ENUMERATION OF (s^n, s^k) CONFOUNDED SYMMETRICAL FACTORIAL DESIGNS

by

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1. Introduction.

A series of papers by Bose (1) and by Bose and Kishen (2) showed how the theory of symmetrical factorial experiments can be developed when the treatments of the experiment are identified with the points of a Finite Euclidean Geometry. In particular a method of completely enumerating the various possibilities of confounding for designs of the class (s^n, s^k) was described; a design of class (s^n, s^k) is a s^n factorial design with n factors, each at s levels, these being s^k blocks each containing s^{n-k} plots, in a complete replication.

This report describes a new geometrical method of enumerating designs of class (s^n, s^2), (s^n, s^3), and (s^n, s^4). The actual enumeration for designs of the class (s^6, s^3) is carried out. It is shown here that we can establish a 1:1 correspondence between designs of class (s^n, s^r) and designs of class (s^n, s^{n-r}) which preserve the equivalence relation defined by two designs being of the same type. The number of types of designs in class (s^n, s^2) is derived.

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2. Definitions

Consider a s^m factorial design involving n factors F_1, F_2, $\ldots F_n$ each at s levels, s a prime number or power of a prime. Any treatment in which these factors occur at levels $x_1, x_2, \ldots, x_n$ respectively is denoted by $T(x_1, x_2, \ldots, x_n)$. A linear function of the treatments may be written in the form,

$$L = \sum_{x_1, x_2, \ldots, x_n} C_{x_1, x_2, \ldots, x_n} \; T(x_1, x_2, \ldots, x_n),$$

x_i taking all the s possible values.

Contrasts, orthogonality relation and degree of freedom are defined as usual. The s levels at which a factor occurs are identified within the elements of a Galois Field GF(s). The treatments $T(x_1, x_2, \ldots x_n)$ may be represented by the points $(x_1, x_2, \ldots, x_n)$ of the n-dimensional Finite Euclidean Space EG(n,s). A 1:1 correspondence is thus established between the s^n treatments and the s^n points of the space EG(n,s). Splitting and confounding of degrees of freedom can be translated in geometrical terms.

A symmetrical factorial design s^n is said to be of class (s^n, s^k) if each replication is laid out in s^k blocks $(k \leq n)$ of s^{n-k} treatments each. In paper (1) a way to look at the problem of construction of an (s^n, s^k) design is to consider k independent parallel pencils P_1, $P_2, \ldots, P_k$ in EG(n,s) with:

$$\begin{aligned} P_1 &= P(a_{11}, a_{12}, \ldots a_{1n}) \\ P_2 &= P(a_{21}, a_{22}, \ldots a_{2n}) \\ &\;\vdots \\ P_k &= P(a_{k1}, a_{k2}, \ldots a_{kn}) \end{aligned} \qquad \begin{aligned} & a_{ij} \in GF(s) \\ & A = (a_{ij}) \text{ has rank } k \end{aligned}$$

and think of P_i as the main effect pencil of a general factor Φ_i, $(i=1,2,\ldots,k)$.

Consider another set $P_{k+1}, \ldots, P_n$ of n-k independent pencils in EG(n,s):

$$P_{k+1} = P(b_{k+1,1}, b_{k+1,2}, \ldots, b_{k+1,n})$$
$$P_{k+2} = P(b_{k+2,1}, b_{k+2,2}, \ldots, b_{k+2,n})$$
$$\vdots$$
$$P_n = P(b_{n1}, b_{n2}, \ldots, b_{nn})$$

$b_{ij} \in GF(s)$ $\qquad B = (b_{ij})$ has rank k $\qquad \binom{A}{B}$ has rank n

and think of P_i as the main effect pencil of a general factor $\Phi_i (i=k+1,..,n)$.

The intersection of the k pencils $P_1, P_2, \ldots, P_k$ defines a partition of EG(n,s) in s^k sets of s^{n-k} elements each, and correspondingly a partition of the s^n treatments in s^k sets of s^{n-k} treatments. If each set of treatments is assigned to a block, the design obtained is of the class (s^n, s^k). $P_1, P_2, \ldots, P_k$ are said to be the generating pencils of the design.

In such a design each of the general factor $\Phi_1, \Phi_2, \ldots, \Phi_k$ remains at constant level in each of the blocks. Thus the s^k-1 degrees of freedom belonging to all possible generalized interactions of $\Phi_1, \Phi_2, \ldots, \Phi_k$ are confounded.

In terms of the original factors the s^k-1 degrees of freedom confounded belong to the pencils with coefficients

$$(2.1) \qquad (\lambda_1, \lambda_2, \ldots, \lambda_k) \cdot \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{bmatrix} \qquad (\lambda_1, \lambda_2. \ldots, \lambda_k) \neq (0, 0, \ldots, 0) \quad \lambda i \in GF(s)$$

each of these pencils carrying (s-1) degrees of freedom.

There are $(s^k-1)/(s-1)$ confounded pencils. It is known that if the $i_1^{th}, i_2^{th}, \ldots, i_e^{th}$ coefficients of a pencil are the only non-null coefficients, the (s-1) degrees of freedom carried by this pencil belong to the (e-1)-order interaction between F_{i_1}, F_{i_2}, ..., F_{i_e} factors, (i_1, i_2, ..., i_e, a subset of 1, 2, ..., n). This characterizes the degrees of freedom confounded by a design of type (s^n, s^k) completely.

3. Designs of same type and similar designs

Consider two designs of class (s^n, s^k) identical if the $(s^k-1)/(s-1)$ pencils confounded in each are the same. Let D_1 be a (s^n, s^k) design generated by the k independent pencils P_1', P_2', ...P_k' the coefficients of which make up a k x n matrix A:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{bmatrix}$$

The elements of the i^{th} row of A are the coefficients of P_i. The rank of A equals k. A is the generating matrix of the design P.

Let D_2 be a (s^n, s^k) design generated by k independent pencils P_1^2, P_2^2, ...,P_k^2 the coefficients of which make up a k x n generating matrix B of rank k:

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ b_{k1} & b_{k2} & \cdots & b_{kn} \end{bmatrix}$$

Then we have the obvious:

<u>Lemma</u> 3.1 Two (s^n, s^k) designs D_1 and D_2 with generating matrix A and B respectively, are identical if and only if we can write $A = CB$ and $|C| \neq 0$.

We will not make a distinction between identical designs as mathematical objects.

Two designs of class (s^n, s^k) are defined as of the <u>same type</u> if we can make them identical by permuting in the same way the coefficients of the confounded pencils of one design. In other words if D_1 is an (s^n, s^k) design with confounded pencils $\{P(\underline{a}_i') \; i = 1, 2, \ldots, m\}$ and D_2 another (s^n, s^k) design with confounded pencils $\{P(\underline{b}_j') \; j = 1, 2, \ldots, m\}$, $m = (s^k-1)/(s-1)$, we say D_1 and D_2 to be of the same type if there exists a permutation matrix E_n such that

$$\{P(\underline{a}_i') \; ; \; i = 1, \ldots, m\} \equiv \{P(\underline{b}_j' E_n) \; ; \; j = 1, \ldots, m\}.$$

We have immediately the following characterization of designs of the same type in:

<u>Lemma</u> 3.2 Two (s^n, s^k) designs D_1 and D_2 with generating matrix A and B respectively, are of the same type if and only if there exists a permutation matrix E_n and a non-singular matrix ${}_kC_k$ such that

$$A = CBE_n$$

Before defining similar designs the following notions must be introduced. Two n-vectors $\underline{a}$ and $\underline{b}$ are said to be of the <u>same structure</u> when their non-null coefficients occur at the same position. e.g. (1,0,0,3,2,0) and (2,0,0,4,1,0) are of the same structure.

Two finite sets of n-vectors are said to be of the same structure when it is possible to establish a 1:1 correspondence between both sets such that two corresponding vectors are of the same structure.

Two (s^n,s^k) designs D_1 and D_2 with a set of confounded pencils $\{P(\underline{a}_i) \quad i = 1, 2, \ldots, m\}$ and $\{P(\underline{b}_j) \quad j = 1, 2, \ldots, m\}$ respectively, are said to be similar when the sets of vectors $\{\underline{a}_i \quad i = 1, 2, \ldots, m\}$ and $\{\underline{b}_j \quad j = 1, 2, \ldots, m\}$ have the same structure.

The problem of enumeration of (s^n,s^k) confounded designs is to divide the (s^n,s^k) designs into classes of similar designs of the same type, and produce the confounded pencils in each class.

4. Enumeration of (s^n,s^k) designs

Let D be a (s^n,s^k) design with generating matrix A. By lemma 3.2 we know that the design associated with CAE_n ($|C| \neq 0$ and E_n a permutation matrix) is possibly a different (s^n,s^k) design of the same type as D. Thus there is no loss of generality in considering the generating matrix in a canonical form, i.e., of the form:

$$\left[\begin{array}{cccc|cccc} 1 & 0 & \ldots & 0 & p_{11} & p_{12} & \cdots & p_{1n-k} \\ 0 & 1 & \ldots & 0 & p_{21} & p_{22} & \cdots & p_{2n-k} \\ \cdot & \cdot & \ldots & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \ldots & 1 & p_{k1} & p_{k2} & \cdots & p_{kn-k} \end{array}\right] = [I \vdots P]$$

Given $\{p_{ij}\}$ the design associated with such a matrix will have $(s^k-1)/(s-1)$ confounded pencils, the coefficients of which are given by $(s^k-1)/(s-1)$ different vectors of $EG(n,s)$:

$$(4.2) \qquad (\lambda_1 \; \lambda_2 \; \dots \; \lambda_k) \begin{bmatrix} 1 & 0 & \dots & 0 & \vdots & p_{11} & p_{12} & \dots & p_{1n-k} \\ 0 & 1 & \dots & 0 & \vdots & p_{21} & p_{22} & \dots & p_{2n-k} \\ \cdot & \cdot & \dots & \cdot & \vdots & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & 1 & \vdots & p_{k1} & p_{k2} & \dots & p_{kn-k} \end{bmatrix}$$

for different choices of $(\lambda_1, \lambda_2, \dots, \lambda_k) \neq (0, 0, \dots, 0)$.

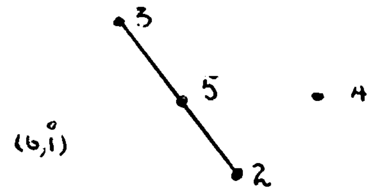
Consider $(\lambda_1, \lambda_2, \dots, \lambda_k)$ as coefficients of a (k-2)-flat in PG(k-1,s) and the non-null columns of $[I \vdots P]$ as points in PG(k-1,s). The pencil with coefficients given by (4.2) will belong to the interaction $F_{i_1}, F_{i_2}, \dots F_{i_r}$ if and only if the $i_1^{th}, i_2^{th}, \dots, i_r^{th}$ coefficients of the vector (4.2) are the only ones which are non-null, or if and only if among the n points represented by the columns of $[I \vdots P]$, only the $i_1^{th}, i_2^{th}, \dots, i_r^{th}$ belong to the (k-2)-flat considered.

Given a generating matrix of the form (4.1) we associate a graph in PG(k-1,s), each of the n points on it being given by a column of (4.1) and thus corresponding to a factor. I have shown how each (k-2)-flat of PG(k-1,s) corresponds to a pencil. Its degrees of freedom belong to the interaction between the factors whose corresponding points do not lie on the (k-1)-flat.

This can be illustrated by a design of class (s^6, s^3) whose generating matrix is

$$H = \begin{bmatrix} 1 & 0 & 0 & \vdots & p_1 & 0 & p_2 \\ 0 & 1 & 0 & \vdots & q_1 & q_2 & 0 \\ 0 & 0 & 1 & \vdots & r_1 & r_2 & 0 \end{bmatrix} \quad \text{and} \quad \begin{vmatrix} q_1 & q_2 \\ r_1 & r_2 \end{vmatrix} \neq 0$$

This gives rise to the following generating graph:



The i^{th} point represents the i^{th} column of H. Points 2, 3, 5 are collinear. Points 6 and 1 are identical. Let us consider all the possible $(s^3-1)/(s-1)$ lines in PG(2,s).

The line containing 2 and 4 corresponds to a set of $(\lambda_1, \lambda_2, \lambda_3) = \underline{\lambda}'$ such that $\underline{\lambda}'H$ has null coefficients 2 and 4; all the others are non-zero because this line does not contain points other than 2,4. The pencil with coefficient $\underline{\lambda}'H$ belongs to $F_1F_3F_5F_6$ interaction.

The line containing 2 and 3 contains 5 also. Such a line has $\underline{\lambda}' = (\lambda_1, \lambda_2, \lambda_3)$ as coefficients and $\underline{\lambda}'H$ is a vector with 2^{nd}, 3^{rd}, and 5^{th} coefficient null; the pencil with such coefficient belongs to the 2^{nd} order interaction $F_1F_4F_6$... and so on for the other lines of PG(2,s).

From such a graph we immediately realize that no main effect factors are confounded because no lines can meet in 5 points simultaneously. There is no 1^{st} order interaction confounded; no lines pass through any 4 points of such graph.

It is obvious that any graph with the same structure will correspond to a similar design. The number of similar designs is equal to the number of w ways of placing the points 3, 4, and 5 in PG(2,s) such that the graph obtained is of the same structure as (4.3), multiplied by a factor of $(s-1)^3$ (each (s-1) representation of a point gives rise to a different design).

The number of designs of the same type is seen to be the number of permutations of the numbers 1 to 6 which change the structure of the graph. Permuting 1 and 6 does not change the structure; the same holds for permutation of 2, 3, and 5. But permuting 1 and 4 gives rise to a different design of the same type. Here the number of types is $6!/2!3! = 60$.

The case of a null column is a degenerate one. The corresponding factor is always unconfounded. In the graphic representation we simply omit such points.

5. Construction of (s^n, s^k) design for the case $k = 2,3,4$

Case $k = 2$: a (s^n, s^2) design corresponds to a choice of n points in the geometry PG(1,s). The (s+1) pencils confounded correspond to the (s+1) 0-flat. A confounded pencil belongs to $F_{i_1}, F_{i_2}, \ldots, F_{i_r}$ interaction if the points $i_1, i_2, \ldots, i_r$ do not lie in the corresponding 0-flat. The 23 types of designs of class (s^6, s^2) were enumerated and will be produced in another paper.

Case $k = 3$: a (s^n, s^3) design corresponds to a choice of n points in PG(2,s). The s^2+s+1 pencils confounded correspond to the (s^2+s+1) lines in PG(2,s). A confounded pencil belongs to $F_{i_1}, F_{i_2}, \ldots, F_{i_r}$ interaction if the points $i_1, i_2, \ldots i_r$ do not lie in the corresponding line. The designs of class (s^6, s^3) are enumerated. They divide in 38 types.

Case $k = 4$: a (s^n, s^4) design corresponds to a choice of n points in PG(3,s). Here the $(s^4-1)/(s-1)$ pencils confounded correspond to the $(s^4-1)/(s-1)$ planes of PG(3,s). A confounded pencil belongs to

$F_{i_1}, F_{i_2}, \ldots, F_{i_r}$ interaction if the points $i_1, i_2, \ldots, i_r$ do not lie in the corresponding plane. The 23 types of (s^6, s^4) designs were enumerated and will be produced later.

In each class of designs, every type is described in the following synopsis. The generating matrix is given in a canonical form; the "x"'s in the matrix represent the non-null element in GF(s) chosen such that all possibly non-vanishing sub-determinants of the matrix must indeed be non-vanishing. Special cases of vanishing sub-determinants are indicated. The generating graph is produced. The various pencils confounded are grouped under the interaction they belong to. The number of similar designs and designs of the same types are produced.

6. Types of similar designs

We will need the following lemmas:

Lemma 1 If $\underline{a}$ and $\underline{b}$ are two n-vectors of the same structure then the sets

$$\{\underline{x} \mid \underline{a}'\underline{x} = 0\} \text{ and } \{\underline{y} \mid \underline{b}'\underline{y} = 0\} \text{ are of the same structure.}$$

The proof is by induction on the number of non-null elements of $\underline{a}'$ and $\underline{b}'$. If $\underline{a}$ and $\underline{b}$ have only one non-null element in the p^{th} position then $\{\underline{x} \mid \underline{a}'\underline{x} = 0\} = \{\underline{x} \mid x_p = 0\}$ is obviously of the same structure as $\{\underline{y} \mid \underline{b}'\underline{y} = 0\} = \{\underline{y} \mid y_p = 0\}$.

Let us suppose the lemma true when $\underline{a}$ and $\underline{b}$ are of the same structure and possess (r-1) or fewer non-null coefficients. Let us investigate the case when exactly r coefficients of $\underline{a}$ and $\underline{b}$ are non-null. In that case:

$$\{\underline{x} \mid \underline{a}'\underline{x} = 0\} = \{\underline{x} \mid a_{i_1}x_{i_1} + \ldots + a_{i_r}x_{i_r} = 0\}$$

$$\{\underline{y} \mid \underline{b}'\underline{y} = 0\} = \{\underline{y} \mid b_{i_1}y_{i_1} + \ldots + b_{i_r}y_{i_r} = 0\}$$

But by the hypothesis of induction the subset $\{\underline{x} \mid a_{i_1} x_{i_1} + \ldots + a_{i_r} x_{i_r} = 0,$ at least one $\underline{x}_{i_j} = 0,\ j = 1,2,\ldots r\}$ is of the same structure as $\{\underline{y} \mid b_{i_1} y_{i_1} + \ldots + b_{i_r} y_{i_r} = 0,$ at least one $y_{i_j} = 0,\ j = 1,2,\ldots r\}$; in particular both sets contain the same number of elements. But the number of solutions of $\underline{a}'\underline{x} = 0$ and $\underline{b}'\underline{y} = 0$ is the same, hence $\{x \quad a'x = 0$ and $x_{i_j} \neq 0,$ $j = 1,2,\ldots r\}$ and $\{y \quad b'y = 0$ and $y_{i_j} \neq 0,\ j = 1,2,\ldots r\}$ are of the same structure. This terminates the proof.

In a similar way we may prove the

<u>Theorem</u> 1 If $\{\underline{a}_i\}$ and $\{\underline{b}_j\}$ are two sets of vectors of the same structure and dimension then the sets $\{\underline{x} \mid \underline{a}_i'\underline{x} = 0$ all $i\}$ and $\{\underline{y} \mid \underline{b}_j'\underline{y} = 0$ all $j\}$ are of the same structure and dimension.

Assume a (s^n, s^k) design D, with generating matrix H. The (s^n, s^{n-k}) design D* with generating matrix H* such that rank H* = n-k and HH* = 0, is said to be the dual design of D. In other words D* is the design the confounded pencils of which have their coefficients orthogonal to H.

<u>Lemma</u> 2 The correspondence of a (s^n, s^k) design to its (s^n, s^{n-k}) dual establishes a 1:1 correspondence between designs of class (s^n, s^k) and of class (s^n, s^{n-k}).

Proof: From the definition, we notice that for every design D of class (s^n, s^k) there is only one corresponding dual design. Conversely every design D* of class (s^n, s^{n-k}) is seen to be the dual of only one (s^n, s^k) design, namely the design having the coefficients of their confounded pencils orthogonal to the generating matrix D*.

<u>Lemma</u> 3 The dual designs of two similar (s^n, s^k) designs are similar (s^n, s^{n-k}) designs.

Proof : Let D_1 and D_2 be two similar (s^n,s^k) designs with generating matrix H_1 and H_2 respectively. The row vector space of H_1 and H_2 have the same structure and equal dimension. The orthogonal spaces spanned respectively by the rows of H_1^* and H_2^* will be of the same structure and the same dimension from theorem 1. Hence the designs D_1^* and D_2^* associated with H_1^* and H_2^* are similar.

Lemma 4 The dual designs of two (s^n,s^k) designs of the same type are (s^n,s^{n-k}) designs of the same type.

Proof: If D_1 and D_2 are (s^n,s^k) designs of the same type, their generating matrices H_1 and H_2 are such that

$$H_1 = CH_2E_n \quad |C| \neq 0 \text{ and } E_n \text{ is a permutation matrix.}$$

Their duals are D_1^* and D_2^* with generating matrix H_1^* and H_2^* such that $H_2H_2^{*'} = 0$; let us take $H_2^*E_n = H_1^*$. Then $H_1H_1^{*'} = CH_2E_nE_n'H_2^{*'} = CH_2H_2^{*'} = 0$. Such a choice of H_1^* is valid; this shows D_1^* and D_2^* to be the same type of designs.

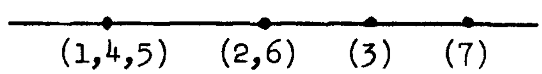
Corollary The class of designs (s^n,s^k) and (s^n,s^{n-k}) can each be partitioned in an equal number of corresponding type of similar designs; two corresponding classes have the same number of designs.

7. The number of different types of similar (s^n, s^2) designs

In section 5 we saw how the different (s^n, s^2) designs correspond to the different ways of choosing n points on PG(1,s) and how to work out the confounded pencils in each case. PG(1,s) may be represented as (s+1) points (x_1, x_2) on a line, two points (x_1, x_2) and (y_1, y_2) being identical when $x_1 = \rho y_1$, $x_2 = \rho x_2$ (ρ a non-null element in GF(s)).
As example let the design with generating matrix

$$\begin{pmatrix} 1 & 0 & x & x & x & 0 & x \\ 0 & 1 & x & 0 & 0 & x & x \end{pmatrix} = H$$

x being any non-null element in GF(s) such that all the possible non-singular 2 x 2 matrices are non-singular. Such a design can be represented by the graph in PG(1,s):



where the i^{th} point is the one corresponding to the i^{th} column of H. We saw that the confounded pencils are the ones with coefficients given by

$$(\lambda_1, \lambda_2) \cdot \begin{pmatrix} 1 & 0 & x & x & x & 0 & x \\ 0 & 1 & x & 0 & 0 & x & x \end{pmatrix} \qquad (\lambda_1, \lambda_2) \neq (0, 0)$$

Taking (λ_1, λ_2) as coefficients of an 0-flat in PG(1,s), the confounded pencil for this (λ_1, λ_2) will belong to interaction $F_{i_1}, F_{i_2}, \ldots, F_{i_r}$ if the $i_1{}^{th}, i_2{}^{th}, \ldots, i_r{}^{th}$ points do not lie in the 0-flat in question.

From this equivalence of a design with a repartition of factors on a line we have the following observations: two designs will be similar and of the same type when the number of points corresponding to k factors are equal in both graphic representations of the design (k = 1,2,...n-1). The number of different types of similar matrices will be the number of ways we can arrange n objects in at least two different cells (when the n column is non-null) plus the number of ways we can arrange n-1 objects in at least two different cells (when only one column is null) and so on.

Two arrangements are identical if the number of cells with k objects is the same in both arrangements for $k = 1,\ldots,n-1$. Let V_n be the number of ways we can arrange n objects in at least two different cells. The number of different types of similar (s^n, s^2) is thus $\sum_{n=2}^{n} V_n$. The quantity $(V_n + 1)$ is the number of ways we can arrange n objects in one or more cells, two arrangements being identical when the number of cells with k objects is the same in both arrangements for $k = 1,\ldots,n$. The last quantity is well computed (see Riordan's book, page 122,(3) as example).

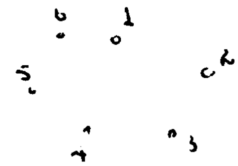
The number of type of (s^n, s^2) designs is:

3	for	$n = 3$
7	for	$n = 4$
13	for	$n = 5$
23	for	$n = 6$
37	for	$n = 7$
58	for	$n = 8$
87	for	$n = 9$
128	for	$n = 10$

(s^6, s^3) Designs

Type 1

$$\begin{bmatrix} 1 & 0 & 0 & X & X & X \\ 0 & 1 & 0 & X & X & X \\ 0 & 0 & 1 & X & X & X \end{bmatrix}$$



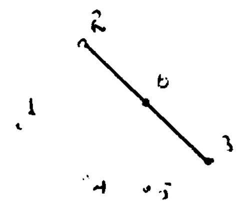
main effect	first order	second order	third order	fourth order	fifth order
---	---	---	$F_1F_2F_3F_4$ 1	$F_1F_2F_4F_5F_6$ (s-4)	$(s^2-5s+10)$
			$F_1F_2F_3F_5$ 1	$F_1F_3F_4F_5F_6$ (s-4)	
			$F_1F_2F_3F_6$ 1	$F_2F_3F_4F_5F_6$ (s-4)	
			$F_1F_2F_4F_5$ 1	$F_1F_2F_3F_4F_5$ (s-4)	
			$F_1F_2F_4F_6$ 1	$F_1F_2F_3F_4F_6$ (s-4)	
			$F_1F_2F_5F_6$ 1	$F_1F_2F_3F_5F_6$ (s-4)	
			$F_1F_3F_4F_5$ 1		
			$F_1F_3F_4F_6$ 1		
			$F_1F_3F_5F_6$ 1		
			$F_1F_4F_5F_6$ 1		
			$F_2F_3F_4F_5$ 1		
			$F_2F_3F_4F_6$ 1		
			$F_2F_3F_5F_6$ 1		
			$F_2F_4F_5F_6$ 1		
			$F_3F_4F_5F_6$ 1		

Number of similar designs: $(s-1)^5(s-2)(s-3)^2(s-6)$

Number of analogous types: 1

Type 2

$$\begin{bmatrix} 1 & 0 & 0 & X & X & 0 \\ 0 & 1 & 0 & X & X & X \\ 0 & 0 & 1 & X & X & X \end{bmatrix}$$



main effect	first order	second order		third order		fourth order		fifth order
- - -	- - -	$F_1F_4F_5$	1	$F_2F_4F_5F_6$	1	$F_2F_3F_4F_5F_6$	(s-4)	(s^2-5s+9)
				$F_3F_4F_5F_6$	1	$F_1F_3F_4F_5F_6$	(s-3)	
				$F_1F_2F_4F_6$	1	$F_1F_2F_4F_5F_6$	(s-3)	
				$F_1F_2F_5F_6$	1	$F_1F_2F_3F_5F_6$	(s-4)	
				$F_1F_3F_4F_6$	1	$F_1F_2F_3F_4F_6$	(s-4)	
				$F_1F_3F_5F_6$	1	$F_1F_2F_3F_4F_5$	(s-3)	
				$F_1F_2F_3F_4$	1			
				$F_1F_2F_3F_5$	1			
				$F_2F_3F_5F_6$	1			
				$F_2F_3F_4F_6$	1			
				$F_2F_3F_4F_5$	1			
				$F_1F_2F_3F_6$	1			

Number of similar designs: $(s-1)^5(s-2)(s-3)(s-4)$

Number of analogous types: 20

Type 3

$$\begin{bmatrix} 1 & 0 & 0 & X & X & 0 \\ 0 & 1 & 0 & X & X & 0 \\ 0 & 0 & 1 & X & X & X \end{bmatrix}$$

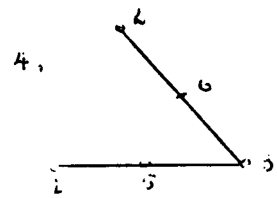
main effect	first order	second order		third order		fourth order		fifth order
---	---	$F_1F_2F_5$	1	$F_1F_2F_4F_5$	(s-3)	$F_1F_2F_3F_5F_6$	(s-3)	(s^2-4s+6)
		$F_1F_2F_4$	1	$F_1F_2F_3F_6$	1	$F_1F_2F_3F_4F_6$	(s-3)	
		$F_2F_4F_5$	1	$F_1F_3F_5F_6$	1	$F_2F_3F_4F_5F_6$	(s-3)	
		$F_1F_4F_5$	1	$F_1F_3F_4F_6$	1	$F_1F_3F_5F_6F_6$	(s-3)	
				$F_2F_3F_4F_6$	1			
				$F_2F_3F_5F_6$	1			
				$F_4F_3F_5F_6$	1			

Number of similar designs: $(s-1)^5(s-2)(s-3)$

Number of analogous types: 15

Type 4

$$\begin{bmatrix} 1 & 0 & 0 & X & X & 0 \\ 0 & 1 & 0 & X & 0 & X \\ 0 & 0 & 1 & X & X & X \end{bmatrix}$$



main effect	first order	second order		third order		fourth order		fifth order
---	---	$F_2F_6F_4$	1	$F_1F_2F_5F_6$	1	$F_1F_2F_3F_5F_6$	(s-4)	(s^2-5s+8)
		$F_1F_5F_4$	1	$F_1F_3F_5F_6$	1	$F_1F_3F_4F_5F_6$	(s-3)	
				$F_2F_3F_5F_6$	1	$F_2F_3F_4F_5F_6$	(s-3)	
				$F_1F_2F_3F_6$	1	$F_1F_2F_4F_5F_6$	(s-2)	
				$F_1F_2F_3F_5$	1	$F_1F_2F_3F_4F_6$	(s-3)	
				$F_3F_4F_5F_6$	1	$F_1F_2F_3F_4F_5$	(s-3)	
				$F_1F_3F_4F_6$	1			
				$F_1F_2F_3F_4$	1			
				$F_2F_3F_4F_5$	1			

Number of similar designs: $(s-1)^5(s-2)(s-3)$

Number of analogous types: 90

Type 5

$$\begin{bmatrix} 1 & 0 & 0 & X & X & 0 \\ 0 & 1 & 0 & X & X & 0 \\ 0 & 0 & 1 & X & X & 0 \end{bmatrix}$$

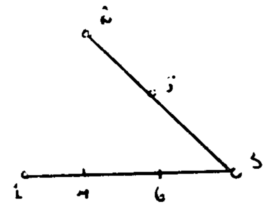
main effect	first order	second order		third order	fourth order	fifth order
---	---	$F_1F_2F_3$	1	$F_2F_3F_4F_5(s-3)$	$F_1F_2F_3F_4F_5(s^2-4s+6)$	---
		$F_1F_2F_4$	1	$F_1F_3F_4F_5(s-3)$		
		$F_1F_2F_5$	1	$F_1F_2F_4F_5(s-3)$		
		$F_1F_3F_4$	1	$F_1F_2F_3F_5(s-3)$		
		$F_1F_3F_5$	1	$F_1F_2F_3F_4(s-3)$		
		$F_1F_4F_5$	1			
		$F_2F_3F_4$	1			
		$F_2F_3F_5$	1			
		$F_2F_4F_5$	1			
		$F_3F_4F_5$	1			

Number of similar designs: $(s-1)^4(s-2)(s-3)$

Number of analogous types: 6

Type 6

$$\begin{bmatrix} 1 & 0 & 0 & X & 0 & X \\ 0 & 1 & 0 & 0 & X & 0 \\ 0 & 0 & 1 & X & X & X \end{bmatrix}$$



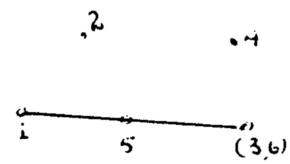
main effect	first order	second order	third order	fourth order	fifth order
---	F_2F_5 1	$F_1F_4F_5$ 1	$F_3F_4F_5F_6$ 1	$F_2F_3F_4F_5F_6(s-2)$	(s^2-5s+6)
			$F_2F_3F_4F_6$ 1	$F_1F_3F_4F_5F_6(s-3)$	
			$F_1F_3F_5F_6$ 1	$F_1F_2F_4F_5F_6(s-1)$	
			$F_1F_2F_3F_6$ 1	$F_1F_2F_3F_5F_6(s-2)$	
			$F_1F_3F_4F_5$ 1	$F_1F_2F_3F_4F_6(s-3)$	
			$F_1F_2F_3F_4$ 1	$F_1F_2F_3F_4F_5(s-2)$	

Number of similar designs: $(s-1)^5(s-2)$

Number of analogous types: 60

Type 7

$$\begin{bmatrix} 1 & 0 & 0 & X & X & 0 \\ 0 & 1 & 0 & X & 0 & 0 \\ 0 & 0 & 1 & X & X & X \end{bmatrix}$$



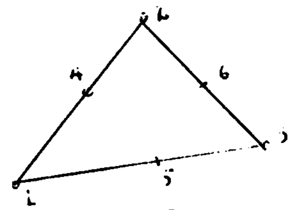
main effect	first order	second order	third order	fourth order	fifth order
---	F_2F_4 1	$F_1F_2F_5$ 1	$F_3F_4F_5F_6$ 1	$F_2F_3F_4F_5F_6(s-2)$	(s^2-4s+5)
		$F_1F_4F_5$ 1	$F_2F_3F_5F_6$ 1	$F_1F_3F_4F_5F_6(s-3)$	
			$F_1F_3F_4F_6$ 1	$F_1F_2F_3F_5F_6(s-3)$	
			$F_1F_2F_3F_6$ 1	$F_1F_2F_3F_4F_6(s-2)$	
			$F_1F_2F_4F_5(s-2)$		
			$F_1F_3F_5F_6$ 1		

Number of similar designs: $(s-1)^5(s-2)$

Number of analogous types: 90

Type 8

$$\begin{bmatrix} 1 & 0 & 0 & X & X & 0 \\ 0 & 1 & 0 & X & 0 & X \\ 0 & 0 & 1 & 0 & X & X \end{bmatrix}$$



main effect	first order	second order		third order		fourth order	fifth order
---	---	$F_3F_5F_6$	1	$F_2F_3F_4F_5$	1	$F_2F_3F_4F_5F_6(s-2)$	(s^2-5s+7)
		$F_2F_4F_6$	1	$F_1F_3F_4F_6$	1	$F_1F_3F_4F_5F_6(s-2)$	
		$F_1F_4F_5$	1	$F_1F_2F_5F_6$	1	$F_1F_2F_4F_5F_6(s-2)$	
				$F_1F_2F_3F_6$	1	$F_1F_2F_3F_5F_6(s-3)$	
				$F_1F_2F_3F_5$	1	$F_1F_2F_3F_4F_6(s-3)$	
				$F_1F_2F_3F_4$	1	$F_1F_2F_3F_4F_5(s-3)$	

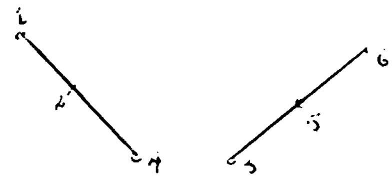
Number of similar designs: $(s-1)^5(s-2)$

Number of analogous types: 120

Type 9

$$\begin{bmatrix} 1 & 0 & 0 & X & X & X \\ 0 & 1 & 0 & X & X & X \\ 0 & 0 & 1 & 0 & X & X \end{bmatrix}$$

singular (the 2×2 submatrix $\begin{pmatrix} X & X \\ X & X \end{pmatrix}$ in rows 1–2, columns 5–6)



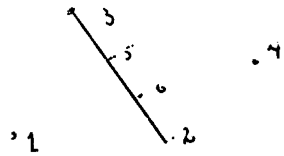
main effect	first order	second order	third order	fourth order	fifth order
---	---	$F_1F_2F_3$ 1	$F_1F_2F_4F_5$ 1	$F_2F_3F_4F_5F_6(s-3)$	(s^2-5s+8)
		$F_4F_5F_6$ 1	$F_1F_2F_4F_6$ 1	$F_1F_3F_4F_5F_6(s-3)$	
			$F_1F_2F_5F_6$ 1	$F_1F_2F_4F_5F_6(s-3)$	
			$F_1F_3F_4F_5$ 1	$F_1F_2F_3F_5F_6(s-3)$	
			$F_1F_3F_4F_6$ 1	$F_1F_2F_3F_4F_6(s-3)$	
			$F_1F_3F_5F_6$ 1	$F_1F_2F_3F_4F_5(s-3)$	
			$F_2F_3F_4F_5$ 1		
			$F_2F_3F_4F_6$ 1		
			$F_2F_3F_5F_6$ 1		

Number of similar designs: $(s-1)^5(s-2)^2$

Number of analogous types: 10

Type 10

$$\begin{bmatrix} 1 & 0 & 0 & X & 0 & 0 \\ 0 & 1 & 0 & X & X & X \\ 0 & 0 & 1 & X & X & X \end{bmatrix}$$



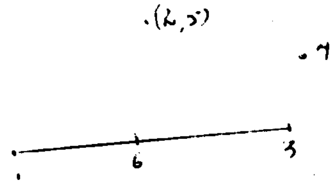
main effect	first order	second order	third order	fourth order	fifth order
---	F_1F_4 1	---	$F_2F_3F_5F_6$ 1	$F_2F_3F_4F_5F_6(s-4)$	(s^2-5s+7)
			$F_2F_4F_5F_6$ 1	$F_1F_3F_4F_5F_6(s-2)$	
			$F_2F_3F_4F_5$ 1	$F_1F_2F_4F_5F_6(s-2)$	
			$F_2F_3F_4F_6$ 1	$F_1F_2F_3F_5F_6(s-4)$	
			$F_3F_4F_5F_6$ 1	$F_1F_2F_3F_4F_6(s-2)$	
			$F_1F_2F_5F_6$ 1	$F_1F_2F_3F_4F_5(s-2)$	
			$F_1F_2F_3F_6$ 1		
			$F_1F_2F_3F_5$ 1		
			$F_1F_3F_5F_6$ 1		

Number of similar designs: $(s-1)^5(s-2)(s-3)$

Number of analogous types: 15

Type 11

$$\begin{bmatrix} 1 & 0 & 0 & X & 0 & X \\ 0 & 1 & 0 & X & X & 0 \\ 0 & 0 & 1 & X & 0 & X \end{bmatrix}$$



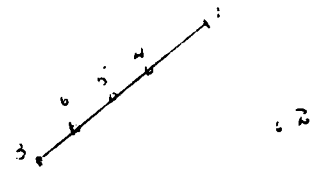
main effect	first order	second order		third order		fourth order		fifth order
---	---	$F_1F_3F_6$	1	$F_1F_2F_5F_6$	1	$F_2F_3F_4F_5F_6$	$(s-2)$	(s^2-4s+5)
		$F_1F_4F_6$	1	$F_1F_2F_3F_5$	1	$F_1F_2F_4F_5F_6$	$(s-2)$	
		$F_1F_3F_4$	1	$F_2F_3F_5F_6$	1	$F_1F_2F_3F_5F_6$	$(s-3)$	
		$F_3F_4F_6$	1	$F_1F_3F_4F_6$	$(s-3)$	$F_1F_2F_3F_4F_5$	$(s-2)$	
		$F_2F_4F_5$	1					

Number of similar dwsigns: $(s-1)^5(s-2)$

Number of analogous types: 60

Type 12

$$\begin{bmatrix} 1 & 0 & 0 & X & X & X \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & X & X & X \end{bmatrix}$$



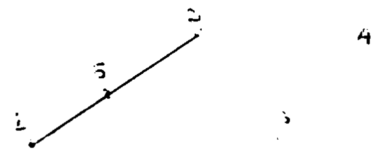
main effect	first order	second order	third order	fourth order	fifth order
F_2 1	---	---	$F_1F_4F_5F_6$ 1	$F_2F_3F_4F_5F_6(s-1)$	(s^2-5s+5)
			$F_1F_3F_5F_6$ 1	$F_1F_3F_4F_5F_6(s-4)$	
			$F_1F_3F_4F_6$ 1	$F_1F_2F_4F_5F_6(s-1)$	
			$F_1F_3F_4F_5$ 1	$F_1F_2F_3F_5F_6(s-1)$	
				$F_1F_2F_3F_4F_6(s-1)$	
				$F_1F_2F_3F_4F_5(s-1)$	

Number of similar designs: $(s-1)^4(s-2)(s-3)$

Number of analogous types: 6

Type 13

$$\begin{bmatrix} 1 & 0 & 0 & X & X & 0 \\ 0 & 1 & 0 & X & X & 0 \\ 0 & 0 & 1 & X & 0 & 0 \end{bmatrix}$$



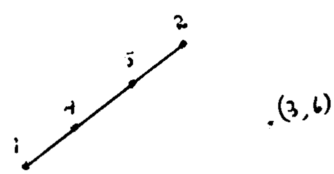
main effect	first order	second order	third order	fourth order	fifth order
---	F_3F_4	$F_1F_2F_5$ 1	$F_2F_3F_4F_5(s-2)$	$F_1F_2F_3F_4F_5(s^2-4s+5)$	---
		$F_1F_2F_3$ 1	$F_1F_3F_4F_5(s-2)$		
		$F_1F_3F_5$ 1	$F_1F_2F_4F_5(s-3)$		
		$F_1F_4F_5$ 1	$F_1F_2F_3F_5(s-3)$		
		$F_2F_3F_5$ 1	$F_1F_2F_3F_4(s-2)$		
		$F_2F_4F_5$ 1			

Number of similar designs: $(s-1)^4(s-2)$

Number of analogous types: 60

Type 14

$$\begin{bmatrix} 1 & 0 & 0 & X & X & 0 \\ 0 & 1 & 0 & X & X & 0 \\ 0 & 0 & 1 & 0 & 0 & X \end{bmatrix}$$



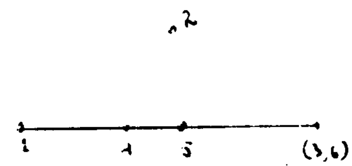
main effect	first order	second order	third order	fourth order	fifth order
---	F_3F_6 1	$F_1F_4F_5$ 1	$F_1F_2F_4F_5(s-3)$	$F_2F_3F_4F_5F_6(s-1)$	(s^2-4s+3)
		$F_1F_2F_4$ 1		$F_1F_3F_4F_5F_6(s-1)$	
		$F_1F_2F_5$ 1		$F_1F_2F_3F_5F_6(s-1)$	
		$F_2F_4F_5$ 1		$F_1F_2F_3F_4F_6(s-1)$	

Number of similar designs: $(s-1)^4(s-2)$

Number of analogous types: 15

Type 15

$$\begin{bmatrix} 1 & 0 & 0 & X & X & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & X & X & X \end{bmatrix}$$



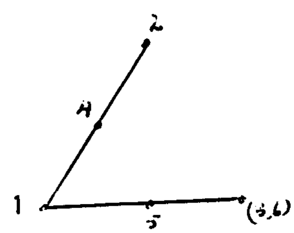
main effect	first order	second order	third order	fourth order	fifth order
F_2 1	---	$F_1F_4F_5$ 1	$F_1F_2F_4F_5$ (s-1)	$F_1F_3F_4F_5F_6$ (s-3)	(s^2-4s+3)
			$F_3F_4F_5F_6$ 1	$F_2F_3F_4F_5F_6$ (s-1)	
			$F_1F_3F_4F_6$ 1	$F_1F_2F_3F_5F_6$ (s-1)	
			$F_1F_3F_5F_6$ 1	$F_1F_2F_3F_4F_6$ (s-1)	

Number of similar designs: $(s-1)^4(s-2)$

Number of analogous types: 60

Type 16

$$\begin{bmatrix} 1 & 0 & 0 & X & X & 0 \\ 0 & 1 & 0 & X & 0 & 0 \\ 0 & 0 & 1 & 0 & X & X \end{bmatrix}$$



main effect	first order	second order	third order	fourth order	fifth order
---	F_2F_4 1	$F_1F_2F_5$ 1	$F_1F_3F_4F_6$ 1	$F_2F_3F_4F_5F_6$ (s-1)	(s^2-4s+4)
		$F_1F_4F_5$ 1	$F_1F_2F_3F_6$ 1	$F_1F_3F_4F_5F_6$ (s-2)	
		$F_5F_3F_6$ 1	$F_1F_2F_4F_5$ (s-2)	$F_1F_2F_3F_5F_6$ (s-2)	
				$F_1F_2F_3F_4F_6$ (s-2)	

Number of similar designs: $(s-1)^5$

Number of analogous types: 180

Type 17

$$\begin{bmatrix} 1 & 0 & 0 & X & X & 0 \\ 0 & 1 & 0 & X & 0 & 0 \\ 0 & 0 & 1 & X & 0 & X \end{bmatrix}$$

main effect	first order		second order		third order		fourth order	fifth order
---	F_2F_4	1	$F_3F_4F_6$	1	$F_1F_3F_5F_6$	1	$F_1F_3F_4F_5F_6(s-2)$	(s^2-3s+3)
			$F_2F_3F_6$	1	$F_1F_2F_4F_5(s-2)$		$F_1F_2F_3F_5F_6(s-2)$	
			$F_1F_2F_5$	1	$F_2F_3F_4F_6(s-2)$			
			$F_1F_4F_5$	1				

Number of similar designs: $(s-1)^5$

Number of analogous types: 45

Type 18

$$\begin{bmatrix} 1 & 0 & 0 & X & 0 & 0 \\ 0 & 1 & 0 & X & 0 & 0 \\ 0 & 0 & 1 & X & X & X \end{bmatrix}$$

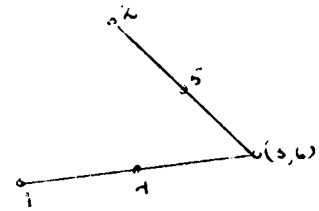
main effect	first order		second order	third order		fourth order	fifth order
---	F_1F_2	1	$F_1F_2F_4(s-2)$	$F_1F_3F_5F_6$	1	$F_2F_3F_4F_5F_6(s-2)$	(s^2-3s+3)
	F_2F_4	1		$F_2F_3F_5F_6$	1	$F_1F_3F_4F_5F_6(s-2)$	
	F_1F_4	1		$F_3F_4F_5F_6$	1	$F_1F_2F_3F_5F_6(s-2)$	

Number of similar designs: $(s-1)^5$

Number of analogous types: 20

Type 19

$$\begin{bmatrix} 1 & 0 & 0 & X & 0 & 0 \\ 0 & 1 & 0 & 0 & X & 0 \\ 0 & 0 & 1 & X & X & X \end{bmatrix}$$



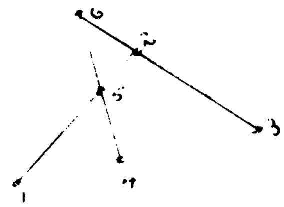
main effect	first order	second order	third order	fourth order	fifth order
---	F_1F_4 1	---	$F_1F_2F_3F_6$ 1	$F_2F_3F_4F_5F_6(s-2)$	(s^2-4s+4)
	F_2F_5 1		$F_1F_2F_4F_5(s-1)$	$F_1F_3F_4F_5F_6(s-2)$	
			$F_1F_3F_5F_6$ 1	$F_1F_2F_3F_5F_6(s-2)$	
			$F_2F_3F_4F_6$ 1	$F_1F_2F_3F_4F_6(s-2)$	
			$F_3F_4F_5F_6$ 1		

Number of similar designs: $(s-1)^5$

Number of analogous types: 45

Type 20

$$\begin{bmatrix} 1 & 0 & 0 & X & X & 0 \\ 0 & 1 & 0 & 0 & X & X \\ 0 & 0 & 1 & X & 0 & X \end{bmatrix}$$



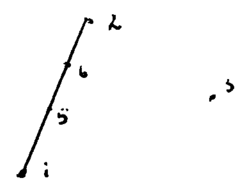
main effect	first order	second order		third order		fourth order	fifth order
---	---	$F_1F_2F_3$	1	$F_2F_3F_4F_5$	1	$F_2F_3F_4F_5F_6(s-2)$	(s^2-5s+7)
		$F_3F_4F_6$	1	$F_1F_3F_5F_6$	1	$F_1F_3F_4F_5F_6(s-2)$	
		$F_1F_4F_5$	1	$F_1F_2F_4F_6$	1	$F_1F_2F_4F_5F_6(s-2)$	
						$F_1F_2F_3F_5F_6(s-2)$	
						$F_1F_2F_3F_4F_6(s-2)$	
						$F_1F_2F_3F_4F_5(s-2)$	

Number of similar designs: $(s-1)^5$

Number of analogous types: 30

Type 21

$$\begin{bmatrix} 1 & 0 & 0 & 0 & X & X \\ 0 & 1 & 0 & 0 & X & X \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$



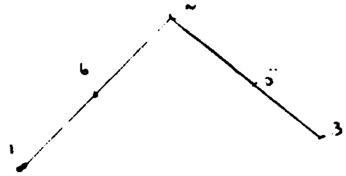
main effect	first order	second order		third order	fourth order	fifth order
F_3	---	$F_1F_5F_6$	1	$F_1F_3F_5F_6(s-1)$	s^2-4s+3	----
		$F_1F_2F_6$	1	$F_2F_3F_5F_6(s-1)$		
		$F_1F_2F_5$	1	$F_1F_2F_5F_6(s-3)$		
		$F_2F_5F_6$	1	$F_1F_2F_3F_6(s-1)$		
				$F_1F_2F_3F_5(s-1)$		

Number of similar designs: $(s-1)^3(s-2)$

Number of analogous types: 30

Type 22

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & X \\ 0 & 1 & 0 & 0 & X & X \\ 0 & 0 & 1 & 0 & X & 0 \end{bmatrix}$$



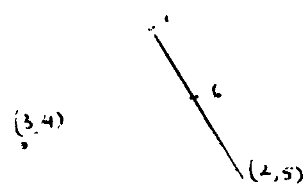
main effect	first order	second order	third order	fourth order	fifth order
---	F_3F_5 1	$F_1F_2F_5$ 1	$F_2F_3F_5F_6(s-2)$	(s^2-4s+4)	---
	F_1F_6 1	$F_1F_2F_3$ 1	$F_1F_3F_5F_6(s-1)$		
		$F_2F_5F_6$ 1	$F_1F_2F_5F_6(s-2)$		
		$F_2F_3F_6$ 1	$F_1F_2F_3F_6(s-2)$		
			$F_1F_2F_3F_5(s-2)$		

Number of similar designs: $(s-1)^4$

Number of analogous types: 90

Type 23

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & X \\ 0 & 1 & 0 & 0 & X & X \\ 0 & 0 & 1 & X & 0 & 0 \end{bmatrix}$$



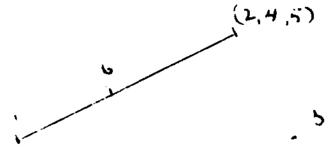
main effect	first order	second order	third order	fourth order	fifth order
---	F_3F_4 1	$F_1F_2F_5$ 1	$F_1F_3F_4F_6(s-1)$	$F_1F_2F_3F_4F_5(s-1)$	(s^2-3s+2)
	F_1F_6 1	$F_2F_5F_6$ 1	$F_1F_2F_5F_6(s-2)$	$F_2F_3F_4F_5F_6(s-1)$	

Number of similar designs: $(s-1)^4$

Number of analogous types: 90

Type 24

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & X \\ 0 & 1 & 0 & X & X & X \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$



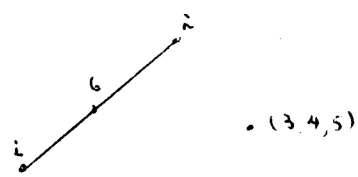
main effect	first order	second order	third order	fourth order	fifth order
F_3 1	F_1F_6 1	$F_1F_3F_6$ (s-1)	$F_1F_2F_4F_5$ 1	$F_2F_3F_4F_5F_6$ (s-1)	(s^2-3s+2)
			$F_2F_4F_5F_6$ 1	$F_1F_2F_4F_5F_6$ (s-2)	
				$F_1F_2F_3F_4F_5$ (s-1)	

Number of similar designs: $(s-1)^4$

Number of analogous types: 60

Type 25

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & X \\ 0 & 1 & 0 & 0 & 0 & X \\ 0 & 0 & 1 & X & X & 0 \end{bmatrix}$$



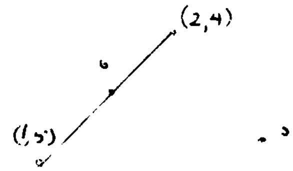
main effect	first order	second order	third order	fourth order	fifth order
---	F_1F_6 1	$F_3F_4F_5$ 1	---	$F_2F_3F_4F_5F_6$ (s-1)	(s^2-3s+2)
	F_2F_6 1	$F_1F_2F_6$ (s-2)		$F_1F_3F_4F_5F_6$ (s-1)	
	F_1F_2 1			$F_1F_2F_3F_4F_5$ (s-1)	

Number of similar designs: $(s-1)^4$

Number of analogous types: 20

Type 26

$$\begin{bmatrix} 1 & 0 & 0 & 0 & X & X \\ 0 & 1 & 0 & X & 0 & X \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$



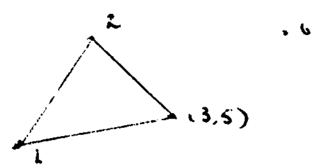
main effect	first order	second order	third order	fourth order	fifth order
F_3 1	---	$F_1F_5F_6$ 1	$F_1F_3F_5F_6$ $(s-1)$	$F_1F_2F_4F_5F_6$ $(s-2)$	(s^2-3s+2)
		$F_2F_4F_6$ 1	$F_2F_3F_4F_6$ $(s-1)$	$F_1F_2F_3F_4F_5$ $(s-1)$	
			$F_1F_2F_4F_5$ 1		

Number of similar designs: $(s-1)^4$

Number of analogous types: 90

Type 27

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & X \\ 0 & 1 & 0 & 0 & 0 & X \\ 0 & 0 & 1 & 0 & X & X \end{bmatrix}$$



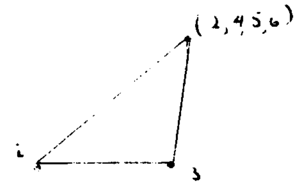
main effect	first order	second order	third order	fourth order	fifth order
---	F_1F_2 1	$F_1F_3F_5$ 1	$F_2F_3F_5F_6$ $(s-2)$	(s^2-3s+3)	---
	F_1F_6 1	$F_2F_3F_5$ 1	$F_1F_3F_5F_6$ $(s-2)$		
	F_2F_6 1	$F_3F_5F_6$ 1	$F_1F_2F_3F_5$ $(s-2)$		
		$F_1F_2F_6$ $(s-2)$			

Number of similar designs: $(s-1)^4$

Number of analogous types: 60

Type 28

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & X & X & X \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$



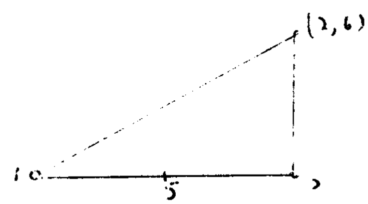
main effect	first order	second order	third order	fourth order	fifth order
F_1 1	$F_1F_3(s-1)$	---	$F_2F_4F_5F_6$ 1	$F_1F_2F_4F_5F_6(s-1)$	(s^2-2s+1)
F_3 1				$F_2F_3F_4F_5F_6(s-1)$	

Number of similar designs: $(s-1)^3$

Number of analogous types: 15

Type 29

$$\begin{bmatrix} 1 & 0 & 0 & 0 & X & 0 \\ 0 & 1 & 0 & 0 & 0 & X \\ 0 & 0 & 1 & 0 & X & 0 \end{bmatrix}$$



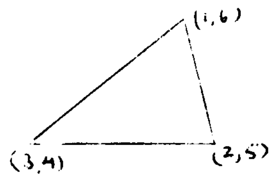
main effect	first order	second order	third order	fourth order	fifth order
---	F_1F_5 1	$F_1F_3F_5(s-2)$	$F_2F_3F_5F_6(s-1)$	(s^2-3s+3)	---
	F_1F_3 1		$F_1F_2F_5F_6(s-1)$		
	F_3F_5 1		$F_1F_2F_3F_6(s-1)$		

Number of similar designs: $(s-1)^3$

Number of analogous types: 60

Type 30

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & X \\ 0 & 1 & 0 & 0 & X & 0 \\ 0 & 0 & 1 & X & 0 & 0 \end{bmatrix}$$



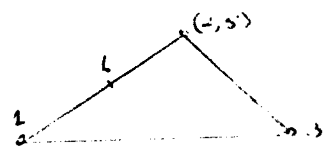
main effect	first order		second order	third order	fourth order	fifth order
---	F_1F_6	1	---	$F_1F_2F_5F_6(s-1)$	---	(s^2-2s+1)
	F_2F_5	1		$F_1F_3F_4F_6(s-1)$		
	F_3F_4	1		$F_2F_3F_4F_5(s-1)$		

Number of similar designs: $(s-1)^3$

Number of analogous types: 15

Type 31

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & X \\ 0 & 1 & 0 & 0 & X & X \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$



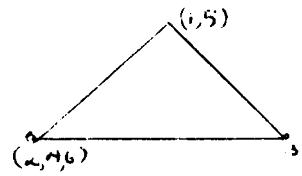
main effect		first order		second order		third order	fourth order	fifth order
F_3	1	F_1F_6	1	$F_1F_3F_6(s-1)$		$F_1F_2F_5F_6(s-2)$	(s^2-3s+2)	---
				$F_1F_2F_5$	1	$F_2F_3F_5F_6(s-1)$	$F_1F_2F_3F_5F_6$	
				$F_2F_5F_6$		$F_1F_2F_3F_5(s-1)$		

Number of similar designs: $(s-1)^3$

Number of analogous types: 180

Type 32

$$\begin{bmatrix} 1 & 0 & 0 & 0 & X & 0 \\ 0 & 1 & 0 & X & 0 & X \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$



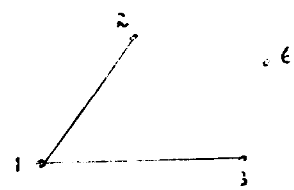
main effect		first order		second order		third order		fourth order		fifth order
F_3	1	F_1F_5	1	$F_1F_3F_5$	$(s-1)$	$F_2F_3F_4F_6$	$(s-1)$	$F_1F_2F_4F_5F_6$	$(s-1)$	(s^2-2s+1)
				$F_2F_4F_6$	1					

Number of similar designs: $(s-1)^3$

Number of analogous types: 60

Type 33

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & X \\ 0 & 1 & 0 & 0 & 0 & X \\ 0 & 0 & 1 & 0 & 0 & X \end{bmatrix}$$



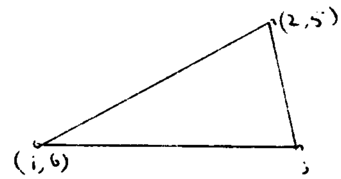
main effect	first order		second order		third order		fourth order	fifth order
	F_1F_6	1	$F_1F_2F_3$	$(s-2)$	$F_1F_2F_3F_6$	(s^2-3s+3)	---	---
	F_1F_3	1	$F_1F_2F_6$	$(s-2)$				
	F_1F_2	1	$F_1F_3F_6$	$(s-2)$				
	F_2F_3	1	$F_2F_3F_6$	$(s-2)$				
	F_2F_6	1						
	F_3F_6	1						

Number of similar designs: $(s-1)^3$

Number of analogous types: 15

Type 34

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & X \\ 0 & 1 & 0 & 0 & X & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$



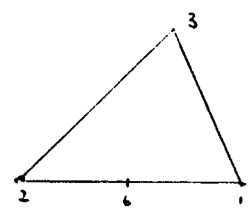
main effect	first order	second order	third order	fourth order	fifth order
F_3 1	F_1F_6 1	$F_1F_3F_6(s-1)$	$F_1F_2F_5F_6(s-1)$	$F_1F_2F_3F_5F_6(s^2-2s+1)$	---
	F_2F_5 1	$F_2F_3F_6(s-1)$			

Number of similar designs: $(s-1)^2$

Number of analogous types: 90

Type 35

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & X \\ 0 & 1 & 0 & 0 & 0 & X \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$



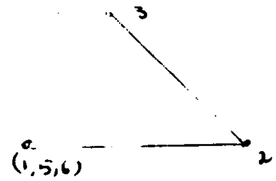
main effect	first order	second order	third order	fourth order	fifth order
F_3	F_6F_1 1	$F_1F_2F_6(s-2)$	$F_1F_2F_3F_6(s^2-3s+2)$	---	---
	F_1F_2 1	$F_1F_2F_3(s-1)$			
	F_2F_6 1	$F_1F_3F_6(s-1)$			
		$F_2F_3F_6(s-1)$			

Number of similar designs: $(s-1)^2$

Number of analogous types: 60

Type 36

$$\begin{bmatrix} 1 & 0 & 0 & 0 & X & X \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$



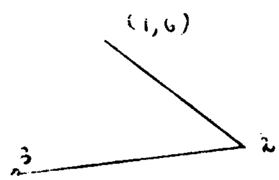
main effect	first order	second order	third order	fourth order	fifth order
F_3 1	$F_2F_3(s-1)$	$F_1F_5F_6$ 1	$F_1F_2F_5F_6(s-1)$	$F_1F_2F_3F_5F_6(s^2-2s+1)$	---
F_2 1			$F_1F_3F_5F_6(s-1)$		

Number of similar designs: $(s-1)^2$

Number of analogous types: 60

Type 37

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & X \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$



main effect	first order	second order	third order	fourth order	fifth order
F_2 1	F_1F_6 1	$F_1F_2F_6(s-1)$	$F_1F_2F_3F_6(s^2-2s+1)$	---	---
F_3 1	$F_2F_3(s-1)$	$F_1F_3F_6(s-1)$			

Number of similar designs: $(s-1)$

Number of analogous types: 90

Type 38

$$\begin{bmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \end{bmatrix}$$

main effect		first order	second order	third order	fourth order	fifth order
F_1	1	$F_1F_2(s-1)$	$F_1F_2F_3(s^2-2s+1)$	---	---	---
F_2	1	$F_1F_3(s-1)$				
F_3	1	$F_2F_3(s-1)$				

Number of similar designs: 1

Number of analogous types: 20

Reference:

(1) Bose, R. C. (1947): Mathematical theory of the symmetrical factorial design. SANKHYA, 8, 107-166..

(2) Bose, R. C. and Kishen, K. (1940): On the problem of confounding in the general Symmetrical Factorial design. SANKHYA, 5, 21-36.

(3) Riordan, J. (1958): An introduction to Combinatorial Analysis, John Wiley and Sons, Inc.

I thank Professor I. M. Chakravarti for suggesting this problem and for his help and guidance.