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AN ASYMPTOTIC DISTRIBUTION FOR THE j-th esf OF THE GENERALISED HOTELLING'S BETA MATRIX

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An Asymptotic Distribution for the j-th esf of the Generalised Hotelling's Beta Matrix

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1. INTRODUCTION. Let $X_\alpha^{(i)}(p \times 1)$, $\alpha = 1, \ldots, N_i$, be i.i.d. $N(\mu^{(i)}, \Sigma)$, $i = 1, \ldots, r$. Let

$$\overline{X}^{(i)} = \sum_{\alpha=1}^{N_i} X_\alpha^{(i)}/N_i$$

$$A_i = \sum_{\alpha=1}^{N_i} \left(X_\alpha^{(i)} - \overline{X}^{(i)}\right)\left(X_\alpha^{(i)} - \overline{X}^{(i)}\right)'$$

$$A = \sum_{i=1}^{r} A_i \ , \quad \overline{X} = \sum_{i=1}^{r} N_i \overline{X}^{(i)}/N \ , \quad \overline{\mu} = \sum_{i=1}^{r} N_i \mu^{(i)}/N \ ,$$

$$\overline{X}^* = \left(\sqrt{N_1}\,(\overline{X}^{(1)} - \overline{X}), \ldots, \sqrt{N_q}\,(\overline{X}^{(r)} - \overline{X})\right) \ ,$$

$$B = \overline{X}^*\overline{X}^{*\prime} \ ,$$

$$M^* = \left(\sqrt{N_1}\,(\mu^{(1)} - \overline{\mu}), \ldots, \sqrt{N_q}\,(\mu^{(r)} - \overline{\mu})\right) \ .$$

Let $M(p \times q)$ be a transform of M^* and let $\Omega = \Sigma^{-1}MM'$, $q = r - 1$, then it follows that for $p \le r \le N$, A is distributed $W(\Sigma, n)$, $n = N - q$ independent of B which is distributed $W(\Sigma, q, \Omega)$. Let

$$V = B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}} \ ,$$

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$B^{\frac{1}{2}}$ being the symmetric square root of B, then $n\,trV$ is the well known Hotelling's generalised T_0^2 statistic for testing the hypothesis

$$H: \mu^{(i)} = \mu^{(j)} \quad \text{for all} \quad i,j$$

against $\mu^{(i)} \neq \mu^{(j)}$ for some i,j, $i \neq j$. Fujikoshi (1970) has derived the asymptotic distribution for T_0^2 in the noncentral case to order terms n^{-2}, which also holds if $q < p \leq N$. The equivalent of V in the case $q < p \leq N$ is defined as

$$F = Z'A^{-1}Z$$

where $Z(p \times q) \sim N(M, \sum \otimes I_q)$. For this case Ω is defined as $\Omega = M'\Sigma^{-1}M$.

We shall consider here the asymptotic distribution of the j-th esf of tr_jF. The result reduces to the result given by Fujikoshi if $j = 1$. We shall refer to V and F as Hotelling's generalised beta matrices, since they are distributed as multivariate beta distributions and Hotelling's generalised T_0^2 are related to them.

2. PRELIMINARY RESULTS.

Lemma 2.1 (de Waal (1974))

2.1
$$\frac{\partial tr_j\Sigma}{\partial\sigma^*_{rs}} = tr\Lambda_j E^*_{rs}$$

where $E^*_{rs} = \dfrac{\partial\Sigma}{\partial\sigma^*_{rs}}$

$$\Lambda_j = \frac{\partial tr_j\Sigma}{\partial\Sigma} = (-1)^{j-1} \sum_{i=1}^{j} (-1)^{j-i}\, \Sigma^{i-1} tr_{j-1}\Sigma$$

and

$$\left(\frac{\partial}{\partial\sigma^*_{rs}}\right) = \left(\frac{1}{2}(1 + \delta_{rs}) \frac{\partial}{\partial\sigma_{rs}}\right) = \partial \ ; \quad r,s = 1, \ldots ,q \ .$$

<u>Lemma 2.2</u> Let

$$\Gamma_j = \Lambda_j \Big|_{\Sigma=R^{-1}} ,$$

then

2.2 $$\operatorname{etr}(B\partial)\exp(\nu^{1-j}\Lambda \operatorname{tr}_j\Sigma)\Big|_{\Sigma=\nu R^{-1}} = \operatorname{etr}(\lambda B\Gamma_j)\exp(\nu\lambda \operatorname{tr}_j R^{-1})\{1 + O(\nu^{-1})\} \ .$$

<u>Proof</u>

$$\operatorname{tr}(B\partial)\exp(\nu^{1-j}\lambda \operatorname{tr}_j\Sigma)\Big|_{\Sigma=\nu R^{-1}} = \sum_{r,s} b_{rs}\frac{\partial}{\partial\sigma^*_{sr}} \exp(\nu^{1-j}\lambda \operatorname{tr}_j\Sigma)\Big|_{\Sigma=\nu R^{-1}}$$

$$= \nu^{1-j}\lambda\exp(\nu\lambda \operatorname{tr}_j\Lambda_j)\sum_{r,s} b_{rs}\operatorname{tr}\Lambda_j E^*_{sr}\Big|_{\Sigma=\nu R^{-1}}$$

$$= \lambda\exp(\nu\lambda \operatorname{tr}_j\Gamma_j)\operatorname{tr}(B\Gamma_j) \ .$$

Also

$$\operatorname{tr}^2(B\partial)\exp(\nu^{1-j}\lambda \operatorname{tr}_j\Sigma)\Big|_{\Sigma=\nu R^{-1}} = \sum_{m,n}\sum_{r,s} b_{mn}b_{rs} \frac{\partial^2}{\partial\sigma^*_{nm}\partial\sigma^*_{sr}} \exp(\nu^{1-j}\lambda \operatorname{tr}_j\Sigma)\Big|_{\Sigma=\nu R^{-1}}$$

$$= \sum_{m,n}\sum_{r,s} b_{mn}b_{rs} \frac{\partial}{\partial\sigma^*_{nm}}\left\{ \nu^{1-j} \exp(\nu^{1-j}\lambda \operatorname{tr}_j\Sigma) \operatorname{tr}\Lambda_j E^*_{sr}\right\}\Big|_{\Sigma=\nu R^{-1}}$$

(cont. on next page)

$$= \sum_{m,n} \sum_{r,s} b_{mn} b_{rs} \; \nu^{2(1-j)} \lambda^2 \exp(\nu^{1-j} \lambda \mathrm{tr}_j \Sigma) \mathrm{tr} \Lambda_j E^*_{nm} \; \mathrm{tr} \; \Lambda_j E^*_{sr} \Big|_{\Sigma = \nu R^{-1}} + O(\nu^{-1})$$

$$= \lambda^2 \mathrm{tr}^2 (B\Gamma_j) \exp(\nu\lambda \mathrm{tr}_j \Gamma_j) + O(\nu^{-1}) \; .$$

Continuing in this way, it is clear that the lemma holds.

Lemma 2.3 (Crowther (1974))

Let $Z(p \times q) \sim N(M, I_p \otimes I_q)$, $q < p$, then for $A(p \times p)$ fixed, the density of $S = Z'A^{-1}Z$ is given by

2.3
$$\frac{1}{2^{\frac{1}{2}qp} \Gamma_q\left(\frac{1}{2}p\right)} |A|^{\frac{1}{2}q} \mathrm{etr}\left(-\tfrac{1}{2}\Omega\right) |S|^{\frac{1}{2}(p-q-1)}$$
$$\sum_{k=0}^{\infty} \sum_K \frac{1}{\left(\frac{1}{2}p\right)_K k!} E_Y C_K\left[-\tfrac{1}{2}S\left(Y - \tfrac{i}{\sqrt{2}} M\right)' A\left(Y - \tfrac{i}{\sqrt{2}} M\right)\right] , \quad S > 0$$

where the expectation is taken w.r.t. the density

2.4
$$\pi^{-\frac{1}{2}pq} \mathrm{etr}(-YY') , \quad Y(p \times q) \; .$$

Crowther (1974) proved this lemma for the case $\Sigma \neq I_p$.

Lemma 2.4 (Fujikoshi (1970))

2.5
$$\frac{\mathrm{etr}\left(-\frac{1}{2}\eta'\eta\right) \Gamma_p\left(\frac{1}{2}(n+q)\right)}{(\mathrm{int})^{\frac{1}{2}pq} \Gamma_p\left(\frac{1}{2}n\right)} E_Y \left| I_q + \frac{1}{itn}\left(Y - \frac{i\eta}{\sqrt{2}}\right)' \left(Y - \frac{i\eta}{\sqrt{2}}\right) \right|^{-\frac{1}{2}(n+q)}$$
$$= (1 - 2it)^{-\frac{1}{2}pq} \mathrm{etr}\left(\frac{it}{1-2it} \eta'\eta\right) \Big[\; 1 + \frac{1}{4n}\Big\{ pq(q - p - 1)$$
$$- \frac{2q}{1-2it} (pq - \mathrm{tr}\eta'\eta) + \frac{1}{(1-2it)^2}\left(pq(p + q + 1) - 2(p + 2q + 1)\mathrm{tr}\eta'\eta\right.$$
$$\left. + \mathrm{tr}(\eta'\eta)^2 + \frac{2}{(1-2it)^3}\left((p + q + 1)\mathrm{tr}\eta'\eta - \mathrm{tr}(\eta'\eta)^2\right)\right.$$

(cont. on next page)

$$+ \frac{tr(\eta'\eta)^2}{(1-2it)^4}\Big\} + \frac{1}{96n^2}\Big\{ \sum_{\alpha=0}^{8} (1-2it)^{-\alpha} B_\alpha\left(\tfrac{1}{2}\eta'\eta\right)\Big\} + O(n^{-3}) \Big]$$

$$= g(t|\eta'\eta) \text{ , say}$$

where the expectation is taken w.r.t. the density 2.4, $\eta(p \times q)$ any fixed matrix and $B_\alpha\left(\frac{1}{2}\eta'\eta\right)$, $\alpha = 0,1, \ldots ,8$, given in Fujikoshi (1970) p. 104.

Proof Fujikoshi (1970) showed that the characteristic function of Hotelling's generalised $T_0^2 = n\mathrm{tr}F$ is given by

2.6 $$C(t) = g(t|\Omega) .$$

On the other hand for $Z(p \times q) \sim N(M, I_p \otimes I_q)$ and $A(p \times p) \sim W(I,n)$ independent of each other and using Lemma 2.3, the character function of $n\mathrm{tr}RF$, $R(q \times q)$ fixed nonsingular

2.7 $$E\,\mathrm{etr}(int RF) = \frac{2^{-\frac{1}{2}p(n+q)}}{\Gamma_q\left(\frac{1}{2}p\right)\Gamma_p\left(\frac{1}{2}n\right)} \mathrm{etr}\left(-\tfrac{1}{2}\Omega\right) \int_{A>0}\int_{S>0} \mathrm{etr}(intRS)\ |S|^{\frac{1}{2}(p-q-1)}$$

$$\mathrm{etr}\left(-\tfrac{1}{2}A\right) |A|^{\frac{1}{2}(n+q-p-1)} \sum_{k=0}^{\infty} \sum_K \frac{1}{\left(\frac{1}{2}p\right)_K k!} E_Y C_K\left[-\tfrac{1}{2}S\left(Y - \tfrac{i}{\sqrt{2}}M\right)'\right.$$

$$\left. A\left(Y - \tfrac{i}{\sqrt{2}}M\right)\right] dS dA$$

$$= \frac{\mathrm{etr}\left(-\frac{1}{2}\Omega\right)\Gamma_p\left(\frac{1}{2}(n+q)\right)}{(int)^{\frac{1}{2}pq}\,\Gamma_p\left(\frac{1}{2}n\right)} E_Y \left| I_q + \frac{1}{int}\left(Y - \tfrac{i}{\sqrt{2}}MR^{-\frac{1}{2}}\right)'\left(Y - \tfrac{i}{\sqrt{2}}MR^{-\frac{1}{2}}\right)\right|^{-\frac{1}{2}(n+q)} .$$

Equating 2.6 and 2.7 for $R = I_q$ proves the lemma.

The following theorem generalises the result given by Fujikoshi (1970):

Theorem 2.1 For $R(q \times q)$ fixed p.d.s. and $F(q \times q)$ Hotelling's generalised beta matrix

$$2.8 \quad P(ntrRF < x) = etr\left(-\tfrac{1}{2}\Omega(I - R^{-1})\right)\Big[P\left(\chi_f^2(\delta^2) < x\right) + \frac{1}{4n}\Big\{pq(q - p - 1)$$

$$- 2q(pq - tr\Omega R^{-1})P\left(\chi_{f+2}^2(\delta^2) < x\right) + \left(pq(p + q + 1)\right.$$

$$- 2(p + 2q + 1)tr\Omega R^{-1} + tr(\Omega R^{-1})^2\Big)P\left(\chi_{f+4}^2(\delta^2) < x\right)$$

$$+ 2\left((p + q + 1)tr\Omega R^{-1} - tr(\Omega R^{-1})^2\right)P\left(\chi_{f+6}^2(\delta^2) < x\right)$$

$$+ tr(\Omega R^{-1})^2 P\left(\chi_{f+8}^2(\delta^2) < x\right)\Big\} + \frac{1}{96n^2}\Big\{\sum_{\alpha=0}^{8} B_\alpha\left(\tfrac{1}{2}\Omega R^{-1}\right)$$

$$P\left(\chi_{f+2\alpha}^2(\delta^2) < x\right)\Big\} + O(n^{-3}) \Big] \quad \text{where} \quad \delta^2 = tr\left(\tfrac{1}{2}\Omega R^{-1}\right) ,$$

$f = \frac{1}{2}pq$ and $\chi_f^2(\delta^2)$ denotes the noncentral chi-square variable with f degrees of freedom and noncentrality parameter δ^2 .

Proof From 2.7 and Lemma 2.4 the c.f. of $ntrR^{-1}F$ is given by

$$2.9 \quad E\, etr(itntrRF) = \frac{etr\left(-\frac{1}{2}\Omega\right)\Gamma_p\left(\frac{1}{2}(n+q)\right)}{(int)^{\frac{1}{2}pq}\,\Gamma_p\left(\frac{1}{2}n\right)} E_Y\left|I_q + \frac{1}{int}\left(Y - \frac{i}{\sqrt{2}}MR^{-\frac{1}{2}}\right)'\left(Y - \frac{i}{\sqrt{2}}MR^{-\frac{1}{2}}\right)\right|^{-\frac{1}{2}(n+q)}$$

$$= etr\left(-\tfrac{1}{2}\Omega\right)etr\left(\tfrac{1}{2}\Omega R^{-1}\right)g(t|\Omega R^{-1}) .$$

Inverting 2.9 gives the theorem. This result holds for $q \gtrless p$ using the argument given by Fujikoshi (1970).

3. THE ASYMPTOTIC DISTRIBUTION OF $tr_j F$.

We shall now consider the asymptotic distribution of

$$\gamma = ntr_j(F/\tau)$$

where τ is some fixed constant.

Theorem 3.1 Let R be any fixed p.d.s. matrix and $\xi = n\left(tr_j(F/\tau) + trR^{-1}\Gamma_j - tr_j R^{-1}\right)$, then

3.1 $$P(\xi < x) = P\left(\chi_f^2(\delta^2) < x\right) + O(n^{-1})$$

where

$$f = \tfrac{1}{2}pq\ , \quad \delta^2 = \tau tr\left(\tfrac{1}{2}\Omega\Gamma_j^{-1}\right)\ , \quad \tau = tr\Omega/tr\Omega\Gamma_j^{-1} \quad \text{and}$$

$$\Gamma_j = (-1)^{j-1}\sum_{i=1}^{j}(-1)^{j-i}R^{1-i}tr_{j-i}R^{-1}\ .$$

Proof The characteristic function of

$$\gamma = ntr_j(F/\tau)$$

is given by

$$\phi(t) = E\ \exp\left(int\ tr_j(F/\tau)\right)\ .$$

Expanding $\exp(int\ tr_j F/\tau)$ as a Taylor series at

$$\frac{int\ F}{\tau} = int\ R^{-1}$$

for some p.d.s. matrix R

i.e. for $\nu = \text{int}$

$$\exp\left(\nu \operatorname{tr}_j(F/\tau)\right) = \exp\left(\nu^{1-j}\operatorname{tr}_j(\nu F/\tau)\right)$$

$$= \operatorname{etr}\left(\nu(F/\tau - R^{-1})\partial\right)\exp(\nu^{1-j}\operatorname{tr}_j\Sigma)\Big|_{\Sigma=\nu R^{-1}} .$$

Let $B = \nu(F/\tau - R^{-1})$ in Lemma 2.2, then $\phi(t)$ can be written as

3.2 $$\phi(t) = E \operatorname{etr}\left(\nu(F/\tau - R^{-1})\Gamma_j\right)\exp(\nu\operatorname{tr}_j R^{-1}) + O(n^{-1})$$
$$= \exp\left(\nu(\operatorname{tr}_j R^{-1} - \operatorname{tr}R^{-1}\Gamma_j)\right)E \operatorname{etr}\left(\frac{\nu}{\tau} F\Gamma_j\right) + O(n^{-1})$$

where Γ_j is given in Lemma 2.2.

Let

$$R = \frac{1}{\tau}\Gamma_j$$

in 2.9, then using 2.5 the characteristic function becomes

3.3 $$\phi(t) = \exp\left(\nu(\operatorname{tr}_j R^{-1} - \operatorname{tr}R^{-1}\Gamma_j\right)\operatorname{etr}\left(-\frac{1}{2}\Omega\right)$$
$$\operatorname{etr}\left(\frac{1}{2}\tau\Omega\Gamma_j^{-1}\right)(1 - 2it)^{-\frac{1}{2}pq}\operatorname{etr}\left(\frac{it\tau}{1-2it}\Omega\Gamma_j^{-1}\right) + O(n^{-1}) .$$

Let

3.4 $$\tau = \operatorname{tr}\Omega/\operatorname{tr}\Omega\Gamma_j^{-1}$$

then it follows from 3.3 that the characteristic function of

$$\xi = n\left\{\operatorname{tr}_j(F/\tau) + \operatorname{tr}R^{-1}\Gamma_j - \operatorname{tr}_j R^{-1}\right\}$$

can be written as

3.5 $$h(t) = (1 - 2it)^{-\frac{1}{2}pq} \operatorname{etr}\left(\frac{it\tau}{1-2it}\, \Omega\Gamma_j^{-1}\right) + O(n^{-1})$$

which proves the theorem.

Since R is arbitrary, it may be useful to let $R = I$. In this case Γ_j becomes (see Gradshteyn and Ryzhik (1965))

3.6 $$\begin{aligned} \Gamma_j &= \Lambda_j \Big|_{\Sigma=I_q} \\ &= (-1)^{j-1} \sum_{i=1}^{j} (-1)^{j-i} \binom{q}{j-i} I_q \\ &= (-1)^{j-1} \sum_{x=0}^{j-1} (-1)^x \binom{q}{x} I_q \\ &= \binom{q-1}{j-1} I_q \, . \end{aligned}$$

Then

$$\tau = \binom{q-1}{j-1}$$

and ξ becomes

3.7 $$\xi = n\left\{\operatorname{tr}_j\left[F/\binom{q-1}{j-1}\right] + (j - 1)\binom{q}{j}\right\} .$$

The noncentrality parameter will then be

$$\delta^2 = \operatorname{tr} \tfrac{1}{2}\Omega \, .$$

In the special case $j = 1$ it follows that

$$\xi = n\operatorname{tr}F$$

and the result by Fujikoshi (1970) follows immediately.

REFERENCES

Crowther, N. A. S. (1974): The exact non-central distribution of a quadratic form in normal vectors. Technical report no. 87, Stanford University.

de Waal, D. J. (1974): An asymptotic distribution for the j-th esf of the characteristic roots of the non-central Wishart matrix. Institute of Statistics Mimeo Series no. 960.

Fujikoshi, Y. (1970): Asymptotic expansions of tests statistics in multivariate analysis. *J. Sci. Hiroshima Univ.* Ser. A-I, 34, 73-144.

Gradshteyn, I. S. and Ryzhik, I. M. (1965): *Table of Integrals, Series and Products*. Academic Press, N.Y.