This research was supported in part by the Air Force Office of Scientific Research under Contract AFOSR-68-1415 and by the Office of Naval Research under Contract N00014-67-A-0321-0006 (NRO42-69).

ZER0-ONE LAWS FOR GAUSSIAN MEASURES ON BANACH SPACE

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Institute of Statistics Mimeo Series No. 785

November, 1971
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Abstract

Let $B$ be a real separable Banach space, $\mu$ a Gaussian measure on the Borel $\sigma$-field of $B$, and $B_\mu[B]$ the completion of the Borel $\sigma$-field under $\mu$. If $G \in B_\mu[B]$ is a subgroup, we show that $\mu(G) = 0$ or $1$, extending a result due to Kallianpur and Jain. Necessary and sufficient conditions are given for $\mu(G) = 1$ for the case where $G$ is the range of a bounded linear operator. These results are then applied to obtain a number of 0-1 statements for the sample function properties of a Gaussian stochastic process. The zero-one law is then extended to a class of non-Gaussian measures, and applications are given to some non-Gaussian stochastic processes.
1. Introduction

Kallianpur [1] and Jain [2] have proved the following result. Let $T$ be a complete separable metric space, $\chi$ a linear space of real-valued functions on $T$, and $B[\chi]$ the $\sigma$-field of $\chi$ sets generated by sets of the form

$\{x: (x(t_1), \ldots, x(t_n)) \in C\}$, $t_1, \ldots, t_n \in T$ and $C$ a Borel set in $R^n$. Suppose that $P$ is a Gaussian probability measure on $B[\chi]$ with continuous covariance function $K$ and zero mean, and that $\chi$ contains the reproducing kernel Hilbert space of $K$. Let $B_0[\chi]$ denote the completion of $B[\chi]$ under $P$. With these assumptions, Kallianpur and Jain have shown that $P(G) = 0$ or $1$ for every $B_0[\chi]$-measurable subgroup of $\chi$.

This result does not immediately apply to the case of a Gaussian measure on the Borel sets of an arbitrary real separable Banach space, even in the case of $L_p$, since on $L_p$ the co-ordinate maps $\pi_t: x \mapsto x_t$ are not well-defined.

We first show that this zero-one law holds for Gaussian measures on a Banach space, and without the assumption of zero mean. Some necessary and sufficient conditions for the alternatives are also given. The 0-1 law is extended to a class of non-Gaussian measures. A number of applications are given on path properties of Gaussian stochastic processes and some non-Gaussian processes.

As noted in [1], Cameron and Graves [3] first considered this problem for the case of Wiener measure. In another direction, Pitcher [4] essentially proved the following result. Let $\mu$ be a zero-mean Gaussian measure on a real separable Hilbert space $H$ with covariance operator $K$. Suppose $K = TST^*$ for $S$ and $T$ Hilbert-Schmidt linear operators. Then $\mu[\text{range}(T)] = 0$ or $1$.

Our interest in this problem was motivated by Pitcher's work. We extended his result to any bounded operator $T$, and conjectured from this that a similar 0-1 law holds for any measurable linear manifold in Hilbert space. We are indebted to the appearance of [1] for the key step in completing our proof (Lemma 2, below), and to [2] for enabling us to extend our result to subgroups.
2. DEFINITIONS

\( B \) will denote a separable Banach space, with norm \(||\cdot||\), \( H \) a separable Hilbert space with inner product \( \langle \cdot, \cdot \rangle \). All linear spaces are defined over the real numbers. For a given topological space \( A \), \( B[A] \) will denote the Borel \( \sigma \)-field of \( A \) sets. \( I_F \) is the characteristic function of the set \( F \).

A Gaussian measure \( \mu \) on \( B[\mathcal{B}] \) is by definition a probability measure such that all bounded linear functionals on \( B \) are Gaussian with respect to \( \mu \). \( B^* \) is the space of all bounded linear functionals on \( B \). We will usually not distinguish between \( H \) and \( H^* \).

3. THE ZERO-ONE LAW FOR BANACH SPACE

In this section we obtain a zero-one law for any Gaussian measure on a real and separable Banach space. In particular, the measure need not have zero mean element.

We first obtain the zero-one law for \( H \). Let \( \mu_1 \) be a Gaussian measure on \( B[H] \) with null mean element and covariance operator \( K; \langle Ku, v \rangle = \int_H \langle x, u \rangle \langle x, v \rangle d\mu_1(x) \), all \( u, v \) in \( H \). \( K \) is linear, bounded, non-negative, self-adjoint, trace-class, and uniquely determines \( \mu_1 \) [5]. Since \( H \) is a linear topological space, \( \{ x: x + z \in A \} \) is a Borel set in \( H \) if \( A \) is a Borel set, any \( z \in H \). We define the Gaussian measure \( \mu \) on \( B[H] \) by \( \mu(A) = \mu_1 \{ x: x + m \in A \} \). \( \mu \) has covariance operator \( K \) and mean element \( m \).

\( \{ e_n \}, n = 1, 2, \ldots \) will denote a complete orthonormal set of eigenvectors for \( K \); we require only that \(||Ke_n|| \neq 0 \) for at least one value of \( n \).
We will obtain the zero-one law for $\mu$. Let $B^\mu[H]$ be the completion of $B[H]$ under $\mu$. $G$ is a subgroup of $H$, $G \subseteq B^\mu[H]$.

For any $v$ in $H$, define the Gaussian measure $\nu_v$ by

$$\nu_v(A) = \mu(x: x + Kv \in A).$$

The following result is well-known.

**Lemma 1** [6]: $\nu_v$ and $\mu$ are mutually absolutely continuous ($\nu_v \sim \mu$), and have Radon-Nikodym derivative $[d\nu_v/d\mu] = \ell_v$ given by

$$\ell_v(x) = \exp(\langle x, v \rangle - \langle m, v \rangle - \frac{1}{2} |K_v|^2).$$

Comparing $\ell_v^2$ with $\ell_{2v}$, it is clear that $\ell_v \in L_2[H, B^\mu[H], \mu]$ ($\equiv L_2[\mu]$). Since $\nu_v \sim \mu$, $B^\mu[H] = B^\mu[H]$, $L_2[\mu] = L_2[\mu_v]$, and from the definition of $\nu_v$ we obtain that $\{x: x + Kv \in A\} \in B^\mu[H]$ for $A \in B^\mu[H]$.

**Lemma 2**: $\{1, 1 - \ell_v, v \in H\}$ is dense in $L_2[\mu]$.

**Proof**: Suppose $\int_H g(x)d\mu(x) = \int_H g(x)[1 - \ell_v(x)]d\mu(x) = 0$ for all $v$ in $H$, some $g$ in $L_2[\mu]$. Then $\int_H g(x)\exp(\langle x - m, v \rangle - \frac{1}{2} |K_v|^2)d\mu(x) = 0$, all $v \in H$. Let $v = \sum_{1}^{N} \alpha_i e_i$, where the $\alpha_i$ are scalars. Proceeding as in the proof of Lemma 3 of [1], one obtains $E[g|B[H^N]] = 0$ a.e. $d\mu$, where $B[H^N]$ is the $\sigma$-field generated by $f_n : x \rightarrow \langle x, e_n \rangle$, $n = 1, \ldots, N$. Since $B[H]$ is the smallest $\sigma$-field such that $f_n$ is measurable for all $e_n$, one obtains $E[g|B[H]] = 0$ a.e. $d\mu$, and thus $g(x) = 0$ a.e. $d\mu(x)$ (see Appendix 1 for details).

**Lemma 3**: For any $F \in B^\mu[H]$, let $F_{\alpha n} = \{x: x + \alpha_n Kv \in F\}$, where $\alpha_n$ are scalars such that $\lim_{n} \alpha_n = 0$. Then $\lim_{n} \mu(F_{\alpha n}) = \mu(F)$.

**Proof**: If $\mu(F) = 0$, then $\mu(F_{\alpha n}) = \mu(F_{\alpha n}) = 0$, since $\mu \sim \nu_{\alpha n}$. If $\mu(F) > 0$, $\mu(F_{\alpha n}) = \int_F \ell_{\alpha n}(x)d\mu(x)$. Noting that $\ell_{\alpha n}$ is positive for each $\alpha_n$, that $\ell_{\alpha n}(x) \rightarrow 1$ as $\alpha_n \rightarrow 0$, for all $x \in H$, and that $\int_H \ell_{\alpha n}(x)d\mu(x) = 1$, all $n$, we see that $\{\ell_{\alpha n}\}$ is uniformly integrable with respect to $\mu$ [7, p. 18]. Hence $\{I_F \ell_{\alpha n}\}$ is uniformly integrable,
\[ I_\mu(x) \mathcal{E}_{\alpha_n} \mu(x) \rightarrow I_\mu(x) \] as \( \alpha_n \rightarrow 0 \), all \( x \) in \( H \), and this implies
\[ \int_H I_\mu(x) \mathcal{E}_{\alpha_n} \mu(x) \, d\mu(x) \rightarrow \mu(F) \] as \( \alpha_n \rightarrow 0 \).

Lemma 4: If \( \mu(G) > 0 \), then \( G \supset \text{range}(K) \).

Proof: We apply a proof used to establish Theorem 1 of [2]. Suppose \( \mu(G) > 1/s \), \( s \) a positive integer. For \( n = 1, 2, \ldots \) define \( G(n)_k \),
\[ 0 \leq k \leq s, \] by \( G(n)_0 = G \), \( G(n)_k = \{ x : x + (s!kn)^{-1}Kx \in G \} \), \( 1 \leq k \leq s \), where \( Kx \in G \). For each \( n \), \( G(n)_k \) and \( G(n)_j \) are disjoint if \( j \neq k \) (see [2]).
This implies that \( \sum_{k=0}^{s} \mu(G(n)_k) \leq 1 \), all \( n \), and hence by Lemma 3
\( (s+1)\mu(G) \leq 1 \), a contradiction.

Lemma 5: \( \mu(G) = 0 \) or 1.

Proof: Suppose \( \mu(G) > 0 \). From Lemma 4, \( G \supset Kx \) for all \( x \in H \), so that
\[ \mu_v(G) = \mu(\{ x : x + Kx \in G \}) = \mu(G), \] and \[ \int_H I_G(x)[1 - \mathcal{E}_v(x)] \, d\mu(x) = 0 \], all \( v \) in \( H \).
We note that \( 1 \bot \{ 1 - \mathcal{E}_v, v \in H \} \), so that by Lemma 2 \( \text{L}_2[\mu] \) has the orthogonal decomposition \( \text{Span}(1) \oplus \text{Span}(1 - \mathcal{E}_v, v \in H) \). Hence, \( I_G(x) = \text{constant a.e.} \, d\mu(x) \).

Corollary: If \( \mu(G) = 1 \), then \( G \supset \text{range}(K^\perp) \).

Proof: Let \( \mu'_v \) be the Gaussian measure on \( B[H] \) defined by
\[ \mu'_v(A) = \mu(\{ x : x + K^\perp x \in A \}) \]. \( \mu'_v \sim \mu \) [6], so that \( B_{\mu'_v}(H) = B_{\mu}(H) \). If \( K^\perp x \notin G \),
then \( \mu(G) = 1 \Rightarrow \mu'_v(G) = 0 \), a contradiction.

Now let \( \mu \) be a Gaussian measure on \( B[B] \) with mean element \( m \), let \( B_{\mu}[B] \) be the completion of \( B[B] \) under \( \mu \), and suppose that \( G \in B_{\mu}[B] \) is a
subgroup of \( B \).

Theorem 1: \( \mu(G) = 0 \) or 1.

Proof: We can construct a separable Hilbert space \( H \) such that the elements of \( B \) constitute a dense linear manifold in \( H \), \( B \in B[H] \), and
\( B[B] = B \cap B[H] \); see [8]. Let \( \nu \) be the Gaussian measure on \( B[H] \) defined by
\( v(A) = \mu(B \cap A) \). Let \( B_v[H] \) be the completion of \( B[H] \) under \( v \). Since all sets in \( B[B] \) of \( \mu \)-measure zero belong to \( B[H] \) and have \( v \)-measure zero, it is clear that \( B_v[H] \) contains all elements of \( B_\mu[B] \). Hence, if \( G \) is a subgroup in \( B, \ G \in B_\mu[B] \), then \( G \) is a subgroup in \( H, \ G \in B_v[H] \). The result now follows from Lemma 5 and the definition of \( v \).

Remarks: 1. Theorem 1 holds for Gaussian measures with non-zero mean element. This does not appear to be obvious if one has only the zero-one law for zero-mean Gaussian measures.

2. If one limits attention to manifolds, there are several ways to prove Lemma 4. For example, let \( G_n = \{ x : x + \frac{1}{n} K \mu \in G \} \) \( n = \pm 1, \pm 2, \ldots \) and suppose \( K \mu \perp G \). Then \( G_n \) and \( G_m \) are disjoint if \( n \neq m \), so that \( \mu(G_n) \to 0 \) as \( |n| \to \infty \). Also, \( \mu(G_n) = \mu_{v/n}(G) \), and the family \( \{ \ell_{v/n} \} \) is easily seen to be uniformly integrable. Now suppose \( \mu(G) > 0 \). For any \( \delta > 0 \), there exists \( n \) such that \( \mu(G_n) < \delta \), while \( \int_{G_n} \ell_{v/n}(x) d\mu(x) = \mu_{v/n}(G_n) = \mu(G) \). This contradicts the uniform integrability of \( \{ \ell_{v/n} \} \).

4. MEASURABLE SUBGROUPS

The following two lemmas can often be used to show that a given subgroup is measurable.

Lemma 6 [9]: Let \( T \) be a map from a complete separable metric space \( M_1 \) into a complete separable metric space \( M_2 \). Suppose \( T \) is one-to-one and \( B[M_1]/B[M_2] \) measurable. Then \( T[A] \in B[M_2] \) when \( A \in B[M_1] \).
In our applications, we deal with a bounded linear operator $T$ between two Banach spaces. The following lemma extends Lemma 6 (Kuratowski's theorem) to the case where $N_T$ (the null space of $T$) contains elements other than the null element.

Lemma 7: Let $B_1$ and $B_2$ be two separable Banach spaces, with $B_1$ reflexive. Suppose $T: B_1 \to B_2$ is a bounded linear operator. Then $T[A] \in B[B_2]$ if $A \in B[B_1]$.

Proof: See Appendix 2.

We note that Lemma 7 can be used if $B_1$ is $\ell_p$ or $L_p$ (Lebesgue) for $p \in (1, \infty)$.

5. NECESSARY AND SUFFICIENT CONDITIONS FOR $\mu(G) = 1$

There are many problems in which one will wish to determine if $\mu(G) = 1$ for some measurable subgroup $G$. Theorem 2 gives necessary and sufficient conditions for $\mu(G) = 1$ for a large class of subgroups.

Theorem 2: Suppose $T: H + B$ is a bounded linear operator. If $\mu$ is a Gaussian measure on $B[B]$, with mean element $m$, then

(a) $\mu[\text{range}(T)] = 0$ or 1.

(b) $\mu[\text{range}(T)] = 1$ if and only if there exists a Gaussian measure $\nu$ on $B[H]$ such that $\mu[A] = \nu(x:Tx \in A)$ for all $A$ in $B[B]$.

(c) Let $H_1$ be a separable Hilbert space containing $B$ as a dense linear manifold and such that $B \in B[H]$, $B[B] = B \cap B[H]$. Let $\mu_1$ be the Gaussian measure on $B[H_1]$ defined by $\mu_1[A] = \mu(B \cap A)$, $A \in B[H_1]$. Then $\mu[\text{range}(T)] = 1$ if and only if $m \in \text{range}(T)$ and there exists a covariance operator $S$ in $H$ such that $K_1 = TST_1^*$,
where $K_1$ is the covariance operator of $\mu_1$ and $T_1^*$ is the restriction of $T^*$ to $H_1$.

**Proof:** Part (a) follows from Theorem 1 and Lemma 7. To prove (b), we first note that if such a measure $\nu$ exists, then obviously $\mu[\text{range}(T)] = 1$. Thus, suppose $\mu[\text{range}(T)] = 1$. Define a map $Y : B \to H$ by $Yx = \nu$ if $x = Tv$ and $\nu \perp H_T$; $Yx = 0$ if $x \not\in \text{range}(T)$. For $A \in \mathcal{B}[H]$, $Y^{-1}(A) = \{x : x = Tv, \nu \in A \cap H_T^{-1}\}$ if $0 \notin A$. If $0 \in A$, then $Y^{-1}(A) = T[A \cap H_T^{-1}] \cup \{x : x \not\in \text{range}(T)\}$. In both cases, $Y^{-1}(A)$ belongs to $\mathcal{B}[B]$, by Lemma 7, and hence $Y$ is $\mathcal{B}[B]/\mathcal{B}[H]$ measurable. $Y$ thus induces a measure $\nu$ on $\mathcal{B}[H]$ from $\mu$; $\nu(A) = \mu(\{x : Yx \in A\})$. $\nu(H) = \mu(\{x : Yx \in H\}) = 1$; also, $\nu[H_T^{-1}] = \mu(\{x : Yx \perp H_T\}) = \mu[\text{range}(T)] = 1$. To see that $\nu$ is Gaussian, let $u$ be any element of $\text{range}(T^*)$; then there exists $\{u_n\} \subset \mathcal{B}$ such that $T^*u_n + u$. Thus $\nu(y : \langle y, u \rangle \text{<} k) = \mu(x : \langle Yx, u \rangle \text{<} k) = \mu(x : \lim_n \langle Yx, T^*u_n \rangle \text{<} k) = \mu(x : \lim_n \langle Yx, T^*u \rangle \text{<} k) = \mu(x : \lim_n u_n \langle Tyx \rangle \text{<} k)$, since $\text{range}(T) = \{x : Tyx = x\}$. As the a.e. limit of a sequence of Gaussian random variables, the random variable $f_u : x \to \langle x, u \rangle$ is Gaussian with respect to $\nu$. $\nu$ is thus a Gaussian probability measure. To see that $\mu$ is induced from $\nu$ by $T$, one notes that $\mu(x : x \in A) = \mu(x : x \in A \cap \text{range}(T)) = \mu(x : Tyx \in A) = \nu(x : Tyx \in A)$.

To prove (c), we note that $\mu[\text{range}(T)] = 1 \iff \mu_1[\text{range}(T)] = 1$. Thus, if $\mu[\text{range}(T)] = 1$, let $\nu$ be the Gaussian measure on $\mathcal{B}[H]$ such that $\mu_1 = \nu \circ T^{-1}$. Let $S$ and $y$ be the covariance operator and mean element of $\nu$. We first note that $Ty = m$, since $\int_{H_1} \langle x, u \rangle_1 du_1(x) = \int_{H} \langle x, T_1^*u \rangle dv(x)$, all $u$ in $H_1$, where $\langle \cdot, \cdot \rangle_1$ is the inner product on $H_1$ and $\langle \cdot, \cdot \rangle$ is the inner product on $H$. Now, for all $u, v$ in $H_1$, $\langle K_1 u, v \rangle = \int_{H_1} \langle x - m, u \rangle_1 \langle x - m, v \rangle_1 du_1(x) = \int_{H} \langle x - y, T_1^*u \rangle \langle x - y, T_1^*v \rangle dv(x) = \langle T^*_1 u, v \rangle_1$. Hence, $K_1 = T^*_1$. Conversely, if $K_1 = T^*_1$, $S$ a covariance operator in $H$, and $m = Ty$ for $y \in H$, we define $\nu$ to be the Gaussian measure on $\mathcal{B}[H]$ with covariance operator $S$ and mean element $y$. The map $T : H \to H_1$ induces from $\nu$ a Gaussian measure.
on $B[H_1]$ having covariance operator $\text{TST}_1^*$ and mean element $m$; since a Gaussian measure on $H_1$ is uniquely specified by its covariance operator and mean element, this measure must be $\psi_1$.

Some applications of Theorem 2 are given in the next section. However, it may be of interest to note here the following. Suppose $\mu$ is a Gaussian measure on $B[H]$, with covariance operator $K$. Part (c) of Theorem 2 shows that $\mu(x:x+m \in \text{range}(K^*)) = 0$ for all $m \in H$. This property is not shared, for arbitrary Gaussian $\mu$, by all subgroups $G \in B[H]$ such that $\mu(G) = 0$. For example, if $\mu$ is a zero-mean Gaussian measure such that $\mu(\text{range}(T)) = 1$ for $T$ a bounded and linear map in $H$, then $\mu_1(\text{range}(T)) = 0$,

$\mu_1 = \{x:x-m \in \text{range}(T)\}$ = 1, where $\mu_1$ is the translate of $\mu$ by $m$, $m \in \text{range}(T)$. However, it is clear from Theorem 2 that if $\mu$ is a zero-mean Gaussian measure and $T: H \rightarrow H$ is bounded and linear, then $\mu(\text{range}(T)) = 0$ if and only if $\mu(x:x+m \in \text{range}(T)) = 0$ for all $m \in H$. It would be interesting to know whether this holds for all subgroups in $B[H]$. If so, the method used to prove Theorem 1 shows that a similar result would hold for any subgroup in $B[B]$.

Part (c) of Theorem 2 is an extension of a result by Pitcher [4], who proved the following. Suppose $\mu$ is a zero-mean Gaussian measure on $B[H]$, with covariance operator $K$. Suppose $K = \text{TST}$, where $T$ and $S$ are linear, bounded, non-negative, self-adjoint, and Hilbert-Schmidt operators in $H$, with $T$ strictly positive. Then $\mu(\text{range}(T)) = 0$ or 1, and $\mu(\text{range}(T)) = 1$ if and only if $S$ is trace-class. His method of proof requires the assumptions that $K = \text{TST}$, that $\overline{\text{range}(T)} = \overline{\text{range}(S)} = H$, and that $S$ is self-adjoint and compact.

The following theorem generalizes (a) and (b) of Theorem 2 to two Banach spaces.
Theorem 3: Suppose $B_i$ ($i=1,2$) is a real separable Banach space, and that $T: B_1 \to B_2$ is a bounded linear operator. Suppose that either $B_1$ is reflexive, or else $T$ is one-to-one and $\text{range}(T^*) = B_1^*$. Let $\mu$ be a Gaussian measure on $B[B_2]$. Then $\text{range}(T) \in B[B_2]$, and $\mu[\text{range}(T)] = 0$ or 1. Moreover, $\mu[\text{range}(T)] = 1$ if and only if there exists a Gaussian measure $\nu$ on $B[B_1]$ such that $\mu$ is induced from $\nu$ by $T$.

**Proof:** First consider the case where $\text{range}(T^*) = B_1^*$ and $T$ is one-to-one. It is clear that we need only prove the existence of $\nu$ as defined in the theorem when $\mu[\text{range}(T)] = 1$. Since $T^{-1}$ exists, it is obvious that there exists a probability measure $\nu$ on $B[B_1]$ such that $\nu(A) = \mu(x: T^{-1}x \in A)$, and that $T$ induces $\mu$ from $\nu$. We show $\nu$ is Gaussian. Let $B_3$ denote the separable Banach space consisting of $\text{range}(T)$ and the $B_2$ norm. Define $T: B_1 \to B_3$ by $T_1x = Tx$. Let $\mu_1$ be the restriction of $\mu$ to $B[B_3]$; if $\nu_1 \in B_3^*$, then by the Hahn-Banach theorem $\nu_1$ has a norm-preserving extension to $B_2$; $\mu_1$ is thus a Gaussian measure. Since $\text{range}(T_1) = B_3$, $T_1^{-1}$ exists ([10], p. 44). Let $y$ be any element of $B_1^*$ belonging to $\text{range}(T^*)$; say $y = T^*\nu$. If $y$ is not the null element, it is easy to see that $y = T_1^*\nu_1$, $\nu_1$ the restriction of $\nu$ to $B_3$. Now $\nu(x: y(x) < k) = \mu_1(x): y(T^{-1}_1x) < k) = \mu_1(x: y_1(x) < k)$. Hence, $y(x)$ is Gaussian with respect to $\nu$, all $y \in \text{range}(T^*)$. The assumption that $\text{range}(T^*) = B_1^*$ thus implies that $\nu$ is a Gaussian measure.

For the case where $B_1$ is reflexive, with $T$ not necessarily one-to-one, suppose that $\mu[\text{range}(T)] = 1$. The proof of Lemma 7 (Appendix 2) shows that one can imbed $B_1$ as a dense Borel-measurable linear manifold in a separable Hilbert space $H_1$ in such a way that $B[B_1] \subset B[H_1]$, $B[B_1] = B_1 \cap B[H_1]$, and $T$ can be extended by continuity to a bounded linear operator $T_1: H_1 \to H_2$. It is clear that $\text{range}(T) = T_1[B_1] = T_1[B_1 \cap N_{T_1}]$. Define $Y: H_2 \to H_1$ by $Yx = \nu$ if $x = T_1^*\nu$ and $\nu \notin N_{T_1}$, while $Yx = 0$ if $x \notin \text{range}(T_1)$. Define
μ₀ on B[B₂] by μ₀(A) = μ(B₂ ∩ A). The procedure used to prove (b) of Theorem 2 shows that there exists a Gaussian measure ν on B[H₁] such that μ₀(A) = ν(x: T₁x ∈ A) for A in B[B₂]. ν(B₁) = ν(x: T₁x ∈ B₂) = μ₀(B₂) = 1. Let ν₀ be the restriction of ν to B[B₁]. If μ ∈ B₁ is dense in H₁, then μ(x) is a Gaussian random variable with respect to ν₀, since ν is Gaussian. Hence ν₀ is Gaussian provided H₁ is dense in B₁. Suppose there exists f in B₁** such that f(ν) = 0 for all ν in H₁. Since B₁ is dense in H₁ and B₁ is isometrically isomorphic to B₁**, B₁** is dense in H₁** and hence f must be the null element of B₁**. Thus H₁ is dense in B₁, so that ν₀ is Gaussian, with μ(A) = ν₀{x: Tν ∈ A} for A ∈ B[B₂]. The remainder of the proof is obvious.

Theorems 2 and 3 extend the following well-known result: If μ is a Gaussian measure on B[B₁], and T: B₁ → B₂ is bounded and linear, then the measure on B[B₂] induced from μ by T is also Gaussian. Thus, if B₁ is reflexive, and T⁻¹ exists, then Theorem 3 shows that for a Gaussian measure μ on B[B₂], μ[domain(T⁻¹)] = 0 or 1; if μ[domain(T⁻¹)] = 1, then the measure induced from μ by T⁻¹ is Gaussian. If T⁻¹ does not exist, we still have that μ[range(T)] = 1 if and only if there exists a Gaussian measure on B[B₁] from which μ is induced by T.
G. APPLICATIONS TO GAUSSIAN MEASURES AND GAUSSIAN STOCHASTIC PROCESSES

In this section we apply the preceding theorems to obtain a number of zero-one statements for Gaussian measures and Gaussian stochastic processes. Statements regarding analytical properties of sample functions are contained in Theorems 4, 7, and 8. Theorems 5 and 6 deal with the convergence of certain series, and with integrability properties of the sample functions of measurable Gaussian stochastic processes.

In addition to the results on Gaussian measures and processes presented in this section, Theorem 10 of Section 7 contains results on the convergence of sequences and series of Gaussian random variables.

We use $L^p_p$ or $L^p_p[T]$ to denote the set of equivalence classes of all real-valued Lebesgue-measurable functions $f$ on $T$ (an interval) such that $\int_T |f_t|^p dt < \infty$, together with the norm $||f||_p = [\int_T |f_t|^p dt]^{1/p}$. We do not distinguish between a function and its equivalence class.

$\ell^p_p$ denotes the set of all sequences of real numbers $\mathbf{x} = (x_1, x_2, \ldots)$ such that $\sum_n |x_n|^p < \infty$, together with the norm $||\mathbf{x}||_p = [\sum_n |x_n|^p]^{1/p}$. We use either $C$ or $C[T]$ to denote the space of real-valued functions defined and continuous on the compact interval $T$, together with the norm $||f|| = \sup_{t \in T} |f_t|$.
Theorem 4. Suppose $\mu$ is a Gaussian measure on the Borel $\sigma$-field of $C[T]$. The following sets belong to $B[C[T]]$ and have $\mu$-measure zero or one:

1. $AC^p[T] \equiv \{x: x$ is absolutely continuous with $L^p$ derivative, any fixed $p \in [1, \infty)\}$
2. $C^n[T] = \{x: x$ is $n$-times continuously differentiable on $T\}$.

Proof: (1) Let $Q_p$ denote the real separable Banach space with elements $(\alpha, x)$, $\alpha$ a real scalar, $x$ in $L_p[T]$, and with norm defined by $\| (\alpha, x) \| = |\alpha| + \| x \|_p$. Let $S: Q_p \to C[T]$ be defined by $S(\alpha, x)_t = \alpha + \int_a^t x(s) \, ds$, where $T = [a, b]$. $S$ is a one-to-one bounded linear map, so that by Lemma 6 range($S$) is a Borel set in $C[T]$. Noting that range($S$) is a linear manifold and that range($S$) = $AC^p[T]$, one sees that $\mu(AC^p[T]) = 0$ or $1$.

(2) $C^n[T]$ is a real separable Banach space under the norm $\| x \| = \sum_{i=0}^n \sup_{t \in T} |x^{(i)}_t|$. The natural injection of $C^n[T]$ into $C[T]$ is a bounded, linear, and one-to-one map. Thus, $C^n[T]$ is a Borel-measurable linear manifold in $C[T]$, and the result follows.

If $p > 1$ and $\mu(AC^p) = 1$, then Theorem 3 shows that the operation of differentiation induces from $\mu$ a Gaussian measure on $L_p$. For this, one notes that $S^{-1}f = (f(\alpha), f^{(1)})$ for $f \in AC^p$, $S$ as in the proof of Theorem 4. This induces a Gaussian measure on $B[Q_p]$, and projection into $L_p$ gives a Gaussian measure $\nu$ on $B[L_p]$, $\nu(A) = \mu\{x \in AC^p: x^{(1)} \in A\}$ for $A \in B[L_p]$.

The following theorems deal with various properties of the sample functions of Gaussian stochastic processes. In each case the process is assumed real-valued. We first establish a lemma essentially used by Pitcher [4].

Lemma 8: Suppose $(X_t)$, $t \in T$ (a compact interval), is a measurable Gaussian stochastic process on a probability space $(\Omega, \mathcal{F}, P)$. If almost all
sample paths of \((X_t)\) belong to \(L_p[T]\), \(1 \leq p < \infty\), then \((X_t)\) induces a Gaussian measure \(\mu\) on \(B[L_p]\), defined by \(\mu(A) = P(\omega: X(\omega) \in A)\), where \(X(\omega)\) is the sample path of \((X_t)\) evaluated at \(\omega\).

Proof: To see that \(X: \Omega \rightarrow L_p\) is \(\beta/B[L_p]\) measurable, one notes that by Fubini's theorem \(\{\omega: [\int_T |X_t(\omega)|^p dt]^{1/p} < \epsilon\}\) belongs to \(\beta\). Since sets of the form \(\{y: ||y||_p < \epsilon\}\), \(\epsilon > 0\), form a neighborhood base at zero for the norm topology in \(L_p\), \(X^{-1}[A] \in \beta\) for \(A \in B[L_p]\). Hence, \(X\) induces a probability measure \(\mu\) on \(B[L_p]\). By a result of Doob [11, pp.64-65], every bounded linear functional on \(L_p\) is Gaussian with respect to \(\mu\), so that \(\mu\) is a Gaussian measure.

The problems treated in Theorem 5 have a long history (see [1], and the references given there). The proof is remarkably simple, given Theorem 1.

Theorem 5: Suppose \((X_t)\), \(t \in T\) (a compact interval), is a measurable Gaussian stochastic process on a probability space \((\Omega, \beta, P)\), with covariance function \(K(t,s)\) and mean element \(m(t)\). Let \(X(\omega)\) denote the sample path of \((X_t)\), evaluated at \(\omega\). Then

1. \(P(\omega: X(\omega) \in L_1) = 0\) or 1, any fixed \(p \geq 1\).

2. Let \(\{e_n\}\) denote a set of functions continuous on \(T\), with \(\{f_n\}\) any set of functions belonging to \(L_2\). If \(P(\omega: X(\omega) \in L_2) = 1\), then

\[
P(\omega: \sum_{n=1}^{\infty} \int_T f_n(s) X(s) ds \text{ converges uniformly}) = 0 \text{ or } 1.
\]

Proof: As shown in [12], one can define a function \(g \in L_1\) such that \(gX(\omega) \in L_1\) a.e. \(dP(\omega)\); e.g., \(g(t) = 1\) if \(K(t,t)+m^2(t) \leq 1\), and \(g(t) = [K(t,t)+m^2(t)]^{-1}\) if \(K(t,t)+m^2(t) > 1\). Let \((Y_t) = (g_t X_t)\); \((Y_t)\) is a measurable Gaussian process with almost all sample paths in \(L_1\). Thus \(Y\) induces from \(P\) a Gaussian measure \(\mu_Y\) on \(B[L_1]\). Now let \(G: L_1 \rightarrow L_1\) be defined by \(Gv = gv\), \(g\) as above. \(G\) is bounded, linear, and one-to-one. Clearly, \(\{\omega: X(\omega) \in L_1\} = \{\omega: Y(\omega) \in \text{range}(G)\}\). Hence, \(\{\omega: X(\omega) \in L_1\} \in \beta\), and
\( P(\omega: X(\omega) \epsilon L^1_1) = \mu_X(\text{range}(G)) = 0 \) or \( 1 \). For \( p > 1 \), one uses the fact that \( L_1 \supset L_p \), and that the natural injection of \( L_p \) into \( L_1 \) is a bounded, linear, and one-to-one operator.

(2) The set of \( \omega \) for which the series converges uniformly is

\[
A = \bigcap_{N \geq 1} \bigcup_{M \geq 1} \bigcap_{m > n > M} \bigcap_{t \in S} \{ \omega: \left| \sum_{k=n+1}^m e_k(t) \int_T f_k(s) X_s(\omega) \, ds \right| < \frac{1}{N} \},
\]

where \( S \) is a countable dense subset of \( T \). But \( A = X^{-1}[D] \), where

\[
D = \bigcap_{N \geq 1} \bigcup_{M \geq 1} \bigcap_{m > n > M} \bigcap_{t \in S} \{ x: \left| \sum_{k=n+1}^m e_k(t) \int_T f_k(s) x_s \, ds \right| < \frac{1}{N} \}.
\]

\( D \) is obviously in \( B[L_2] \), since \( \{ x: \left| \sum_{k \in I \gamma_k \int_T f_k(s)x_s \, ds \right| < \frac{1}{M} \} \in B[L_2] \) for any finite index set \( I \) and scalars \( \{ \gamma_k \} \). \( D \) is also a linear manifold. Hence,

\[ P[A] = \mu_X[D] = 0 \) or \( 1 \), \( \mu_X \) the measure induced from \( P \) by \( X \). This completes the proof.

Part (1) of Theorem 5, for zero-mean \((X_t)\), has been previously announced by B. Rajput. Part (2) contains the orthogonal series (Karhunen-Loeve) representation of \((X_t)\) as a special case. It is shown in [13,14] that the orthogonal series converges uniformly with probability one when \((X_t)\) has continuous paths.

The following theorem generalizes Part (2) of Theorem 5, and gives the analogue of Part (1) for \( \ell_p \) spaces.

Theorem 6: Let \( \{ Z_n \}, n = 1,2,... \) be a family of jointly Gaussian random variables on a probability space \((\Omega, \beta, P)\). Let \( \{ e_n \} \) be a set of functions defined and continuous on the interval \( T \). Define, for \( 1 \leq p < \infty \),

\[
A = \{ \omega: \int_T^{\infty} Z_n(\omega)e_n(t) \, dt \text{ converges uniformly on } T \},
\]

\[
B = \{ \omega: \int_T^{\infty} Z_n(\omega)P e_n(t) \, dt \text{ converges absolutely and uniformly on } T \},
\]

\[ C_p = \{ \omega: \int_T^{\infty} |Z_n(\omega)|^p < \infty \}.
\]

Then \( A, B_p, \) and \( C_p \) each belongs to \( \beta \), each has probability one or zero, and \( P[C_p] = 1 \) if and only if \( \int_T^{\infty} \left| E(Z_n^2) \right|^{p/2} < \infty \).
Proof: Let \( g_n = n^{-1} \) if \( E Z_n^2(\omega) \leq 1 \), and \( g_n = n^{-1}[E Z_n^2(\omega)]^{-1} \) otherwise. Let \( Y_n = g_n Z_n; \) then \( Y(\omega) = (Y_1(\omega), Y_2(\omega), \ldots) \in \ell_2 \) a.e. \( dP(\omega). \) Let \( \Omega_1 = \{ \omega: Y(\omega) \in \ell_2 \}. \) \( Y: \Omega_1 \to \ell_2 \) induces a probability measure \( \mu_Y \) on \( B[\ell_2]; \) \( \mu_Y \) is Gaussian, since if \( \sum |x_k|^2 < \infty, \sum x_k Y_k \) is a Gaussian random variable as the a.e. limit of a sequence of Gaussian random variables. If \( \sum_n |x_n|^p < \infty, p \geq 1, \) then \( \sum_n n^{-2} |x_n|^2 \leq \sup_m |x_m|^2 \sum_n n^{-2} \leq 2\|x\|_p^2. \) Hence we define the bounded, linear, and one-to-one operator \( G: \ell_p \to \ell_2 \) by \( (Gx)_n = g_n x_n \) for \( x \in \ell_p. \) Define the continuous function \( e_n' \) by \( e_n'(t) = g_n^{-1} e_n(t), \) and define \( e_n'' \) by \( e_n''(t) = g_n^{-1} e_n(t). \) Note that

\[
A = Y^{-1}\{x \in \ell_2: \sum_1^n x_n e_n''(t) \text{ converges uniformly on } T\},
\]

\[
B_p = Y^{-1}\{x \in \ell_2: \sum_1^n |x_n|^p e_n'(t) \text{ converges absolutely and uniformly on } T\}, \quad \text{and}
\]

\[
C_p = Y^{-1}[\text{range}(G_p)].
\]

Using the method of proof for part (b) of Theorem 5, one sees easily that \( A \) is in \( \beta \) and \( P(A) = 0 \) or \( 1. \) Similarly, using the inequality 

\[
|a + b|^P \leq 2^P(|a|^P + |b|^P), \quad P(B_p) = 0 \text{ or } 1.
\]

It is clear that \( C_p \) belongs to \( \beta \) and that \( P(C_p) = 0 \) or \( 1. \) Finally, \( P(C_p) = 1 \) if and only if the paths of \( (Z_n) \) induce a Gaussian measure on \( B[\ell_p], \) the results of \([15, 16]\) show that this occurs if and only if \( \sum_n [EZ_n^2(\omega)]^{p/2} < \infty. \)

For the next result, we make the following definitions. \( C_{eq}[T] \) is the set of functions on \( T \) that are equal a.e. dt to an element of \( C[T]; \)

\( C^n_{eq}[T] \) and \( ACP_{eq}[T] \) are defined similarly.

Theorem 7: Suppose \( (X_t), t \in T \) (a compact interval), is a measurable Gaussian stochastic process on a complete probability space \( (\Omega, \beta, P). \) The following sets belong to \( \beta \) and have probability zero or one:

1. \( \{ \omega: X(\omega) \in C_{eq}[T] \} \)
2. \( \{ \omega: X(\omega) \in C^n_{eq}[T] \} \)
3. \( \{ \omega: X(\omega) \in ACP_{eq}[T] \} \).
Proof: It is clear that $C_{eq}[T]$, $C^n_{eq}[T]$ and $AC^p_{eq}[T]$ are subsets of $L_1[T]$. From Theorem 5, either almost all paths of $(X_t)$ belong to $L_1[T]$, or almost all do not belong to $L_1[T]$. If almost all paths lie outside $L_1$, then $X^{-1}[A] \in \beta$ for all subsets $A$ of $L_1$, since $(\Omega, \beta, P)$ is complete. Thus, to prove the theorem, it is sufficient to show that the sets indicated in (1)-(3) of the theorem are elements of $\beta$, and that each set has probability one or zero, when almost all paths of $(X_t)$ belong to $L_1[T]$.

Thus, suppose that $P(\omega: X(\omega) \in L_1) = 1$, and let $\mu_X$ be the Gaussian measure induced on $B[L_1]$ by $(X_t)$; $\mu_X(A) = P(\omega: X(\omega) \in A)$. Let $W$ be the map taking $x$ in $C$ into its equivalence class in $L_1$. Since

$$\int_T |x_t| dt \leq T \sup_{t \in T} |x_t|,$$

$W$ is bounded and linear; $W$ is also one-to-one. Thus range($W$) $\in B[L_1]$ by Lemma 6, and hence $\mu_X[\text{range}(W)] = 0$ or 1. Since range($W$) $= C_{eq}[T]$, this shows that $X^{-1}(C_{eq}) \in \beta$ and that $P[X^{-1}(C_{eq})] = 0$ or 1.

To prove the remainder of the theorem, we note that in the proof of Theorem 4 both $C^n[T]$ and $AC^p[T]$ were shown to be Borel linear manifolds in $C[T]$. Defining $W: C \to L_1$ as above, noting that $C^n_{eq} = W[C^n]$ and that $AC^p_{eq} = W[AC^p]$, the remainder of the proof is clear.

The preceding theorems contain various zero-one laws for Gaussian processes. The problem of determining necessary and sufficient conditions for the alternatives is usually more difficult. Some general results in this direction are given in Theorem 2. The following theorem is an application of Theorem 1 and Theorem 2.

**Theorem 8:** Suppose that $(X_t)$, $t \in (-\infty, \infty)$, is a measurable mean-square continuous Gaussian process on a complete probability space $(\Omega, \beta, P)$. Suppose that $(X_t)$ is stationary with rational spectral density function $f$ and with zero mean. Let
A \equiv \{ \omega: X(\omega) \text{ is absolutely continuous on } (-\infty, \infty), \text{ with derivative belonging to } L_2[T] \text{ for every compact interval } T \}

A_{eq} \equiv \{ \omega: X(\omega) \text{ is equal a.e. (Lebesgue) to an element of } A \}.

Then,

1. \( X^{-1}[A_{eq}] \in \beta \), and \( P(X^{-1}[A_{eq}]) = 0 \) or 1.

2. If \( (X_t) \) is separable, then
   \( X^{-1}[A] \in \beta \), and \( P(X^{-1}[A]) = 0 \) or 1.

3. \( P(X^{-1}[A_{eq}]) = 1 \) if and only if \( \int_{-\infty}^{\infty} \lambda^2 f(\lambda) d\lambda < \infty \). If \( (X_t) \) is separable, then this condition is also necessary and sufficient for \( P(X^{-1}[A]) = 1 \).

Proof: A function is absolutely continuous on \((-\infty, \infty)\) if and only if it is absolutely continuous on every compact interval. The fact that \( X^{-1}[A_{eq}] \in \beta \) and has probability zero or one thus follows from Theorem 7.

Let \( T \) be any fixed compact interval. Let \( R_T \) and \( R_{0T} \) be the integral operators in \( L_2[T] \) having kernels \( R(t,s) \) and \( R_0(t,s) \), defined by

\[
R(t,s) = \int_{-\infty}^{\infty} f(\lambda) \exp(i\lambda(t-s)) \, d\lambda,
\]

\[
R_0(t,s) = \int_{-\infty}^{\infty} (\lambda^2 + 1)^{-1} \exp(i\lambda(t-s)) \, d\lambda.
\]

Using a result of Hajek [17, Sec. 7], one can show that \( R_T = R_{0T}^\delta W R_{0T}^\delta \) for \( W \) trace-class if and only if \( \int_{-\infty}^{\infty} (\lambda^2 + 1) f(\lambda) d\lambda < \infty \). In [18] it is shown that the range of \( (R_{0T}^\delta) \) consists of all elements of \( L_2[T] \) that are equal a.e. dt to an absolutely continuous function with \( L_2[T] \) derivative. Hence, one concludes that \( P(X^{-1}[A_{eq}]) = 1 \) if and only if \( \int_{-\infty}^{\infty} \lambda^2 f(\lambda) d\lambda < \infty \).

To prove the statements regarding \( A \), we first note that \( X^{-1}[A] \subset X^{-1}[A_{eq}] \). Hence, if \( \int_{-\infty}^{\infty} \lambda^2 f(\lambda) d\lambda = \infty \), then \( X^{-1}[A] \in \beta \), since \( (\Omega, \beta, P) \) is complete, and \( P(X^{-1}[A]) = 0 \). To complete the proof, it is sufficient to show that \( \int_{-\infty}^{\infty} \lambda^2 f(\lambda) d\lambda < \infty \) and \( (X_t) \) separable imply that almost
all paths of \((X_t)\) are absolutely continuous. This has been proved by Doob [11, pp. 535-537].

As noted in the proof of the theorem, this result is an extension in one direction of a result of Doob, which states that \((X_t)\) has absolutely continuous paths if \(\int_\infty^\infty \lambda^2 dF(\lambda) < \infty\), F the spectral distribution function of \((X_t)\), and \((X_t)\) separable [11, pp. 535-537]. However, the result given here is restricted to the case where \((X_t)\) has a spectral density, and this spectral density must be rational. The requirement that \((X_t)\) be Gaussian is not assumed by Doob; the process need only be separable, measurable, mean-square continuous, and wide-sense stationary. Under these same assumptions, with the process also assumed to have a spectral density function, which is rational, the proof of Theorem 7 can be used to show that almost all sample path derivatives belong to \(L_2[T]\) for all compact intervals \(T\) whenever \(\int_\infty^\infty \lambda^2 f(\lambda) d\lambda < \infty\); i.e., normality is not required.

Theorem 8 is proved for zero-mean Gaussian processes. If the process has mean function \(m\), then application of Theorem 2 shows that the necessary and sufficient condition for \(A_{eq}\) (or A, if separability of \((X_t)\) is assumed) to have probability one is that \(\int_\infty^\infty \lambda^2 f(\lambda) d\lambda < \infty\) and \(m_T \in \text{range}(R_{OT}^k)\) for each compact interval \(T\), where \(m_T(t) = m(t), t \in T, m_T(t) = 0, t \notin T\). A sufficient condition for \(m_T\) to be in \(\text{range}(R_{OT}^k)\) is that \(\int_\infty^\infty |\hat{m_T}(\lambda)|^2 \lambda^2 d\lambda < \infty\) [19], where \(\hat{m_T}\) is the Fourier transform of \(m_T\).

We discuss an application to information theory. Suppose that \((S_t), (N_t), t \in T\) (compact interval), are measurable zero-mean Gaussian stochastic processes on a probability space \((\Omega, \beta, P)\). We assume that \((S_t)\) and \((N_t)\) are statistically independent, and that almost all sample paths belong to \(L_2[T]\) for each process. Let \(\mu_S, \mu_N\) and \(\mu_{S+N}\) denote the Gaussian measures induced on \(L_2[T]\) by \((S_t), (N_t)\) and \((S_t+N_t)\), respectively; e.g. \(\mu_S(A) = \ldots\)
$P(\omega: S(\omega) \epsilon A)$, $S(\omega)$ the sample path of $(S_t)$ evaluated at $\omega$. Let $\mu_{S,S+N}$ be the Gaussian measure induced on $B[L_2 \times L_2]$ by $(S_t, S_t+N_t)$. The average mutual information (AMI) of $(S_t)$ and $(S_t+N_t)$ is defined as [20]

$$\text{AMI}(S,S+N) = \int_{L_2 \times L_2} \log \frac{d\mu_{S,S+N}}{d\mu_{S+N}}(x,y) \ d\mu_{S,S+N}(x,y)$$

if $\mu_{S,S+N} \sim \mu_S \otimes \mu_{S+N}$, and equal to $+\infty$ otherwise. This quantity is of much interest in many Communication Theory problems, where it represents the average information about the "signal" $(S_t)$, obtained by observing "signal plus noise" $(S_t+N_t)$. Hajek [17] has shown that for the case considered here, $\text{AMI}(S,S+N) < \infty \iff K_S = K_N^G K_N^{1/2}$ for a trace-class operator $G$, where $K_S$ and $K_N$ denote the covariance operators of $(S_t)$ and $(N_t)$. From this and Theorem 2 above, one sees that $\text{AMI}(S,S+N) < \infty$ if and only if almost all sample paths of $(S_t)$ belong to the range of $K_N^{1/2}$. This result has been previously obtained by Pitcher [4].

7. Zero-One Laws for Non-Gaussian Measures

Our results so far involve only Gaussian measures. However, if $\mu$ is a Gaussian measure on $B[\mathbb{R}]$, and $\mu'$ is any probability measure on $B[\mathbb{R}]$ such that $\mu$ and $\mu'$ are mutually absolutely continuous, then obviously $\mu'(G) = 0$ or 1 for any subgroup $G$ belonging to $B[\mu]$. In this section, we consider
a class of non-Gaussian measures for which this holds. Our results are based on the following lemma. The notation $\mu_1 \ll \mu_2$ means that $\mu_1$ is absolutely continuous with respect to $\mu_2$.

**Lemma 9** [21]: Suppose $\mu_X$ and $\mu_Y$ are two probability measures on $B(B)$. Let $\mu_X \otimes \mu_Y$ denote product measure on $(B \times B, B(B) \times B(B))$. The set \{(x,y): x+y \in A\} belongs to $B(B) \times B(B)$ for all $A \in B(B)$. Define a probability measure $\mu_{X+Y}$ on $B(B)$ by $\mu_{X+Y}(A) = \mu_X \otimes \mu_Y \{(x,y): x+y \in A\}$, and for each fixed $v$ in $B$, a measure $\mu_{X+v}$ by $\mu_{X+v}(A) = \mu_X \{x: x+v \in A\}$. We then have the following results:

1. If $\mu_{X+Y} \ll \mu_X$ a.e. $d\mu_Y(y)$, then $\mu_{X+Y} \ll \mu_X$.
2. If $\mu_X \ll \mu_{X+Y}$ a.e. $d\mu_Y(y)$, then $\mu_X \ll \mu_{X+Y}$.
3. If $\mu_X \sim \mu_{X+Y}$ a.e. $d\mu_Y(y)$, then $\mu_X \sim \mu_{X+Y}$.

We thus consider Gaussian measures $\mu_X$ and the class of measures $\mu_{X+Y}$ defined as in the lemma, where $\mu_Y$ need not be Gaussian. An obvious consequence of this lemma is the following. If $B$ is a Hilbert space, so that $\mu_X$ has a covariance operator $K$, then $\mu_{X+Y} \sim \mu_X$ if $\mu_Y[\text{range}(K^2)] = 1$. For the case where $B$ is not a Hilbert space, one can use the methods of Kuelbs [8] to imbed $B$ as a dense measurable linear manifold in a separable Hilbert space $H$, and work with the covariance operator of the extension of $\mu_X$. There are thus obvious extensions of the results of the preceding section. Two such extensions are given below.

**Theorem 9**: Suppose \{\(Z_n\), \(n = 1,2,\ldots\) and \{\(V_n\), \(n = 1,2,\ldots\) are two families of real random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. Suppose that \{\(Z_n\) is Gaussian, and that \{\(Z_n\) is independent of \{\(V_n\). Let $K(n,m)$ denote the covariance of $Z_n$ and $Z_m$, and let $R(n,m)$ denote the correlation of $V_n$ and $V_m$. Let \{\(e_n\) be any set of real-valued functions defined and continuous on the interval $T$. Define
\[ A' = \{ \omega: \sum_{n=1}^{N} (Z_n(\omega) + V_n(\omega)) \, e_n(t) \text{ converges uniformly on } T \}, \]
\[ B'_p = \{ \omega: \sum_{n=1}^{N} (Z_n(\omega) + V_n(\omega))^p \, e_n(t) \text{ converges absolutely and uniformly on } T \}, \]
\[ C'_p = \{ \omega: \sum_{n=1}^{\infty} |Z_n(\omega) + V_n(\omega)|^p < \infty \}, \]
where \( p \) is a real scalar, \( p \geq 1 \).

Let \( g = (g_1, g_2, \ldots) \) be a sequence of real numbers such that \( g_i > 0 \), all \( i \), and \( \sum_{i=1}^{\infty} g_i^2 \mathbb{E}[Z_1^2(\omega)] < \infty \). Suppose that one of the following conditions is satisfied:

1. \( \sum_{i,j} g_i g_j K(i,j) < \infty \) for a fixed random variable \( k \), all \( x \in \ell_2 \), a.e. \( dP(\omega) \);

2. For some matrix operator \( S \) in \( \ell_2 \) such that \( \sum_{i} S_{ii} < \infty \),
   \[ R' = K_{1/2} S K_{1/2} \]
   where the matrix operators \( R' \) and \( K_1 \) are defined by \( R'(i,j) = g_i g_j R(i,j) \) and \( K_1(i,j) = g_i g_j K(i,j) \).

3. (a) \( \sum_{i \neq j} \frac{K(i,j)^{-1}}{K(i,i)K(j,j)} < 1 \) (summation over \( i, j \) such that \( K_{ii} \neq 0 \), \( K_{jj} \neq 0 \) and \( R(i,i) \leq kK(i,i) \) for all \( i \) and some finite scalar \( k \),
   \( \sum_{i} V_i^2(\omega) < \infty \) a.e. \( dP(\omega) \).

Then, \( A', B'_p, \) and \( C'_p \) all belong to \( \beta \), each has probability one or zero, and \( P(C'_p) = 1 \) if and only if \( \sum_{i} \mathbb{E}[Z_i^2(\omega)]^{p/2} < \infty \).

**Proof:** Let \( (g_i) \) be defined as in the theorem, and set %\( Y_n(\omega) = g_n Z_n(\omega) \). Then \( Y(\omega) = (Y_1(\omega), Y_2(\omega), \ldots) \) is in \( \ell_2 \) a.e. \( dP(\omega) \), and hence induces from \( P \) a Gaussian measure \( \mu_Y \) on \( B[\ell_2] \). \( \mu_Y \) has covariance operator \( K_Y \), \( K_Y(i,j) = g_i g_j K(i,j) \). Consider condition (1) of the theorem. This condition implies that \( V'(\omega) = (g_1 V_1(\omega), g_2 V_2(\omega), \ldots) \) belongs to \( \ell_2 \) a.e. \( dP(\omega) \), and thus \( V' \) induces from \( P \) a probability measure \( \mu' \) on \( B[\ell_2] \).

Moreover, condition (1) is a necessary and sufficient condition that \( V'(\omega) \in \text{range}(K_Y^{1/2}) \) for almost all \( \omega \) \([18]\). Let \( \mu_1 \) be the probability measure induced on \( B[\ell_2] \) by \( Y + V' \). Since \( \mu'[\text{range}(K_Y^{1/2})] = 1 \), \( \mu_1 \sim \mu_Y \), by Lemma 9.
The assertions of the theorem regarding $A', B', \text{ and } C'_p$ now follow from Theorem 6. For example, defining $e_n'''(t) = \frac{1}{n} e_n(t)$,

$$P(A') = \mu_y \left\{ x \in \ell_2 : \sum_{n=1}^{N} x_n e_n'''(t) \right\} \text{ converges uniformly on } T$$

$$= \mu_y \left\{ x \in \ell_2 : \sum_{n=1}^{N} x_n e_n(t) \right\} \text{ converges uniformly on } T,$$

because the latter quantity is 0 or 1, by Theorem 6, and $\mu_y \sim \mu_Y$.

Condition (2) implies condition (1) [18].

Condition (3a) of the theorem is equivalent to $\sum_{i \neq j} \frac{K_{y^2(i,j)}}{K_{Y}(i,i)K_{Y}(j,j)} < 1$ and $g_{i}^{2}R(i,i) \leq k K_{Y}(i,i)$, all $i$, some finite $k$. The assumption that $g_{i}^{2}R(i,i) \leq k K_{Y}(i,i)$ implies that $V'(\omega) \in \ell_2$ a.e. $dP(\omega)$, $V'_1(\omega) = g_1 V_1(\omega)$.

Kuelbs has shown [8] that the first part of (3a) implies that $\mu_y$ and $\mu'_y$ are mutually absolutely continuous, where $\mu'_y$ is the Gaussian measure on $B(\ell_2)$ having the same mean element as $\mu_y$, but with diagonal covariance matrix $Q$, $Q_{ii} = K_{y}(i,i)$. A necessary condition for mutual absolute continuity of $\mu'_y$ and $\mu_y$ is that $Q_{y}^{2}$ and $K_{y}^{2}$ have the same range space [21].

Note that $\sum_{i} \frac{V_{i}^{2}(\omega)}{K(1,1)} = \sum_{i} \frac{[V_1'(\omega)]^2}{K_1(1,1)}$. We now show that if this sum is finite for almost all $\omega$, and if condition (3a) is satisfied, then $V'(\omega)$ belongs to $\text{range}(Q_{y}^{2})$ a.e. $dP(\omega)$. First, $V'(\omega)$ belongs almost surely to the closure of the range of $R'$, $R'_{ij} = g_1 g_j R(i,j)$. To see this, we note that if $\{h_1\}$ are the complete orthonormal (in $\ell_2$) eigenvectors of $R'$, and $h_{1}'$ are those eigenvectors corresponding to the zero eigenvalues of $R'$, then

$$\int_{\Omega} \sum_{k,j} h_{k}' h_{j}' V'(\omega) V_j'(\omega) dP(\omega) = 0 \text{ for each } h_{1}''. \text{ Hence,}$$

$$P\{\omega: V'(\omega) h_{1}' = 1, \text{ each } h_{1}'', \text{ so that } P(\omega: V'(\omega) \text{ is not orthogonal to } h_{1}'') = 0, \text{ all } h_{1}''. \text{ Noting that } \{\omega: V'(\omega) \overline{\text{range}(R')}\} = \bigcup_{h_{1}'} \{\omega: V'(\omega) \text{ is not orthogonal to } h_{1}'\}, \text{ we have that } P(\omega: V'(\omega) \overline{\text{range}(R')}\} = 1. \text{ Next we show that}$$

$$\overline{\text{range}(Q) \supset \overline{\text{range}(R')}}. \text{ Suppose } Q_{11} = 0. \text{ Then the element } e_{1} e_{2}, e_{1} = \delta_{11}' \text{ is in the null space of } Q, \text{ and } \sum_{k,j} e_{j} e_{k} R'(j,k) = R'(i,i) = 0, \text{ since}$$
$R'(i,i) \leq kK_Y(i,i)$, all $i$, and $Q_{ii} = K_Y(i,i)$. Hence $e_i$ is in the null space of $R'$. The elements $e_i$ such that $e_i$ is in $N_Q$ span $N_{Q'}$, so that the null space of $Q'$ is contained in the null space of $R'$. This implies that $\text{range}(R') \subseteq \text{range}(Q')$. Hence, $V'(\omega)\text{range}(Q') = \text{range}(Q')$ for almost all $\omega$. We now note that if $x \in \ell_2$ and $x \in \text{range}(Q)$, then $x \in \text{range}(Q')$ if and only if $\sum_i x_i^2 < \infty$. Hence, the assumption that $R(i,i) \leq kK_Y(i,i)$, all $i$, and the assumption that $\sum_i [V'_i(\omega)]^2 < \infty$ a.e. $dP(\omega)$ imply that $V'(\omega) \in \text{range}(Q')$ a.e. $dP(\omega)$. The assumption that $\sum_{i \neq j} \frac{K^2(i,j)}{K(i,i)K(j,j)} < 1$ implies that $\text{range}(Q') = \text{range}(K_Y^\frac{1}{2})$. Thus, the assumptions of (3) imply that $V'(\omega) \in \text{range}(K_Y^\frac{1}{2})$ a.e. $dP(\omega)$. Hence, assumption (3) implies assumption (1), and the proof is completed.

Remark: It is clear that $\sum_i \frac{V'_1^2(\omega)}{K(1,1)} < \infty$ a.e. $dP(\omega)$ if $\sum_i \frac{R(i,i)}{K(i,i)} < \infty$.

Corollary: Suppose $(X_t)$ and $(Y_t)$, $t \in T$ (a bounded interval), are measurable stochastic processes on the complete probability space $(\Omega, \mathcal{F}, P)$. Suppose that $(X_t)$ and $(Y_t)$ are independent, that both have almost all sample paths in $L_2[T]$, that $(X_t)$ is Gaussian with covariance operator $K$, and that $(Y_t)$ has covariance operator $R$ and mean element $m$. Let $\{e_n\}$ be a set of complete orthonormal eigenfunctions for $K$, and let $\{f_n\}$ be any set of functions defined and continuous on $T$. Let $\{\gamma_n\}$ and $\{\lambda_n\}$ be any two sets of real scalars, and define, for any $p \in [1, \infty)$,

\[ A'' = \{\omega: \sum_1^N \lambda_n [\langle X(\omega) + Y(\omega), e_n \rangle + \gamma_n] f_n(t) \text{ converges uniformly on } T\}, \]

\[ B''_p = \{\omega: \sum_1^N \lambda_n^p [\langle X(\omega) + Y(\omega), e_n \rangle + \gamma_n] f_n(t) \text{ converges absolutely and uniformly on } T\}, \]

\[ C''_p = \{\omega: \sum_{n=1}^\infty |\lambda_n|^p [\langle X(\omega) + Y(\omega), e_n \rangle + \gamma_n]^p < \infty\}, \]

where \langle \cdot, \cdot \rangle denotes the $L_2$ inner product. If $\langle Re_n, e_n \rangle + \langle m, e_n \rangle^2 \leq k\langle K e_n, e_n \rangle$ for all $n$ and some finite scalar $k$, and if $\sum_n [\langle Y(\omega), e_n \rangle^2 < \infty$

a.e. $dP(\omega)$, then $A''$, $B''$ and $C''$ each belongs to $\mathcal{F}$, each has probability
one or zero, and \( \Pr(C^p = 1) \) if and only if

\[
\sum_n |\lambda_n| \Pr(E(\langle X(\omega) + Y(\omega), e_n \rangle + \gamma_n)^2)^{p/2} < \infty.
\]

**Proof:** We note that \( \langle K_n e_n, e_n \rangle = 0 \) if \( n \neq m \); condition (3) of Theorem 7 is thus satisfied for the random variables \( Z_n = \lambda_n \langle X, e_n \rangle + \lambda_n \gamma_n, \quad V_n = \lambda_n \langle Y, e_n \rangle \).

Sufficient conditions for \( \sum_n \frac{\langle Y(\omega), e_n \rangle^2}{\langle K_n e_n, e_n \rangle} < \infty \) a.e. \( dP \) and \( \langle R_n e_n, e_n \rangle \leq k \langle K_n e_n, e_n \rangle \), all \( n \) (\( k \) finite), are given in [21], where a number of examples are also presented. For instance, if \( T = [0, b], \quad b < \infty, \) and \( K \) is the integral operator with kernel either \( \exp(-\alpha |t-s|), \quad \alpha > 0, \) or \( b-|t-s|, \) then \( Y(\omega) \) belongs to \( \text{range}(K^2) \) if and only if \( Y(\omega) \) is equal a.e. (Lebesgue) on \( T \) to an absolutely continuous function with \( L_2(T) \) derivative. We also note that the corollary has straightforward extensions to processes whose sample paths belong (a.s.) to \( L_p \) for any \( p \in [1, \infty) \).

The following theorem gives additional results on the convergence of sequences and sums of random variables.

**Theorem 10.** Let \( \{Z_n\}, \ n = 1, 2, \ldots, \) and \( \{V_n\}, \ n = 1, 2, \ldots, \) be two families or real random variables on a probability space \( (\Omega, \mathcal{F}, \Pr) \). Suppose that \( \{Z_n\} \) is Gaussian, and that \( \{Z_n\} \) is independent of \( \{V_n\} \). Let \( \{g_n\} \) be a set of real scalars such that \( \sum_n g_n^2 E[Z_n^2] < \infty, \sum_n g_n^2 < \infty, \) and \( g_i > 0 \) for \( i = 1, 2, \ldots \)

1. The following sets have probability zero or one:

\[
A_1 = \{\omega: Z_n(\omega) \text{ converges to some real scalar}\}
\]

\[
A_2 = \{\omega: Z_n(\omega) \text{ converges to } k\}, \text{ where } k \text{ is a fixed real number.}
\]

2. Suppose that any of the conditions (1)-(3) of Theorem 9 is satisfied for \( \{g_n\} \) defined as above. Then the following sets have probability zero or one:
\[ B_1 = \{ \omega : Z_n(\omega) + V_n(\omega) \text{ converges} \} , \]
\[ B_2 = \{ \omega : Z_n(\omega) + V_n(\omega) \text{ converges to } k \} . \]
Moreover, \( P(B_1) = 1 \) if and only if \( P(A_1) = 1 \), for \( i = 1, 2 \).

Proof: Define \( Y(\omega) \) by \( Y_1(\omega) = g_1 Z_1(\omega) \). Let \( \mathcal{C} \) denote the space of all convergent sequences of real numbers \( x = (x_1, x_2, \ldots) \). \( \mathcal{C} \) is a separable Banach space under the norm \( ||x|| = \sup_n |x_n| \) ([22], p. 68). Suppose \( x \in \mathcal{C} \). Then \( \sum_1^n (g_1 x_1)^2 \leq (\sum_1^n g_1^2) |x|^2 \). Thus we can define a bounded, linear, and one-to-one operator \( G : \mathcal{C} \to \ell_2 \) by \( (Gx)_i = g_1 x_1 \). We see that \( \{ \omega : Z(\omega) \in \mathcal{C} \} = \{ \omega : Y(\omega) \in \text{range}(G) \} \). Now we note that \( Y \) induces from \( \mu \) a Gaussian measure \( \mu_1 = Y^{-1}[\text{range}(G)] \), and \( P(A_1) = \mu(\text{range}(G)) \), proving the assertion for \( A_1 \).

For the set \( A_2 \), we consider the space \( \mathcal{C}_0 \) of real sequences that are convergent to zero. We note that \( \omega \in A_2 \) if and only if \( Z_n(\omega) - k \to 0 \).
\( \mathcal{C}_0 \) is a separable Banach space under the norm \( ||x|| = \sup_n |x_n| \) ([22], p. 68). Defining the operator \( G : \mathcal{C}_0 \to \ell_2 \) by \( (Gx)_i = g_1 x_1 \), one sees that \( G \) is bounded, linear, and one-to-one. Let \( Y(\omega) \) be defined by \( Y_1(\omega) = g_1 Z_1(\omega) - k \). \( Y(\omega) \) is in \( \ell_2 \) a.e. \( dP(\omega) \), thus induces a Gaussian measure on \( B[\ell_2] \). Moreover, \( Z_n(\omega) - k \to 0 \) if and only if \( Y(\omega) \in \text{range}(G) \). The remainder of the proof is clear.

We now consider \( B_1 \). Define \( Y(\omega) \) by \( Y_1(\omega) = g_1 Z_1(\omega) \) and \( Q(\omega) \) by \( Q_1(\omega) = g_1[Z_1(\omega) + V_1(\omega)] \). Under the assumption that one of the conditions (1) - (3) of Theorem 9 is satisfied, both \( Y(\omega) \) and \( Q(\omega) \) belong to \( \ell_2 \) for almost all \( \omega \), and thus induce probability measures \( \mu_Y \) and \( \mu_Q \) on \( B[\ell_2] \). The proof of Theorem 9 also shows that \( \mu_Y \sim \mu_Q \). The assertion on \( B_1 \) is an obvious consequence of the previous result for \( A_1 \).
The assertions on $B_2$ follow in the same way.

As a corollary of this theorem, we note that if $\{Z_n\}, n = 1, 2, \ldots$ is a family of Gaussian random variables, then $n^{-1} \sum_{1}^{n} [Z_1(\omega) - EZ_1] \to 0$ with probability one or zero. Similarly, $\alpha_n \sum_{1}^{n} Z_i(\omega)$ converges with probability one or zero, and $\alpha_n \sum_{1}^{n} Z_i(\omega) \to k$ with probability one or zero, for any set of scalars $\{\alpha_n\}$.

ACKNOWLEDGEMENT

The author thanks S. Cambanis for helpful discussions on Theorem 5.
In this appendix, we supply the missing steps in the proof of Lemma 2, as outlined in the proof of Lemma 3 of [1]. The notation is as in the proof of Lemma 2 above; we suppose that $g \in L_2[\mu]$, and that

$$I^N_\alpha = \int g(x) \exp\{\sum_1^N \alpha_i \langle x, e_i \rangle\} \, d\mu(x) = 0$$

for any $N$ and scalars $(\alpha_1, \ldots, \alpha_N)$.

Now

$$I^N_\alpha = \int_H E[g(x) \exp\{\sum_1^N \alpha_i \langle x, e_i \rangle\} \mid B[H^N]] \, d\mu(x)$$

$$= \int_H \exp\{\sum_1^N \alpha_i \langle x, e_i \rangle\} \, E[g(x) \mid B[H^N]] \, d\mu(x),$$

from the definition of $B[H^N]$. The conditional expectation $E[g(x) \mid B[H^N]]$ can be written as $E[g(x) \mid B[H^N]] = h[\langle x, e_1 \rangle, \ldots, \langle x, e_N \rangle] = h[q_N(x)]$, where $q_N: x \to (\langle x, e_1 \rangle, \ldots, \langle x, e_N \rangle)$ and $h$ is $B[R^N]/B[R]$ measurable, since $B[H^N]$ is generated by sets of the form $\{x: q_N(x) \in A\}$, $A \in B[R^N]$.

Since $q_N$ is $B[H]/B[R^N]$ measurable, the usual change of variable formula gives

$$I^N_\alpha = \int_H h(q_N(x)) \exp\{\sum_1^N \alpha_i \langle x, e_i \rangle\} \, d\mu(x)$$

$$= \int_{R^N} h(r_1, \ldots, r_N) \exp\{\sum_1^N \alpha_i r_i\} \, d\mathcal{P}(r_1, \ldots, r_N)$$

where $\mathcal{P}$ is the Gaussian measure on $B[R^N]$ having covariance matrix $K^1$, $K^1_{ij} = \langle Ke_i, e_j \rangle = \delta_{ij} \langle Ke_j, e_j \rangle$ and mean vector $m^1$, $m^1_i = \langle m, e_i \rangle$. We now have a family of probability measures $\{Q_\alpha\}$ on $B[R^N]$, defined by

$$dQ_\alpha(r_1, \ldots, r_N) = C(\alpha) \exp\{\sum_1^N \alpha_i r_i\} \, d\mathcal{P}(r_1, \ldots, r_N)$$

where $\alpha = (\alpha_1, \ldots, \alpha_N)$ is any element of $R^N$, and $C(\alpha)$ is a normalizing factor. By Theorem 1, p. 132 of [23], the family $\{Q_\alpha\}$ is complete; i.e.,

$$\int_{R^N} f(x) dQ_\alpha(x) = 0,$$

all $\alpha$ in $R^N$, implies $f(x) = 0$ a.e. $dQ_\alpha(x)$, all $\alpha$ in $R^N$.

Thus, $h(r_1, \ldots, r_N) = 0$ a.e. $d\mathcal{P}(r_1, \ldots, r_N)$. Since $\mathcal{P}(\{r_1, \ldots, r_N: h(r_1, \ldots, r_N) = 0\})$
\[ \mu\{x: \text{h}[q_N(x)]=0\} = \mu\{x: \text{h}[\langle x,e_1\rangle, \ldots, \langle x,e_N\rangle]\} = 0, \] we see that
\[ \text{h}[^{\langle x,e_1\rangle, \ldots, \langle x,e_N\rangle}] = 0 \text{ a.e. } \mu(x), \text{ or, } \mathbb{E}[g(x)|\mathcal{B}[H^N]] = 0 \text{ a.e. } \mu(x). \]
By the martingale convergence theorem ([7], pp. 84-85), \( \lim_N \mathbb{E}[g(x)|\mathcal{B}[H^N]] \) exists and is equal to \( \mathbb{E}[g(x)|\mathcal{B}[H]] \), since \( \mathcal{B}[H] \) is the smallest \( \sigma \)-field containing \( \mathcal{B}[H^N] \) for all \( N \). Thus \( \mathbb{E}[g(x)|\mathcal{B}[H]] = 0 \text{ a.e. } \mu(x), \)
\( \mathbb{E}[g(x)|\mathcal{B}_\mu[H]] = 0 \text{ a.e. } \mu(x), \) and thus \( g(x) = 0 \text{ a.e. } \mu(x). \)
Let $B_3$ be the separable Banach space consisting of range$(T)$ and the $B_2$ norm. Define $T_1 : B_1 \rightarrow B_3$ by $T_1 x = T x$. Since range$(T)$ is dense in $B_3$, $T_1 \cdot (\cdot )$ exists ([10], p.44). Let $\{x_n\}$, $n = 1, 2, \ldots$ be a dense subset of $B_1$. If there exists $u$ in $B_3^*$ such that $u(T_1 x_n) = 0$ for all $n$, then $[T_1^*(\cdot) x_n] = 0$ for all $n$; since $\{x_n\}$ is dense in $B_1$ and $T_1 \cdot (\cdot )$ exists, this implies that $u$ is the null element of $B_3^*$. Hence, $\{T x_n\}$ is dense in $B_3$.

For each $x_n$, define an element $F_n$ in $B_3^*$ as follows. If $||T x_n||_2 = 0$, $F_n$ is the null element. If $||T x_n||_2 \neq 0$, pick $F_n$ so that $||F_n|| = 1$ and $F_n(T x_n) = ||T x_n||_2$. Let $\{\alpha_n\}$ be a set of real scalars such that $\alpha_n = 0$ if $||T x_n||_2 = 0$, while otherwise $\alpha_n > 0$, and $\sum \alpha_n = 1$. Define a norm $||\cdot||_3$ on $B_3$ by $||u||_3^2 = \sum \alpha_n F_n^2(u)$. This norm is obviously weaker than the $||\cdot||_2$ norm, since $||T u||_3^2 \leq \sup_n F_n^2(T u) \leq ||T u||_2^2$. Let $H_3$ denote the completion of $B_3$ in the $||\cdot||_3$ norm. Kuelbs has shown [8] that $B[B_3] \subseteq B[H_3]$, so that $B[B_3] = B_3 \cap B[H_3]$.

Consider $T_1^* F_n$. $||T_1^* F_n|| \leq ||T||$ since $||F_n|| = 1$. Moreover, $||T x_n||_2 = ||T_1^* F_n(x_n)|| \leq ||T_1^* F_n|| \cdot ||x_n||_1$, so that $||T|| \leq ||T_1^* F_n||$. Hence $||T_1^* F_n|| = ||T||$, each $F_n$. Since $B_1$ is reflexive, there exists an element $y_n$ in $B_1$ such that $||y_n||_1 = 1$ and $T_1^* F_n(y_n) = ||T||$ ([23], p.103). Define $G_n = \frac{1}{||T||} T_1^* F_n$; $||G_n|| = 1$ and $G_n(y_n) = 1$. Choose $L_n$ in $B_1^*$, $n = 1, 2, \ldots$ such that $||L_n|| = 1$ and $L_n(x_n) = ||x_n||$. Let $\{\beta_n\}$, $n = 1, 2, \ldots$ be a set of strictly positive real scalars such that $\sum \beta_n = \frac{1}{2}$. Define an inner product $\langle \cdot, \cdot \rangle_1$ on $B_1$ by $\langle u, v \rangle_1 = \frac{1}{2} \sum \alpha_n G_n(u) G_n(v) + \sum \beta_n L_n(u) L_n(v)$. Let $H_1$ be the completion of $B_1$ under the norm obtained from this inner product. Using the results of [8], we conclude that $B[B_1] \subseteq B[H_1]$, so that $B[B_1] = B_1 \cap B[H_1]$. 


Consider $T_1$ as a map from $H_1$ into $H_3$. For $u$ in $B_1$, one has $||T_1u||_3^2 = \sum_n \alpha_n F_n^2 (T_1u) \leq 2||T||^2 \langle u, u \rangle_1$. Thus $T_1$ is a bounded linear operator from $B_1 (\subset H_1)$ into $H_3$, and can be extended by continuity to a bounded linear operator on $H_1$. Denote this extension as $T_2$.

We show that $T_2$ maps Borel sets in $H_1$ into Borel sets in $H_3$. Note that range($T_2$) is in $B[H_3]$, since $T_2[H_1] = T_2[N_{T_2}^\perp]$ and $T_2$ is a one-to-one bounded linear map of the separable Hilbert space $N_{T_2}^\perp$ (with the $H_1$ inner product) into $H_3$. Lemma 6 thus implies that $T_2[N_{T_2}^\perp]$ is in $B[H_3]$. Now let $A$ be any Borel set in $H_1$. We have $T_2[A] = T_2[A \cap N_{T_2}^\perp]$. Since $A \cap N_{T_2}^\perp$ is in $B[N_{T_2}^\perp]$, $T_2[A \cap N_{T_2}^\perp]$ must belong to $B[H_3]$.

REFERENCES


FOOTNOTES

AMS Subject Classifications:  Primary 28A40, 60G15, 60G17;
Secondary 60G30.

Key Phrases:  Gaussian measures, Gaussian stochastic processes,
Zero-one laws

1. This research was supported in part by the Air Force Office of Scientific Research under Contract AFOSR-68-1415 and by the Office of Naval Research under Contract N00014-67-A-0321-0006 (NRO42-69).