

CONVERGENCE OF SEQUENCES OF REGULAR FUNCTIONALS OF
EMPIRICAL DISTRIBUTIONS TO PROCESSES OF BROWNIAN MOTION

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For partial cumulative sums of independent and identically distributed random variables (iidrv) having a positive (finite) variance, weak convergence to Brownian motion processes has been established by Donsker (1951, 1952). The result is extended here to von Mises' (1947) differentiable statistical functions and Hoeffding's (1948) U-statistics. Few applications are sketched.

1. Introduction. Let $\omega = \{X_1, X_2, \dots\}$ be a sequence of iidrv with each X_i having a distribution function (d.f.) $F(x)$, $x \in \mathbb{R}^p$, the $p (> 1)$ -dimensional Euclidean space. Let $g(X_1, \dots, X_m)$, symmetric in its arguments, be a Borel-measurable kernel of degree $m (> 1)$, and consider the regular functional

$$(1.1) \quad \theta(F) = \int_{\mathbb{R}^{pm}} \dots \int g(x_1, \dots, x_m) dF(x_1) \dots dF(x_m),$$

defined over $F \in \mathcal{F} = \{F: \theta(F) < \infty\}$. For a sample (X_1, \dots, X_n) , consider the empirical d.f.

$$(1.2) \quad F_n(x, \omega) = n^{-1} \sum_{i=1}^n c(x - X_i), \quad x \in \mathbb{R}^p,$$

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where for $u = (u_1, \dots, u_p)$, $c(u) = 1$ if $u_i \geq 0$, $1 \leq i \leq p$, and $c(u) = 0$, otherwise.

Then, the corresponding regular functional of $F_n(x, \omega)$ is defined by

$$(1.3) \quad \theta(F_n, \omega) = \int_{\mathbb{R}^{pm}} \cdots \int g(x_1, \dots, x_m) dF_n(x_1, \omega) \cdots dF_n(x_m, \omega).$$

Under certain regularity conditions, von Mises (1947) derived the asymptotic normality of $n^{1/2}[\theta(F_n, \omega) - \theta(F)]$; a comparatively simpler proof was subsequently devised by Hoeffding (1948), who primarily considered the unbiased estimators (U-statistics)

$$(1.4) \quad U_n(\omega) = n^{-[m]} \sum_{P_{n,m}} g(X_{i_1}, \dots, X_{i_m}), \quad n \geq m,$$

(where $P_{n,m} = \{1 \leq i_1 \neq \dots \neq i_m \leq n\}$ and $n^{-[m]} = \{n \cdots (n-m+1)\}^{-1}$), and showed that under essentially the same conditions, $n^{1/2}[U_n(\omega) - \theta(F)]$ is asymptotically normal and $n^{1/2}[\theta(F_n, \omega) - U_n(\omega)] \xrightarrow{P} 0$, as $n \rightarrow \infty$.

In particular when $m=1$, $\theta(F_n, \omega) = U_n(\omega) = n^{-1} \sum_{i=1}^n g(X_i) = \bar{g}_n(\omega)$, say. If we assume that $E\{[g(X) - \theta(F)]^2\} = \sigma^2$ is positive (finite), and let

$$(1.5) \quad Y_n(t, \omega) = \{[nt] \bar{g}_{[nt]}(\omega) + (nt - [nt])g(X_{[nt]+1}) - nt \theta(F)\} / \{\sigma n^{1/2}\},$$

where $[s]$ denotes the integral part of $s (> 0)$ and $t \in I = \{t: 0 \leq t \leq 1\}$, then by the Donsker theorem [cf. Billingsley (1968, p. 68)], as $n \rightarrow \infty$

$$(1.6) \quad Y_n(\omega) = [Y_n(t, \omega), t \in I] \xrightarrow{\mathcal{D}} W = [W_t, t \in I],$$

where $W_t, t \geq 0$, is a standard Brownian motion, and $\xrightarrow{\mathcal{D}}$ stands for convergence in distribution.

In the present paper, the above result is extended to general $\theta(F_n, \omega)$ and $U_n(\omega)$, and some possible applications are briefly sketched. Throughout the paper, we assume that $m \geq 2$.

~~The main theorem.~~ For every $h(0 < h < m)$, let

$$(2.1) \quad g_h(x_1, \dots, x_h) = E\{g(x_1, \dots, x_h, X_{h+1}, \dots, X_m)\}, \quad g_0 = \theta(F);$$

$$(2.2) \quad \zeta_h(F) = E\{[g_h(X_1, \dots, X_h) - \theta(F)]^2\} (\leq \zeta_m(F)), \quad \zeta_0(F) = 0.$$

Our basic assumptions are (I, IIa) or (I, IIb), where (I) $\theta(F)$ is stationary of order 0 i.e., $\zeta_1(F) > 0$, (IIa) $\zeta_m(F) < \infty$, and (IIb)

$$(2.3) \quad \max_{1 \leq i_1 < \dots < i_m} \{g^2(X_{i_1}, \dots, X_{i_m})\} = \zeta^*(F) < \infty.$$

[Note that (IIb) \Rightarrow (IIa).] Let then, for every $n \geq 1$,

$$(2.4) \quad Y_n(t, \omega) = \{m[n\zeta_1(F)]^{\frac{1}{2}}\}^{-1} \{([nt]+1)(nt-[nt])\theta(F_{[nt]+1}, \omega) \\ + [nt]([nt]+1-nt)\theta(F_{[nt]}, \omega) - nt\theta(F)\}, \quad t \in I.$$

Also, we define $Y_n^*(t, \omega)$ as in (2.4), with $\theta(F_k, \omega)$ being replaced by $U_k(\omega)$ for $m \leq k \leq n$, and $\theta(F_k, \omega)$ by 0 for $k < m$. Let then

$$(2.5) \quad Y_n(\omega) = [Y_n(t, \omega), t \in I], \quad Y_n^*(\omega) = [Y_n^*(t, \omega), t \in I];$$

$$(2.6) \quad \rho(Y_n(\omega), Y_n^*(\omega)) = \sup_{t \in I} |Y_n(t, \omega) - Y_n^*(t, \omega)|.$$

Then, we have the following theorem:

Theorem 2.1. Under (I) and (IIa), as $n \rightarrow \infty$,

$$(2.7) \quad Y_n^*(\omega) \xrightarrow{P} W,$$

while under (I) and (IIb), as $n \rightarrow \infty$,

$$(2.8) \quad Y_n(\omega) \xrightarrow{P} W, \text{ and } \rho(Y_n(\omega), Y_n^*(\omega)) \xrightarrow{P} 0.$$

The proof of the theorem is postponed to section 4.

~~3. Some results on $\theta(F_n, \omega)$ and $U_n(\omega)$.~~ For every $h(1 \leq h \leq m)$, define

$$(3.1) \quad V_{n,h}(\omega) = \int_{R^{ph}} \dots \int g_h(x_1, \dots, x_h) \prod_{j=1}^h d[F_n(x_j, \omega) - F(x_j)],$$

so that

$$(3.2) \quad nV_{n,1}(\omega) = \sum_{i=1}^n [g_1(X_i) - \theta(F)] = S_n(\omega), \text{ say.}$$

Now, writing $dF_n(x_j, \omega) = dF(x_j) + d[F_n(x_j, \omega) - F(x_j)]$, $i \leq j \leq m$, we obtain from (1.1),

(1.3), (3.1) and some simplifications that

$$(3.3) \quad [\theta(F_n, \omega) - \theta(F) - mV_{n,1}(\omega)] = \sum_{h=2}^m \binom{m}{h} V_{n,h}(\omega), \quad n \geq 1.$$

Let then \mathcal{B}_n be the σ -field generated by $\{X_i, i \leq n\}$, and let

$$(3.4) \quad V_{n,2}^*(\omega) = n^2 V_{n,2}(\omega), \quad \theta_2^*(F) = \int g_2(x, x) dF(x).$$

Lemma 3.1. $\{V_{n,2}^*(\omega) + n[\theta(F) - \theta_2^*(F)], \mathcal{B}_n\}$ forms a martingale sequence.

Proof. By (1.2), (3.1) and (3.4),

$$(3.5) \quad \begin{aligned} V_{n,2}^*(\omega) &= V_{n-1,2}^*(\omega) + 2 \sum_{i=1}^{n-1} \int_{\mathbb{R}^{2p}} g_2(x_1, x_2) d[c(x_1 - X_i) - F(x_1)] d[c(x_2 - X_n) - F(x_2)] \\ &+ \int_{\mathbb{R}^{2p}} g_2(x_1, x_2) d[c(x_1 - X_n) - F(x_1)] d[c(x_2 - X_n) - F(x_2)]. \end{aligned}$$

Also, for all $i \leq n-1$,

$$(3.6) \quad E\{d[c(x_1 - X_i) - F(x_1)] d[c(x_2 - X_n) - F(x_2)] \mid \mathcal{B}_n\} = 0,$$

$$(3.7) \quad \begin{aligned} &E\{d[c(x_1 - X_n) - F(x_1)] d[c(x_2 - X_n) - F(x_2)] \mid \mathcal{B}_n\} \\ &= (\delta_{x_1 x_2}) dF(x_1) - dF(x_1) dF(x_2), \end{aligned}$$

where $\delta_{x_1 x_2}$ is 1 or 0 according as $x_1 = x_2$ or not. Hence,

$$(3.8) \quad E\{V_{n,2}^*(\omega) \mid \mathcal{B}_{n-1}\} = V_{n-1,2}^*(\omega) + \theta_2^*(F) - \theta(F), \quad n \geq 2,$$

and the lemma follows.

Now, for all $r_j \geq 1$, $j=1, \dots, \ell (\geq 1)$, $\sum_{j=1}^{\ell} r_j = 2s$, $s \geq 1$,

$$(3.9) \quad \begin{aligned} &|E\{[d[c(x_1 - X_1) - F(x_1)]]^{r_1} \dots [d[c(x_\ell - X_\ell) - F(x_\ell)]]^{r_\ell}\}| \\ &= 0, \text{ if at least one of } r_1, \dots, r_\ell = 1, \\ &\leq dF(x_1) \dots dF(x_\ell), \text{ otherwise;} \end{aligned}$$

and hence, the maximum l for which (3.9) is different from zero is s , where $r_1 = \dots = r_s = 2$.

Lemma 3.2. Under (2.3), for every $\epsilon > 0$,

$$(3.10) \quad \lim_{n \rightarrow \infty} P\{\omega: \max_{1 \leq k \leq n} k |V_{k,2}(\omega)| > \epsilon n^{\frac{1}{2}}\} = 0.$$

Proof. For every $\epsilon > 0$, there exists an $n_0(\epsilon)$, such that

$$(3.11) \quad |\theta(F) - \theta_2^*(F)| < \frac{1}{2} \epsilon n^{\frac{1}{2}} \text{ for all } n \geq n_0(\epsilon).$$

Hence, for $n \geq n_0(\epsilon)$, by definition in (3.4),

$$(3.12) \quad \begin{aligned} & P\{\omega: \max_{1 \leq k \leq n} k |V_{k,2}(\omega)| > \epsilon n^{\frac{1}{2}}\} \\ & \leq P\{\omega: \max_{1 \leq k \leq n} k^{-1} |V_{k,2}^*(\omega) + k[\theta(F) - \theta_2^*(F)]| > \frac{1}{2} \epsilon n^{\frac{1}{2}}\}. \end{aligned}$$

Now, under (2.3), by (3.1) and (3.9),

$$(3.13) \quad \begin{aligned} E\{V_{k,2}^*(\omega) + k[\theta(F) - \theta_2^*(F)]\}^2 & \leq 2\{E[V_{k,2}^*(\omega)]^2 + k^2[\theta(F) - \theta_2^*(F)]^2\} \\ & \leq C_1 k^2 \zeta^*(F), \text{ where } C_1 (< \infty) \text{ does not depend on } k. \end{aligned}$$

Hence, by theorem 1 of Chow (1960), (i.e., the semi-martingale extension of the Hájek-Rényi inequality), Lemma 3.1, (3.12) and (3.13), it follows that for $n \geq n_0(\epsilon)$,

$$\begin{aligned}
(3.14) \quad & P\{\omega: \max_{1 \leq k \leq n} k |V_{k,2}(\omega)| > \varepsilon n^{\frac{1}{2}}\} \\
& \leq (4C_1 \zeta^*(F)/n\varepsilon^2) (1 + \sum_{j=1}^{n-1} \{j^{-2} - (j+1)^{-2}\} j^2) \\
& \leq (8C_1 \zeta^*(F) [\log n]/n\varepsilon^2) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad \text{Q.E.D.}
\end{aligned}$$

Lemma 3.3. If $m > 3$ and (2.3) holds, then for every $\varepsilon > 0$,

$$(3.15) \quad \lim_{n \rightarrow \infty} P\{\omega: \max_{1 \leq k \leq n} k \left| \sum_{h=3}^m \binom{m}{h} V_{K,h}(\omega) \right| > \varepsilon n^{\frac{1}{2}}\} = 0.$$

Proof. By (2.3), (3.1) and (3.9), for all $k \geq 1$,

$$(3.16) \quad E\left\{ \left[\sum_{h=1}^3 \binom{m}{h} V_{K,h}(\omega) \right]^2 \right\} \leq C_2 k^{-3} \zeta^*(F), \quad C_2 < \infty.$$

Hence, by the Bonferroni and the Markov inequality,

$$\begin{aligned}
(3.17) \quad & P\{\omega: \max_{1 \leq k \leq n} k \left| \sum_{h=3}^m \binom{m}{h} V_{k,h}(\omega) \right| > \varepsilon n^{\frac{1}{2}}\} \\
& \leq \sum_{k=1}^n P\{\omega: k \left| \sum_{h=3}^m \binom{m}{h} V_{k,h}(\omega) \right| > \varepsilon n^{\frac{1}{2}}\} \\
& \leq \sum_{k=1}^n \{C_2 \zeta^*(F)/n\varepsilon^2\} k^{-1} \leq C_2 (\log n) \zeta^*(F)/n\varepsilon^2,
\end{aligned}$$

which converges to 0 as $n \rightarrow \infty$. Q.E.D.

From (3.2), (3.3), (3.10) and (3.15), we have for every $\varepsilon' > 0$,

$$(3.18) \quad \lim_{n \rightarrow \infty} P\{\omega: \max_{1 \leq k \leq n} k |\theta(F_k, \omega) - \theta(F) - mV_{k,1}(\omega)| > \varepsilon' n^{\frac{1}{2}}\} = 0.$$

Now, by (1.4), for all $n \geq m$,

$$\begin{aligned}
 (3.19) \quad U_n(\omega) &= n^{-[m]} \sum_{\mathcal{P}_{n,m}} \int_{\mathbb{R}^{pm}} \cdots \int g(x_1, \dots, x_m) \prod_{j=1}^m d[c(x_j - X_{i_j})] \\
 &= \theta(F) + \sum_{h=1}^m \binom{m}{h} U_{n,h}(\omega),
 \end{aligned}$$

where $U_{n,1}(\omega) = V_{n,1}(\omega)$ is given by (3.2), and for $2 \leq h \leq m$,

$$(3.20) \quad U_{n,h}(\omega) = n^{-[h]} \sum_{\mathcal{P}_{n,h}} \int_{\mathbb{R}^{kh}} \cdots \int g(x_1, \dots, x_h) \prod_{j=1}^h d[c(x_j - X_{i_j}) - F(x_j)].$$

Lemma 3.4. $\{n^{[h]} U_{n,h}(\omega), \mathcal{B}_n\}$ forms a martingale sequence for ever $1 \leq h \leq m$.

The proof is similar to Lemma 3.1 and may be found in Hoeffding (1961).

Lemma 3.5. If $\zeta_m(F) < \infty$, then for every $h (2 \leq h \leq m)$ and $\varepsilon > 0$,

$$(3.21) \quad \lim_{n \rightarrow \infty} P\{\omega: \max_{1 \leq k \leq n} k |U_{k,h}(\omega)| > \varepsilon n^{\frac{1}{2}}\} = 0.$$

By virtue of Lemma 3.4 and the Chow-Hájek-Rényi inequality, the proof follows exactly as in Lemma 3.2 after noting that $E\{U_{k,h}^2(\omega)\} = O(k^{-h})$, $2 \leq h \leq m$. By (3.19) and Lemma 3.5, we obtain that for every $\varepsilon' > 0$,

$$(3.22) \quad \lim_{n \rightarrow \infty} P\{\omega: \max_{1 \leq k \leq n} k |U_k(\omega) - \theta(F) - m U_{k,1}(\omega)| > \varepsilon' n^{\frac{1}{2}}\} = 0.$$

Now from (3.1) and (3.20), for $n \geq m$,

$$\begin{aligned}
 (3.23) \quad V_{n,2}^*(\omega) - n^{[2]} U_{n,2}(\omega) &= \sum_{i=1}^n \int \int_{\mathbb{R}^{2p}} g_2(x_1, x_2) d[c(x_1 - X_i) - F(x_1)] d[c(x_2 - X_i) - F(x_2)] \\
 &= Z_n^*(\omega), \quad \text{say.}
 \end{aligned}$$

Then, by the same technique as in Lemmas 3.1 and 3.2, we have the following.

Lemma 3.6. $\{Z_n^*(\omega) + n[\theta(F) - \theta_2^*(F)], \mathcal{G}_n\}$ forms a martingale sequence, and hence, under (2.3), for every $\epsilon > 0$,

$$(3.24) \quad \lim_{n \rightarrow \infty} P\{\omega: \max_{1 \leq k \leq n} k^{-1} |Z_k^*(\omega)| > \epsilon n^{\frac{1}{2}}\} = 0.$$

Finally, from (3.18), (3.21), Lemma 3.3 and 3.6, we obtain the following.

Lemma 3.7. Under (2.3), for every $\epsilon > 0$,

$$(3.25) \quad \lim_{n \rightarrow \infty} P\{\omega: \max_{m \leq k \leq n} k |\theta(F_k, \omega) - U_k(\omega)| > \epsilon n^{\frac{1}{2}}\} = 0.$$

4. The proof of the main theorem. For $t \in I$, define

$$(4.1) \quad Y_n^0(t, \omega) = \{S_{[nt]}(\omega) + (nt - [nt])[g_1(X_{[nt]+1}) - \theta(F)]\} / \{n\zeta_1(F)\}^{\frac{1}{2}},$$

where $S_k(\omega)$ is defined by (3.2). Then, by the Donsker theorem [cf. Billingsley (1968, p. 68)], as $n \rightarrow \infty$

$$(4.2) \quad Y_n^0(\omega) = [Y_n^0(\omega, t), t \in I] \xrightarrow{\mathcal{D}} W = [W_t, t \in I].$$

Now, by (2.5), (3.2), (3.18) and (4.1), under (2.3),

$$\begin{aligned}
 \rho(Y_n(\omega), Y_n^0(\omega)) &= \sup_{t \in I} |Y_n(t, \omega) - Y_n^0(t, \omega)| \\
 (4.3) \quad &= \max_{1 \leq k \leq n} k |\theta(F_k, \omega) - \theta(F) - mV_{k,1}(\omega)| / \{m[n\zeta_1(F)]^{\frac{1}{2}}\} \xrightarrow{P} 0, \text{ as } n \rightarrow \infty.
 \end{aligned}$$

Hence, by (4.2) and (4.3), under (2.3),

$$(4.4) \quad Y_n(\omega) \xrightarrow{P} Y_n^0(\omega) \xrightarrow{\mathcal{D}} W.$$

Similarly, by (3.19) and (4.1), as $n \rightarrow \infty$,

$$(4.5) \quad Y_n^*(\omega) \xrightarrow{P} Y_n^0(\omega) \xrightarrow{\mathcal{D}} W.$$

Further, by definition, $U_k(\omega) = 0$ for $k < m$, and hence,

$$\begin{aligned}
 \rho(Y_n(\omega), Y_n^*(\omega)) &\leq \max_{1 \leq k < m} k |\theta(F_k, \omega)| / \{m[n\zeta_1(F)]^{\frac{1}{2}}\} \\
 (4.6) \quad &+ \max_{m \leq k \leq n} k |\theta(F_k, \omega) - U_k(\omega)| / \{m[n\zeta_1(F)]^{\frac{1}{2}}\},
 \end{aligned}$$

where by the Chebyshev and the Bonferroni inequalities, for every $\epsilon > 0$,

$$\begin{aligned}
 &P\{\omega: \max_{1 \leq k < m} k |\theta(F_k, \omega)| / \{m[n\zeta_1(F)]^{\frac{1}{2}}\} > \epsilon\} \\
 (4.7) \quad &\leq \sum_{k=1}^{m-1} P\{\omega: |\theta(F_k, \omega)| > \epsilon [n\zeta_1(F)]^{\frac{1}{2}}\} \\
 &\leq [(m-1)\zeta^*(F)] / [n\epsilon^2 \zeta_1(F)] \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ [by (2.3)]}
 \end{aligned}$$

and hence, by Lemma 3.7 and (4.6), $\rho(Y_n, Y_n^*) \xrightarrow{P} 0$ as $n \rightarrow \infty$. Q.E.D.

5. Some applications.

(I) Asymptotic normality for random sample sizes. For every index variable $r(\geq 1)$, let N_r be an integer valued (non-negative) random variable and n_r is a real (positive) number, such that

$$(5.1) \quad \lim_{r \rightarrow \infty} n_r = \infty \text{ and } \lim_{r \rightarrow \infty} (N_r/n_r) = 1, \text{ in probability.}$$

This implies that for every $\delta > 0$,

$$(5.2) \quad \lim_{r \rightarrow \infty} P\{|n_r^{-1} N_r - 1| > \delta\} = 0.$$

Now, by the tightness property [cf. Billingsley (1968, p. 54)] of W_t , $t \geq 0$, for every $\epsilon > 0$ and $\eta > 0$, there exists a $\delta > 0$, such that

$$(5.3) \quad P\{\sup_{|t-s| < \delta} |W_t - W_s| > \epsilon \mid t, s \in I\} < \eta.$$

Hence, by theorem 2.1, (5.2) and (5.3), we have as $r \rightarrow \infty$,

$$\begin{aligned} N_r^{\frac{1}{2}}[\theta(F_{N_r}, \omega) - \theta(F)] &\stackrel{P}{\sim} n_r^{\frac{1}{2}}[\theta(F_{N_r}, \omega) - \theta(F)] \\ &\sim n_r^{\frac{1}{2}}[\theta(F_{n_r}, \omega) - \theta(F)], \end{aligned}$$

and the asymptotic normality readily follows.

A similar result for $U_{N_r}(\omega)$ has been obtained by Sproule (1969) by an indirect method involving some elaborate analysis. Our proof follows from theorem 2.1, (5.2) and (5.3).

(II) Limiting distributions of $\sup_{t \in I} Y_n(t, \omega)$ etc. By virtue of theorem 2.1 and well-known results on Brownian motion processes [viz., Billingsley (1968, p. 79)], we obtain that for all $\lambda > 0$,

$$(5.5) \quad \lim_{n \rightarrow \infty} P\{\sup_{t \in I} Y_n(t, \omega) > \lambda\} = \lim_{n \rightarrow \infty} P\{\sup_{t \in I} Y_n^*(t, \omega) > \lambda\} = \sqrt{\frac{2}{\pi}} \int_{\lambda}^{\infty} e^{-\frac{1}{2}t^2} dt,$$

$$(5.6) \quad \begin{aligned} \lim_{n \rightarrow \infty} P\{\omega: \sup_{t \in I} |Y_n(t, \omega)| > \lambda\} &= \lim_{n \rightarrow \infty} P\{\omega: \sup_{t \in I} |Y_n^*(t, \omega)| > \lambda\} \\ &= 1 - \sum_{k=-\infty}^{\infty} (-1)^k [\Phi((2k+1)\lambda) - \Phi((2k-1)\lambda)], \end{aligned}$$

where $\Phi(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^x \exp(-\frac{1}{2}t^2) dt$, $-\infty < x < \infty$.

(III) An application to rank statistics. Consider the kernel

$$(5.7) \quad \phi(x_1, x_2) = 1, 0 \text{ or } -1 \text{ according as } x_1 + x_2 \text{ is } >, = \text{ or } < 0,$$

and assume that the X_i have a continuous d.f. $F(x)$, $x \in \mathbb{R}^1$. Then

$$(5.8) \quad \theta(F) = \int_{-\infty}^{\infty} [1 - 2F(-x)] dF(x) \quad (=0 \text{ when } F \text{ is symmetric about } 0).$$

The corresponding $\theta(F_n, \omega)$ is given by

$$(5.9) \quad n\theta(F_n, \omega) = 2n^{-1} \sum_{i=1}^n R_{ni} \operatorname{sgn}(X_i) = 2W_n, \text{ say,}$$

where R_{ni} = Rank of $|X_i|$ among $|X_1|, \dots, |X_n|$, $\operatorname{sgn} u = 1, 0$ or -1 according as u is $>, =$ or < 0 , and W_n is the classical Wilcoxon signed rank statistic. Also, here

$$(5.10) \quad \zeta_1(F) = \int_{-\infty}^{\infty} [1-2F(-x)]^2 dF(x) - \theta^2(F),$$

which equals to $1/3$ when F is symmetric about 0. Thus, if we let

$$(5.11) \quad Y_n(t, \omega) = \{(nt - [nt])W_{[nt]+1} + ([nt]+1 - nt)W_{[nt]}^{-1/2nt\theta(F)}\} / [n\zeta_1(F)]^{1/2}.$$

$0 \leq t \leq 1$, we have from theorem 2.1 that if $\zeta_1(F) > 0$,

$$(5.12) \quad Y_n(\omega) \xrightarrow{\mathcal{D}} W,$$

and thus the sequence $\{W_k^{-1/2k\theta(F)}; k \geq 1\}$ though not linear in the basic random variables is attracted by the Brownian motion process.

REFERENCES

- [1] BILLINGSLEY, P. (1968). Convergence of Probability Measures. John Wiley & Sons, New York.
- [2] CHOW, Y. S. (1960). A martingale inequality and the law of large numbers. Proc. Amer. Math. Soc. 11, 107-111.
- [3] DONSKER, M. (1951). An invariance principle for certain probability limit theorems. Mem. Amer. Math. Soc. 6, 1-12.
- [4] DONSKER, M. (1952). Justification and extension of Doob's heuristic approach to the Kolmogoroll-Smirnov theorems. Ann. Math. Statist. 23, 277-281.
- [5] HOEFFDING, W. (1948). A class of statistics with asymptotically normal distribution. Ann. Math. Statist. 19, 293-325.
- [6] HOEFFDING, W. (1961). The strong law of large numbers for U-statistics. Inst. Statist. Univ. N. Carolina Mimeo Rep. No. 302.
- [7] MISES, R. V. (1947). On the asymptotic distribution of differentiable statistical functions. Ann. Math. Statist. 18, 309-348.
- [8] SPROULE, R. N. (1969). A sequential fixed-width confidence interval for the mean of a U-statistic. Doctoral Dissertation, Univ. N. Carolina, Chapel Hill.