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STEINER TRIEDERS AND STRONGLY REGULAR GRAPHS

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1. INTRODUCTION

A finite, undirected linear graph $G$ is strongly regular if it is regular of valence $n_1$, each adjacent pair of vertices is adjacent to exactly $p_{11}$ other vertices, and each non-adjacent pair of vertices is adjacent to exactly $p_{12}$ other vertices.

We shall consider the Steiner configuration of 45 triangles formed by 27 straight lines contained in a general cubic surface. The graph associated with this configuration will be proved to be strongly regular.

The Steiner configuration has an interesting property about its triangles, and we shall see under which conditions graphs satisfying this property are strongly regular.

2. THE STEINER CONFIGURATION

The Steiner configuration, determined by the 27 straight lines of any cubic surface has the following properties (see [2] and [4]):

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P.2.1. Each straight line \( A \) meets ten others.

P.2.2. The ten lines which meet any line \( A \) cut in pairs to form five triangles with \( A \).

P.2.3. Given two triangles \( A_1B_1C_1, A_2B_2C_2 \) with no side in common, there exists a unique triangle \( A_3B_3C_3 \), such that \( A_1A_2A_3, B_1B_2B_3, C_1C_2C_3 \) are three more triangles, and \( A_3B_3C_3 \) has no side in common with \( A_1B_1C_1 \) and \( A_2B_2C_2 \).

These nine lines forming six triangles are called the Steiner trieder.

Let us consider the graph \( G \) associated with the Steiner configuration and defined as follows:

1. The vertices of the graph \( G \) are the lines of the configuration.
2. Two vertices of the graph \( G \) are adjacent if their corresponding lines intersect in the configuration.

Thus each triangle in the configuration determines a triangle in the graph (the sides of a triangle become its vertices).

Let us rewrite the properties P.2.1, P.2.2, P.2.3 for the graph \( G \):

P.1: Each vertex is adjacent to ten others.

P.2: The ten vertices adjacent to any vertex \( x \) of the graph are associated in adjacent pairs to form five triangles with \( x \).

P.3: Given two triangles \( x_1y_1z_1 \) and \( x_2y_2z_2 \) of \( G \) with no vertex in common, there exists a unique triangle \( x_3y_3z_3 \), such that \( x_1x_2x_3, y_1y_2y_3, z_1z_2z_3 \) are three more triangles and \( x_3y_3z_3 \) has no vertex in common with \( x_1y_1z_1 \) and \( x_2y_2z_2 \).

These nine vertices will also be said to form a Steiner trieder.

The graph \( G \) has 27 vertices and is regular of valance 10 by P.1.
Proposition 1.

The graph $G$ associated with the Steiner configuration is strongly regular with the parameters: $v = 27$, $n_1 = 10$, $p^{1}_{11} = 1$, $p^{2}_{11} = 5$.

Proof:

First, let us notice that the property P.3 implies obviously the following property:

P.2.3 Given two triangles $x_1y_1z_1$ and $x_2y_2z_2$ of the graph $G$, with no vertex in common, each vertex of one triangle is adjacent at least to one vertex of the other triangle.

Given any pair $x,y$ of adjacent vertices of $G$, there exists, by P.2, a vertex $z$ such that $xyz$ is a triangle. There are 24 vertices adjacent to $x,y$ and $z$. If they are distinct, we have, with $x,y$ and $z$, the 27 vertices of the graph $G$. If not, there exists a vertex $u$ which is adjacent neither to $x$ nor to $y$, nor to $z$ and a triangle $uvw$ which has no vertex in common with $xyz$. But, by the property P.2.3, each vertex of the triangle $uvw$ has to be adjacent to one vertex of $xyz$, which contradicts our hypothesis that $u$ is adjacent neither to $x$ nor to $y$ nor to $z$.

Then, the 24 vertices adjacent to $x,y$ and $z$ are distinct and $z$ is the unique vertex which is adjacent to $x$ and $y$.

Thus we have $p^{1}_{11} = 1$. The ten vertices adjacent to any vertex $x$ form with $x$ five triangles, two of them having only $x$ as a common vertex.

Any pair $x,y$ of nonadjacent vertices of $G$ determine 10 distinct triangles.
Let \( xx_1x_2 \) and \( yy_1y_2 \) be two of these triangles. They cannot have two vertices in common, since \( p_{11}^1 = 1 \). If they have one vertex in common, say \( x_1 = y_1 \), and if the third vertex \( x_2 \) is adjacent to \( y \), we will have a triangle \( yx_1x_2 \) which is impossible since \( p_{11}^1 = 1 \). Hence, \( x_2 \) is nonadjacent to \( y \) and \( y_2 \) is nonadjacent to \( x \) for the same reason.

If they have no vertex in common, the property P.2.3 must hold, that is that at least one of \( x_1x_2 \) is adjacent to \( y \) and one of \( y_1, y_2 \) is adjacent to \( x \). But if both of \( x_1x_2 \) are adjacent to \( y \), or if both of \( y_1, y_2 \) are adjacent to \( x \), then \( xy_1x_2 \) or \( xy_1y_2 \) will be a triangle which contradicts that \( p_{11}^1 = 1 \). Thus each of the five triangles determined by the vertex \( x \) has one and only one vertex which is adjacent to \( y \). That is there are exactly five vertices adjacent to \( x \) and \( y \) and we have \( p_{11}^2 = 5 \).

3. Graphs satisfying the property P.3.

Consider a graph \( G \), with at least two triangles with no vertex in common and satisfying P.3. \( G \) has at least nine distinct vertices \( x_i, y_i, z_i \), \( i = 1 \) to \( 3 \) which form a Steiner trieder \( T_0 \), that is, which satisfy the following:

P.3.3 Given two triangles which are two rows of

\[
\begin{align*}
&x_1 y_1 z_1 \\
x_2 y_2 z_2 \\
x_3 y_3 z_3
\end{align*}
\]

the third row is the unique triangle such that each column is a triangle

then we have the following property:

P.3.4 for \( i = j, i, j = 1 \) to \( 3 \).
\[ x_1 \text{ and } y_j \text{ are nonadjacent} \]
\[ y_1 \text{ and } z_j \text{ are nonadjacent} \]
\[ z_1 \text{ and } x_j \text{ are nonadjacent} \]

**PROOF:**

Suppose \( x_1 \) adjacent to \( y_2 \), then \( x_1 x_2 y_2 \) and \( x_1 y_1 y_2 \) will be two triangles with no vertex in common with \( z_1 z_2 z_3 \). Then \( z_3 \) has to be adjacent to \( x_1 \), or to \( y_2 \), or to \( x_2 \) and \( y_1 \).

If \( z_3 \) is adjacent to \( x_1 \), then \( z_1 x_1 z_3 \) is a triangle and we have:
\[
\begin{array}{c}
  x_1 y_1 z_1 \\
  x_2 y_2 x_1 \\
  x_3 y_3 z_3
\end{array}
\]

where each row and each column is a triangle which contradicts P.3.3 since \( x_1 \neq z_2 \).

If \( z_3 \) is adjacent to \( y_2 \), then \( y_2 z_2 z_3 \) is a triangle and we have:
\[
\begin{array}{c}
  x_1 y_1 y_2 \\
  x_2 y_2 z_2 \\
  x_3 y_3 z_3
\end{array}
\]

where each row and each column is a triangle which contradicts P.3.3 since \( z_1 \neq y_2 \).

If \( z_3 \) is adjacent to \( y_1 \) and \( x_2 \), then \( x_2 z_3 z_2 \) and \( y_1 z_3 y_3 \) are two triangles and we have:
\[
\begin{array}{c}
  x_1 y_1 z_1 \\
  x_2 z_3 z_2 \\
  x_3 y_3 z_3
\end{array}
\]

where each row and each column is a triangle which contradicts P.3.3 since \( y_2 \neq z_3 \).
Thus \( x_1 \) is not adjacent to \( y_2 \) and we can reason similarly for any other pair and P.3.4 holds. This implies three more properties:

**P.3.5** Any triangle \( xyz \) of \( G \) which is not contained in the triezer \( T_0 \) has at most one vertex in common with it.

**P.3.6** Any triangle \( xyz \) of \( G \) is contained in a Steiner triezer.

**P.3.7** For any pair \( xy \) of adjacent vertices of \( G \), there is at most one vertex \( z \) adjacent to \( x \) and \( y \).

If \( xyz \) is a triangle of \( G \) not contained in \( T_0 \), which has two vertices \( xy \) in common with \( T_0 \), say \( x = x_1 \), then \( y_2 z_2 y_3 z_3 \) are distinct from \( y \) since they are not adjacent to \( x_1 \). Suppose \( y = y_1 \), then \( z \neq z_1 \) and \( xyz = x_1 y_1 z \) has no vertex in common with \( x_2 y_2 z_2 \) and \( x_3 y_3 z_3 \) and P.2.3 implies that \( z_2 \) and \( z_3 \) nonadjacent to \( x_1 \) and \( y_1 \) are adjacent to \( z \), that is \( zz_2 z_3 \) is a triangle. We have \( x_1 y_1 z \)

\[
\begin{array}{ccc}
   & x_2 & y_2 z_2 \\
   x_1 & y_1 & z \\
   x_3 & y_3 & z_3 \\
\end{array}
\]

Where each row and each column is a triangle which contradicts P.3.3 since \( z \neq z_1 \). Thus P.3.5 is satisfied.

It follows that any pair of adjacent vertices in \( T_0 \) determine a unique triangle. It also follows that for any triangle \( xyz \) of \( G \) there is a triangle of \( T_0 \) which has no vertex in common with it and by P.3 they must form with a third triangle a Steiner triezer. Thus P.3.6 and P.3.7 are satisfied.

Let us introduce now one more property:

**P.3.8** Each vertex of the graph \( G \) is contained in at least one triangle.
A graph $G$ with at least two triangles with no vertex in common, satisfying P.3.8 and P.3 also satisfies:

P.3.9 Each vertex of $G$ is contained in a Steiner trieder.

**Theorem 1.**

A finite, undirected linear graph $G$, containing at least two triangles with no vertex in common satisfies P.3.9 iff it is strongly regular. Its parameters are:

$$v = 3(2r-1), \quad n_1 = 2r, \quad p_{11}^1 = 1, \quad p_{11}^2 = r, \quad r \geq 2$$

**Proof:**

Let $xy$ be a pair of adjacent vertices of $G$, then P.3.7 and P.3.8 imply that there is a unique vertex $z$ which is adjacent to $x$ and $y$. Then $p_{11}^1 = 1$. The valence $n_x$ of each vertex $x$ of $G$ is even and $x$ is contained in $\frac{n_x}{2}$ triangles. Let $xy$ be a pair of nonadjacent vertices in $G$, and $x_1x_2$, $yy_1y_2$ be two triangles. If they have one vertex in common, $x_1 = y_1$, and if $x_2$ is adjacent to $y$, or $y_2$ adjacent to $x$, then $yx_1x_2$ or $xy_1y_2$ will be a triangle which contradicts that $p_{11}^1 = 1$. If they have no vertex in common they must form with a third triangle a Steiner trieder and P.3.4 implies, since $x$ is not adjacent to $y$ that one and only one of $x_1, x_2$ is adjacent to $y$, and one and only one of $y_1, y_2$ is adjacent to $x$. And this for any pair of triangles containing respectively $x$ and $y$. But $x$ is contained in $\frac{n_x}{2}$ triangles and $y$ in $\frac{n_y}{2}$ triangles. That implies $n_x = n_y = n_{xy}$ and there are $\frac{n_{xy}}{2}$ vertices of $G$ adjacent to $xy$.

Now let $x$ be a vertex of valence $n_1 = 2r$ in $G$, $r$ must be $> 2$. 
For any vertex \( y \) adjacent to \( x \), there is a vertex \( z \) such that \( xyz \) is a triangle. Then, for every \( z_1 \) adjacent to \( z \), \( z_1 \) is not adjacent to \( x \) nor to \( y \) and then \( n_y = n_z = n_1 = 2r \). Thus any vertex adjacent or not to \( x \) has the same valence \( n_1 = 2r \), and the graph \( G \) is regular. It also is strongly regular with \( p_{11} = \frac{n_1}{2} = r \). Then the number of vertices in \( G \) is \( 3(2r-1) \). The converse is obvious.

**Corollary**

A strongly regular graph \( G \), with at least two triangles with no vertex in common satisfies P.3 iff its parameters are

\[ v = 3(2r-1), \quad n_1 = 2r, \quad p_{11}^1 = 1, \quad p_{11}^2 = r, \quad r \geq 2. \]

Strongly regular graphs with such parameters have been proved by Seidel in [3] to exist only for \( r = 1, 2, 3 \), and 5.

I would like to express my thanks to I. M. Chakravarti who suggested to me the study of the property P.3. see [1].
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[1] Chakravarti, I. M., Some properties and applications of Hermitian varieties in a finite projective space \( \text{PG}(N,q^2) \) in the construction of strongly regular graphs (two-class association schemes) and block designs. *Institute of Statistics Mimeo Series No. 600.23.* March 1970, U.N.C., Chapel Hill.

