critical numbers. In this section we will show that if the demand distributions are ordered in some stochastic sense, then the $i^{th}$ period critical numbers will have the same solutions as those of a process where demand distributions are identically $\{\phi_i, \psi_i\}$. For mathematical convenience, we will consider the demand distribution sequence $\{\phi_1, \psi_1, \phi_2, \psi_2, \phi_3, \psi_3, \ldots, \phi_n, \psi_n, \phi_n, \psi_n, \ldots\}$. Namely the demand distributions are identical from the $n^{th}$ period on where $n$ is an arbitrarily large number. For practical purpose, it does little harm to approximate any demand distribution sequence by our specified one while allowing $n$ to be sufficiently large.

Theorem 4.12: Under the assumptions of Theorem 4.7, and furthermore for $i = 1, 2, \ldots, n-1$,

$$\phi_i \left(\frac{\epsilon}{p} + a_i\right) > \phi_{i+1}(\epsilon) \quad \text{for all } \epsilon$$

$$\psi_i \left(\frac{\epsilon^*}{p}\right) > \psi_{i+1}(\epsilon^*) \quad \text{for all } \epsilon^*$$

and for $i \geq n$

$$\phi_i = \phi_{i+1}, \psi_i = \psi_{i+1}, \text{ then the following is true.}$$

(a) When $c + h_1 - \frac{r+p+h_q-rq}{p-q} \leq 0$, then

$$Z_i = Z_{i+1}, Y_i = Y_{i+1}, \text{ i = 1,2,3,...}$$
(b) When \( c + h_2 \frac{r^*p + h_2q - rpq}{p-q} > 0 \), then

\[
Y_i = Y_i^-, \quad i = 1, 2, 3, \ldots
\]

Proof:

(a) Since demand distributions are identical from \( n^{th} \) period on,

\[
Z_i = Z_i^-, \quad Y_i = Y_i^- \quad \text{for all } i \geq n \tag{4.4.1}
\]

by definition of \( Z_i^- \), \( Y_i^- \). From \( Z_i = Z_i^- \) and (4.2.2)

\[
\phi_{n-1}(\frac{Z}{p} + a_{n-1}) = \phi_{n-1}(\frac{n}{p} + a_{n-1}) \geq \phi_{n}(\frac{Z}{n}) = \phi_{n-1}(\frac{Z_{n-1}}{n-1}).
\]

Hence

\[
\frac{Z}{p} + a_{n-1} \geq \frac{Z_{n-1}}{n-1} \geq Z_{n-1} \tag{4.4.2}
\]

while the last inequality is by Corollary of Lemma 4.9. Now

\[
V_{n-1}(Z_{n-1}) = r^* + h_2 + sq - rp - sp
\]

\[
+ \int_{a_{n-1}}^{Z_{n-1}} (p-q)[r + s + \alpha f_n(Z_{n-1} - \varepsilon)] \phi_{n-1}(\varepsilon) d\varepsilon
\]

\[
= r^* + h_2 + sq - rp - sp + (p-q)(r+s-\alpha c) \phi_{n-1}(Z_{n-1})
\]

\[
= V_{n-1}(Z_{n-1}) \tag{4.4.3}
\]
where \( f'(Z_{n-1} - \varepsilon) = -c \) since

\[
Z_{n-1} - \varepsilon < \left( \frac{Z_n}{p} + a_{n-1} \right) - a_{n-1} = \frac{Z_n}{p} = Y_n
\]

from (4.4.2) and Lemma 4.10.

But \( V_{n-1}(Z_{n-1}) = V_{n-1}(Z_{n-1}) = V_{n-1}(Z_{n-1}) \) implies \( Z_{n-1} \leq Z_{n-1} \).

And hence:

\[
Z_{n-1} = Z_{n-1}
\]  \hspace{1cm} (4.4.4)

Using (4.4.4) then (4.4.2), (4.4.3), (4.4.4) can be proved when \( n \) is replaced by \( n-1 \). Repeat this argument, we obtain:

\[
Z_i = Z_i \quad \text{for all } i = 1, 2, 3, \ldots
\]  \hspace{1cm} (4.4.5)

For \( Y_i \) part, we will first show \( W' \frac{Y + a_{n+1}}{p} > W'(Y_n) \).

From Lemma 4.10,

\[
\frac{Z_n}{p} < Y_n
\]  \hspace{1cm} (4.4.6)

From (4.4.2),

\[
\frac{Z_n}{p} > Z_{n-1} - a_{n-1}
\]  \hspace{1cm} (4.4.7)
Hence

\[ \frac{Y_n + a_{n-1}}{p} > \frac{Z_n}{p} + \frac{a_{n-1}}{p} \]

\[ \geq \frac{(Z_{n-1} - a_{n-1} + a_{n-1})}{p} \]

\[ = \frac{Z_{n-1}}{p} \]  \hspace{1cm} (4.4.8)

From (4.4.6) and (4.4.8), we have to deal with the following four possible cases:

\[ \frac{Z}{q} < \frac{Y}{n}, \quad \frac{Z_{n-1}}{q} < \frac{Y_n + a_{n-1}}{p} \]  \hspace{1cm} (4.4.9)

\[ \frac{Z}{q} < \frac{Y}{n}, \quad \frac{Z_{n-1}}{p} - \frac{Y_n + a_{n-1}}{p} < \frac{Z_{n-1}}{q} \]  \hspace{1cm} (4.4.10)

\[ \frac{Z}{p} - \frac{Y}{n} - \frac{Z}{q}, \quad \frac{Z_{n-1}}{q} < \frac{Y_n + a_{n-1}}{p} \]  \hspace{1cm} (4.4.11)

\[ \frac{Z}{p} - \frac{Y}{n} - \frac{Z}{q}, \quad \frac{Z_{n-1}}{p} - \frac{Y_n + a_{n-1}}{p} < \frac{Z_{n-1}}{p} \]  \hspace{1cm} (4.4.12)

First consider (4.4.9), then:
\[ W'_n(Y_n) = D_1g_n(Y_n, 0), \]

\[ W'_{n-1}(\frac{Y_{n-1} + a_{n-1}}{p}) = D_1g_{n-1}(\frac{Y_{n-1} + a_{n-1}}{p}, 0) \]

Now

\[
\begin{align*}
D_1g_{n-1}(\frac{Y_{n-1} + a_{n-1}}{p}, 0) & = \int_{\frac{Y_{n-1} + a_{n-1}}{p}} \left\{ \frac{c + h_1 + h_2 - rp - sp}{a^*_n} \right\} \\
& + \int_{\frac{Y_{n-1} + a_{n-1} - p\varepsilon^*}{a^*_n}} \int_{\frac{\varepsilon}{\varepsilon^*}} \int_{\frac{r + s + a\varepsilon(Y_{n-1} + a_{n-1} - p\varepsilon^* - \varepsilon)}{\phi_{n-1}(\varepsilon + q\varepsilon^*)d\varepsilon \cdot \psi_{n-1}(\varepsilon^*)d\varepsilon}} \\
& + \int_{\frac{Y_{n-1} + a_{n-1}}{p}} \int_{\infty} \left[ (c + h_1 - r^* - sq)\psi_{n-1}(\varepsilon^*)d\varepsilon \right] \\
& + \int_{\frac{Y_{n-1} + a_{n-1}}{p}} \int_{\frac{r^* + h_2 + sq - rp - sp + p(r + s - ac)}{a^*_n}} \\
& \cdot \phi_{n-1}(Y_{n-1} + a_{n-1} - p\varepsilon^* + q\varepsilon^*)\psi_{n-1}(\varepsilon^*)d\varepsilon + (c + h_1 - r^* - sq) 
\end{align*}
\]
\[(\text{since } f'(Y + a_{n-1} - p\varepsilon - \varepsilon) \geq -c)\]

\[
= \int_{\frac{Y + a_{n-1}}{p} + a_{n-1}} [r^* + h_2 + sq - rp - sp + p(r+s-ac)]
\]

\[
\psi_{n-1}(\frac{Y - pu^* + qu^*}{p}) \psi_{n-1}(\frac{u^*}{p}) \, du^* + (c+h_1-r^*-sq)
\]

(obtained by the transformation \(u^* = p\varepsilon^*)\)

\[
\geq \int_{\frac{Y + a_{n-1}}{p} + a_{n-1}} [r^* + h_2 + sq - rp - sp + p(r+s-ac)]
\]

\[
\psi_{n-1}(\frac{Y - pu^* + qu^*}{p}) \psi_{n-1}(\frac{u^*}{p}) \, du^* + (c+h_1-r^*-sq)
\]
\[
\phi_n(pY_n - pu^* + qu^*) \psi_n(u^*) du^* + (c + h_1 - r^* - sq)
\]

(by Lemma 4.4)

\[
\geq \int_{a_n^*}^{Y_n} [r^* + h_2 + sq - rp - sp + p(r+s-ac)]
\]

\[
\phi_n(pY_n - pu^* + qu^*) \psi_n(u^*) du^* + (c + h_1 - r^* - sq)
\]

(since \( p_{a_n^*-1} \leq a_n^* \) and \( 0 \leq a_n^*-1 \))

\[
= \int_{a_n^*}^{Y_n} \left\{ c + h_1 + h_2 - rp - sp \right\}
\]

\[
+ \int_{a_n^*-qu^*}^{p(Y_n - u^*)} p[r + s + af'_{n-1}(pY_n - pu^* - \epsilon)] \phi_n(\epsilon + qu^*) d\epsilon \psi_n(u^*) du^*
\]

\[
+ \int_{Y_n}^{\infty} (c + h_1 - r^* - sq) \psi_n(u^*) du^*
\]

(\text{where} \( f'_{n+1}(pY_n - pu^* - \epsilon) = -c \) \text{ since} \( pY_n - pu^* - \epsilon \leq Y_n = Y_{n+1} \))

\[
= D_{1g_n}(Y_n, 0) \hspace{1cm} (4.4.13)
\]
\[ \therefore W'_{n-1}\left(\frac{Y^{n-a_{n-1}}}{p}\right) \geq W'_n(Y_n) \text{ under (4.4.9).} \quad (4.4.14) \]

Next consider (4.4.10). Then:

\[ W'_n(Y_n) = D_1 g_n(Y_n, 0) \]

\[ W'_{n-1}\left(\frac{Y^{n-a_{n-1}}}{p}\right) = D_1 g_{n-1}\left(\frac{Y^{n-a_{n-1}}}{p}, \frac{Z_{n-1} - q \cdot Y^{n-a_{n-1}}}{p-q}\right) \]

\[ \geq D_1 g_{n-1}\left(\frac{Y^{n-a_{n-1}}}{p}, 0\right), \]

since \( D_1 g_1(y, z) \) is monotone-nondecreasing in \( z \) for the range \( z \leq \frac{Z_1 - qy}{p-q} \). So again from (4.4.13), we obtain:

\[ W'_{n-1}\left(\frac{Y^{n-a_{n-1}}}{p}\right) \geq W'_n(Y_n) \text{ under (4.4.10) } \quad (4.4.15) \]

Next consider (4.4.12). Then:

\[ W'_n(Y_n) = D_1 g_n(Y_n, \frac{Z_{n-q}Y_n}{p-q}) \]

\[ W'_{n-1}\left(\frac{Y^{n-a_{n-1}}}{p}\right) = D_1 g_{n-1}\left(\frac{Y^{n-a_{n-1}}}{p}, \frac{Z_{n-1} - q \cdot Y^{n-a_{n-1}}}{p-q}\right) \]
Now:

\[ D_{\text{ign}} n_{-1}\left(\frac{Y_{n_{-1}} + a_{n_{-1}}}{p}, \frac{Z_{n_{-1}} - q \cdot Y_{n_{-1}}}{p-q}\right) = \left\{ \begin{array}{c} Y_{n_{-1}} + a_{n_{-1}} - Z_{n_{-1}} \frac{p-q}{p-q} \\
\begin{array}{c} c + h_{1} + h_{2} - rp - sp \\
a_{n_{-1}}^{*}
\end{array}
\right\} \psi_{n_{-1}}(\epsilon^{*})d\epsilon^{*}
\]

\[ + \left\{ \begin{array}{c} Y_{n_{-1}} + a_{n_{-1}} - p\epsilon^{*} \\
\begin{array}{c} a_{n_{-1}}^{*} - q\epsilon^{*}
\end{array}
\right\} p[r + s + qf_{n}(Y_{n_{-1}} + a_{n_{-1}} - p\epsilon^{*} - \epsilon)]\phi_{n_{-1}}(\epsilon + q\epsilon^{*})d\epsilon \psi_{n_{-1}}(\epsilon^{*})d\epsilon^{*}
\]

\[ + \left\{ \begin{array}{c} c + h_{1} - r^{*} - sq \\
\frac{Y_{n_{-1}} + a_{n_{-1}} - Z_{n_{-1}}}{p-q}
\right\} \psi_{n_{-1}}(\epsilon^{*})d\epsilon^{*}
\]

\[ + \left\{ pZ_{n_{-1}} - qY_{n} - qa_{n_{-1}} \frac{p-q}{p-q} \\
\begin{array}{c} a_{n_{-1}} - q \cdot \frac{Y_{n_{-1}} + a_{n_{-1}} - Z_{n_{-1}}}{p-q}
\end{array}
\right\} q[r + s + f_{n}(pZ_{n_{-1}} - qY_{n} - qa_{n_{-1}} - \epsilon)]
\]

\[ \times \phi_{n_{-1}}(\epsilon + q \cdot \frac{Y_{n_{-1}} + a_{n_{-1}} - Z_{n_{-1}}}{p-q})d\epsilon \psi_{n_{-1}}(\epsilon^{*})d\epsilon^{*} \]
\[
\begin{align*}
\exists \int_{\psi_{n-1}(\epsilon^*)}^{\psi_{n-1}(\epsilon^*)} \left[ c + h_1 + h_2 - rp - sp + p(r+s-ac) \right] d\epsilon^* \\
+ \int_{\infty}^{\infty} \left[ c + h_1 - r^* - sq + q(r+s-ac) \phi_{n-1}(Z_{n-1}) \right] \psi_{n-1}(\epsilon^*) d\epsilon^* \\
= \int_{a^*_{n-1}}^{Y_{n-1} + a_{n-1} Z_{n-1}} \left[ r^* + h_2 + sq - rp - sp - q(r+s-ac) \right] d\epsilon^* \\
\ast \phi_{n-1}(Z_{n-1}) + p(r+s-ac) \phi_{n-1}(Y_{n-1} + a_{n-1} - p\epsilon^* + q\epsilon^*) \\
+ [c + h_1 - r^* - sq + q(r+s-ac) \phi_{n-1}(Z_{n-1})] \\
= \int_{p - \frac{Y_{n-1} + a_{n-1} Z_{n-1}}{p-q}}^{p - \frac{Y_{n-1} + a_{n-1} Z_{n-1}}{p-q}} \left[ r^* + h_2 + sq - rp - sp - q(r+s-ac) \phi_{n-1}(Z_{n-1}) \right] d\epsilon^* \\
+ \frac{pY_{n-1} - pu^* + qu^*}{p} + \phi_{n-1}(\frac{Y_{n-1} + a_{n-1}}{p}) - \phi_{n-1}(\frac{a_{n-1}}{p}) \right] \psi_{n-1}(\epsilon^*) d\epsilon^* \\
\end{align*}
\]
+ [c + h_1 - r* - sq + q(r+s-ac)*_n-1(Z_{n-1})]

(obtained by the transformation u* = pε*).

\[
\geq \int p \cdot \frac{Y + a_{n-1} - Z_{n-1}}{p-q} \cdot \frac{pa_{n-1}^*}{p,q} \cdot [r* + h_2 + sq - rp - sp - q(r+s-ac)*_{n-1}(Z_{n-1})

+ p(r+s-ac)*_n(pY_n - pu* + qu*)] \psi_n(u*) du*

+ [c + h_1 - r* - sq + q(r+s-ac)*_{n-1}(Z_{n-1})]

(by Lemma 4.4)

\[
\geq \int \frac{pY_n - Z_n}{p-q} \cdot \frac{a^*_n}{a^*_n} \cdot [r* + h_2 + sq - rp - sp - q(r+s-ac)*_n(Z_n)

+ p(r+s-ac)*_n(pY_n - pu* + qu*)] \psi_n(u*) du*

+ [c + h_1 - r* - sq + q(r+s-ac)*_n(Z_n)]
(since \( p(Z_{n-1} - a_{n-1}) \leq Z_n \) from (4.4.7), \( p a_{n-1} \leq a_n \) and \( \phi_n(Z_{n-1}) = \phi_n(Z_{n-1}) = \phi_n(Z_n) = \phi_n(Z_n) \) from (4.4.4))

\[
\begin{align*}
&= \int_{a_n}^{Y_n} \frac{Z_n - qY_n}{p-q} \left\{ c + h_1 + h_2 \right\} - rp - sp \, d\varepsilon^* \\
&+ \int_{a_n - qu^*}^{p(Y_n - u^*)} p[r + s + \alpha f'_{n+1}(pY_n - pu^* - \varepsilon)] \phi_n(\varepsilon+qu^*) \, d\varepsilon^* \\
&+ \int_{Y_n - \frac{qY_n}{p-q}}^{\infty} \left\{ c + h_1 - r^* - sq \right\} \psi_n(\varepsilon^*) \, d\varepsilon^* \\
&+ \int_{a_n - q(Y_n - \frac{Z_n - qY_n}{p-q})}^{p \cdot \frac{Z_n - qY_n}{p-q}} q[r + s + \alpha f'_{n-1}(p \cdot \frac{Z_n - qY_n}{p-q} - \varepsilon)] \\
&\quad \cdot \phi_n(\varepsilon + qY_n - q \cdot \frac{Z_n - qY_n}{p-q}) \, d\varepsilon^* \\
&\cdot \psi_n(\varepsilon^*) \, d\varepsilon^*
\end{align*}
\]

(\text{where } f'_{n+1}(pY_n - pu^* - \varepsilon) = -c \text{ since } pY_n - pu^* - \varepsilon \leq Y_n = Y_{n+1} \text{ and} \n f'_{n-1}(p \cdot \frac{Z_n - qY_n}{p-q} - \varepsilon) = -c \text{ since } p \cdot \frac{Z_n - qY_n}{p-q} - \varepsilon \leq pY_n \leq Y_n = Y_{n+1} \text{ for} \n \frac{Z_n}{p} \leq Y_n)
\[ D \ll_n (Y, \frac{Z - q Y_n}{p - q}) \quad (4.4.16) \]

\[ W'_n (n) \quad (4.4.17) \]

Finally consider (4.4.11). Then

\[ W'(Y_n) = D \ll_n (Y_n, \frac{Z_n - q Y_n}{p - q}) \]

\[ W'_n (n) \quad (4.4.18) \]

\[ W'_n (\frac{Y + a n - 1}{n} - p) = D \ll_n (\frac{Y + q n - 1}{p}, 0) \]

\[ = D \ll_n (\frac{Y + a n - 1}{p}, \frac{Z_n - q Y_n}{p - q}) \]

since \( D \ll_n (y, z) \) is independent of \( z \) when \( y - z \geq a_k^* \) and

\[ \frac{Y + a n - 1}{p} - \frac{Z_n - q Y_n}{p - q} > \frac{Y + a n - 1}{p} - 0 > \frac{Z_n}{q} - \frac{a_n}{n} > a_k^*. \]

From (4.4.16), therefore

\[ W'_n (\frac{Y + a n - 1}{p}) \quad (4.4.18) \]

From (4.4.14), (4.4.15), (4.4.17) and (4.4.18), we conclude that for all values of \( Y_n \):

\[ W'_n (\frac{Y + a n - 1}{p}) \geq W'(Y_n) = W'_n (n). \]
Hence:

\[ \frac{Y_{n+1}}{\frac{a_{n-1}}{p}} > Y_{n-1} \quad \text{or} \quad pY_{n-1} - a_{n-1} \leq Y_n \] (4.4.19)

Note that in proving (4.4.13) and (4.4.16), the crucial facts used are:

\[ f_{n+1}'(p \frac{Y_n}{n} - pu^*-\varepsilon) = -c \quad u^* > a_n \quad \varepsilon > a_n \]

and

\[ f_{n+1}'(p \cdot \frac{Z - qY_n}{p-q} - \varepsilon) = -c \quad \varepsilon > a_n \]

From (4.4.6) and (4.4.19)

\[ f_{n}'(pY_n - pu^*-\varepsilon) < f_{n}'(pY_{n-1} - a_{n-1}) < f_{n}'(Y_n) = -c \]

and

\[ f_{n}'(p \cdot \frac{Z_{n-1} - qY_{n-1}}{p-q} - \varepsilon) < f_{n}'(pY_{n-1} - a_{n-1}) < f_{n}'(Y_n) = -c \]

when \( u^* \geq a_{n-1} \) and \( \varepsilon \geq a_{n-1} \)

Thus we can prove inductively

\[ W'_{n-2} \left( \frac{Y_{n-1}+a_{n-2}}{p} \right) > W'_{n-1}(Y_{n-1}) \quad \text{... and in general} \quad W'_i \left( \frac{Y_{i+1}+a_i}{p} \right) \geq W'_i(Y_i). \]
Hence: \( p_{Y_1} - a_1 \leq Y_{i+1} \)

\[
f_{i+1}(p_{Y_1} - pu^* - \varepsilon) = -c \quad u^* \geq a_1 \quad \varepsilon \geq a_1
\]

\[
f_{i+1}(p \cdot \frac{Z_i - qY_i}{p-q} - \varepsilon) = -c \quad \varepsilon \geq a_1
\]

for all \( i = 1, 2, 3 \ldots \) \( (4.4.20) \)

Utilizing \((4.4.20)\), we note that \( W_i(Y_i) = W_{i-1}(Y_i) \), and therefore \( W_i(Y_i) = W_{i-1}(Y_i) \) gives \( Y_i = Y_{i-1} \) for \( i = 1, 2, 3 \ldots \)

(b) From \((4.4.1)\) and \((4.2.6)\)

\[
\phi_{n-1}(Y_n + a_{n-1}) = \phi_{n-1}(\frac{pY_n}{p} + a_{n-1}) \geq \phi_n(p_{Y_n})
\]

\[
= \phi_{n-1}(p_{Y_{n-1}})
\]

Hence

\[
Y_n + a_{n-1} \geq p_{Y_{n-1}} \geq pY_{n-1} \quad \text{or} \quad p_{Y_{n-1}} - a_{n-1} \leq Y_n \quad (4.4.21)
\]

Now from \((4.4.21)\):

\[
W_{n-1}(Y_{n-1}) = D_{1}g_{n-1}(Y_{n-1}, Y_{n-1}) + D_{2}g_{n-1}(Y_{n-1}, Y_{n-1})
\]

\[
= \int_{0}^{\infty} \left\{ c + h_1 - r^* - sq \right\}
\]
\[ + \int_{a_{n-1}}^{p_{n-1}} q[r + s + \alpha f_n(p_{n-1} - \varepsilon)] \phi_{n-1}(\varepsilon) d\varepsilon \left\{ \psi_{n-1}(\varepsilon) d\varepsilon \right\} \\
+ r^* + h_2 + s q - r p - s p + \int_{a_{n-1}}^{q_{n-1}} (p-q) \\
. [r + s + \alpha f_n(p_{n-1} - \varepsilon)] \phi_{n-1}(\varepsilon) d\varepsilon \]

\[ = \int_0^\infty [c + h_1 - r^* - s q + q(r+s-\alpha c) \phi_{n-1}(p_{n-1})] \psi_{n-1}(\varepsilon) d\varepsilon \]

\[ + r^* + h_2 + s q - r p - s p + (p-q)(r+s-\alpha c) \phi_{n-1}(q_{n-1}) \]

\[ = \bar{W}_{n-1} (Y_{n-1}) \]

But

\[ \bar{W}_{n-1} (Y_{n-1}) = \bar{W}_{n-1} (Y_{n-1}) \]

\[ \therefore \ Y_{n-1} = Y_{n-1} \quad (4.4.22) \]

From (4.4.22) and an inductive argument, we obtain \( Y_i = Y_{-i} \)

\[ i = 1, 2, 3, \ldots \] Theorem 4.12 is proved.
When $\phi_i(\varepsilon) \geq \phi_2(\varepsilon)$ for all $\varepsilon$, the variable of $\phi_2$ is said to be stochastically larger than the variable of $\phi_i$. The condition in Theorem 4.12 is of the type $\phi_i(\frac{e}{p} + a_i) \geq \phi_{i+1}(\varepsilon)$. The variable of $\phi_{i+1}$ is stochastically larger than a linear transformation of the variable of $\phi_i$. Since $a_i \geq 0$ and $0 < p \leq 1$, the condition $\phi_i(\frac{e}{p} + a_i) \geq \phi_{i+1}(\varepsilon)$ is more general than the condition $\phi_i(\varepsilon) \geq \phi_{i+1}(\varepsilon)$, and contains the latter as a special case.

4.5 A Generalization

Supposing that demand at each period comes from $n$ independent customers (or $n$ groups of customers), each has a demand sequence with structure as suggested in Chapter 3 of $D_i$ and $D^*_i$ (except now we need a subscript $j$ to index customer $j$), then what can we say of the optimal policy with respect to the aggregated demand?

Let $D^*_i = \sum_{i=1}^{n} D^*_{ij}$, $D_i = \sum_{j=1}^{n} D_{ij}$, and $I_i = \min(D^*_i, y_i - z_i)$

Then $D_i(I_i) = \sum_{j=1}^{n} D_{ij}(I_{ij}) = \sum_{j=1}^{n} D_{ij} + q \sum_{j=1}^{n} I_{ij} = D_i + qI_i$.

So the same pattern of dependence holds for the aggregated demand.

$D_i$ is independent of $D^*_i$ since each $D_{ij}$ is independent of $D^*_{ij}$.

Apparently $D_i$ are also independent of each other. Hence results in Chapter 2 and Chapter 3 apply to the aggregated demand.

Again, if for each $j$ $D_{ij}$ (or $D^*_{ij}$) are identical random variables for all $i$, then $D_i$ (or $D^*_i$) are identical for all $i$. Results in Section 4.1, Section 4.2 hold.

Finally, when $D_{ij}$ ($D^*_{ij}$) are not identical among $i$, but for each $j$: 
\[ \phi_{ij}(\varepsilon + a_{ij}) \geq \phi_{i+1,j}(\varepsilon) \quad \text{for all } \varepsilon, \quad i = 1,2,3,\ldots \]

\[ \psi_{ij}(\varepsilon^*) \geq \psi_{ij}(-\varepsilon^*) \quad \text{for all } \varepsilon^*, \quad i = 1,2,3,\ldots \]

then we will show:

\[ \phi_i(\varepsilon + a_{i1}) \geq \phi_{i+1}(\varepsilon) \quad \text{for all } \varepsilon, \quad i = 1,2,3,\ldots \]

\[ \psi_i(\varepsilon^*) \geq \psi_{i+1}(\varepsilon^*) \quad \text{for all } \varepsilon^*, \quad i = 1,2,3,\ldots \]

and consequently, results in Sections 4.3, 4.4 hold.

We will only prove the case when \( j = 1,2 \) (and for \( \phi \) only), extension to the general case is immediate.

**Lemma 4.13:** Let \( \phi_1 = \phi_{11} \ast \phi_{12} \) and \( \phi_2 = \phi_{21} \ast \phi_{22} \), where "\( \ast \)" denotes convolution, then:

\[ \phi_{11}(\varepsilon + a_{11}) \geq \phi_{21}(\varepsilon) \quad \text{for all } \varepsilon, \]

and

\[ \phi_{12}(\varepsilon + a_{12}) \geq \phi_{22}(\varepsilon) \quad \text{for all } \varepsilon, \]

implies

\[ \phi_1(\varepsilon + a_{11}) \geq \phi_2(\varepsilon) \quad \text{for all } \varepsilon, \]
where $a_1 = a_{11} + a_{12}$ is the lower range of the distribution function $\phi_1$.

Proof:

$$\phi_1\left(\frac{c}{p} + a_{11} + a_{12}\right) = \int_{-\infty}^{\infty} \phi_{11}\left(\frac{c}{p} + a_{11} + a_{12} - u\right) \phi_{12}(u) du$$

$$= \int_{-\infty}^{\infty} \phi_{11}\left(\frac{c-v}{p} + a_{11}\right) \phi_{12}\left(\frac{v}{p} + a_{12}\right) \frac{1}{p} dv$$

(obtained by the transformation $v = p(u-a_{12})$)

$$\geq \int_{-\infty}^{\infty} \phi_{21}(c-v) \phi_{22}(v) dv$$

(obtained by using Lemma 4.4)

$$= \phi_2(c)$$
CHAPTER 5: CONCLUSION

5.1 Summary

In Chapter 1, we introduce the inventory model which has two different modes of sale. Then we discuss the realistic aspect of the model and give some examples. A discussion on motivation and methodology and a review of literature is included in this chapter.

In Chapter 2, the single-stage process is thoroughly studied and optimal policies corresponding to each possible parametric structure are either derived or indicated. Some effort is accorded to the physical interpretations of parametric structure. The hope is to give some insight to the inventory process which mathematical formulations and derivations fail to reveal.

Chapter 3 is an extension of the single-stage process to the multistage process. Though the extension works in general, we give our attention to a particular parametric structure (which is more or less a normal type) only. The avoidance of discussion on the other situations is mainly due to fear of redundancy.

Chapter 4 investigates the infinite-stage process. First, an existence and uniqueness result in given to validate further discussions. Then we solve the critical numbers in the optimal policy when demand distributions are identical from period to period. When demand distributions vary, we establish the result: if for all $i$, $\phi_i(\epsilon + a_i) > \phi_{i+1}(\epsilon)$ for all $\epsilon$ and $\psi_i(\epsilon^*) > \psi_{i+1}(\epsilon^*)$ for all $\epsilon^*$, where $\phi_i$, $\psi_i$ are the $i^{th}$ period regular demand distribution and special demand distribution respectively, then the $i^{th}$ period critical numbers
are equal to the critical numbers in a process where demand distributions are identical to the \( i \)th period demand distributions.

5.2 Suggestions for Future Study

Since this work is a first attempt to investigate inventory models with special sales, many simplifying assumptions are made to facilitate the presentation of the main feature in a distinct manner. Generalizing some of the assumptions will be a natural second step. Among them, we mention the following:

a. The classical generalization as followed by many other inventory models, has several directions here. We may introduce a delivery lag, usually a fixed number \( \lambda \), to account for the time difference between placing a procurement order and its actual delivery. Or we may introduce a procurement setup cost. Another possible generalization is to assume the existence of a transferment cost for goods transferred back to the regular stock after special sale.

b. In our model, special sale is assumed to precede regular sale of the same period. The model assuming the reverse, namely, that regular sale precedes special sale, and a comparison between this model and our model for optimality will certainly be interesting to study. The difficulty of the new model is, if we retain the assumption that the regular demand depends on the amount sold in the previous special sale, then the inventory process at each period, will not be only characterized by the stock level, but
also the amount of goods sold in the special sale of
the preceding period.

c. The particular dependence between the regular demand
and the special demand assumed in our model carries
along with it the restrictive condition \( a^* \leq \frac{a_i}{q} \). It
is desirable to introduce other kind of dependence
relationship between ordinary demand distribution and
special demand distribution where such restrictions
are not necessary to impose.
CHAPTER 6. LIST OF REFERENCES


CHAPTER 7: APPENDICES

7.1 Derivatives of the First and Second Derivatives of the Function $g_{1N}(y,z)$

$g_{1N}(y,z) = cy + h_1 y$

\[ + \int_{y-z}^{y-z} \left\{ - r* \varepsilon + h_2 (y-\varepsilon) \right\} \psi_1 (\varepsilon + q \varepsilon) d\varepsilon \]

\[ + \int_{a^*}^{p(y-\varepsilon*)} \left[ - r \varepsilon + a f_{2N} (p y - p \varepsilon*) \right] \phi_1 (\varepsilon + q \varepsilon) d\varepsilon \]

\[ + \int_{p(y-\varepsilon*)}^{\infty} \left[ - r p(y-\varepsilon*) + s (\varepsilon - p(y-\varepsilon*)) \right] \psi_1 (\varepsilon*) d\varepsilon \]

\[ + \int_{a^*}^{p(y-\varepsilon*)} \left[ - r \varepsilon + a f_{2N} \right] \right\} \phi_1 (\varepsilon + q \varepsilon) d\varepsilon \]

\[ + \int_{y-z}^{\infty} \left\{ - r* (y-z) + h_2 z \right\} \psi_1 (\varepsilon) d\varepsilon \]

\[ + \int_{a^*}^{p(y-\varepsilon*)} \left[ - r \varepsilon + a f_{2N} (p y - \varepsilon) \right] \phi_1 (\varepsilon + q y - q z) d\varepsilon \]
\[ + \int_{pz}^{\infty} [- \, r \varepsilon + \alpha f_{2N}(pz-\varepsilon)] \phi_{1}(\varepsilon+qy+qz) d\varepsilon \psi_{1}(\varepsilon^{*}) d\varepsilon^{*} \]  \quad (7.1.1)

The taking of partial derivatives will be a straightforward task except for the two terms:

\[ A = \int_{a_1}^{p(z+qy-qz)} [- \, r \varepsilon + \alpha f_{2N}(pz-\varepsilon)] \phi_{1}(\varepsilon+qy+qz) d\varepsilon \]  \quad (7.1.2)

and

\[ B = \int_{pz+qy-qz}^{\infty} [ - \, r p + s \varepsilon - pz + \alpha f_{2N}(0)] \phi_{1}(\varepsilon+qy-qz) d\varepsilon \]  \quad (7.1.3)

We will compute \( \frac{\partial A}{\partial y}, \frac{\partial A}{\partial z}, \frac{\partial B}{\partial y}, \frac{\partial B}{\partial y} \) first and then substitute them back into \( D_1 g_{1N}(y,z), D_2 g_{1N}(y,z) \)
\[ \frac{\partial A}{\partial y} = \int_{a_1}^{p_z+q_y-q_z} \left[ r_q + \alpha f_{2N}^r(p_z-e+q_y-q_z) \right] \phi_1(\varepsilon) d\varepsilon \]

\[ + q[- r_p z + \alpha f_{2N}(0)] \phi_1(p_z+q_y-q_z) \]

\[ \frac{\partial A}{\partial z} = \int_{a_1}^{p_z+q_y-q_z} \left[ - r_q + \alpha(p-q)f_{2N}^r(p_z-e+q_y-q_z) \right] \phi_1(\varepsilon) d\varepsilon \]

\[ + (p-q)[- r_p z + \alpha f_{2N}(0)] \phi_1(p_z+q_y-q_z) \]

\[ \frac{\partial B}{\partial y} = \int_{p_z+q_y-q_z}^{\infty} - sq \phi_1(\varepsilon) d\varepsilon \]

\[ - q[- r_p z + \alpha f_{2N}(0)] \phi_1(p_z+q_y-q_z) \]
\[
\frac{3R}{\delta z} = \int_{pz+qy-qz}^{\infty} (-rp-sp+sq)\phi_1(\epsilon)d\epsilon
\]
\[
- (p-q)[-rpz + af_{2N}(0)]\phi_1(pz+qy-qz) \quad (7.1.7)
\]

Now considering (7.1.2), (7.1.3), (7.1.4), (7.1.5), (7.1.6) and (7.1.7),

\[
D_{1g_{1N}}(y,z)
\]
\[
= c + h_1 + \int_{a_1-q\epsilon^*}^{y-z} \left\{ h_2 + \int_{a_{1-q\epsilon^*}}^{p(y-\epsilon^*)} \right\}
\]
\[
\text{apf}_{2N}(py-p\epsilon^*-\epsilon)\phi_1(\epsilon+q\epsilon^*)d\epsilon
\]
\[
+ p[rp(y-\epsilon^*) + af_{2N}(0)]\phi_1(py-p\epsilon^*+q\epsilon^*)
\]
\[
+ \int_{p(y-\epsilon^*)}^{\infty} (-rp-sp)\phi_1(\epsilon+q\epsilon^*)d\epsilon - p[-rp(y-\epsilon^*) + af_{2N}(0)]
\]
\[
\left\{ \phi_1(py-p\epsilon^*+q\epsilon^*) \right\} \psi_1(\epsilon^*)d\epsilon^*
\]
\[
+ \left\{ -r^*(y-z) + h_2z + \int_{a_{1-q(y-z)}}^{pz} [-r\epsilon + af_{2N}(pz-\epsilon)] \right\}
\]
\[
\phi_1(\epsilon+qy-qz)d\epsilon
\]
\[ + \int_{yz}^{\infty} \left[ -rpz + s(\varepsilon-pz) + af_{2N}(0) \right] \phi_1(\varepsilon+qy-qz) d\varepsilon \psi_1(y-z) \]

\[ + \int_{yz}^{\infty} \left\{ -r* + \int_{a-q(y-z)}^{pz} [qr - qaf_{2N}(pz-\varepsilon)] \phi_1(\varepsilon+qy-qz) d\varepsilon \right\} \]

\[ + q[-rpz + af_{2N}(0)] \phi_1(pz+qy-qz) \]

\[ + \int_{pz+qy-qz}^{\infty} -sq\phi_1(\varepsilon)d\varepsilon - q[-rpz + af_{2N}(0)] \phi_1(pz+qy-qz) \]

\[ \psi_1(\varepsilon*)d\varepsilon^* \]

\[ - \left\{ -r*(y-z) + h_2z + \int_{a-q(y-z)}^{pz} [-r\varepsilon + af'_{2N}(pz-\varepsilon)] \phi_1(\varepsilon+qy-qz) d\varepsilon \right\} \]

\[ + \int_{pz}^{\infty} [-rpz + s(\varepsilon-pz) + af_{2N}(0)] \phi_1(\varepsilon+qy-qz) d\varepsilon \psi_1(y-z) \]

\[ = c + h_1 + \int_{a_1}^{y-z} \left\{ h_2 - rp - sp \right\} \]

\[ + \int_{a_1-q\varepsilon*}^{p(y-\varepsilon*)} p[r + s + af'_{2N}(py-p\varepsilon*-\varepsilon)] \phi_1(\varepsilon+q\varepsilon*) \psi_1(\varepsilon*) d\varepsilon^* \]
\[ D_{2g_{1N}}(y, z) = - \left\{ -r^*(y, z) + h_z z + \int_{a_1-q(y-z)}^{pz} \left[ -r\epsilon + \alpha f_{2N}^*(py-pe^*-\epsilon) \right] \right\} \] 

\[ + \int_{y-z}^{\infty} \left\{ -r^* - sq + \int_{a_1-q(y-z)}^{pz} q[r + s + \alpha f_{2N}^*(pz-\epsilon)] \right\} \phi_1(\epsilon+qy-qz)d\epsilon \] 

\[ \psi_1(\epsilon^*)d\epsilon^* \] 

\[ (7.1.8) \]
\[ + \int_{a_1-q(y-z)}^{p^2} [-r \epsilon + af_{2N}(pz-\epsilon)] \phi_1(\epsilon+qy-qz) d\epsilon \]

\[ + \int_{p^2}^{\infty} [-rpz + s(\epsilon-pz) + af_{2N}(0)] \phi_1(\epsilon+qy-qz) d\epsilon \psi_1(y-z) \]

\[ = \int_{y-z}^{\infty} \left\{ r^* + h_2 + sq - rp - sp \right\} \]

\[ + \int_{a_1-q(y-z)}^{p^2} (p-q)[r + s + af'_{2N}(pz-\epsilon)] \phi_1(\epsilon+qy-qz) d\epsilon \psi_1(\epsilon^*) d\epsilon^* \]

(7.1.9)

Again it is straightforward to compute \( D_{11} g_{1N}(y,z) \) from (6.1.8) and \( D_{12} g_{1N}(y,z), D_{22} g_{1N}(y,z) \) from (6.1.9) except possibly the term:

\[ C = \int_{a_1-q(y-z)}^{p^2} [r + s + af'_{2N}(pz-\epsilon)] \phi_1(\epsilon+qy-qz) d\epsilon \]

(7.1.10)
Now

\[
\frac{\partial C}{\partial y} = \int_a^{p_z + q_y - q_z} q a f''_{2N}(p_z - \epsilon + q_y - q_z) \phi_1(\epsilon) d\epsilon
\]

\[+ q[r + s + a f'_{2N}(0)] \phi_1(p_z + q_y - q_z)\]

\[= \int_a^{p_z} q a f''_{2N}(p_z - \epsilon) \phi_1(\epsilon + q_y - q_z) d\epsilon\]

\[+ q[r + s + a f'_{2N}(0)] \phi_1(p_z + q_y - q_z)\]

(7.1.11)

\[
\frac{\partial C}{\partial z} = \int_a^{p_z + q_y - q_z} (p-q) f''_{2N}(p_z - \epsilon + q_y - q_z) \phi_1(\epsilon) d\epsilon
\]

\[+ (p-q)[r + s + a f'_{2N}(0)] \phi_1(p_z + q_y - q_z)\]

\[= \int_a^{p_z} (p-q) f''_{2N}(p_z - \epsilon) \phi_1(\epsilon + q_y - q_z) d\epsilon\]

\[+ (p-q)[r + s + a f'_{2N}(0)] \phi_1(p_z + q_y - q_z)\]

(7.1.12)
Then:

\[ D_{11} g_{1N}(y,z) \]

\[ = \int_{a^*}^{y-z} \left\{ \int_{a_{1-q\epsilon}}^{p(y-\epsilon*)} p^2 a f''_{2N}(py-p\epsilon*-\epsilon) \phi_1(\epsilon+q\epsilon*) d\epsilon \right. \]

\[ + p^2 [r + s + af'_{2N}(0)] \phi_1(py-p\epsilon+q\epsilon*) \psi_1(\epsilon*) d\epsilon^* \]

\[ - \left\{ \int_{h_2 - rp - sp}^{pz} \left[ \int_{a_{1-q(y-z)}}^{p(z-\epsilon)} p[r + s + af'_{2N}(pz-\epsilon)] \phi_1(\epsilon+qy-qz) d\epsilon \right] \psi_1(y-z) \right. \]

\[ + \int_{y-z}^{\infty} \left\{ \int_{a_{1-q(y-z)}}^{pz} q^2 a f''_{2N}(pz-\epsilon) \phi_1(\epsilon) d\epsilon \right. \]

\[ + q^2 [r + s + af'_{2N}(0)] \phi_1(qy+pz-qz) \psi_1(\epsilon*) d\epsilon^* \]

\[ - \left\{ - r* - sq + \int_{a_{1-q(y-z)}}^{pz} q[r + s + af'_{2N}(pz-\epsilon)] \phi_1(\epsilon+qy-qz) d\epsilon \right. \psi_1(y-z) \]

\[ = \int_{a^*}^{y-z} \left\{ \int_{a_{1-q\epsilon}}^{p(y-\epsilon*)} p^2 a f''_{2N}(py-p\epsilon*-\epsilon) \phi_1(\epsilon+q\epsilon*) d\epsilon \right. \]

\[ + p^2 [r + s + af'_{2N}(0)] \phi_1(py-p\epsilon*-q\epsilon*) \psi_1(\epsilon*) d\epsilon^* \]
\[
+ \int_{-\infty}^{\infty} \left\{ \int_{a_{1-q(y-z)}}^{p_{z}} q^2 \alpha f''_{2N}(p_{z}-\varepsilon) \phi_1(\varepsilon) \, d\varepsilon \right. \\
\left. + q^2(\varepsilon+s-\varepsilon c) \phi_1(q_y+p_{z}-q_{z}) \right\} \psi_1(\varepsilon) \, d\varepsilon^* \\
+ \left\{ h^* + h + sq - sp + \int_{a_{1-q(y-z)}}^{p_{z}} (p-q) \left[ r + s + \alpha f'_{2N}(p_{z}-\varepsilon) \right] \right. \\
\left. \phi_1(\varepsilon+q_y-q_{z}) \, d\varepsilon \right\} \psi_1(y-z) \quad (7.1.13) 
\]

\[
D_{22} S_1^N(y,z) 
\]

\[
= \int_{-\infty}^{\infty} \left\{ \int_{a_{1-q(y-z)}}^{p_{z}} (p-q)^2 \alpha f''_{2N}(p_{z}-\varepsilon) \phi_1(\varepsilon+q_y-q_{z}) \, d\varepsilon \right. \\
\left. + (p-q)^2 \left[ r + s + \alpha f'_{2N}(0) \right] \phi_1(q_y+p_{z}-q_{z}) \right\} \psi_1(\varepsilon) \, d\varepsilon^* \\
+ \left\{ h^* + h + sq - sp + \int_{a_{1-q(y-z)}}^{p_{z}} (p-q) \left[ r + s + \alpha f'_{2N}(p_{z}-\varepsilon) \right] \right. \\
\left. \phi_1(\varepsilon+q_y-q_{z}) \, d\varepsilon \right\} \psi_1(y-z) \quad (7.1.14) 
\]
\[ D_{12 \& 1N}(y, z) \]

\[
= \int_{y-z}^{\infty} \left\{ \int_{a_1-q(y-z)}^{p_2} \left[ q(p-q)af''_{2N}(pz-\epsilon)\phi_1(\epsilon+qy-qz)d\epsilon \\
+ q(p-q)[r + s + af_{2N}'(0)]\phi_1(qy-pz-qz) \right] \right\} \psi_1(\epsilon)d\epsilon^* \\
- \left\{ \psi_1 \ast h_2 + sq - rp - sp + \int_{a_1-q(y-z)}^{p_2} (p-q)[r + s + af'_{2N}(pz-\epsilon)] \phi_1(\epsilon + qy-qz)d\epsilon \right\} \psi_1(y-z) \ (7.1.15)
\]

For the single-period process, \( N = 1 \). By our convention \( f_{1N} = 0 \) for \( i > N \).

For the infinite-stage process, then replace \( f_{1N} \) by \( f_1 \) in all above equations.

7.2 Glossary of Terms

In this section, we will list some of the important terms and functions frequently appearing in the text. Again we will deal with the \( N \)-stage functions only.

\( X_i \) = initial stock level at period \( i \);

\( Y_i - X_i \) = amount of goods procured at period \( i \);

\( Y_i - Z_i \) = amount of goods allocated to the special sale of the \( i^{th} \) period;
\( \phi_i = i^{th} \) period regular demand distribution when nothing is sold in the special sale of the same period;

\( \psi_i = i^{th} \) period special-sale demand distribution;

\( \overline{z}_i \) = optimum \( i^{th} \) period allocation policy given \( y_i \);

\( \overline{y}_i \) = optimum \( i^{th} \) period procurement policy;

\( \overline{z}_i \) = optimum \( i^{th} \) period allocation policy;

\( \underline{y}_i \) = \( i^{th} \) period procurement policy critical number;

\( Z_i \) = \( i^{th} \) period allocation policy critical number;

\( p \) = the leakage factor;

\( q \) = parameter reflecting the dependence of the regular demand on the amount sold in the special sale;

\( c \) = procurement cost;

\( h_1 \) = holding cost applying to all goods available at the beginning of the period;

\( h_2 \) = holding cost applying only to goods not sold in the special sale;

\( r \) = regular sale revenue;

\( r^* \) = special sale revenue; and

\( s \) = shortage cost.

\[
\nu_{1N}(\theta) = r^* + h_2 + sq - rp - sp + (p-q)[r + s + f'_2N(\theta-e)]\phi_1(e)\,de
\]

(7.2.1)

\( g_{1N}(y,z) \) and its first and second derivatives has been listed in the previous section.
\[ \bar{z}_y = 0 \quad \text{for } \frac{Z_{1N}}{q} < y < \infty \]

\[ = \frac{Z_{1N} - qy}{p - q} \quad \text{for } \frac{Z_{1N}}{p} \leq y \leq \frac{Z_{1N}}{q} \]

\[ = y \quad \text{for } 0 \leq y < \frac{Z_{1N}}{p} \]  
(7.2.2)

\[ \mathcal{W}_{1N}(y) = g_{1N}(y, \bar{z}_y) \]  
(7.2.3)

\[ f_{1N}(x) = -cx + \mathcal{W}_{1N}(x) \quad Y_{1N} \leq x \]

\[ = -cx + \mathcal{W}'_{1N}(Y_{1N}) \quad x < Y_{1N} \]  
(7.2.4)

\[ f'_{1N}(x) = -c + \mathcal{W}'_{1N}(x) \quad Y_{1N} \leq x \]

\[ = -c \quad x < Y \]  
(7.2.5)

also

\[ f_{1N}(x_1) = L_{x_1}(y, z) + \alpha \int_0^\infty f_{2N}(x_2) dH_{y, x_1}(x_2) \]  
(7.2.6)

where

\[ L_{x_1}(y, x) = -cx + g_{11}(y, z) \]  
(7.2.7)

is the single period cost given \( x_1 \) and a policy \((y, z)\).