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AN ASYMPTOTIC DISTRIBUTION FOR THE j-th esf
OF THE GENERALISED HOTELLING'S BETA MATRIX

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BIOMATHMATICS TRAINING PROGRAM
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1. **INTRODUCTION.** Let $X^{(i)}_{\alpha} (p \times 1), \, \alpha = 1, \ldots, N_i$, be i.i.d. $N(\mu^{(i)}, \Sigma)$, $i = 1, \ldots, r$. Let

$$\bar{X}^{(i)} = \frac{\sum_{\alpha=1}^{N_i} X^{(i)}_{\alpha}}{N_i}$$

$$A_i = \sum_{\alpha=1}^{N_i} (X^{(i)}_{\alpha} - \bar{X}^{(i)})(X^{(i)}_{\alpha} - \bar{X}^{(i)})$$

$$A = \sum_{i=1}^{r} A_i, \quad \bar{x} = \sum_{i=1}^{r} \frac{N_i \bar{X}^{(i)}}{N}, \quad \bar{\mu} = \sum_{i=1}^{r} \frac{N_i \mu^{(i)}}{N},$$

$$\bar{X}^* = (\sqrt{N_1} (\bar{X}^{(1)} - \bar{x}), \ldots, \sqrt{N_q} (\bar{X}^{(r)} - \bar{x})),$$

$$B = \bar{X}^* \bar{X}^{*\prime},$$

$$M^* = (\sqrt{N_1} (\mu^{(1)} - \bar{\mu}), \ldots, \sqrt{N_q} (\mu^{(r)} - \bar{\mu})).$$

Let $W(p \times q)$ be a transform of $M^*$ and let $\Omega = \Sigma^{-1}MM^\prime$, $q = r - 1$, then it follows that for $p \leq r \leq N$, $A$ is distributed $W(\Sigma, n)$, $n = N - q$ independent of $B$ which is distributed $W(\Sigma, q, \Omega)$. Let

$$r^2 = \frac{1}{B'[A^{-1}B]^{-\frac{1}{2}}},$$

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\[ \frac{1}{\mathbf{B}^2} \] being the symmetric square root of \( \mathbf{B} \), then \( \mathrm{ntrV} \) is the well known Hotelling's generalised \( T^2_0 \) statistic for testing the hypothesis

\[ H: \mu^{(i)} = \mu^{(j)} \text{ for all } i, j \]

against \( \mu^{(i)} \neq \mu^{(j)} \) for some \( i, j \), \( i \neq j \). Fujikoshi (1970) has derived the asymptotic distribution for \( T^2_0 \) in the noncentral case to order terms \( n^{-2} \), which also holds if \( q < p \leq N \). The equivalent of \( V \) in the case \( q < p \leq N \) is defined as

\[ F = Z'A^{-1}Z \]

where \( Z(p \times q) \sim N(M, \sum \Theta I_q) \). For this case \( \Omega \) is defined as \( \Omega = M'\Sigma^{-1}M \).

We shall consider here the asymptotic distribution of the \( j \)-th esf of \( \text{tr}_j F \). The result reduces to the result given by Fujikoshi if \( j = 1 \). We shall refer to \( V \) and \( F \) as Hotelling's generalised beta matrices, since they are distributed as multivariate beta distributions and Hotelling's generalised \( T^2_0 \) are related to them.

2. PRELIMINARY RESULTS.

Lemma 2.1 (de Waal (1974))

2.1

\[ \frac{\partial \text{tr}_j \Sigma}{\partial \sigma^*_{rs}} = \text{tr}_j A^* \Sigma \]

where \( E^*_{rs} = \frac{\partial \Sigma}{\partial \sigma^*_{rs}} \)

\[ A_j = \frac{\partial \text{tr}_j \Sigma}{\partial \Sigma} = (-1)^{j-1} \sum_{i=1}^{j} (-1)^{j-i} \Sigma^{i-1} \text{tr}_{j-1} \Sigma \]
and

\[
\left( \frac{\partial}{\partial \sigma^*_{rs}} \right) = \left( \frac{1}{2} (1 + \delta_{rs}) \frac{\partial}{\partial \sigma_{rs}} \right) = \phi ; \quad r, s = 1, \ldots, q .
\]

**Lemma 2.2** Let

\[
\Gamma_j = \Lambda_j \bigg|_{\Sigma = \nu R^{-1}}
\]

then

\[
2.2 \quad \text{etr}(B \Lambda \exp (\nu^{1-j} \Lambda \nu) \Sigma) \bigg|_{\Sigma = \nu R^{-1}} = \text{etr}(\Lambda \Gamma_j \exp (\nu^2 \Lambda \nu R^{-1}) (1 + O(\nu^{-1})) .
\]

**Proof**

\[
\text{tr}(B \Lambda \exp (\nu^{1-j} \Lambda \nu) \Sigma) \bigg|_{\Sigma = \nu R^{-1}} = \sum_{r, s} b_{rs} \phi_{sr} \exp (\nu^{1-j} \Lambda \nu) \Sigma \bigg|_{\Sigma = \nu R^{-1}}
\]

\[
= \nu^{1-j} \lambda \exp (\nu^2 \nu R^{-1}) \sum_{r, s} b_{rs} \text{tr} \Lambda \nu \Sigma \bigg|_{\Sigma = \nu R^{-1}}
\]

\[
= \lambda \exp (\nu \nu R^{-1}) \text{tr}(B \Lambda) .
\]

Also

\[
\text{tr}^2(B \Lambda \exp (\nu^{1-j} \Lambda \nu) \Sigma) \bigg|_{\Sigma = \nu R^{-1}} = \sum_{m, n} \sum_{r, s} b_{mn} b_{rs} \phi_{nm} \phi_{sr} \exp (\nu^{1-j} \Lambda \nu) \Sigma \bigg|_{\Sigma = \nu R^{-1}}
\]

\[
= \sum_{m, n} \sum_{r, s} b_{mn} b_{rs} \phi_{nm} \phi_{sr} \left\{ \nu^{1-j} \exp (\nu^{1-j} \Lambda \nu) \Sigma \right\}
\]

\[
\text{tr} \Lambda \nu \Sigma \bigg|_{\Sigma = \nu R^{-1}}
\]

(cont. on next page)
\[-4-\]

\[
= \sum_{m,n} \sum_{r,s} b_{m,n} b_{r,s} v^{2(1-j)} \lambda^2 \exp(v^{1-j} \lambda r_{j}) \text{tr} \Lambda_{j}^{E^*} \text{tr} \Lambda_{j}^{E^*} \left| \Sigma_{v} = v R^{-1} + O(v^{-1}) \right.
\]

\[
= \lambda^2 \text{tr}^2(\Sigma_{v} R_{j}) \exp(v \lambda r_{j} \Sigma_{v} R_{j}) + O(v^{-1}).
\]

Continuing in this way, it is clear that the lemma holds.

**Lemma 2.3** *(Crowther (1974))*

Let \( Z(p \times q) \sim N(M, I_p \otimes I_q) \), \( q < p \), then for \( A(p \times p) \) fixed, the density of \( S = Z'A^{-1}Z \) is given by

\[
2.3 \quad \frac{1}{2^{p+q} \Gamma_{p}(\frac{1}{2})} \left| A \right|^{\frac{1}{2}q} \text{etr}(\frac{1}{2} \Omega) \left| S \right|^{\frac{1}{2}(p-q)}
\]

\[
= \sum_{k=0}^{\infty} \Gamma_{K} \frac{1}{(\frac{1}{2})^K} K! E_{Y, K} \left( -\frac{1}{2} \Omega \right) \left( Y - \frac{i}{\sqrt{2}} M \right) \Lambda \left( Y - \frac{i}{\sqrt{2}} M \right), \quad S > 0
\]

where the expectation is taken w.r.t. the density

\[
2.4 \quad \pi^{-\frac{1}{2}pq} \text{etr}(-YY'), \quad Y(p \times q).
\]

Crowther (1974) proved this lemma for the case \( \Sigma = I_p \).

**Lemma 2.4** *(Fujikoshi (1970))*

\[
2.5 \quad \frac{\text{etr}(\frac{1}{2}n'\eta) \Gamma_{p}(\frac{1}{2}(n+q))}{\pi^{\frac{1}{2}pq}} \left| I_q + \frac{1}{i\pi}(Y - \frac{i\eta}{\sqrt{2}}) (Y - \frac{i\eta}{\sqrt{2}}) \right|^{\frac{1}{2}(n+q)}
\]

\[
= \left( 1 - 2it \right)^{-\frac{1}{2}pq} \text{etr}(\frac{1}{1 - 2it}n'\eta) \left[ 1 + \frac{1}{4n}(pq(q - p - 1)
\]

\[
- \frac{2q}{1 - 2it}(pq - trn'\eta) + \frac{1}{(1 - 2it)^2}(pq(p + q + 1) - 2(p + 2q + 1)trn'\eta)
\]

\[
+ tr(\eta'\eta)^2 - \frac{2}{(1 - 2it)^3}((p + q + 1)trn'\eta - tr(\eta'\eta))^2)
\]

(cont. on next page)
\[ \begin{align*}
&= \frac{\text{tr}(n^*n)^2}{(1-2it)^2} + \frac{1}{96n^2} \left\{ \sum_{\alpha=0}^{8} (1 - 2it)^{-\alpha} B_\alpha \left( \frac{1}{2} n^*n \right) \right\} + o(n^{-3}) \\
&= g(t|n^*n), \text{ say}
\end{align*} \]

where the expectation is taken w.r.t. the density 2.4, \( n(p \times q) \) any fixed matrix and \( p_\alpha \left( \frac{1}{2} n^*n \right), \alpha = 0,1, \ldots, 8 \), given in Fujikoshi (1970) p. 104.

**Proof** Fujikoshi (1970) showed that the characteristic function of Hotelling's generalised \( T_0^2 = ntrF \) is given by

\[ C(t) = g(t|\Omega). \]

On the other hand for \( Z(p \times q) \sim N(M, I_p \otimes I_q) \) and \( A(p \times p) \sim W(I,n) \) independent of each other and using Lemma 2.3, the character function of \( ntrF \), \( R(q \times q) \) fixed nonsingular

\[ 2.7 \quad E \etr(\text{int}RF) = \frac{\frac{1}{2} p(n+q)}{\Gamma_{\frac{1}{2}(p)} \Gamma_{\frac{1}{2}(n)}} \etr(-\frac{1}{2} M) \int_{A>0} \int_{S>0} \etr(\text{int}RS) |S|^\frac{1}{2}(n-p-1) \]

\[ \etr(-\frac{1}{2} A |A|^2)^{\frac{1}{2}(n+q-p-1)} \sum_{k=0}^{\infty} \frac{1}{(\frac{1}{2} p)^k k!} E_Y C_K \left\{ \frac{1}{2} c(Y - \frac{i}{\sqrt{2}}) \right\}, \]

\[ A(Y - \frac{i}{\sqrt{2}})dSdA \]

\[ = \frac{\etr(-\frac{1}{2} \alpha) \Gamma_{\frac{1}{2}(p)} \Gamma_{\frac{1}{2}(q)}}{\frac{1}{2} pq \Gamma_{\frac{1}{2}(n)}} \int_{\text{int}} \frac{1}{\int_{\text{int}} \left( Y - \frac{i}{\sqrt{2}} \right)^2, \left( Y - \frac{i}{\sqrt{2}} \right)^2 \frac{1}{2} \frac{1}{2} (n+q)} \]

Equating 2.6 and 2.7 for \( R = I_q \) proves the lemma.
The following theorem generalizes the result given by Fujikoshi (1970):

**Theorem 2.1** For $R(q \times q)$ fixed p.d.s. and $F(q \times q)$ Hotelling's generalised beta matrix

$$P(ntrRF < x) = \text{etr}(\frac{1}{2} \Omega (I - R^{-1})) \left[ \sum p(\chi^2_f(\delta^2) < x) + \frac{1}{4n} pq(p - q - 1) ight. $$

$$- 2q(pq - tr\Omega^{-1}p(\chi^2_{f+2}(\delta^2) < x) + (pq(q + q + 1)) $$

$$- 2(p + 2q + 1)tr\Omega^{-1} + tr(\Omega^{-1})^2 p\chi^2_{f+4}(\delta^2) < x) $$

$$+ 2((p + q + 1)tr\Omega^{-1} - tr(\Omega^{-1})^2 p\chi^2_{f+6}(\delta^2) < x) $$

$$+ tr(\Omega^{-1})^2 p\chi^2_{f+8}(\delta^2) < x) \right) + \frac{1}{96n^2} \sum_{\alpha=0}^{8} \binom{1}{\alpha} \Omega^{-1}(\alpha) $$

$$p(\chi^2_{f+2\alpha}(\delta^2) < x) \right\} + o(n^{-3}) \right] \text{ where } \delta^2 = \text{tr}(\frac{1}{2} \Omega^{-1}) $$

where $f = \frac{1}{2}pq$ and $\chi^2_f(\delta^2)$ denotes the noncentral chi-square variable with $f$ degrees of freedom and noncentrality parameter $\delta^2$.

**Proof** From 2.7 and Lemma 2.4 the c.f. of $ntrR^{-1}F$ is given by

$$E \text{etr}(intR^{-1}F) = \frac{\text{etr}(\frac{1}{2} \Omega) \Gamma_p\left(\frac{1}{2}(n+q)\right)}{1pq} \text{etr}(\frac{1}{2} \Omega^{-1}) \left[ \text{et}_{Y,\frac{1}{2}}(Y - \frac{1}{2} \Omega^{-1}) \right] $$

$$= \frac{\text{etr}(\frac{1}{2} \Omega) \text{etr}(\frac{1}{2} \Omega^{-1})}{1pq} g(\Omega^{-1}).$$

Inverting 2.9 gives the theorem. This result holds for $q > p$ using the argument given by Fujikoshi (1970).
3. THE ASYMPTOTIC DISTRIBUTION OF $\text{tr}_j F$. We shall now consider the asymptotic distribution of

$$\gamma = n \text{tr}_j (F/\tau)$$

where $\tau$ is some fixed constant.

**Theorem 3.1** Let $R$ be any fixed p.d.s. matrix and

$$\xi = n(\text{tr}_j (F/\tau) + \text{tr} R^{-1} \Gamma_j - \text{tr}_j R^{-1})$$

then

$$P(\xi < x) = P\left(\chi^2_\nu (\delta^2) < x\right) + o(n^{-1})$$

where

$$\delta^2 = \tau \text{tr} \left(\frac{1}{2} \Lambda \Gamma_j \right), \ \tau = \text{tr} \Lambda / \text{tr} \Lambda \Gamma_j^{-1}$$

and

$$\Gamma_j = (-1)^{j-1} \sum_{i=1}^{j} (-1)^{j-i} \Lambda^{i-1} \text{tr}_j^{-1} R^{-1}.$$  

**Proof** The characteristic function of

$$\gamma = n \text{tr}_j (F/\tau)$$

is given by

$$\phi(t) = E \exp \left( \text{int} \text{tr}_j (F/\tau) \right).$$

Expanding $\exp(\text{int} \text{tr}_j F/\tau)$ as a Taylor series at

$$\frac{\text{int} F}{\tau} = \text{int} R^{-1}$$

for some p.d.s. matrix $R$. 

i.e. for \( \nu = \text{int} \)

\[
\exp(\nu \, \text{tr}_j(F/\tau)) = \exp(\nu^{1-j} \, \text{tr}_j(\nu^{\frac{p}{q}}/\tau)) = \exp(\nu \, \text{tr}(F/\tau - R^{-1}) \delta \exp(\nu^{1-j} \, \text{tr}_j(\delta)) \bigg|_{\Sigma = \nu R^{-1}}.
\]

Let \( B = \nu(F/\tau - R^{-1}) \) in Lemma 2.2, then \( \phi(t) \) can be written as

\[
\phi(t) = E \, \text{etr}(\nu(F/\tau - R^{-1}) \Gamma_j) \exp(\nu \, \text{tr}_j(R^{-1})) + o(n^{-1})
\]

\[
= \exp(\nu(\text{tr}_j(R^{-1} - \text{tr}R^{-1} \Gamma_j)) E \, \text{etr}(\nu \, \Gamma_j) + o(n^{-1})
\]

where \( \Gamma_j \) is given in Lemma 2.2.

Let

\[
R = \frac{1}{\text{tr}_j}
\]

in 2.9, then using 2.5 the characteristic function becomes

\[
\phi(t) = \exp(\nu(\text{tr}_j R^{-1} - \text{tr}R^{-1} \Gamma_j) \text{etr}(\frac{1}{2} \Omega)) \text{etr}(\frac{1}{2} \Omega_{pq}) \text{etr}(\frac{1}{2} \Omega_{pq} \Omega^{-1} \Gamma_j) (1 - 2it)^{-2} \text{etr}(\frac{it \Gamma_j}{1 - 2it \Gamma_j^{-1}}) + o(n^{-1}).
\]

Let

\[
\tau = \text{tr} \Omega / \text{tr} \Omega_{pq}
\]

then it follows from 3.3 that the characteristic function of

\[
\xi = n \{ \text{tr}_j(F/\tau) + \text{tr}R^{-1} \Gamma_j - \text{tr}_j R^{-1} \}
\]

can be written as
3.5 \[ h(t) = (1 - 2it)^{-\frac{1}{2}} n_{pq} \text{etr}(\frac{it}{1-2it} \Omega R^{-1}_j) + o(n^{-1}) \]

which proves the theorem.

Since \( R \) is arbitrary, it may be useful to let \( R = I \). In this case \( \Gamma_j \) becomes \( \text{(see Gradshteyn and Ryzhik (1965))} \)

3.6 \[
\Gamma_j = \Lambda_j \sum_{\Sigma=1}^{q} \\
= (-1)^{j-1} \sum_{i=1}^{j} (-1)^{j-i} \binom{q}{j-i} I_q \\
= (-1)^{j-1} \sum_{x=0}^{j-1} (-1)^x \binom{q}{x} I_q \\
= \binom{q-1}{j-1} I_q.
\]

Then \( \tau = \binom{q-1}{j-1} \)

and \( \xi \) becomes

3.7 \[ \xi = n \left\{ \text{tr} \left\{ \frac{F/\binom{q-1}{j-1}}{j-1} \right\} + (j - 1) \binom{q}{j} \right\}. \]

The noncentrality parameter will then be \( \delta^2 = \text{tr} \frac{1}{2} \Omega \).

In the special case \( j = 1 \) it follows that \( \xi = n \text{tr} F \),

and the result by Fujikoshi (1970) follows immediately.
REFERENCES


