ESTIMATORS BASED ON A CLOSENESS CRITERION
OVER A RESTRICTED PARAMETER SPACE
FOR THE GENERAL LINEAR MODEL OF FULL RANK

BY

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### TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>LIST OF TABLES</th>
<th>Page vi</th>
</tr>
</thead>
<tbody>
<tr>
<td>LIST OF FIGURES</td>
<td>Page viii</td>
</tr>
<tr>
<td>1. INTRODUCTION</td>
<td>Page 1</td>
</tr>
<tr>
<td>1.1 Motivation</td>
<td>Page 1</td>
</tr>
<tr>
<td>1.2 First Procedure</td>
<td>Page 2</td>
</tr>
<tr>
<td>1.3 Second Procedure</td>
<td>Page 6</td>
</tr>
<tr>
<td>1.4 Full Model</td>
<td>Page 7</td>
</tr>
<tr>
<td>1.5 Development</td>
<td>Page 7</td>
</tr>
<tr>
<td>2. ESTIMATOR FOR THE SINGLE $\beta$, FIXED $\sigma^2$ CASE</td>
<td>Page 9</td>
</tr>
<tr>
<td>2.1 General Structure</td>
<td>Page 9</td>
</tr>
<tr>
<td>2.2 First Maxi-min Solution</td>
<td>Page 10</td>
</tr>
<tr>
<td>3. ESTIMATOR FOR THE SINGLE $\beta$, $\sigma^2 \varepsilon [\sigma_0^2, \sigma_1^2]$ CASE</td>
<td>Page 16</td>
</tr>
<tr>
<td>3.1 General Structure</td>
<td>Page 16</td>
</tr>
<tr>
<td>3.2 Minimization of $\phi$ with Respect to $\beta$ and $\sigma^2$</td>
<td>Page 16</td>
</tr>
<tr>
<td>3.3 Second Maxi-min Solution</td>
<td>Page 18</td>
</tr>
<tr>
<td>4. GENERALIZATION TO THE CASE OF $p \beta$'S</td>
<td>Page 25</td>
</tr>
<tr>
<td>4.1 General Structure</td>
<td>Page 25</td>
</tr>
<tr>
<td>4.2 Extension: Fixed $\sigma^2$ Case</td>
<td>Page 27</td>
</tr>
<tr>
<td>4.3 Extension: $\sigma^2 \varepsilon [\sigma_0^2, \sigma_1^2]$ Case</td>
<td>Page 29</td>
</tr>
<tr>
<td>5. AN ALTERNATE ESTIMATOR</td>
<td>Page 31</td>
</tr>
<tr>
<td>5.1 General Structure</td>
<td>Page 31</td>
</tr>
<tr>
<td>5.2 Form of the Estimator: $K\beta$</td>
<td>Page 31</td>
</tr>
<tr>
<td>5.3 Properties of $T(K, \sigma^2)$ for $K$ Fixed</td>
<td>Page 33</td>
</tr>
<tr>
<td>5.4 Solution for $K$</td>
<td>Page 36</td>
</tr>
<tr>
<td>6. SOME PROPERTIES OF THE ESTIMATORS</td>
<td>Page 40</td>
</tr>
<tr>
<td>6.1 Dependence on $K_i$</td>
<td>Page 40</td>
</tr>
<tr>
<td>6.2 Bounds on $K_i$</td>
<td>Page 40</td>
</tr>
<tr>
<td>6.3 Bias</td>
<td>Page 41</td>
</tr>
<tr>
<td>6.4 Variances and Covariances</td>
<td>Page 41</td>
</tr>
<tr>
<td>6.5 On Limiting Properties</td>
<td>Page 41</td>
</tr>
<tr>
<td>7. A NUMERICAL EXAMPLE</td>
<td>Page 43</td>
</tr>
<tr>
<td>7.1 General Problem</td>
<td>Page 43</td>
</tr>
<tr>
<td>7.2 Fixed $\sigma^2$ Case</td>
<td>Page 44</td>
</tr>
<tr>
<td>7.3 $\sigma^2 \varepsilon [\sigma_0^2, \sigma_1^2]$ Case</td>
<td>Page 52</td>
</tr>
</tbody>
</table>
## TABLE OF CONTENTS (Continued)

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>8. SUMMARY, CONCLUSIONS, AND SUGGESTIONS FOR FUTURE RESEARCH</td>
<td>72</td>
</tr>
<tr>
<td>8.1 Summary</td>
<td>72</td>
</tr>
<tr>
<td>8.2 Conclusions</td>
<td>74</td>
</tr>
<tr>
<td>8.3 Suggestions for Future Research</td>
<td>75</td>
</tr>
<tr>
<td>8.3.1 X'X Matrix of Less than Full Rank</td>
<td>75</td>
</tr>
<tr>
<td>8.3.2 Transformation of Parameters</td>
<td>75</td>
</tr>
<tr>
<td>8.3.3 Alternate Decision Procedure</td>
<td>76</td>
</tr>
<tr>
<td>9. LIST OF REFERENCES</td>
<td>77</td>
</tr>
<tr>
<td>10. APPENDICES</td>
<td>78</td>
</tr>
<tr>
<td>10.1 Appendix A - Computer Program to Calculate K for σ² Fixed</td>
<td>79</td>
</tr>
<tr>
<td>10.2 Appendix B - Computer Program to Calculate K for Type A Estimator</td>
<td>83</td>
</tr>
<tr>
<td>10.3 Appendix C - Computer Program to Calculate K for Type B Estimator</td>
<td>88</td>
</tr>
</tbody>
</table>
LIST OF TABLES

7.1. Values of $K$ and corresponding probability increases:
Pr[M - $\varepsilon \hat{K} \hat{\beta} + M + \varepsilon$] - Pr[M - $\varepsilon \hat{K} \hat{\beta} + M + \varepsilon$] for fixed $\sigma^2$ case --
$\sigma^2 = .0001$ .................................................... 45

7.2. Values of $K$ and corresponding probability increases:
Pr[M - $\varepsilon \hat{K} \hat{\beta} + M + \varepsilon$] - Pr[M - $\varepsilon \hat{K} \hat{\beta} + M + \varepsilon$] for fixed $\sigma^2$ case --
$\sigma^2 = .001$ ....................................................... 46

7.3. Values of $K$ and corresponding probability increases:
Pr[M - $\varepsilon \hat{K} \hat{\beta} + M + \varepsilon$] - Pr[M - $\varepsilon \hat{K} \hat{\beta} + M + \varepsilon$] for fixed $\sigma^2$ case --
$\sigma^2 = .01$ .......................................................... 47

7.4. Values of $K$ and corresponding probability increases:
Pr[M - $\varepsilon \hat{K} \hat{\beta} + M + \varepsilon$] - Pr[M - $\varepsilon \hat{K} \hat{\beta} + M + \varepsilon$] for fixed $\sigma^2$ case --
$\sigma^2 = .05$ .......................................................... 48

7.5. Values of $K$ and corresponding probability increases:
Pr[M - $\varepsilon \hat{K} \hat{\beta} + M + \varepsilon$] - Pr[M - $\varepsilon \hat{K} \hat{\beta} + M + \varepsilon$] for fixed $\sigma^2$ case --
$\sigma^2 = 1.0$ .......................................................... 49

7.6. Values of $K$ and corresponding probability increases:
Pr[M - $\varepsilon \hat{K} \hat{\beta} + M + \varepsilon$] - Pr[M - $\varepsilon \hat{K} \hat{\beta} + M + \varepsilon$] for fixed $\sigma^2$ case --
$\sigma^2 = 10.0$ ......................................................... 50

7.7. Values of $K$ and corresponding probability increases:
Pr[M - $\varepsilon \hat{K} \hat{\beta} + M + \varepsilon$] - Pr[M - $\varepsilon \hat{K} \hat{\beta} + M + \varepsilon$] for fixed $\sigma^2$ case --
$\sigma^2 = 100.0$ ....................................................... 51

7.8. Percent increase in Pr[M - $\varepsilon \hat{K} \hat{\beta} + M + \varepsilon$] over Pr[M - $\varepsilon \hat{K} \hat{\beta} + M + \varepsilon$] for
M=10 and $\sigma^2$ fixed .............................................. 52

7.9. Values of $K$, probability deviations, and BLUE probabilities
for Type A estimator -- $\sigma^2\varepsilon[.0001, 10.0]$ ......................... 54

7.10. Values of $K$, probability deviations, and BLUE probabilities
for Type A estimator -- $\sigma^2\varepsilon[.001, 1.0]$ ......................... 55

7.11. Values of $K$, probability deviations, and BLUE probabilities
for Type A estimator -- $\sigma^2\varepsilon[.01, .1]$ ......................... 56

7.12. Values of $K$, probability deviations, and BLUE probabilities
for Type A estimator -- $\sigma^2\varepsilon[.0001, .01]$ ......................... 57

7.13. Values of $K$, probability deviations, and BLUE probabilities
for Type A estimator -- $\sigma^2\varepsilon[.1, 10]$ ......................... 58

7.14. Values of $K$, probability deviations, and BLUE probabilities
for Type A estimator -- $\sigma^2\varepsilon[.03, .07]$ ......................... 59
### LIST OF TABLES (Continued)

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.15</td>
<td>Values of K, probability deviations, and BLUE probabilities for Type A estimator -- $\sigma^2e[1.0, 10.0]$</td>
<td>60</td>
</tr>
<tr>
<td>7.16</td>
<td>Values of K, probability deviations, and BLUE probabilities for Type A estimator -- $\sigma^2e[10, 20]$</td>
<td>61</td>
</tr>
<tr>
<td>7.17</td>
<td>Values of K, probability deviations, and BLUE probabilities for Type B estimator -- $\sigma^2e[0.0001, 10.0]$</td>
<td>62</td>
</tr>
<tr>
<td>7.18</td>
<td>Values of K, probability deviations, and BLUE probabilities for Type B estimator -- $\sigma^2e[0.001, 1.0]$</td>
<td>63</td>
</tr>
<tr>
<td>7.19</td>
<td>Values of K, probability deviations, and BLUE probabilities for Type B estimator -- $\sigma^2e[0.01, 0.1]$</td>
<td>64</td>
</tr>
<tr>
<td>7.20</td>
<td>Values of K, probability deviations, and BLUE probabilities for Type B estimator -- $\sigma^2e[0.0001, 0.01]$</td>
<td>65</td>
</tr>
<tr>
<td>7.21</td>
<td>Values of K, probability deviations, and BLUE probabilities for Type B estimator -- $\sigma^2e[1, 10]$</td>
<td>66</td>
</tr>
<tr>
<td>7.22</td>
<td>Values of K, probability deviations, and BLUE probabilities for Type B estimator -- $\sigma^2e[0.03, 0.07]$</td>
<td>67</td>
</tr>
<tr>
<td>7.23</td>
<td>Values of K, probability deviations, and BLUE probabilities for Type B estimator -- $\sigma^2e[1, 10]$</td>
<td>68</td>
</tr>
<tr>
<td>7.24</td>
<td>Values of K, probability deviations, and BLUE probabilities for Type B estimator -- $\sigma^2e[10, 20]$</td>
<td>69</td>
</tr>
<tr>
<td>7.25</td>
<td>Percent increase in probability over minimum BLUE probability for Type A estimator -- $M = 10$</td>
<td>71</td>
</tr>
<tr>
<td>7.26</td>
<td>Percent increase in probability over minimum BLUE probability for Type B estimator -- $M = 10$</td>
<td>71</td>
</tr>
</tbody>
</table>
## LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Partition of K-H space into the four regions A, B, C, and D</td>
<td>12</td>
</tr>
<tr>
<td>3.1</td>
<td>The two possible relationships between $\Phi(M, \sigma_0^2, K)$ and $\Phi(M, \sigma_1^2, K)$</td>
<td>24</td>
</tr>
<tr>
<td>5.1</td>
<td>Typical representation of $T(K, \sigma^2)$ for $K &gt; 1 - \frac{\epsilon}{M}$ and fixed</td>
<td>36</td>
</tr>
</tbody>
</table>
1. INTRODUCTION

1.1 Motivation

The standard approach to estimating the parameter vector $\beta$ in the general linear model is based on a criterion of minimum variance unbiasedness. That is, the estimator chosen is the one which gives rise to the smallest diagonal terms in its variance-covariance matrix, within the class of all linear unbiased estimators. While this requirement of unbiasedness is, in a general sense, quite reasonable, there can be certain experimental situations in which it could be less desirable than a criterion of "closeness" of the estimator to the parameter being estimated.

That is, an experimenter might be less interested in unbiasedness than in obtaining an estimate within a certain distance of the true parameter. In particular, given the simple (one $\beta$) linear model:

$$Y_i = X_i \beta + E_i \quad i = 1, 2, \ldots, n$$

with

$$E_i \sim N(0, \sigma^2)$$

$E_i$ and $E_j$ independent for $j \neq i$.

The experimenter might feel that if he could get an estimate of $\beta$ which was within $\pm \varepsilon$ of the true value, he would be satisfied. Any estimate in the interval $[\beta - \varepsilon, \beta + \varepsilon]$ is all the same to him, but he strongly desires one in this interval. While we can never guarantee that an estimate be contained in this interval, we can, for fixed $\varepsilon$, consider the probability:

$$Pr[\beta - \varepsilon \leq \beta \leq \beta + \varepsilon]$$
for different linear estimators $B$ of $\beta$, and use it as our measure of "closeness" of an estimator to the parameter estimated: the greater the probability, the greater the "closeness". Hence if one estimator is found to be closer to $\beta$ than another estimator, the former will be referred to as "better" than the latter.

Since we are considering linear estimators only we have:

$$B = C'y$$

where $C$ is an $n$ by 1 vector of constants, and $Y$ is the $n$ by 1 vector of observations. Then the distribution of $B$ will be normal with mean $C'X\beta$, and variance $C'C\sigma^2$, where $X$ is the vector whose $i^{th}$ component is $X_i$. Symbolically:

$$B \sim N(C'X\beta, C'C\sigma^2)$$

We will proceed by first considering the single $\beta$ case, for two different formulations of the problem, and finally extending the results to the general $p \beta$ case.

1.2 First Procedure

Webster [4] considered this criterion, and referred to it as an "$\varepsilon$-neighbourhood" criterion. He considers $\hat{\mu}_1$ better than $\hat{\mu}_2$ for estimating $\mu$ if, for fixed $\varepsilon$:

$$\Pr[|\hat{\mu}_1 - \mu| < \varepsilon] > \Pr[|\hat{\mu}_2 - \mu| < \varepsilon]$$

Webster compares two estimators, $X_1$ and $X_2$ where:

$$X_1 \sim N(\mu, \sigma^2_1)$$
\[ X_2 \sim N(\mu - \delta, \sigma_2^2) \]

\[ \sigma_1^2 > \sigma_2^2 \quad ; \quad \delta > 0 \]

with correlation \( \rho \).

He determines that \( X_2 \) is a better "\( \varepsilon \)-neighbourhood" estimator of \( \mu \) than \( X_1 \) when:

\[ |\delta| \leq \varepsilon [1- \frac{\sigma_2^2}{\sigma_1^2}] \]

In other words, knowledge of the ratio of the variances, as well as the
difference between the means, is necessary (assuming \( \varepsilon \) as given), before
\( X_2 \) can be said to be better than \( X_1 \). This requires a good deal of
knowledge about the parameters at the outset.

Webster's work points up the difficulty of working with biased
estimators: their properties tend to depend on the parameters more than
do those of unbiased estimators. For example defining \( \Phi(\beta, \sigma^2, B) = Pr[\beta - \varepsilon < B < \beta + \varepsilon] \) where \( \varepsilon \) is fixed, we could have two distinct
estimators of \( \beta \): \( B_1 \) and \( B_2 \), which have the property that:

\[ \Phi(\beta_1, \sigma_1^2, B_1) > \Phi(\beta_1, \sigma_2^2, B_2) \]

For all estimators \( B \) except \( B = B_1 \), while at the same time:

\[ \Phi(\beta_2, \sigma_2^2, B_2) > \Phi(\beta_2, \sigma_2^2, B) \]

for all estimators \( B \) except \( B = B_2 \). That is, \( B_1 \) is the best estimator
at the point \( (\beta_1, \sigma_1^2) \) in the parameter space, while \( B_2 \) is best at the
point \( (\beta_2, \sigma_2^2) \).

Dantzig (1) alluded to this problem by proving that there exists
no test of Student's hypothesis which has a power function independent
of $\sigma^2$. This relates to our discussion through the correspondence between confidence intervals and (two-sided) tests, where our estimator can be thought of as defining the center of a confidence interval for $\beta$ of width $2\epsilon$. Following Dantzig's work, Stein (3) developed a two sample sequential procedure for establishing fixed width confidence intervals independent of $\sigma^2$, at prescribed $\alpha$ levels.

Since we wish our closeness measure to be as large as possible, for $\epsilon$ fixed, whatever the "true" values of $\beta$ and $\sigma^2$, we might employ a decision rule which picks as estimator the $B$ which maximizes the infimum of:

$$\Pr[\beta - \epsilon \leq B \leq \beta + \epsilon]$$

over the parameter space. That is, for each candidate $B$ we find the infimum of $\Pr[\beta - \epsilon \leq B \leq \beta + \epsilon]$ over the parameter space: $0 < \sigma^2 < +\infty$, $-\infty < \beta < +\infty$; and then we choose the $B$ which produces the largest of these infimums. This procedure, however, will not work, because each of the above mentioned infimums will be zero. For any biased $B$ and $\sigma^2$ fixed, it will be seen from the proof of Lemma 2.1 that the probability decreases to zero as $|\beta| \to +\infty$. On the other hand, for $\beta$ fixed, if $\beta - \epsilon \leq E[B] \leq \beta + \epsilon$ then the probability will go to zero as $\sigma^2 \to \infty$, while if $E[B] < \beta - \epsilon$ or $E[B] > \beta + \epsilon$ the probability will approach zero as $\sigma^2 \to 0$.

Hence the parameter space must be restricted to make the criterion nontrivial. Restrictions which suggest themselves, and which we shall use are: $|\beta| \leq M$ and either $\sigma^2$ fixed and known or $\sigma^2 [\sigma_0^2, \sigma_1^2]$ where $M, \sigma_0^2$, and $\sigma_1^2$ are known.
Under these restrictions on the parameter space we can now state the First Decision Procedure as: Choose as an estimator of β any B which maximizes the minimum of the closeness probability over the restricted parameter space. That the estimator B so determined does exist, and is unique, will be seen by actually finding, in each case, a unique B which satisfies this formulation.

It should be pointed out that from the point of view of the experimenter, the above restrictions on the parameter space can, on some occasions, be quite reasonable. If parameters have occurred repeatedly in similar contexts, this may occasionally give rise to a general knowledge, or "feeling" of their magnitude. In fact, some experimenters profess that in some of their work, they have complete knowledge of σ^2. Aside from this, sometimes mathematical and/or physical constraints exist which restrict the parameter space. If, however, no such information or conditions exist, the estimators developed here can not be used.

Finally, a completely equivalent procedure would have been to define the loss function L(B, β, σ^2) as:

\[
L(B, \beta, \sigma^2) = \begin{cases} 
0 & \text{if } \beta - \epsilon \leq B \leq \beta + \epsilon \\
1 & \text{otherwise}
\end{cases}
\]

where \( \epsilon \) is understood to be fixed.

Then the risk function, \( R(B, \beta, \sigma^2) \) is:

\[
R(B, \beta, \sigma^2) = E[L(B, \beta, \sigma^2)] = 1 - Pr[\beta - \epsilon \leq B \leq \beta + \epsilon]
\]

and the well known minimax decision procedure applied to this risk function gives exactly our First Procedure.
1.3 Second Procedure

One property of the estimator derived from the first formulation is that this estimator is not necessarily uniformly closer to $\beta$ than the BLUE (best linear unbiased estimator) is over the restricted parameter space. A second decision procedure is presented in this section which is without this potential drawback.

This procedure is to choose as the estimator of $\beta$ the $B$ which maximizes the minimum of:

$$\Pr[\beta - \varepsilon \leq B \leq \beta + \varepsilon] - \Pr[\beta - \varepsilon \leq \hat{\beta} \leq \beta + \varepsilon]$$

over the restricted parameter space. Here, and henceforth, $\hat{\beta}$ will mean the BLUE. As with the previous estimator, the existence and uniqueness of a $B$ which satisfies this criterion will be seen by actually deriving an appropriate $B$, and seeing that it is unique.

Here, as before, a formal decision procedure could have been used with Loss Function:

$$L(B, \beta, \sigma^2) = L_1(\hat{\beta}, \beta, \sigma^2) - L_2(B, \beta, \sigma^2)$$

where

$$L_1(\hat{\beta}, \beta, \sigma^2) = \begin{cases} 1 & \text{if } \beta - \varepsilon \leq \hat{\beta} \leq \beta + \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

$$L_2(B, \beta, \sigma^2) = \begin{cases} 1 & \text{if } \beta - \varepsilon \leq B \leq \beta + \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

giving risk function $R(B, \beta, \sigma^2)$:

$$R(B, \beta, \sigma^2) = E[L(B, \beta, \sigma^2)]$$
\[ E[L(B, \beta, \sigma^2)] = \Pr[\beta - \varepsilon \leq \hat{\beta} \leq \beta + \varepsilon] - \Pr[\beta - \varepsilon \leq B \leq \beta + \varepsilon] \]

The minimax solution of this risk function then is exactly our Second Decision Procedure.

1.4 Full Model

To this point we have considered only the simple linear model with one \( \beta \). For the general linear model of full rank:

\[ Y = X\beta + E \quad \text{where} \ E \sim N(0, I\sigma^2) \]

where \( \beta \) is a \( p \) dimensional vector of parameters, we will estimate separately each component of the \( \beta \) vector, and then consolidate the results. That such an approach is reasonable is seen from the closeness criterion itself: it is concerned with individual \( \beta \)'s. That is, if we have \( p \) different \( \beta \)'s to estimate, we will get from the experimenter \( p \) values of \( \varepsilon \), such that we have the \( p \) closeness measures:

\[ \Pr[\beta_i - \varepsilon_i \leq \hat{\beta}_i \leq \beta_i + \varepsilon_i] \quad i = 1, 2, \ldots, p \]

Each individual \( \varepsilon_i \) is obtained from the experimenter, and is based on his attitudes toward \( \beta_i \) only.

One additional restriction must be made on the estimators in this general case: that the expected value of \( B_i \) not depend on \( \beta_j \) for \( j \neq i \). This restriction should appear intuitively reasonable.

1.5 Development

The development of the remaining chapters is as follows: the single \( \beta \), fixed \( \sigma^2 \) case is considered in Chapter 2. Chapter 3 contains an
extension to the $\sigma^2_{0}, \sigma^2_{1}$ case for the estimator of the first procedure. In Chapter 4 this result is generalized to the full p-dimensional model. The complete development of the second procedure estimator through the general model is considered in Chapter 5. Chapter 6 presents properties of the derived estimators. Chapter 7 presents a numerical example, and Chapter 8 contains the summary, conclusions, and suggestions for future research.
2. ESTIMATOR FOR THE SINGLE $\beta$, FIXED $\sigma^2$ CASE

2.1 General Structure

Consider the linear model:

$$Y = X \beta + \varepsilon$$

where $Y$ is an $n$ by 1 vector of observations

$X$ is an $n$ by 1 vector of fixed values

$\varepsilon$ is an $n$ by 1 vector of random variables

such that $\varepsilon \sim N(0, \sigma^2)$

We are assuming that $\sigma^2$ is known, and that a value $M$ is known such that $|\beta| < M$. Define $\phi$ as:

$$\phi(\beta, \sigma^2, C) = \Pr[\beta - \varepsilon \leq B \leq \beta + \varepsilon]$$

where $B = C'Y = [C_1, C_2, \ldots, C_n]$ and

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \sum_{i=1}^{n} C_i y_i$$

and where $\varepsilon$ and $X$ are though of as being constants, and hence not listed as arguments of $\phi$. We have, however, listed $\sigma^2$ as an argument of $\phi$ although in this case it also is fixed, since in future chapters $\sigma^2$ will not be fixed.

Our decision procedure is to choose as the estimator of $\beta$ the $B$, or equivalently the $C$, which maximizes the minimum of $\phi(\beta, \sigma^2, C)$ over the restricted parameter space. The restricted parameter space is the set of ordered pairs: $(\sigma^2, \beta)$ where the first quantity assumes only one
value, and where the second quantity may assume any value from \(-M\) to \(M\) inclusive.

2.2 First Maxi-min Solution

The desired estimator is determined from a lemma and two theorems. The lemma establishes that for any fixed \(C\), \(\phi(\beta, \sigma^2, C)\) will assume a minimum when \(|\beta| = M\). The first theorem then shows that the \(C\) which maximizes \(\phi(M, \sigma^2, C)\) must be of the form: \(k(X'X)^{-1}X\), or equivalently: \(B = k \hat{\beta}\) where \(\hat{\beta}\) is the BLUE of \(\beta\), for some constant \(k\). Finally the second theorem gives the value of \(k\) which maximizes \(\phi(M, \sigma^2, k(X'X)^{-1}X)\).

Lemma 2.1: For \(\sigma^2\) fixed, \(|\beta| \leq M\), and for any fixed \(C\), \(\phi(\beta, \sigma^2, C)\) is minimized when \(\beta = |M|\).

Proof:
\[B = C'X \sim N(C'X, C'C\sigma^2)\]

Set \(K = C'X\), and \(H = C'C\)

Now:
\[
\phi(\beta, \sigma^2, C) = Pr[\beta - \varepsilon \leq B \leq \beta + \varepsilon] = Pr\left[\frac{\beta - \varepsilon - KB}{\sqrt{\sigma^2}} \leq Z \leq \frac{\beta + \varepsilon - KB}{\sqrt{\sigma^2}}\right]
\]

where \(Z \sim N(0,1)\).

Note that \(\phi(-\beta, \sigma^2, C) = \phi(\beta, \sigma^2, C)\) and that the width of the \(Z\) interval is:
\[
\frac{2\varepsilon}{\sqrt{\sigma^2}}
\]

which does not depend on \(\beta\). The midpoint of the interval is:
\[
\frac{\beta(1-K)}{\sqrt{\sigma^2}}
\]

Hence as \(|\beta|\) increases \(\phi\) decreases and \(\phi\) is minimized when \(\beta = |M|\). QED
It will be seen that the estimator which maximizes \( \phi \) for \( \beta = M \) is
identical to the one which maximizes \( \phi \) for \( \beta = -M \). Hence in the following
we consider only \( \beta = M \).

We now must determine what vector: \( \mathbf{C} \) will maximize \( \phi(M, \sigma^2, \mathbf{C}) \). The
most straightforward procedure might be to use the Lagrangian method of
maximization to determine \( C_i, i = 1,2, \ldots, n \). This was, in fact,
the way the initial solution was obtained, but a different approach is
presented here. Our method is to first show that the \( \mathbf{B} \) which maximizes
\( \phi \) must be of the form \( k\hat{\beta} \), where \( k \in [0,1] \); and then to use the Lagrangian
method to determine the exact value of \( k \). The purpose for this perhaps
unusual approach is that it leads naturally into the approach necessary
for the more general \( \sigma^2 \epsilon [\sigma_0^2, \sigma_1^2] \) case.

**Theorem 2.1:** The \( \mathbf{C} \) which maximizes \( \phi(M, \sigma^2, \mathbf{C}) = \Pr[M-\epsilon \leq \mathbf{B} \leq M + \epsilon] \)
for \( M, \epsilon, \) and \( \sigma^2 \) fixed, is of the form:

\[
\mathbf{C}' = k(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' \quad \text{or} \quad \mathbf{B} = k\hat{\beta}
\]

for some \( k \in [0,1] \).

**Proof:** If \( M < \epsilon \) then the interval \([M-\epsilon, M+\epsilon]\) contains zero, and
the estimator \( \mathbf{B} = 0 \) gives \( \Pr[M-\epsilon < \mathbf{B} < M+\epsilon] = 1 \) which is clearly a
maximum. Since \( \mathbf{B} = 0 \) can be trivially written in the form:

\[
\mathbf{B} = k(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}
\]

for \( k = 0 \), we can limit our consideration to \( \epsilon < M \).

Also, since \( \phi \) is a function of \( \mathbf{C} \) only through \( K = \mathbf{C}'\mathbf{X} \) and \( H = \mathbf{C}'\mathbf{C} \),
we will proceed to seek the point \((\mathbf{K}, \mathbf{H})\) which maximizes \( \phi \). It will
be seen that the determined point will correspond to a unique \( \mathbf{C} \). The
approach will be to consider the space of all possible points: \((\mathbf{K}, \mathbf{H}), \)
referred to henceforth as K-H space, and to eliminate from consideration any point which gives rise to a smaller value of \( \phi \) than does some other point. The boundaries of this space are given by the Cauchy-Schwartz inequality:

\[
-\sqrt{H} \sqrt{x^T x} \leq k \leq \sqrt{H} \sqrt{x^T x}.
\]

Consider the partition of K-H space given in Figure 2.1.

---

Figure 2.1. Partition of K-H space into the four regions A, B, C, and D.
A is the set of points \((K, H)\) such that \(H = K^2 (X'X)^{-1}\) for \(K \in [0, 1]\).

B is the set of points \((K, H)\) such that \(H \leq (X'X)^{-1}\) and \(K < \sqrt{H X'X}\)

C is the set of points \((K, H)\) such that \(H > (X'X)^{-1}\) and \(K \neq 1\).

D is the set of points \((K, H)\) such that \(H > (X'X)^{-1}\) and \(K = 1\).

Now in Lemma 2.1 we saw that if \(C \neq 0\) \(\Phi(M, \sigma^2, C)\) could be written as the probability of a standardized normal variable being within an interval of width: \(\frac{2\sigma}{\sqrt{Ho^2}}\) and center: \(\frac{M(1-K)}{\sqrt{Ho^2}}\).

Hence for any fixed \(H > (X'X)^{-1}\), any point \((K, H)\) in region C can be eliminated in favor of the point \((1, H)\) in region D. Similarly, for any fixed \(H \leq (X'X)^{-1}\), any point \((K, H)\) in region B can be eliminated in favor of the point \((\sqrt{H X'X}, H)\) in region A. Finally, any point in region D can clearly be eliminated in favor of the point \((1, (X'X)^{-1})\) in A. We have thus reduced the possible points to points on the boundary \(H = K^2 (X'X)^{-1}\) for which \(K \in [0, 1]\). However, we know that the Cauchy-Schwartz inequality becomes an equality if and only if one vector is a multiple of the other. So \(C = a\bar{X}\) for some \(a\).

Hence

\[B = C'Y = aX'Y = a(X'X)^{-1}X'Y = k\hat{\beta}\]

with \(k = a(X'X)^{-1}\).

But

\[KM = E[B] = kE[\hat{\beta}] = kM\]

so we must have that \(k = K\).

Hence the \(C\) which maximizes \(\Phi(M, \sigma^2, C)\) can be written in the form: \(C' = K(X'X)^{-1}X'\) or equivalently, \(B = K\hat{\beta}\) for some value of \(K \in [0, 1]\).

\(\text{QED}\)
In order to determine the exact value of $K$ which maximizes $\Phi(M, \sigma^2, K \mathbf{X}(X'X)^{-1})$ we have the following theorem:

**Theorem 2.2:** If $\hat{\beta} \sim N(M, \sigma^2)$, $0 < \varepsilon < M$ then

$\Phi = \text{Pr}[M - \varepsilon \leq K \hat{\beta} \leq M + \varepsilon]$ is maximized when:

$$K = \frac{-1 + \sqrt{1 + 2\delta}}{\delta} \quad \text{where} \quad \delta = \frac{d\sigma^2}{M\varepsilon} \ln \left(\frac{M+\varepsilon}{M-\varepsilon}\right)$$

and $0 < K < 1$.

**Proof:**

$$\Phi = \text{Pr}\left[\frac{M-\varepsilon - K M}{K' \sigma^2} \leq Z \leq \frac{M+\varepsilon - K M}{K' \sigma^2}\right] = \int_{-\infty}^{\frac{M+\varepsilon}{K' \sigma^2} - \frac{M}{K' \sigma^2}} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{\omega^2}{2}\right] d\omega - \int_{-\infty}^{\frac{M-\varepsilon}{K' \sigma^2} - \frac{M}{K' \sigma^2}} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{\omega^2}{2}\right] d\omega$$

So

$$\frac{\partial \Phi}{\partial K} = \frac{1}{\sqrt{2\pi}} \left\{ \exp\left[-\frac{1}{2d\sigma^2} \left(\frac{M+\varepsilon}{K} - M\right)^2\right] - \left[ -\frac{1}{2d\sigma^2} \left(\frac{M-\varepsilon}{K} - M\right)^2\right] - \left[ -\frac{1}{2d\sigma^2} \left(\frac{M-\varepsilon}{K} - M\right)^2\right] \right\}$$

Setting this equal to zero and solving for $K$ we get:

$$\exp\left[-\frac{2M}{Kd\sigma^2} (\frac{1}{K} - 1)\right] = \frac{M-\varepsilon}{M+\varepsilon}$$

which gives: $K = \frac{-1 + \sqrt{1 + 2\delta}}{\delta}$

where $\delta = \frac{d\sigma^2}{M\varepsilon} \ln \left(\frac{M+\varepsilon}{M-\varepsilon}\right)$

We choose the plus sign since otherwise $K$ would be negative. That this value of $K$ must correspond to a maximum in $[0,1]$ is evident since for $K > 1 \Phi$ is less than $\Phi$ at $K = 1$, and also $\Phi = 0$ when $K = 0$. QED
Note that if we had chosen $\beta = -M$ as the value of $\beta$ which minimizes $\phi(\beta, \sigma^2, C)$ for $C$ fixed, we would have gotten exactly the same results as above since

$$\frac{1}{M} \ln \left( \frac{-M + C}{-M - C} \right) = \frac{1}{M} \ln \left( \frac{M + C}{M - C} \right).$$

Hence the maxi-min solution for the single $\beta$, fixed $\sigma^2$ case is:

$$B = K\hat{\beta} \quad \text{where} \quad K = \frac{-1 + \sqrt{1 + 2\delta}}{\delta}$$

$$\delta = \frac{\sigma^2 d}{M\epsilon} \ln \left( \frac{M + C}{M - C} \right)$$

$$d = (X'X)^{-1}$$

$$\hat{\beta} = (X'X)^{-1}X'y$$
3. ESTIMATOR FOR THE SINGLE $\beta$, $\sigma^2 [\sigma_0^2, \sigma_1^2]$ CASE

3.1 General Structure

Suppose we have the linear model:

$$\mathbf{y} = \mathbf{X}\beta + \mathbf{e}$$

where:
- $\mathbf{y}$ is an $n$ by 1 vector of observations
- $\mathbf{X}$ is an $n$ by $l$ vector of fixed values
- $\mathbf{e}$ is an $n$ by 1 vector of random variables

such that $\mathbf{e} \sim N(0, \mathbf{I}\sigma^2)$.

Here $\beta$ and $\sigma^2$ are unknown parameters, but are known to lie in the bounded intervals:

$$|\beta| \leq M \quad \sigma^2 \in [\sigma_0^2, \sigma_1^2]$$

As before, our purpose is to estimate $\beta$, and our criterion is to choose as an estimator of $\beta$ the $\hat{\beta}$ which maximizes the minimum of $\phi(\beta, \sigma^2, \mathbf{C})$ where:

$$\phi(\beta, \sigma^2, \mathbf{C}) = \Pr[\beta - \epsilon \leq \beta \leq \beta + \epsilon]$$

over the restricted parameter space: $|\beta| \leq M$, $\sigma^2 \in [\sigma_0^2, \sigma_1^2]$.

3.2 Minimization of $\phi$ with Respect to $\beta$ and $\sigma^2$

Since $\phi$ depends on $\mathbf{C}$ only through $K$ and $H$, as defined in Lemma 2.1, we henceforth write $\phi$ as $\phi(\beta, \sigma^2, K, H)$.

In Lemma 2.1 we saw that, for $\sigma^2, K$, and $H$ fixed, $\phi$ assumed a minimum when $\beta = M$. Hence we are concerned with what value of $\sigma^2$ in $[\sigma_0^2, \sigma_1^2]$ will minimize $\phi(M, \sigma^2, K, H)$ for $K$ and $H$ (and of course, $M$)
fixed. The following two lemmas prove that for any fixed $K$ and $H$, $\phi$ will be minimized with respect to $\sigma^2$ at either $\sigma^2_0$, $\sigma^2_1$ or at both simultaneously.

**Lemma 3.1:** For $K \notin [1 - \frac{\epsilon}{M}, 1 + \frac{\epsilon}{M}]$ and $K$, $H$, and $M$ fixed; $\phi(M, \sigma^2, K, H)$ will assume a minimum with respect to $\sigma^2$, where $\sigma^2 \in [\sigma^2_0, \sigma^2_1]$ at either $\sigma^2_0$, $\sigma^2_1$ or at both.

**Proof:** The proof will consist of showing that for $K \notin [1 - \frac{\epsilon}{M}, 1 + \frac{\epsilon}{M}]$, $\phi$ is a unimodal function of $\sigma^2$ having a maximum at:

$$
\sigma^2 = \frac{2(1-K)M\epsilon}{H \ln \frac{(1-K)M+\epsilon}{(1-K)M-\epsilon}}
$$

$$
\phi = \left(\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right)dz\right) - \left(\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right)dz\right)
$$

Taking derivatives with respect to $\sigma^2$ and setting them equal to zero we get:

$$
[(1-K)M+\epsilon] \exp\left(-[(1-K)M+\epsilon]^2 / 2H\sigma^2\right) - [(1-K)M-\epsilon] \exp\left(-[(1-K)M-\epsilon]^2 / 2H\sigma^2\right) = 0
$$

or

$$
\exp\left(-2(1-K)M\epsilon / H\sigma^2\right) = \frac{(1-K)M-\epsilon}{(1-K)M+\epsilon}
$$

which gives

$$
\sigma^2 = \frac{2(1-K)M\epsilon}{H \ln \frac{(1-K)M+\epsilon}{(1-K)M-\epsilon}}
$$
as the only solution. But since $\phi \geq 0$, and since $\phi$ approaches zero as $\sigma^2$ approaches either zero or infinity, our above solution must correspond to a maximum. Hence for $\sigma^2$ restricted to the interval: $[\sigma_0^2, \sigma_1^2]$, $\phi$ will be smallest for either $\sigma^2 = \sigma_0^2$ or $\sigma_1^2$, or at both.

**QED**

**Lemma 3.2:** For $K\in[1 - \frac{E}{M}, 1 + \frac{E}{M}]$, $K$, and $M$ fixed, $\phi(M, \sigma^2, K, H)$ is monotone decreasing with respect to the product: $H\sigma^2$.

**Proof:** $\phi(M, \sigma^2, K, H) = Pr[M-\epsilon \leq B \leq M + \epsilon]$ with $B \sim N(KM, H\sigma^2)$.

But $K\in[1 - \frac{E}{M}, 1 + \frac{E}{M}]$ if and only if $E[B] = [M-\epsilon, M+\epsilon]$. Hence $\phi$ is monotone decreasing with respect to the product $H\sigma^2$.

**QED**

As a result of this lemma, for $K\in[1 - \frac{E}{M}, 1 + \frac{E}{M}]$, $H$ fixed, $\phi$ will be minimized with respect to $\sigma^2$ for $\sigma^2 \in [\sigma_0^2, \sigma_1^2]$ when $\sigma^2 = \sigma_1^2$.

### 3.3 Second Maxi-min Solution

Proceeding as in the fixed $\sigma^2$ case, we obtain:

**Theorem 3.1:** The $B$ which maximizes the minimum of $\phi(M, \sigma^2, K, H) = Pr[M-\epsilon \leq B \leq M + \epsilon]$ for $\sigma^2 \in [\sigma_0^2, \sigma_1^2]$ can be written in the form:

$$B = k\hat{\beta}$$

for some $k\in[0,1]$.

where $\hat{\beta}$ is the BLUE of $\beta$.

**Proof:** Consider the partition of the $K-H$ space given in Figure 2.1 of Chapter 2. As in Theorem 2.1 we will proceed to eliminate from consideration any point $(K,H)$ which produces a minimum of $\phi$ that is smaller than the minimum of $\phi$ produced by some other point.

From Lemma 3.2 we have that for any point in region $D$, $\phi$ is minimized at $\sigma_1^2$. Thus consider the point $(K,H)$ in region $C$. If $\phi$ at this point is also minimized at $\sigma_1^2$ then, by Theorem 2.1:

$$\phi(M, \sigma_1^2, K, H) < \phi(M, \sigma_1^2, 1, H)$$

and this point can be eliminated from
consideration. If, on the other hand, \( \phi \) is minimized at \( \sigma_0^2 \) then:

\[
\phi(M, \sigma_0^2, K, H) \leq \phi(M, \sigma_1^2, K, H) < \phi(M, \sigma_1^2, 1, H)
\]

so in either case a point in C can always be eliminated in favor of one in D.

Similarly, if for any point \((K, H)\) in B and the point \((\sqrt{H/X'X}, H)\) in A, if \( \phi \) is minimized for the same value of \( \sigma^2 \) at both points, then by Theorem 2.1 the point in B can be eliminated in favor of the one in A. Otherwise we have either

\[
\phi(M, \sigma_0^2, K, H) \leq \phi(M, \sigma_1^2, K, H) < \phi(M, \sigma_1^2, \sqrt{H/X'X}, H)
\]

or

\[
\phi(M, \sigma_1^2, K, H) \leq \phi(M, \sigma_0^2, K, H) < \phi(M, \sigma_0^2, \sqrt{H/X'X}, H)
\]

depending on whether the point in A is minimized for \( \sigma^2 = \sigma_1^2 \) or \( \sigma_0^2 \), respectively. Thus the points in B can all be eliminated in favor of those in A.

Finally, since for any point in D, \( \phi \) is minimized at \( \sigma_1^2 \), we have by Theorem 2.1 that any point in D can be eliminated in favor of the point \((1, (X'X)^{-1})\) in A.

We have, therefore, successfully restricted our consideration of the whole K-H space to the boundary line: \( H = K^2(X'X)^{-1} \) for \( K \in [0,1] \).

As it was pointed out in Theorem 2.1, this implies that the estimator must be of the form:

\[
B = \hat{K} \hat{\phi}
\]

for \( K \in [0,1] \) QED

As a result of this theorem, we can eliminate \( H \) as an argument of \( \phi \), giving:
\( \phi(\beta, \sigma^2, K) = \Pr[\beta - \varepsilon \leq \hat{\beta} \leq \beta + \varepsilon] \)

where \( \hat{\beta} \sim N(\beta, d\sigma^2) \)
\[ d = (X'X)^{-1} \]

Since for any \( K \in [0,1] \) \( \phi \) is minimized at either \( \sigma_0^2 \) or \( \sigma_1^2 \), we proceed to examine the two functions: \( \phi(M, \sigma_0^2, K) \) and \( \phi(M, \sigma_1^2, K) \) as functions of \( K \) for \( K \in [0,1] \).

This maximizing value was:
\[ K = \frac{-1 + \sqrt{1 + 2\delta}}{\delta} \text{ with } \delta = \frac{\sigma^2d}{M \varepsilon} \ln\frac{M + \varepsilon}{M - \varepsilon} \]

That \( \frac{-1 + \sqrt{1 + 2\delta}}{\delta} \) is a monotone function of \( \delta \) is established in the following lemma:

**Lemma 3.3:** \( \frac{-1 + \sqrt{1 + 2\delta}}{\delta} \) is a strictly decreasing function of \( \delta \).

**Proof:** Taking the derivative of the function with respect to \( \delta \) yields
\[
\frac{d}{d\delta} \left[ \frac{-1 + \sqrt{1 + 2\delta}}{\delta} \right] = \frac{-1 - \delta + \sqrt{1 + 2\delta}}{\delta^2 \sqrt{1 + 2\delta}}
\]

But \( 0 \leq \frac{1}{2} (1 - \sqrt{1 + 2\delta})^2 = 1 + \delta - \sqrt{1 + 2\delta} \)

with equality only when \( \delta = 0 \), but \( \delta > 0 \).

One further lemma is necessary.

**Lemma 3.4:** Set \( \gamma = \frac{-1 + \sqrt{1 + 2\delta}}{\delta}, \delta = \frac{\sigma^2d}{M \varepsilon} \ln\frac{M + \varepsilon}{M - \varepsilon} \);

then \( \phi(M, \sigma^2, \gamma) \) is a strictly decreasing function of \( d\sigma^2 = (X'X)^{-1} \sigma^2 = \text{Var} [\hat{\beta}] \).
Proof: Let \( R = \sqrt{1 + 2\delta} = 1 + \gamma \delta \);

then \( \delta = \frac{(R+1)(R-1)}{2} \) and \( \gamma = \frac{2}{R+1} \)

\[
\frac{d\delta}{dR} = R.
\]

So \( \frac{d\gamma}{d\delta} \cdot \frac{dR}{d\delta} = -\frac{2}{(R+1)^2} \cdot \frac{1}{R} \)

Let \( D = \gamma \delta \), say.

Then \( \frac{d\delta}{dD} = W = \frac{\delta}{D} = \frac{(R+1)(R-1)}{2D} \)

So \( \frac{dy}{dD} = \frac{d\gamma}{d\delta} \cdot \frac{d\delta}{dD} = -\frac{(R-1)}{DR(R+1)} \)

and

\[
\frac{d}{dD} \left( \frac{1}{\gamma \sqrt{D}} \right) = \frac{d}{d\gamma} \left( \frac{1}{\gamma \sqrt{D}} \right) \cdot \frac{dy}{dD} = \frac{(R+1)(R-1)}{4D^{3/2}R}.
\]

Now

\[
\frac{M+\varepsilon}{\gamma D^{1/2}} - \frac{M}{D^{1/2}}
\]

\[
\phi = \int \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{z^2}{2} \right\} \, dz
\]

\[
\frac{M-\varepsilon}{\gamma D^{1/2}} - \frac{M}{D^{1/2}}
\]

Using the above derivatives we have:

\[
\sqrt{2\pi} \frac{d\phi}{dD} = \left[ \frac{(R+1)(R-1)(M+\varepsilon)}{4D^{3/2}R} + \frac{M}{2D^{3/2}} \right] \exp \left\{ -\frac{1}{2}\left[ \frac{M+\varepsilon}{\gamma D^{1/2}} - \frac{M}{D^{1/2}} \right]^2 \right\}
\]

\[
- \left[ \frac{(R+1)(R-1)(M-\varepsilon)}{4D^{3/2}R} + \frac{M}{2D^{3/2}} \right] \exp \left\{ -\frac{1}{2}\left[ \frac{M-\varepsilon}{\gamma D^{1/2}} - \frac{M}{D^{1/2}} \right]^2 \right\}
\]

or
\[ \sqrt{2\pi} \frac{d\phi}{dD} = \frac{1}{4D^{3/2}} \exp \left\{ - \frac{1}{2D} \left( \frac{1}{Y} - 1 \right)^2 M_2^2 + \frac{\epsilon^2}{Y^2} \right\} \frac{1}{R} \]

where \( G = [(R+1)(R-1)(M+\varepsilon) + 2MR] \exp \left\{ - \frac{1}{Y} \left( \frac{1}{Y} - 1 \right) M_\varepsilon \right\} \frac{1}{Y} \]

\[ [(R+1)(R-1)(M-\varepsilon) + 2MR] \exp \left\{ - \frac{1}{Y} \left( \frac{1}{Y} - 1 \right) M_\varepsilon \right\} \frac{1}{Y} \]

Now in the proof of Lemma 2.3 we had:

\[ \exp \left\{ - \frac{2\varepsilon M}{\gamma D} \left( \frac{1}{Y} - 1 \right) \right\} = \frac{M-\varepsilon}{M+\varepsilon} \]

so \( \exp \left\{ - \frac{(1/\gamma - 1)M_\varepsilon}{\gamma D} \right\} = \left( \frac{M-\varepsilon}{M+\varepsilon} \right)^{1/2} \)

and \( \exp \left\{ - \frac{1}{\gamma D} \left( \frac{1}{Y} - 1 \right) M_\varepsilon \right\} = \left( \frac{M+\varepsilon}{M-\varepsilon} \right)^{1/2} \)

Noting that the coefficient of \( G \) in \( \frac{d\phi}{dD} \) is positive, we have:

\[ \frac{d\phi}{dD} = C \left\{ [(R+1)(R-1)(M+\varepsilon) + 2MR] \frac{1}{M+\varepsilon} \left[ \frac{1}{M+\varepsilon} \right]^2 - [(R+1)(R-1)(M-\varepsilon) + 2MR] \frac{1}{M-\varepsilon} \left[ \frac{1}{M-\varepsilon} \right]^2 \right\} \]

where \( C > 0 \)

or

\[ \frac{d\phi}{dD} = C \left( 2MR \right) \left\{ \frac{1}{M+\varepsilon} \left[ \frac{1}{M+\varepsilon} \right]^2 - \frac{1}{M-\varepsilon} \left[ \frac{1}{M-\varepsilon} \right]^2 \right\} \]

which is clearly negative.

Hence \( \phi \) is a strictly decreasing function of \( D = d\sigma^2 \).

QED

Denoting the value of \( K \) which maximizes \( \phi(M, \sigma_i^2, K) \) for \( \sigma_i^2 \) fixed \( i = 0, 1 \) by \( \gamma_i \), we have
\[ \gamma_i = \frac{-1 + \sqrt{1 + 2\delta_i}}{\delta_i} \quad \text{where} \quad \delta_i = \frac{d\sigma_i}{M} \ln\left[\frac{M + \varepsilon}{M - \varepsilon}\right] \]

\[ i = 0, 1 \]

Using this notation in the following theorem we establish the maxi-min solution for this Chapter.

**Theorem 3.2:** The value of \( K \) that maximizes the minimum of \( \phi(M, \sigma^2, K) \) for \( \sigma^2 \in [\sigma_0^2, \sigma_1^2] \) is:

\[
K = \begin{cases} 
\gamma_1 & \text{if } \phi(M, \sigma_1^2, \gamma_1) \leq \phi(M, \sigma_0^2, \gamma_1) \\
\phi(M, \sigma_0^2, k) & \text{if } \phi(M, \sigma_1^2, \gamma_1) > \phi(M, \sigma_0^2, \gamma_1) 
\end{cases}
\]

where \( k \) is the unique value in the open interval \((\gamma_1, \gamma_0)\) for which

\[ \phi(M, \sigma_0^2, k) = \phi(M, \sigma_1^2, k) \]

**Proof:** From Lemma 3.3 we have that \( \gamma_1 < \gamma_0 \) and from Lemma 3.4 that \( \phi(M, \sigma_1^2, \gamma_1) < \phi(M, \sigma_0^2, \gamma_0) \).

If \( \phi(M, \sigma_1^2, \gamma_1) \leq \phi(M, \sigma_0^2, \gamma_1) \)

then \( K = \gamma_1 \) must be the desired maxi-min solution since for this value of \( K \), \( \phi \) is minimized with respect to \( \sigma^2 \) at \( \sigma^2 = \sigma_1^2 \), while for \( \sigma^2 = \sigma_0^2 \), \( \phi \) is maximized at \( K = \gamma_1 \). On the other hand if \( \phi(M, \sigma_1^2, \gamma_1) > \phi(M, \sigma_0^2, \gamma_1) \)

then for any value of \( K < \gamma_1 \), the minimum of \( \phi(M, \sigma^2, K) \) with respect to \( \sigma^2 \) must be less than \( \phi(M, \sigma_0^2, \gamma_1) \) due to the fact that both \( \phi(M, \sigma_0^2, K) \) and \( \phi(M, \sigma_1^2, K) \) decrease as \( K \) decreases from \( K = \gamma_1 \).

For \( \gamma_1 \leq K \leq \gamma_0 \), \( \phi(M, \sigma_0^2, K) \) will be strictly increasing, while \( \phi(M, \sigma_1^2, K) \) will be strictly decreasing. This is due to the fact that \( \phi(M, \sigma^2, K) \) assumes a maximum with respect to \( K \) for \( \sigma^2 \) fixed, at only one point in the interval \([0, 1]\). Hence as \( K \) approaches this value from
the left $\Phi(M, \sigma^2, K)$ must be increasing, while as $K$ continues to increase beyond the maximizing value, $\Phi(M, \sigma^2, K)$ will decrease.

Since we know that $\Phi(M, \sigma^2_1, \gamma_0) < \Phi(M, \sigma^2_0, \gamma_0)$ there must be a unique value of $k$ in the open interval $(\gamma_1, \gamma_0)$ such that:

$$\Phi(M, \sigma^2_0, k) = \Phi(M, \sigma^2_1, k)$$

These two possible situations are depicted in Figure 3.1 as Case A and Case B, respectively.

_QED_
4. GENERALIZATION TO THE CASE OF p \( \beta \)'S

4.1 General Structure

Consider the general linear model of full rank:

\[
Y = X\beta + E
\]

where: 
- \( Y \) is an \( n \) by \( 1 \) vector of observations
- \( X \) is an \( n \) by \( p \) matrix of known values
- \( \beta \) is a \( p \) by \( 1 \) vector of unknown parameters
- \( E \) is an \( n \) by \( 1 \) vector of random variables such that:

\[
E \sim N(0, \sigma^2)
\]

where \( \sigma^2 \) is fixed and known, and where \( \beta_1 \) is known to satisfy:

\[
|\beta_1| \leq M_i \quad i = 1, 2, \ldots, p
\]

with \( M_i, \quad i = 1, \ldots, p \) known constants.

We are considering only estimators which are linear in \( Y \), so any estimator \( \hat{\beta} \) can be written:

\[
\hat{\beta} = C'Y
\]

where \( C \) is an \( n \) by \( p \) matrix.

We can write:

\[
C' = \begin{bmatrix}
C_1 \\
C_2 \\
\vdots \\
C_p
\end{bmatrix}
\]

where \( C'_{ij} \) is the \( ij \)th row of \( C' \).
Then
\[
B = \begin{bmatrix}
B_1 \\
B_2 \\
\vdots \\
B_p
\end{bmatrix} = \begin{bmatrix}
C_i'X_1 \\
\vdots \\
C_i'X_p
\end{bmatrix} = C'Y
\]

and we will proceed to determine \( C' \) by determining each \( C_i' \) separately, and then forming \( C' \).

For this case of \( p \) \( \beta \)'s we also require that \( E[B_i] \) not depend on \( \beta_j \) for \( j \neq i \). Now,

\[
E[B_i] = E[C_i'X] = C_i'X_\beta = [C_i'X_1, C_i'X_2, \ldots, C_i'X_p]\begin{bmatrix}
\beta_1 \\
\beta_2 \\
\vdots \\
\beta_p
\end{bmatrix} = C_i'X_1\beta_1 + C_i'X_2\beta_2 + \ldots + C_i'X_p\beta_p
\]

where \( X_j \) is the \( j \)th column vector in \( X \).

But for \( E[B_i] \) not to involve \( \beta_j \) for \( j \neq i \), we must have:

\[
C_i'X_j = 0 \text{ for } j \neq i.
\]

The closeness criterion for the estimator \( B_i \) of the parameter \( \beta_i \) is as in the single \( \beta \) case:

\[
\phi_i(\beta_i, \sigma^2, K_i, H_i) = \Pr[\beta_i - \varepsilon_i \leq B_i \leq \beta_i + \varepsilon_i]
\]

\[
B_i = C_i'Y \sim N(K_i \beta_i, H_i \sigma^2)
\]

where \( K_i = C_i'X_i \) and \( H_i = C_i'C_i \).
Our decision rule is then to consider only those vectors $C_i'$ which are orthogonal to $X_j$ for $j \neq i$, and within that class choose the one which maximizes the minimum of $\phi_i$ over the restricted parameter space.

4.2 Extension: Fixed $\sigma^2$ Case

In determining the estimator $B_i$ of $\beta_i$, we will proceed exactly as in Chapter 2, except that, where in Chapter 2 we considered K-H space, we now consider $K_i - H_i$ space defined as follows:

Definition 4.1: $K_i - H_i$ space is the set of all ordered pairs $(K_i, H_i)$ for which there exists a vector $C_i$ such that $C_i'X_i = K_i$, $C_i'C_i = H_i$, and $C_i'X_j = 0$ for $j \neq i$.

We can now proceed exactly as in Chapter 2, but we must first determine the boundaries of $K_i - H_i$ space.

Consider an arbitrary $C_i$:

$$C_i = A_iX + A_2X_2 + \ldots + A_pX_p + Z_i$$

where $Z_i'X_j = 0$, $i, j = 1, 2, \ldots, p$.

Writing $A_i' = (A_{i1}, A_{i2}, \ldots, A_{ip})$ we have,

$$C_i = XA_i + Z_i.$$ 

Now $C_i'X = (A_i'X + Z_i')X = A_i'X'X + Z_i'X = A_i'X'X$

But $C_i'X = (C_i'X_1, C_i'X_2, \ldots, C_i'X_p) = (0, 0, \ldots, K_i, 0, \ldots, 0) = K_i e_i'$

where $e_i'$ is a row vector of zeros except for a one in the $i$th position.

So we have,
\[ A_1'X'X = K_1e_1' \]

or

\[ A_1 = (X'X)^{-1}e_1K_1. \]

Now consider

\[ H_1 = C_1'C_1 = (A_1'X'X + Z_1')(XA_1'X + Z_1) \]

so

\[ H_1 = A_1'X'X + Z_1'Z_1 \]

\[ = K_1e_1'(X'X)^{-1}(X'X)(X'X)^{-1}e_1K_1 + Z_1'Z_1 \]

\[ = K_1^2e_1'(X'X)^{-1}e_1 + Z_1'Z_1 \]

\[ = K_1^2d_1 + Z_1'Z_1 \]

where \( d_1 \) is the \( i \)th diagonal term of \((X'X)^{-1}\).

Now \( Z_1'Z_1 \) is the square of the length of \( Z_1 \), and hence is non-negative.

Therefore the boundary of \( K_1-H_1 \) space is \( H_1 = K_1^2d_1 \) and points in \( K_1-H_1 \) space must satisfy: \( H_1 \geq K_1^2d_1 \).

This result is summarized in the following lemma:

**Lemma 4.1:** Given the \( p \) linearly independent \( n \) component vectors: \( X_1, X_2, \ldots, X_p \); the set of all ordered pairs \((K_1, H_1)\) where \( K_1 = C_1'X_1 \), \( H = C_1'C_1 \), and such that \( C_1'X_j = 0 \) for \( j \neq i \) for some \( C_i \), must satisfy:

\[ K_1^2d_1 \leq H_1 \]

where \( d_1 \) is the \( i \)th diagonal term of \((X'X)^{-1}\). QED
On the boundary \( H_1 = K_1^2 d_1 \) we must have \( Z_1 = 0 \), so,

\[
C_1 = XA_1 + 0
\]
or

\[
C_1 = XA_1 = X(X'X)^{-1}e_1 K_1
\]

But

\[
B_1 = C_1'Y = K_1 e_1' (X'X)^{-1} X'Y = K_1 e_1' \hat{\beta} = K_1 \hat{\beta}_1.
\]

In Theorem 2.1 if we replace \( B \) with \( B_1 \), \( \beta \) with \( \hat{\beta}_1 \), \( M \) with \( M_1 \), \( \varepsilon \) with \( e_1 \), \( \hat{\beta} \) with \( \hat{\beta}_1 \), and \( d \) with \( d_1 \), we have immediately that the maxi-min solution is \( B_1 = \gamma_1 \hat{\beta}_1 \) where

\[
\gamma_1 = \frac{-1 + \sqrt{1 + 2 \delta_1}}{\delta_1}, \quad \delta_1 = \frac{\sigma^2 d_1}{M_1 e_i} \ln \left[ \frac{M_1 + e_i}{M_1 - e_i} \right].
\]

Hence the general estimator \( B \) is

\[
B = \begin{bmatrix}
\gamma_1 \\
\gamma_2 \\
\vdots \\
\gamma_p
\end{bmatrix}
\]

4.3 **Extension: \( \sigma^2 \varepsilon [ \sigma_0^2, \sigma_1^2 ] \) Case**

Here we have exactly the same model as in the fixed \( \sigma^2 \) case, only now we have \( \sigma^2 \varepsilon [ \sigma_0^2, \sigma_1^2 ] \). Referring to Theorems 3.1 and 3.2, and replacing \( B \) with \( B_1 \), \( \beta \) with \( \hat{\beta}_1 \), \( M \) with \( M_1 \), \( \varepsilon \) with \( e_1 \), \( \hat{\beta} \) with \( \hat{\beta}_1 \), and \( d \) with \( d_1 \), we have immediately the result:
\[ B = \begin{bmatrix} K_1 & \hat{\beta} \\ K_2 & K_p \end{bmatrix} \]

where \( K_1 \) is determined by the method of Theorem 3.2.
5. AN ALTERNATE ESTIMATOR

5.1 General Structure

A second estimator will be derived based on the second procedure discussed in Chapter 1. Defining $T(\beta, \sigma_1^2, K, H)$ as:

$$T(\beta, \sigma_1^2, K, H) = \Pr[\beta - \epsilon \leq \hat{\beta} \leq \beta + \epsilon] - \Pr[\beta - \epsilon \leq \hat{\beta} \leq \beta + \epsilon]$$

with $B = C'Y \sim N(K\beta, H\sigma^2)$

our decision rule is to choose the $B$ which maximizes the minimum of $T$ over the restricted parameter space. For $\sigma^2$ fixed, $\Pr[\beta - \epsilon \leq \hat{\beta} \leq \beta + \epsilon]$ is completely specified, and the above procedure is no different from the first procedure, giving the same solution. Hence we consider only the $\sigma^2[\sigma_{0}^2, \sigma_{1}^2]$ case. Consequently, the general structure is that of Chapter 3, but with a new measure of goodness of the estimator. That is, we are considering the single $\beta$ linear model:

$$Y = X\beta + E \quad E \sim N(0, I\sigma^2)$$

$\sigma^2[\sigma_{0}^2, \sigma_{1}^2], \; |\beta| \leq M$

5.2 Form of the Estimator: $K\hat{\beta}$

Proceeding as in Chapter 3 we have:

Theorem 5.1: The $B$ which maximizes the minimum of $T(\beta, \sigma_1^2, K, H)$ over the restricted parameter space: $\sigma^2[\sigma_{0}^2, \sigma_{1}^2], \; |\beta| \leq M$; can be written in the form: $B = K\hat{\beta}$ for some $K \in [0, 1]$.

Proof: Since $\Pr[\beta - \epsilon \leq \hat{\beta} \leq \beta + \epsilon]$ does not depend on $\beta$, $T$ will be minimized with respect to $\beta$ when $\beta = M$. 
Consider again the partition of K-H space given in Figure 2.1 of Chapter 2. For \( H > (X'X)^{-1} \) let \( \sigma_2^2 \) be the value of \( \sigma^2 \) in \([\sigma_0^2, \sigma_1^2]\) which minimizes \( T \) at the point \((1,H)\), and let \( \sigma_3^2 \) be the corresponding minimizing value for the point \((K,H)\) \(K \neq 1\) in region C. Then by Theorem 2.1 we have:

\[
T(M, \sigma_3^2, K, H) \leq T(M, \sigma_2^2, K, H) < T(M, \sigma_2^2, 1, H)
\]

Hence any point in C can be eliminated in favor of some point in D.

For \( H \leq (X'X)^{-1} \) let \( \sigma_2^2 \) now be the value of \( \sigma^2 \) in the interval \([\sigma_0^2, \sigma_1^2]\) which minimizes \( T \) at the point \((\sqrt{H}, \sqrt{X'TX}, H)\), and let \( \sigma_3^2 \) be the minimizing value at the point \((K,H)\) for \( K < \sqrt{H} \sqrt{X'TX} \). Hence by Theorem 2.1 we have:

\[
T(M, \sigma_3^2, K, H) \leq T(M, \sigma_2^2, K, H) < T(M, \sigma_2^2, \sqrt{H} \sqrt{X'TX}, H)
\]

so we can eliminate any point in region B in favor of one in region A.

Lemma 3.2 implies that \( T(M, \sigma_2^2, 1, H) < 0 \) for any \( \sigma^2 \) and for \( H > (X'X)^{-1} \), so all points in region D can be eliminated in favor of the point \([1, (X'X)^{-1}]\) at which \( T = 0 \) for all \( \sigma^2 \). Thus there remain only points on the curve: \( H = K^2(X'X)^{-1} \) for \( K \in [0,1] \).

We have seen before that this is sufficient to imply that the estimator is of the form:

\[
B = \hat{K} B \text{ for some } K \in [0,1].
\]

QED

As a consequence of this theorem we can redefine \( T \) to be a function of \( K \) and \( \sigma^2 \) only:

\[
T(\sigma^2, K) = \Pr[M - \varepsilon < \hat{K} B \leq M + \varepsilon] - \Pr[M - \varepsilon \leq \hat{B} \leq M + \varepsilon].
\]
5.3 Properties of $T(K, \sigma^2)$ for K Fixed

The problem which presents itself here is that for any fixed K value, we cannot immediately conclude that $T(K, \sigma^2)$ will assume a minimum at either $\sigma^2_0$ or $\sigma^2_1$. It will therefore be necessary to investigate properties of $T(K, \sigma^2)$ as a function of $\sigma^2$ for K fixed. Important properties are:

5.3.1 By virtue of the one to one correspondence between $\sigma^2$ and the value of K which maximizes $\Phi$ in Chapter 2, we know that for any fixed K there will exist values of $\sigma^2$ for which $T(K, \sigma^2) > 0$.

5.3.2 $\lim_{\sigma^2 \to \infty} T(K, \sigma^2) = 0$.

5.3.3 If $K < 1 - \frac{\bar{\mu}}{M}$ then $\lim_{\sigma^2 \to 0} T(K, \sigma^2) = -1$.

5.3.4 If $K = 1 - \frac{\bar{\mu}}{M}$ then $\lim_{\sigma^2 \to 0} T(K, \sigma^2) = -\frac{1}{2}$.

5.3.5 If $K > 1 - \frac{\bar{\mu}}{M}$ then $\lim_{\sigma^2 \to 0} T(K, \sigma^2) = 0$.

**Lemma 5.1:** For $K > 1 - \frac{\bar{\mu}}{M}$ $T(K, \sigma^2)$ approaches zero through negative values as $\sigma^2 \to 0$.

**Proof:** The derivative of $T(K, \sigma^2)$ with respect to $\sigma^2$ can be written as:

$$\frac{dT(K, \sigma^2)}{\sigma^2} = \frac{D}{\sigma^2} F(\sigma^2),$$

where $F(\sigma^2) = A \exp\{-\frac{\bar{\mu}}{\sigma^2}\} + B \exp\{-\frac{\bar{\nu}}{\sigma^2}\} + C \exp\{-\frac{\bar{\tau}}{\sigma^2}\}$.
and where $A = K - 1 - W; B = 1 - K - W$

$$W = \frac{\xi}{M};\ C = 2KW$$

$$a = \frac{M^2A^2}{2K^2d}; b = \frac{M^2B^2}{2K^2d}; c = \frac{M^2W^2}{2d}.$$ 

Now, since $a > b$ and $c > b$, the sign of $F(\sigma^2)$ as $\sigma^2 \to 0$ will be determined by the sign of $B$, which is negative. Hence $\frac{\partial T(K, \sigma^2)}{\partial \sigma^2}$ is negative as $\sigma^2 \to 0$, and since for $K > 1 - \frac{\xi}{M}$, $\lim_{\sigma^2 \to 0} T(K, \sigma^2) = 0$, $T(K, \sigma^2)$ must approach zero through negative values. QED

Properties 5.3.1, 5.3.2, 5.3.3, and 5.3.4 imply that for $K \leq 1 - \frac{\xi}{M}$, $T(K, \sigma^2)$ will have at least one local maximum, while properties 5.3.1, 5.3.2, 5.3.5, and Lemma 5.1 show that for $K > 1 - \frac{\xi}{M}$, $T(K, \sigma^2)$ will have at least one local maximum, and at least one local minimum.

We now proceed to investigate $T(K, \sigma^2)$ by evaluating its first and second derivatives. Recall from Lemma 5.1 that:

$$\frac{dT(K, \sigma^2)}{d\sigma^2} = \frac{D}{\sigma^2} F(\sigma^2).$$

The second derivative of $T$ with respect to $\sigma^2$ is then:

$$\frac{d^2T(K, \sigma^2)}{d\sigma^2^2} = -\frac{D}{\sigma^4} F(\sigma^2) + \frac{D}{\sigma^2} F'(\sigma^2)$$

where $F'(\sigma^2) = \frac{Aa}{\sigma^4} \exp \left(-\frac{a}{\sigma^2}\right) + \frac{Bb}{\sigma^4} \exp \left(-\frac{b}{\sigma^2}\right) + \frac{Cc}{\sigma^4} \exp \left(-\frac{c}{\sigma^2}\right)$.

We now consider the sign of the second derivative of $T$, at values of $\sigma^2$ which give rise to local maxima or minima. At such values we find that:
\[
\frac{d^2 T(K, \sigma^2)}{d \sigma^2} = \frac{D}{\sigma} \left[ B(b-a) \exp\left( -\frac{b}{\sigma^2} \right) + C(c-a) \exp\left( -\frac{c}{\sigma^2} \right) \right]
\]

where \( b-a = \frac{2M^2(K-1)W}{K^2} < 0 \)

\( c-a = \frac{M^2}{2K^2d} [W^2(K^2-1) - (K-1)^2 + 2(K-1)W] < 0 \)

\( C = 2WK > 0 \)

and \( B = 1-W-K \geq 0 \) if and only if \( K \leq 1-W \).

Hence for \( K \leq 1 - \frac{\varepsilon}{M} \) and fixed, \( \frac{\partial^2 T(K, \sigma^2)}{\partial (\sigma^2)^2} < 0 \). This, together with the earlier result that there exists at least one local maximum implies that for \( K < 1 - \frac{\varepsilon}{M} \), \( T(K, \sigma^2) \) is unimodal.

So for \( K \leq 1 - \frac{\varepsilon}{M} \), \( T(K, \sigma^2) \) will assume a minimum with respect to \( \sigma^2 \) at either \( \sigma^2_0 \), or \( \sigma^2_1 \).

For \( K > 1 - \frac{\varepsilon}{M} \) we will have \( \frac{\partial^2 T(K, \sigma^2)}{\partial (\sigma^2)^2} = 0 \) at one and only one value of \( \sigma^2 \). Call this value \( \sigma^* \). Then for \( \sigma^2 < \sigma^* \), \( \frac{\partial T(K, \sigma^2)}{\partial (\sigma^2)^2} > 0 \), and for \( \sigma^2 > \sigma^* \), \( \frac{\partial^2 T(K, \sigma^2)}{\partial (\sigma^2)^2} < 0 \). This implies that \( T \) can have at most one local maximum, and at most one local minimum. Since we determined earlier that \( T \) will have at least one of each, it has exactly one of each.

Figure 5.1 depicts a representation of this function.

It is seen from Figure 5.1 that for \( K > 1 - \frac{\varepsilon}{M} \) and fixed, that \( T(K, \sigma^2) \) could assume a minimum at a value of \( \sigma^2 \) not equal to \( \sigma^2_0 \) or \( \sigma^2_1 \).

However, that we are still justified in restricting our search to finding the \( K \) which maximizes the minimum of \( \{T(K, \sigma^2_0), T(K, \sigma^2_1)\} \) is seen from the fact that for \( K = 1 \), the above minimum is zero, and we would always, therefore, take this solution before one which would give a negative minimum.
Figure 5.1. Typical representation of $T(K, \sigma^2)$ for $K > 1 - \frac{\epsilon}{M}$ and fixed

5.4 Solution for $K$

We now seek to find the value of $K$ which maximizes the minimum of the two functions: $T(K, \sigma_0^2)$, and $T(K, \sigma_1^2)$. At $K = 1$ both functions are zero. As $K$ decreases from 1, both functions increase (Theorem 2.2, Lemmas 3.3 and 3.4) until $K$ assumes the value which maximizes $\phi(K, \sigma_0^2)$. Call this value $K_0$. If $T(K_0, \sigma_0^2) \leq T(K_0, \sigma_1^2)$, $K_0$ is the maxi-min solution. Otherwise as we continue to decrease $K$ $T(K, \sigma_0^2)$ decreases while $T(K, \sigma_1^2)$ increases. Call $K_1$ the value of $K$ which maximizes $\phi(K, \sigma_1^2)$. If $T(K_1, \sigma_1^2) \leq T(K_1, \sigma_0^2)$, then $K_1$ is the maxi-min solution. Otherwise there will exist a value $K_2$: $K_1 < K_2 < K_0$ for which $T(K, \sigma_0^2) = T(K_2, \sigma_1^2)$ and $K_2$ is the maxi-min solution. That $K_1$ cannot be the maxi-min solution is proved in the following theorem:

Theorem 5.2: In the notation given above,

$$T(K_1, \sigma_1^2) > T(K_1, \sigma_0^2).$$

Proof: It was shown earlier that $T(K_1, \sigma_0^2) > 0$, hence it is sufficient to show that:
\[
\frac{\partial T(K_1, \sigma^2)}{\partial \sigma^2} \bigg|_{\sigma^2 = \sigma_1^2} > 0.
\]

Recall that

\[
\frac{\partial T(K, \sigma^2)}{\partial \sigma^2} = \frac{D}{\sigma^2} \left[ A \exp \left\{ - \frac{a}{\sigma^2} \right\} + B \exp \left\{ - \frac{b}{\sigma^2} \right\} + C \exp \left\{ - \frac{c}{\sigma^2} \right\} \right].
\]

Evaluating at the value of \( \sigma^2 \) which gives \( K \) as the maximum of \( \Phi(K, \sigma^2) \):

\[
\sigma^2 = \frac{\frac{1}{K} (\frac{1}{K-1}) (2M^2W)}{d \ln \left[ \frac{1+W}{1-W} \right]}
\]

so

\[
\exp \left\{ - \frac{b}{\sigma^2} \right\} = \left( \frac{1+W}{1-W} \right) \exp \left\{ - \frac{a}{\sigma^2} \right\}.
\]

Also

\[
\exp \left\{ - \frac{a}{\sigma^2} \right\} = \left[ \frac{1+W}{1-W} \right] \frac{adK^2}{2M^2W(1-K)}
\]

and

\[
\exp \left\{ - \frac{c}{\sigma^2} \right\} = \left[ \frac{1+W}{1-W} \right] \frac{cdK^2}{2M^2W(1-K)}.
\]

So

\[
\frac{\partial T(K_1, \sigma^2)}{\partial \sigma^2} \bigg|_{\sigma^2 = \sigma_1^2} = \frac{D}{\sigma^2} \left[ A + B \left( \frac{1+W}{1-W} \right) \frac{adK^2}{2M^2W(1-K)} \right]
\]

\[
+ C \left[ \frac{1-W}{1+W} \right] \frac{cdK^2}{2M^2W(1-K)} = \frac{D(2KW) \left[ \left( \frac{1}{1-W} \right) \frac{1-W}{1+W} \frac{adK^2}{2M^2W(1-K)} + \left( \frac{1-W}{1+W} \right) \frac{cdK^2}{2M^2W(1-K)} \right]}{\sigma^2}.
\]
\[
\frac{2DKW}{\sigma^2} \left[ \frac{1-W}{1+W} \right] \frac{cdK^2}{2M^2W(1-K)} \left\{ \frac{-1}{(1+W)^2} \left[ \frac{1-W}{1+W} \right] \frac{adK^2-cdK^2}{2M^2W(1-K)} - 1 + 1 \right\}.
\]

Now \( \frac{adK^2-cdK^2}{2M^2W(1-K)} - 1 = \frac{W(1+K)}{4} + \frac{(1-K)}{4W} - \frac{1}{2} \)

which is minimized when \( K \) is as large as possible: \( K = 1, \) so:

\[
\frac{adK^2-cdK^2}{(1-W)^2M^2W(1-K)} - 1 \leq \left[ \frac{1-W}{1+W} \right]^2 \frac{W-1}{2}.
\]

Set

\[ T(W) = \frac{1}{1+W} \left[ \frac{1-W}{1+W} \right]^2 W - 1. \]

then

\[
\frac{dT(W)}{dW} = \frac{1}{2(W+1)} \left[ \frac{1-W}{1+W} \right]^2 \ln \frac{1-W}{1+W} < 0
\]

for \( W \) in the open interval \((0,1)\).

Hence \( T(W) \) for \( W \in (0,1) \) will always be less than \( T(0) = 1; \) i.e.,

\[
\frac{1}{1+W} \left[ \frac{1-W}{1+W} \right]^2 W - 1 < 1.
\]

So

\[
\frac{-1}{1+W} \left[ \frac{1-W}{1+W} \right]^2 + 1 > 0
\]

which implies that

\[
\frac{\partial T(K_1, \sigma^2)}{\partial \sigma^2} \bigg|_{\sigma^2=\sigma^2_1} > 0.
\]

QED

The results of this Chapter are summarized in the following theorem.
Theorem 5.3: The estimator $B = C'Y$ which maximizes the minimum of $T(\beta, \sigma^2, K, H)$ over the restricted parameter space: $|\beta| \leq M$, $\sigma^2 \in [\sigma^2_0, \sigma^2_1]$ is

$$B = K\hat{\beta}$$

where $K = \begin{cases} 
Z_0 & \text{if } T(Z_0, \sigma^2_0) \leq T(Z_0, \sigma^2_1) \\
Z_2 & \text{otherwise}
\end{cases}$

where $Z_0$ is the value of $K$ which maximizes $\phi(K, \sigma^2_0)$, and where $Z_2$ is the largest value less than 1 such that:

$$T(Z_2, \sigma^2_0) = T(Z_2, \sigma^2_1).$$

The generalization of this result to the $p \beta$ case follows exactly as in Chapter 4, where each estimator is obtained separately, and the composite result can be written as:

$$B = D\hat{\beta}$$

where $D$ is the diagonal matrix whose $i$th diagonal term is determined as in this Chapter, with $\beta$ replaced by $\beta_i$, $M$ replaced by $M_i$, $\epsilon$ replaced by $\epsilon_i$, and $d$ replaced by $d_i$. 
6. SOME PROPERTIES OF THE ESTIMATORS

6.1 Dependence on $K_i$

Each of the estimators can be written as: $\hat{R} = D\hat{\beta}$, with $D$ a diagonal matrix having $i$th diagonal term equal to $K_i$, where $K_i$ is determined as in Chapters 2, 3, 4, or 5, depending on the parameter space restrictions, and on the chosen decision procedure. Hence properties of this estimator are dependent exclusively on the $K_i$ values, together with the known properties of the best linear unbiased estimator: $\hat{R}$.

6.2 Bounds on $K_i$

If we define $\gamma_{ij}$ as:

$$\gamma_{ij} = \frac{-1 + \sqrt{1 + 2\delta_{ij}}}{\delta_{ij}}$$

with $\delta_{ij} = \frac{\sigma_{ij}^2}{M_i \varepsilon_i} \ln \left[ \frac{M_i + \varepsilon_i}{M_i - \varepsilon_i} \right]$ for $i = 1, 2, \ldots, p$; and $j = 0, 1$

then we have seen that:

$$0 < \gamma_{i1} \leq K_i \leq \gamma_{i0} < 1$$

for all cases. Hence we present properties of $\gamma_{ij}$:

6.2.1 $\lim_{\varepsilon_i \to M_i} \gamma_{ij} = 0$ for $j = 0, 1$

6.2.2 $\lim_{M_i \varepsilon_i} \gamma_{ij} = 0$ for $j = 0, 1$

6.2.3 $\lim_{M_i \to \infty} \gamma_{ij} = 1$ for $j = 0, 1$
6.2.4 \[ \lim_{\varepsilon_i \to 0} \gamma_{ij} = \sqrt{\frac{1+\frac{4\sigma^2_{
u_1}}{M_1^2}}{2\sigma^2_{
u_1}} j = 0,1} \]

6.3 **Bias**

Since \( B = \hat{\beta} \) has expected value \( E[B] = \hat{\beta} \), or for any individual \( B_i \): \( E[B_i] = K_i \hat{\beta}_i \), the absolute value of the bias is:

\[ |E[B_i] - \hat{\beta}_i| = |K_i \beta_i - \beta_i| = (1-K_i)|\beta_i| \]

which, under our restrictions on the parameter space, is bounded by:

\[ |E[B_i] - \beta_i| \leq (1-K_i)M_1. \]

6.4 **Variances and Covariances**

The variance-covariance matrix of \( B \) is \( Var[B] = D(X'X)^{-1}D \sigma^2 \)

which gives \( Var[B_i] = K_i^2 Var[\hat{\beta}_i] = K_i^2 \sigma^2 \) and \( Cov[B_i, B_j] = K_i K_j Cov[\hat{\beta}_i, \hat{\beta}_j] \).

6.5 **On Limiting Properties**

Considering the \( n \) by \( p \) matrix \( X \) to be fixed, \( m \) replications give:

\[
\begin{bmatrix}
Y_1 \\
Y_2 \\
\vdots \\
Y_m
\end{bmatrix}
= 
\begin{bmatrix}
X \\
X \\
\vdots \\
X
\end{bmatrix}
\begin{bmatrix}
\hat{\beta} \\
\beta
\end{bmatrix}
+ 
\begin{bmatrix}
E_1 \\
E_2 \\
\vdots \\
E_m
\end{bmatrix}
= X\hat{\beta} + \varepsilon
\]

\((mx1) (mxp) (mx1)\)
\[ X'X = mX'X \]

so \((X'X)^{-1} = \frac{1}{m}(X'X)^{-1}\).

Hence the effect of \(m\) replications is to decrease the diagonal term of the \((X'X)^{-1}\) matrix by a factor of \(\frac{1}{m}\). Thus in the expression for \(\gamma_{ij}\), \(d_i\) is reduced by a factor of \(\frac{1}{m}\), which causes \(\gamma_{ij}\) to increase. This limiting properties of \(K\hat{\beta}\) can be established through knowledge of limiting properties of \(\hat{\beta}\), and through the fact that \(K \rightarrow 1\).
7. A NUMERICAL EXAMPLE

7.1 General Problem

In order to obtain a feeling for the values of $K$ which may occur for the derived estimators under different restrictions on the parameter space and different $\varepsilon$ values, as well as for the probabilities under these conditions of the estimator being contained in the interval $[M-\varepsilon, M+\varepsilon]$, we consider an example of a simple linear regression of one independent variable on one dependent variable. The model is then:

$$Y_i = \alpha + \beta X_i + E_i \quad E_i \sim N(0, \sigma^2)$$

where $E_i$ and $E_j$ are independent for $j \neq i$.

For simplicity we will consider estimation of $\beta$ only; we could consider the model as:

$$Y_i - \bar{Y} = \beta (X_i - \bar{X}) + E_i$$

For each estimator of $\beta$, either for $\sigma^2$ fixed, for the estimator of the first procedure, which we will refer to as Type A, or for the estimator of the second procedure, which we will refer to as Type B, we will be interested only in determining the appropriate value of $K$ for which $B = \hat{\beta}$.

Since $\hat{\beta} \sim N(\beta, d\sigma^2)$ where for our above case $d = \frac{1}{\sum (X_i - \bar{X})^2}$, and since $K$ decreases as $d\sigma^2$ increases, we would like, for illustrative purposes, an independent variable for which $\sum_{i=1}^{n} (X_i - \bar{X})^2$ is small. Considering the 10 variables on page 352 of Draper and Smith (2), $X_3$ is found to have the smallest variance, and hence is chosen as the
independent variable. $X_2$ is found to have a significant regression on $X_3$, and is therefore used as the dependent variable. The BLUE of $\beta$ is: 6.1248, and the estimate of $\sigma^2$ is: .0467.

7.2 Fixed $\sigma^2$ Case

We first consider values of $K$ for the fixed (known) $\sigma^2$ case. The following seven values of $\sigma^2$ will be separately considered: .0001, .001, .01, .05, .1, 1, and 10. For the limits of the absolute value of $\beta$ we consider separately: $M = 7, 10, 20, \text{ and } 50$. Also, for the value of $\varepsilon$ we consider $\varepsilon = .05, .10, .50, 1, 5, \text{ and } 10$ where appropriate. Note that since we must have: $\varepsilon < M$, certain combinations of the above stated input values are impossible and are denoted N.A. (for "not applicable") in the following tables.

In the following seven tables for each applicable combination of $M$ and $\varepsilon$ two values are given. The upper value, given to four decimal place accuracy, is the calculated value of $K$, while the lower value is the increase in the probability of being within the interval $[M-\varepsilon, M+\varepsilon]$ which this value of $K$ gives rise to. This latter probability is the improvement over the BLUE probability, which is given for each value of $\varepsilon$ (the BLUE probability does not depend on $\beta$). The superscripted plus and minus signs after a 1 or a 0 mean respectively that the actual number is a little larger, or smaller than that given, but the given number is accurate to 4 decimal places if it is a $K$ value, and to 5 decimal places if it is a probability.

It should be pointed out that in the following tables, the probabilities quoted for the derived estimator are the smallest possible for the given range of $M$ values, and that the true probability could be considerably greater.
Table 7.1. Values of K and corresponding probability increases:
\[ \Pr[M - \epsilon \leq S \leq M + \epsilon] - \Pr[M - \epsilon \leq S \leq M + \epsilon] \] for fixed \( \sigma^2 \) case with \( \sigma^2 = .0001 \)

<table>
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<th>M ( \epsilon )</th>
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Table 7.2. Values of $K$ and corresponding probability increases:
$Pr[M - \varepsilon \leq \hat{\theta} \leq M + \varepsilon] - Pr[M_{\varepsilon} \leq \hat{\theta} - M + \varepsilon]$ for fixed $\sigma^2$ case --
$\sigma^2 = .001$

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Table 7.3. Values of K and corresponding probability increases: 
\( \Pr[\{M - \varepsilon S \leq \varepsilon M + \varepsilon\}] - \Pr[\{M - \varepsilon S \geq \varepsilon M + \varepsilon\}] \) for fixed \( \sigma^2 \) case -- 
\( \sigma^2 = .01 \)

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Table 7.4. Values of $K$ and corresponding probability increases:
$\Pr[M - \varepsilon \leq Z \leq M + \varepsilon] - \Pr[M - \varepsilon < Z < M + \varepsilon]$ for fixed $\sigma^2$ case --
$\sigma^2 = 0.1$

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Table 7.5. Values of K and corresponding probability increases:
\[ \text{Pr}[M-\epsilon \leq \hat{\beta} \leq M+\epsilon] - \text{Pr}[M-\epsilon \leq \hat{\beta} \leq M+\epsilon] \] for fixed \( \sigma^2 \) case. --
\( \sigma^2 = .05 \)

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Table 7.6. Values of K and corresponding probability increases:
\[ \Pr[M - \varepsilon \leq \hat{\beta} \leq M + \varepsilon] - \Pr[M - \varepsilon \leq \hat{\beta} \leq M + \varepsilon] \] for fixed \( \sigma^2 \) case --
\( \sigma^2 = 1.0 \)

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Table 7.7. Values of $K$ and corresponding probability increases:
$Pr[M - \varepsilon \leq \hat{\beta} \leq M + \varepsilon]$ - $Pr[M - \varepsilon \leq \hat{\beta} \leq M + \varepsilon]$ for fixed $\sigma^2$ case -- $\sigma^2 = 10.0$

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As well as the absolute increase in probability for the derived estimator versus that for the BLUE estimator, it is also interesting to have a feeling for the percentage increase in the probability. Table 7.8 gives such percentages for the $M = 10$ case.

Table 7.8. Percent increase in $\Pr[M - \varepsilon \leq \hat{\beta} \leq M + \varepsilon]$ over $\Pr[M - \varepsilon \leq \hat{\beta} \leq M + \varepsilon]$ for $M=10$ and $\sigma^2$ fixed

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7.3 $\sigma^2_{[\sigma^2_0, \sigma^2_1]}$ Case

We now turn to the Type A and Type B estimators. For consistency, the same values of $M$ and $\varepsilon$ are used as in the fixed $\sigma^2$ case. For each $\varepsilon$ value, BLUE probabilities: $\Pr[M - \varepsilon \leq \hat{\beta} \leq M + \varepsilon]$ are calculated at both $\sigma^2_0$ and at $\sigma^2_1$. In the following tables, in the BLUE probability column, for each value of $\varepsilon$ the upper probability was calculated for $\sigma^2_0$, and the lower for $\sigma^2_1$. As before, for each pair of $M, \varepsilon$ values, a value of $K$ is presented, but now there are two probability deviations for each $M, \varepsilon$ pair. For each derived $K$ value the functions $T(M, \sigma^2_0, K)$ and $T(M, \sigma^2_1, K)$ are evaluated and are presented respectively in the tables. Recall that the function $T(M, \sigma^2, K)$ gives the increase
in closeness of the derived estimator over the BLUE estimator when \( \sigma^2 \) is the true value. That some of these deviations are negative reflects the fact that the Type A estimator is not uniformly better than the BLUE estimator over the restricted parameter space.

Recall also that the probability deviations given are the smallest possible over the restricted parameter space. A point in the interior of the restricted parameter space will give rise to even larger deviations for our estimator over the BLUE probability.

The strategy for selecting the \( \sigma^2 \) intervals was that we considered our estimate of \( \sigma^2 \): .0467 as being the "true" value, and then we proceeded to consider intervals which did, or did not, contain this value. We started with an interval which did contain the true value, but which was quite wide, and hence reflected a vagueness as to where the true value of \( \sigma^2 \) was. We then considered two more intervals which were progressively less vague, and finally an interval which was "wrong" on either side. These intervals were: [.0001, 10.0], [.001, 1.0], [.01, .1], [.0001, .01] and [.1, 10]. After these cases were evaluated, it was decided that three additional intervals might be worth investigating, the first reflecting very little vagueness and the following two with relatively large \( \sigma^2 \)'s. These additional intervals were: [.03, .01], [1, 10], and [10, 20]. The tables follow.
Table 7.9. Values of \( K \), probability deviations, and BLUE probabilities for Type A estimator \( \sigma^2 \mathcal{E} \{ .0001, 10.0 \} \)

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Table 7.14. Values of K, probability deviations, and BLUE probabilities for Type A estimator -- \( \sigma^2 \epsilon [0.03, 0.07] \)

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Table 7.17. Values of $K$, probability deviations, and BLUE probabilities for Type B estimator -- $\sigma^2_\varepsilon[.0001, 10.0]$

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Table 7.18. Values of $K$, probability deviations, and BLUE probabilities for Type B estimator $-- \sigma^2_{\epsilon}[.001, 1.0]$  

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<td>1.0000$^-$</td>
<td>1.0000$^-$</td>
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</table>
Table 7.21. Values of $K$, probability deviations, and BLUE probabilities for Type B estimator -- $\sigma^2_e [0.1, 10]$

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>7</th>
<th>10</th>
<th>20</th>
<th>50</th>
<th>$\Pr[M - \varepsilon \leq \hat{\theta} \leq M + \varepsilon]$</th>
</tr>
</thead>
</table>
| .05           | .9905| .9953| .9988| .9998| .07767
|               | .00007 | .00004 | .00001 | .00000$^+$ | .00778
|               | .00007 | .00004 | .00001 | .00000$^+$ |
| .10           | .9905| .9953| .9988| .9998| .15460
|               | .00015 | .00007 | .00002 | .00000$^+$ | .01556
|               | .00015 | .00007 | .00002 | .00000$^+$ |
| .50           | .9911| .9956| .9989| .9998| .67042
|               | .00069 | .00034 | .00008 | .00001 | .07767
|               | .00069 | .00034 | .00008 | .00001 |
| 1             | .9947| .9974| .9993| .9999| .94882
|               | .00062 | .00030 | .00008 | .00001 | .15460
|               | .00081 | .00040 | .00010 | .00002 |
| 5             | .9934| .9971| .9993| .9999| 1.00000
|               | .00000 | .00000 | .00000 | .00000 | .67042
|               | .00320 | .00139 | .00032 | .00005 |
| 10            | N.A. | N.A. | .9993 | .9999 | 1.00000
|               | .00000 | .00000 | .00017 | .00002 | .94882
Table 7.22. Values of $K$, probability deviations, and BLUE probabilities for Type B estimator -- $\sigma^2_{\varepsilon}[.03, .07]$

<table>
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<tr>
<th>$\varepsilon$</th>
<th>M</th>
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<th>20</th>
<th>50</th>
<th>$Pr[\hat{M}-\varepsilon \leq \hat{M}+\varepsilon]$</th>
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Table 7.23. Values of $K$, probability deviations, and BLUE probabilities for Type B estimator -- $\sigma^2_{\varepsilon}[1, 10]$

<table>
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<tr>
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Table 7.24. Values of $K$, probability deviations, and BLUE probabilities for Type B estimator -- $\sigma^2_a[10, 20]$

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<th>50</th>
<th>$\Pr[M - \varepsilon \leq \hat{K} \leq M + \varepsilon]$</th>
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<td></td>
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<td>.00333</td>
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</table>
As in the fixed $\sigma^2$ case, it is interesting to consider also the percent increases in probability which our estimates give. For Type A estimates the appropriate criterion appears to be the percentage increase in closeness over the minimum BLUE probability. Both of these probabilities should then be evaluated at $\sigma_1^2$. For the Type B estimator a relevant criterion appears to be the minimum percentage increase in closeness over the BLUE probability. These probabilities, and hence deviations, are evaluated at $\sigma_0^2$. As before, these calculations are made for the $M = 10$ case. These results are found in Tables 7.25 and 7.26.
Table 7.25. Percent increase in probability over minimum BLUE probability for Type A estimator -- $M = 10$

<table>
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<td>0.01%</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.00%</td>
</tr>
<tr>
<td>[.03, .07]</td>
<td></td>
<td>0.09%</td>
<td>0.09%</td>
<td>0.06%</td>
<td>0.01%</td>
<td>0.00%</td>
</tr>
<tr>
<td>[.01, .1]</td>
<td></td>
<td>0.13%</td>
<td>0.13%</td>
<td>0.09%</td>
<td>0.03%</td>
<td>0.00%</td>
</tr>
<tr>
<td>[.001, 1]</td>
<td></td>
<td>1.06%</td>
<td>1.20%</td>
<td>1.25%</td>
<td>1.13%</td>
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</tr>
<tr>
<td>[.0001, 10]</td>
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<td>0.90%</td>
<td>1.35%</td>
<td>4.88%</td>
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<tr>
<td>[.1, 10]</td>
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</tr>
<tr>
<td>[1, 10]</td>
<td></td>
<td>11.31%</td>
<td>11.25%</td>
<td>11.24%</td>
<td>11.13%</td>
<td>7.98%</td>
</tr>
<tr>
<td>[10, 20]</td>
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</table>

Table 7.26. Percent increase in probability over minimum BLUE probability for Type B estimator -- $M = 10$

<table>
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<td>0.00%</td>
<td>0.00%</td>
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<tr>
<td>[.03, .07]</td>
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<td>0.04%</td>
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<td>0.00%</td>
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<tr>
<td>[.01, .1]</td>
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<td>0.01%</td>
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<td>0.00%</td>
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<td>[.001, 1.0]</td>
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</tr>
<tr>
<td>[.0001, 10]</td>
<td></td>
<td>0.00%</td>
<td>0.00%</td>
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<td>0.00%</td>
</tr>
<tr>
<td>[.1, 10]</td>
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<td>[10, 20]</td>
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<td>11.31%</td>
<td>11.25%</td>
<td>11.24%</td>
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</table>
8. SUMMARY, CONCLUSIONS, AND SUGGESTIONS
FOR FUTURE RESEARCH

8.1 Summary

Suppose an investigator is more interested in the "closeness in probability" of an estimator $\hat{B}$ for $\beta$, as measured by $\Pr[\beta - \varepsilon \leq \hat{B} \leq \beta + \varepsilon]$ for some predetermined value of $\varepsilon$; than he is in unbiasedness of his estimator for $\beta$. Then the methods presented here may be useful. It is, however, necessary that some knowledge of the parameter space $(\sigma^2, \beta)$ be available, either from the investigator's experience with the parameters, or from physical and/or mathematical constraints.

Two cases for $\sigma^2$ are considered separately: Case I, $\sigma^2$ fixed and known; Case II, $\sigma^2 \varepsilon [\sigma_0^2, \sigma_1^2]$ where $\sigma_0^2$ and $\sigma_1^2$ are known. For $\beta$ we must have a value $M$ such that: $|\beta| \leq M$; in the multiple regression situation with $p$ $\beta$'s: $|\beta_i| \leq M_i$, $i = 1, 2, \ldots p$.

Two decision rules based on the closeness probability are considered. The first is to choose as an estimator of $\beta$ that linear estimator $B$ which maximizes the minimum of the closeness probability over the restricted parameter space. This was done for both $\sigma^2$ cases. The second decision rule is to choose as our estimator of $\beta$ the $B$ which maximizes the minimum of the closeness probability minus the corresponding probability for the BLUE estimator $\hat{B}$. In other words, we choose the $B$ which maximizes the minimum of:

$$\Pr[\beta - \varepsilon \leq B \leq \beta + \varepsilon] - \Pr[\beta - \varepsilon \leq \hat{B} \leq \beta + \varepsilon]$$

over the restricted parameter space.
Note that when \( \sigma^2 \) is known, the two estimators are identical. Hence three essentially different estimators are developed: For \( \sigma^2 \) fixed, for \( \sigma^2 \varepsilon [\sigma_0^2, \sigma_1^2] \) with the first decision rule, and for \( \sigma^2 \varepsilon [\sigma_0^2, \sigma_1^2] \) with the second decision rule. The second of these estimators is referred to as the Type A estimator, while the third (and last) is referred to as the Type B estimator.

The extension from the single \( \beta \) case to the multiple \( \beta \) case is accomplished by requiring that the expected value of the estimator for \( \beta_i \), say, not depend on \( \beta_j \) for \( j \neq i, i, j = 1, 2, \ldots, p \). It is shown that, with this restriction, the derived estimator of \( \beta_i \), say, can be written as \( K_i \hat{\beta}_i \) where \( K_i \varepsilon (0, 1) \). Or, in matrix form:

\[
B = \begin{bmatrix}
B_1 \\
B_2 \\
\vdots \\
B_p
\end{bmatrix}
= \begin{bmatrix}
K_1 & & & & \\
& K_2 & & & \\
& & \ddots & & \\
& & & \cdots & \\
& & & & K_p
\end{bmatrix}
\begin{bmatrix}
\hat{\beta}_1 \\
\hat{\beta}_2 \\
\vdots \\
\hat{\beta}_p
\end{bmatrix}
= D\hat{\beta}
\]

where \( D \) is the diagonal \( p \times p \) matrix of \( K \) values.

The value of \( K_i \) for the fixed \( \sigma^2 \) case is:

\[
K_i = \frac{-1+\sqrt{1+2\delta_i}}{\delta_i} \text{ where } \delta_i = \frac{\sigma^2 d_i}{M_i \varepsilon_i} \ln \left( \frac{M_i + \varepsilon_i}{M_i - \varepsilon_i} \right)
\]

where \( d_i \) is the \( i \)th diagonal term of \( (X'X)^{-1} \).

For the Type A estimator \( \sigma^2 \varepsilon [\sigma_0^2, \sigma_1^2] \) and we define \( \delta_{ij} j = 0, 1 \) as the above \( K_i \) for \( \sigma^2 = \sigma_0^2 \), and then for \( \sigma^2 = \sigma_1^2 \), respectively. Then if

\[
\Pr[M_i - \varepsilon_i \leq \hat{\beta}_i \leq M_i + \varepsilon_i]
\]
evaluated at $\sigma^2 = \sigma^2_1$ is less than or equal to this probability evaluated at $\sigma^2 = \sigma^2_0$, the appropriate $K_1$ is $\delta_{11}$. Otherwise there will be a unique value of $K$ in the interval $(\gamma_{11}, \gamma_{10})$ for which this probability is the same for $\sigma^2 = \sigma^2_1$ as for $\sigma^2 = \sigma^2_0$. This $K$ is then the desired value of $K_1$.

For the Type B estimator the appropriate value of $K_1$ is $\delta_{10}$ if

$$\Pr[M_1 - \epsilon_1 \leq \hat{\beta} \leq M_1 + \epsilon_1] - \Pr[M_1 - \epsilon_1 \leq \hat{\beta} \leq M_1 + \epsilon_1]$$

evaluated at $\sigma^2 = \sigma^2_0$ is less than or equal to this probability difference evaluated at $\sigma^2 = \sigma^2_1$. Otherwise $K_1$ will be the unique value in the interval $(\delta_{11}, \delta_{10})$ for which this probability difference is the same for $\sigma^2 = \sigma^2_0$ as for $\sigma^2 = \sigma^2_1$.

8.2 Conclusions

In order for the derived estimator to produce any improvement in closeness, or even any change at all, it is obvious that $K$, the multiple of $\hat{\beta}$, must be appreciably less than 1. Otherwise the estimator will be essentially the same as $\hat{\beta}$, and have essentially the same properties. It therefore appears that the larger is $d\sigma^2$, the variance of $\hat{\beta}$, the greater is the potential improvement which can be obtained. For the fixed $\sigma^2$ case this is seen to hold in the numerical example. As the value of $\sigma^2$ increases, the relative improvement in the closeness probability is seen to increase.

For the Type A and Type B estimators we can say that as $\sigma^2_0$ and $\sigma^2_1$ increase, the relative improvement tends to increase, since $K_1$ must be in or on the boundary of the interval: $[\gamma_{11}, \gamma_{10}]$, and as $\sigma^2_0$ and $\sigma^2_1$ increase these boundary values decrease.
Also, as M decreases the calculated K values decrease as well, which agrees with our intuitive feeling that the more information available on a parameter the better we should be able to estimate it. It is also seen that as we permit our $2\varepsilon$ interval to become wider, the potential relative improvement increases.

In a practical setting $\sigma_0^2, \sigma_1^2$ and M will be fixed, but there may be some flexibility in the choice of $\varepsilon$. If a specific $\varepsilon$ is not required by the investigator, but rather a range of possible $\varepsilon$ values, tables could be compiled to compare percentage gains with different $\varepsilon$ values.

If, after making all the calculations for a fixed set of $\sigma_0^2, \sigma_1^2, M$, and $\varepsilon$, the value of K is calculated to be effectively 1, this does not indicate failure; it simply says that the BLUE estimator does a good job, and in the sense of our criterion, it is best.

8.3 Suggestions for Future Research

8.3.1 X'X Matrix of Less than Full Rank

The analysis performed was dependent on the X matrix being of full rank. If, as if often the case with design matrices, it is not of full rank, we still might wish to obtain an estimator based on closeness probability. This problem is intimately related to the linear constraints usually imposed on the parameters of experimental design models.

8.3.2 Transformation of Parameters

The bounding of the $\beta$'s by absolute values might be improved upon in the situation where one knows that $\beta \in [\beta_0, \beta_1]$, where $0 < \beta_0 < \beta_1$. A linear transformation on $\beta$ such as:
\[ \gamma = \beta - \frac{(\beta_0 + \beta_1)}{2} \]

could possibly reduce the value of M considerably.

8.3.3 **Alternate Decision Procedure**

A different decision procedure might be used, while still being based on the closeness probability. One such procedure might be Bayesian with specified prior distributions over the restricted parameter space.
9. LIST OF REFERENCES


10. APPENDICES
10.1 Appendix A - Computer Program to Calculate $K$ for $\sigma^2$ Fixed

This program, written in G.E. Time-Sharing Fortran as used by Call-a-Computer, Inc., calculates:

1) $\Pr[M-\epsilon \leq \hat{\beta} \leq M+\epsilon]$, denoted as BLUE probability
2) $K$ - the maximizing multiple of $\hat{\beta}$
3) $\Pr[M-\epsilon \leq K\hat{\beta} \leq M+\epsilon] - \Pr[M-\epsilon \leq \hat{\beta} \leq M+\epsilon]$ the improvement in closeness produced by $K$

for all combinations of $n$ different $\epsilon$ values, and $m$ different $M$ values, such that $M > \epsilon$. The number of different $\epsilon$ values is designated on line 2 as $II$. In this case $II = 6$. On lines 6 through 11 the 6 values of $\epsilon$ are defined. Note that in the program, the $i$th $\epsilon$ is written as: ALPHA(I). Any number of $\epsilon$'s up to 20 can be used.

The number of different $M$ values is designated on line 3 as JJ. In this case $JJ = 4$, and the 4 $M$ values are given on lines 14 to 17. Note that the $i$th $M$ is denoted as BL(I). The $i$th diagonal term of the $(X'X)^{-1}$ matrix is given on line 5, and is denoted as DD.

Finally, the number of different $\sigma^2$ values being evaluated is designated on line 4, and is denoted by NI. The $\sigma^2$ values are not written into the program, as the $\epsilon$ and $M$ values are, but are input upon running the program. They could, however, just as well have been written into the program. The program follows.
1 DIMENSION ALPHA(20), SIGO(20), B1(20)
2 4 II=6
3 JJ=4
4 NI=2
5 DD=2.5
6 ALPHA(1)=0.05
7 ALPHA(2)=0.10
8 ALPHA(3)=0.50
9 ALPHA(4)=1.0
10 ALPHA(5)=5.0
11 ALPHA(6)=10.0
12 INPUT, (SIGO(K),K=1,NI)
13 PRINT++++
14 B1(1)=7
15 B1(2)=10
16 B1(3)=20
17 B1(4)=50
18 DO 40 K=1,NI
19 PRINT +++
20 PRINT,"FOR SIGMA SQUARED="SIGO(K)
21 PRINT +++
22 DO 30 J=1,II
23 DO 27 IK=1,JJ
24 IF (B1(IK)-ALPHA(J)) 26,26,67
25 67 SIG=SIGO(K)
26 16 ZAP=DD*SIG*(LOG(B1(IK)+ALPHA(J))-LOG(B1(IK)-ALPHA(J)))
27 ZFAC=B1(IK)*ALPHA(J)
28 AA0=ZAP/ZFAC
29 IF (AA0-.00001) 117,117,121
30 117 DDO=1-((AA0/2)+((AA0**2)/2)-((AA0**3)*5)/8)
31 +=((AA0**4)*7)/8)
32 GO TO 327
33 121 CONTINUE
34 DDO=1-((AA0/2)+((AA0**2)/2)-((AA0**3)*5)/8)
35 +=((AA0**4)*7)/8)-((AA0**5)*21)/16
36 +=((AA0**6)*33)/16)-((AA0**7)*429)/128
37 +=((AA0**8)*715)/128)-((AA0**9)*2431)/256
38 +=((AA0**10)*4199)/256)-((AA0**11)*88179)/3072
39 +=((AA0**12)*52003)/1024)-((AA0**13)*1300075)/14336
40 +=((AA0**14)*334305)/2048)-((AA0**15)*9694845)/32768
41 IF (AA0-.01) 327,327,118
42 118 CONTINUE
43 IF(((AA0**16)*(17678835))/(32768))-.0000000001) 327,327,331
44 331 IEXP=16
45 COEFF=((9694845)*(2*IEXP-1))/(32768)*(IEXP+1)
46 333 DAR=COEFF*(AA0**IEXP)*((-1)**IEXP)
47 335 DDO=DDO+DAR
48 IEXP=IEXP+1
49 D AR=DAR*AA0*(2*IEXP-1)*(-1)/(IEXP+1)
50 DAR=ABS(DAR)
51 IF(DARA-10) 213,214,214
52 213 IF(DARA-.0000001) 327,327,335
53 214 DDO=(ZPAC+SQRT(ZPAC*(ZPAC+2*ZAP)))/ZAP
54 327 D=1
55 IGG=0
56 39 SS=SQR(D(SIGM(K))
57 UO=(B1(IK)+ALPHA(J)-D*B1(IK))/(D*SS)
58 ALO=(B1(IK)-ALPHA(J)-D*B1(IK))/(D*SS)
59 IF(UO-10) 816,817,817
60 817 IF(AL0+10) 814,814,815
61 814 PROBOT=0
62 GO TO 720
63 815 PROBOT=GP(ALO)
64 GO TO 720
65 816 IF(AL0+10) 824,824,825
66 824 PROBOT=1-GP(UO)
67 GO TO 720
68 825 PROBOT=1-GP(UO)+GP(ALO)
69 720 IF(IGG-1) 862,863,863
70 862 PRINT,"FOR EPSILON =",ALPHA(J),"AND M =",B1(IK)
71 PRINT,"BLUE PROBABILITY IS :"
72 PBO=1-PROBOT
73 PRINT 6,PBO
74 IGG=1
75 D=DDO
76 GO TO 39
77 863 PRINT,"MAXI-MIN SOLUTION K ="
78 PRINT 6,DDO
79 PEO=1-PROBOT-PBO
80 PRINT,"IMPROVEMENT IN PROBABILITY FOR DERIVED ESTIMATOR IS :"
81 PRINT 6,PEO
82 6 FORMAT(10X,F11.9)
83 26 CONTINUE
84 27 PRINT ++++
85 30 CONTINUE
86 40 CONTINUE
87 PRINT +++++
88 GO TO 4
89 END
90 FUNCTION GP(X)
91 GP=0
92 Y=X*X
93 IF(324-Y)2
94 \[ Z=0.3989422804*EXP(-Y/2) \]
95 IF(Y-9)3
96 Y=11/X
97 DO1C=1,10
98 1Y=(11-C)/(Y+X)
99 GP=-Z/(Y+X)
100 2IF(X)5
101 GP=1+GP
102 GO TO 5
103 3GP=Z
104 C=1
105 4C=C+2
106 Z=Z*Y/C
107 GP = GP + Z
108 IF (GP - Z * 1E8) ≤ 4
109 GP = .5 + X * GP
110 5
111 END
10.2 Appendix B - Computer Program to Calculate K for Type A Estimator

For this program values of ε and M are introduced exactly as in the fixed \( \sigma^2 \) case of Appendix A. Any number of \( \sigma_0^2 \) and \( \sigma_1^2 \) values can be evaluated at one time. The symbol IN designates the number of \( \sigma_0^2 \) values, while JN designates the number of \( \sigma_1^2 \) values.

This program calculates:

1) \( \Pr[M-\varepsilon \leq \hat{\beta} \leq M+\varepsilon] \) first for \( \sigma^2 = \sigma_0^2 \), and then for \( \sigma^2 = \sigma_1^2 \)

2) The values of K that would maximize: \( \Pr[M-\varepsilon \leq \hat{\beta} \leq M+\varepsilon] \) for \( \sigma^2 = \sigma_0^2 \), and \( \sigma^2 = \sigma_1^2 \), respectively. These values are the upper and lower bounds on K.

3) The value of K which produces the maxi-min solution, iterative or otherwise

4) The probability deviations: \( \Pr[M-\varepsilon \leq \hat{\beta} \leq M+\varepsilon] - \Pr[M-\varepsilon \leq \hat{\beta} \leq M+\varepsilon] \)

for the derived K at respectively \( \sigma_0^2 \) and \( \sigma_1^2 \)

The program follows.
1 DIMENSION ALPHA(15),SIG0(15),SIG1(15),B1(15)
2 4 II=6
3 JJ=4
4 IN=1
5 JN=1
6 DD=2.5
7 ALPHA(1)=0.05
8 ALPHA(2)=0.10
9 ALPHA(3)=0.50
10 ALPHA(4)=1.0
11 ALPHA(5)=5.0
12 ALPHA(6)=10.0
13 INPUT,(SIG0(K),K=1,IN)
14 INPUT,(SIG1(IJ),IJ=1,JN)
15 PRINT ++++
16 B1(1)=7
17 B1(2)=10
18 B1(3)=20
19 B1(4)=50
20 MID=0
21 DO 40 K=1,IN
22 DO 35 IJ=1,JN
23 PRINT +++
24 PRINT,"FOR SIGO =",SIG0(K),"AND SIG1 =",SIG1(IJ)
25 PRINT ++
26 DO 30 J=1,II
27 DO 27 IK=1,JJ
28 IF (B1(IK)-ALPHA(J)) 26,26,67
29 67 IF(SIG0(K)-SIG1(IJ)) 15,35,35
30 15 SIG=SIG0(K)
31 ITT=0
32 16 ZAP=DD*SIG*(LOG(B1(IK)+ALPHA(J))-LOG(B1(IK)-ALPHA(J)))
33 ZFAC=B1(IK)*ALPHA(J)
34 AAO=ZAP/ZFAC
35 IF(AAO-.00001) 117,117,121
36 117 DDO=1-((AAO/2)+((AAO**2)/2)-((AAO**3)/5)/8)
37 ++(((AAO**4)/7)/8)
38 GO TO 327
39 121 CONTINUE
40 DDO=1-((AAO/2)+((AAO**2)/2)-((AAO**3)/5)/8)
41 ++(((AAO**4)/7)/8)-(((AAO**5)/21)/16)
42 ++(((AAO**6)/33)/16)-(((AAO**7)/429)/128)
43 ++(((AAO**8)/715)/128)-(((AAO**9)/2431)/256)
44 ++(((AAO**10)/4199)/256)-(((AAO**11)/88179)/3072)
45 ++(((AAO**12)*52003)/1024)-(((AAO**13)*1300075)/14336)
46 ++(((AAO**14)*334305)/2048)-(((AAO**15)*9694845)/32768)
47 IF(AAO-.01) 327,327,118
48 118 CONTINUE
49 IF(((AAO**16)*(17678835))/(32768))-.0000000001 327,327,331
50 331 IEXP=16
51 COEFF=(((9694845)*(2*IEXP-1))/((32768)*(IEXP+1))
52 333 DAR=COEFF*(AAO**IEXP)*((-1)**IEXP)
53 335 DDO=DDO+DAR
54 IEXP=IEXP+1
55 DAR=DAR*AAO*(2*IEXP-1)*(-1)/(IEXP+1)
56 DARA=ABS(DAR)
57 IF(DARA-10) 213,214,214
58 213 CONTINUE
59 IF(DARA=.0000001) 367,367,335
60 214 DDO=(-ZFAC+SORT(ZFAC*(ZFAC+2*ZAP)))/ZAP
61 367 CONTINUE
62 327 IF(ITT) 81,81,82
63 81 ITT=1
64 SIG=SIG1(IJ)
65 PP=DDO
66 GO TO 16
67 82 DD1=DD0
68 DDO=PP
69 609 IF(DDO-DD1-0.0000001) 616,616,610
70 616 PRINT,"EQUAL"
71 PRINT 6,DDO,DD1
72 GO TO 26
73 610 D=1
74 611 S1=SORT(DD*SIG1(IJ))
75 SO=SORT(DD*SIG0(K))
76 IGG=0
77 39 U1=(B1(IK)+ALPHA(J)-D*B1(IK))/(D*S1)
78 AL1=(B1(IK)-ALPHA(J)-D*B1(IK))/(D*S1)
79 UO=(B1(IK)+ALPHA(J)-D*B1(IK))/(D*S0)
80 ALO=(B1(IK)-ALPHA(J)-D*B1(IK))/(D*S0)
81 IF(U1-10) 716,716,717
82 717 IF(AL1+10) 714,714,715
83 714 PROBIT=0
84 GO TO 719
85 715 PROBIT=GP(AL1)
86 GO TO 719
87 716 IF(AL1+10) 724,724,725
88 724 PROBIT=1-GP(U1)
89 GO TO 719
90 725 PROBIT=1-GP(U1)+GP(AL1)
91 719 CONTINUE
92 IF(UO-10) 816,817,817
93 817 IF(ALO+10) 814,814,815
94 814 PROBOT=0
95 GO TO 720
96 815 PROBOT=GP(AL0)
97 GO TO 720
98 816 IF(ALO+10) 824,824,825
99 824 PROBOT=1-GP(UO)
100 GO TO 720
101 825 PROBOT=1-GP(UO)+GP(AL0)
102 720 CONTINUE
103 721 IF(IGG-1) 862,863,864
104 862 PRINT,"FOR EPSILON=",ALPHA(J),"AND M=",B1(IK)
105 PRINT,"BLUE PROBABILITIES ARE :"
106 PB0=1-PROBOT
107 PBl=1-PROBIT
108 PRINT 6,PBO,PBl
109 IGG=1
110 D=DD1
111 PRINT,"THE UPPER AND LOWER BOUNDS FOR K ARE:"  
112 PRINT 6,DD0,DD1
113 GO TO 39
114 863 CONTINUE
115 IF(PROBOT-PROBIT) 427,427,428
116 427 PRINT,"MAXI-MIN TYPE A SOLUTION : K ="
117 PRINT 7,DD1
118 7 FORMAT(10X,F11.9)
119 PEO=1-PROBOT-PB0
120 PE1=1-PROBIT-PB1
121 PRINT,"DIFFERENCES IN PROBABILITIES FOR DERIVED ESTIMATOR ARE :"
122 PRINT 6,PEO,PE1
123 GO TO 26
124 428 DDU=DD0
125 DDL=DD1
126 868 DDT=(DDU+DDL)/2
127 MID=MID+1
128 IF(DDU-DDL-0.0000001) 898,898,899
129 898 PRINT,"MAXI-MIN ITERATIVE TYPE A SOLUTION : K ="
130 PRINT 6,DDL,DDU
131 6 FORMAT(2(10X,F11.9))
132 PRINT,"NUMBER OF ITERATIONS =",MID
133 998 PEO=1-PROBOT-PB0
134 PE1=1-PROBIT-PB1
135 PRINT,"DIFFERENCES IN PROBABILITIES FOR DERIVED ESTIMATOR ARE :"
136 PRINT 6,PEO,PE1
137 MID=0
138 GO TO 26
139 899 CONTINUE
140 IGG=2
141 D=DDT
142 GO TO 39
143 864 IF(PROBOT-PROBIT) 873,874,875
144 873 DDU=DDT
145 GO TO 868
146 875 DDL=DDT
147 GO TO 868
148 874 PRINT,"PROBABILITY EQUALITY K ="
149 PRINT 7,DDT
150 PRINT,MID
151 MID=0
152 GO TO 998
153 26 CONTINUE
154 27 PRINT +++++
155 30 CONTINUE
156 35 CONTINUE
157 40 CONTINUE
158 PRINT ++++++
159 GO TO 4
160 END
161 FUNCTION GP(X)
162 GP=0
163 Y=\times X
164 IF(324-Y)2
165 Z=.3989422804*EXP(-Y/2)
166 IF(Y-9)3
167 Y=11/X
168 DO1C=1,10
169 CY=(11-C)/(Y+X)
170 GP=-Z/(Y+X)
171 2IIF(X)5
172 GP=1+GP
173 GO TO 5
174 3GP=Z
175 C=1
176 4C=C+2
177 Z=Z*Y/C
178 GP=GP+Z
179 IF(GP-Z*1E8)4
180 GP=.5+X*GP
181 5
182 END
10.3 Appendix C - Computer Program to Calculate K for Type B Estimator

The introduction of ε, M, d, and \( \sigma^2 \) values to this program, as well as the output of the program, is exactly as in the program of Appendix B. The program follows.
1 DIMENSION ALPHA(15),SIGO(15),SIG1(15),B1(15)
2 4 II=6
3 JJ=4
4 IN=1
5 JN=1
6 DD=2.5
7 ALPHA(1)=0.05
8 ALPHA(2)=0.10
9 ALPHA(3)=0.50
10 ALPHA(4)=1.0
11 ALPHA(5)=5.0
12 ALPHA(6)=10.0
13 INPUT,(SIGO(K),K=1,IN)
14 INPUT,(SIG1(IJ),IJ=1,JN)
15 PRINT ++++
16 B1(1)=7
17 B1(2)=10
18 B1(3)=20
19 B1(4)=50
20 MID=0
21 NEX1=0
22 NEX2=0
23 IEZ1=0
24 IEZ2=0
25 DO 40 K=1,IN
26 DO 35 IJ=1,JN
27 PRINT ++
28 PRINT,"FOR SIGO =",SIGO(K),"AND SIG1 =",SIG1(IJ)
29 PRINT ++
30 DO 30 J=1,II
31 DO 27 IK=1,JJ
32 IF (B1(IK)-ALPHA(J)) 26,26,67
33 67 IF(SIGO(K)-SIG1(IJ)) 15,35,35
34 15 SIG=SIGO(K)
35 ITT=0
36 16 ZAP=DD*SIG*(LOG(B1(IK)+ALPHA(J))-LOG(B1(IK)-ALPHA(J)))
37 ZFAC=B1(IK)*ALPHA(J)
38 AAO=ZAP/ZFAC
39 IF(AAO-.00001) 117,117,121
40 117 DDO=1-(AAO/2)+((AAO**2)/2)-((AAO**3)*5)/8
41 ++((AAO**4)*7)/8)
42 GO TO 327
43 121 CONTINUE
44 DDO=1-(AAO/2)+((AAO**2)/2)-((AAO**3)*5)/8
45 ++((AAO**4)*7)/8)-((AAO**5)*21)/16
46 ++((AAO**6)*33)/16)-((AAO**7)*429)/128
47 ++((AAO**8)*715)/128)-((AAO**9)*2431)/256)
48 ++((AAO**10)*4199)/256)-((AAO**11)*88179)/3072)
49 ++((AAO**12)*52003)/1024)-((AAO**13)*1300075)/14336)
50 ++((AAO**14)*334305)/2048)-((AAO**15)*9694845)/32768)
51 IF(AAO-.01) 327,327,118
52 118 CONTINUE
53 IF(((AAO**16)*(17678835))/(32768))-0.0000000001) 327,327,331
54 331 IEXP=16
55 COEFF= ((9694845)*(2*IEXP-1))/((32768)*(IEXP+1))
56 333 DAR=COEFF*(AAO**IEXP)*((-1)**IEXP)
57 335 DDO=DDO+DAR
58 IEXP=IEXP+1
59 DAR=DAR*ZZO*(2*IEXP-1)*(-1)/(IEXP+1)
60 DARA=ABS(DAR)
61 IF(DARA-10) 213,214,214
62 213 CONTINUE
63 IF(DARA-.0000001) 367,367,335
64 214 DDO=(-ZFAC+SORT(ZFAC*(ZFAC+2*ZAP)))/ZAP
65 367 CONTINUE
66 327 IF(ITT) 81,81,82
67 81 ITT=1
68 SIG=SIG1(IJ)
69 PP=DDO
70 GO TO 16
71 82 DDL=DDO
72 DDO=PP
73 609 IF(DDO-DDL-0.0000001) 616,616,610
74 616 PRINT, "EQUAL"
75 PRINT 6,DDO,DD1
76 GO TO 26
77 610 D=1
78 611 S1=SORT(DD*SIG1(IJ))
79 SO=SORT(DD*SIGO(K))
80 IGC=0
81 39 U1=(B1(IK)+ALPHA(J)-D*B1(IK))/(D*S1)
82 A11=(B1(IK)-ALPHA(J)-D*B1(IK))/(D*S1)
83 U0=(B1(IK)+ALPHA(J)-D*B1(IK))/(D*S0)
84 ALO=(B1(IK)-ALPHA(J)-D*B1(IK))/(D*S0)
85 IF(U1-10) 716,717,717
86 717 IF(AL1+10) 714,714,715
87 714 PROBIT=0
88 GO TO 719
89 715 PROBIT=GP(AL1)
90 GO TO 719
91 716 IF(AL1+10) 724,724,725
92 724 PROBIT=1-GP(U1)
93 GO TO 719
94 725 PROBIT=1-GP (AL1) + GP (AL1)
95 719 CONTINUE
96 IF(U0-10) 816,817,817
97 817 IF(AL0+10) 814,814,815
98 814 PROBOT=0
99 GO TO 720
100 815 PROBOT=GP(AL0)
101 GO TO 720
102 816 IF(AL0+10) 824,824,825
103 824 PROBOT=1-GP(U0)
104 GO TO 720
105 825 PROBOT=1-GP(U0) + GP (AL0)
106 720 IF(IGG-1) 41,46,48
107 41 PRINT, "FOR EPSILON =",ALPHA(J), "AND M =",B1(IK)
108 PRINT,"BLUE PROBABILITIES ARE :"
109 PBO=1-PROBOT
110 PB1=1-PROBIT
111 PRINT 6,PBO,PB1
112 IGG=1
113 D=DDO
114 PRINT,"THE UPPER AND LOWER BOUNDS FOR K ARE:"
115 PRINT 6,DDO,DD1
116 GO TO 39
117 46 CONTINUE
118 PZO=1-PROBOT-PBO
119 PZ1=1-PROBIT-PB1
120 IF(PZO-PZ1) 238,238,239
121 238 CONTINUE
122 PRINT,"MAXI-MIN TYPE B SOLUTION: K="
123 PRINT 7,DDO
124 PRINT,"FOR THE DERIVED ESTIMATOR, THE PROBABILITY DIFFERENCES ARE:"
125 PRINT 6,PZO,PZ1
126 GO TO 26
127 239 IGG=2
128 DDU=DDO
129 DDL=DD1
130 888 DDT=(DDU+DDL)/2
131 MID=MID+1
132 IF(DDU-DDL-0.000000001) 874,874,875
133 874 CONTINUE
134 PRINT,"THE MAXI-MIN ITERATIVE TYPE B SOLUTION IS BETWEEN :"
135 PRINT 6,DDU,DDL
136 PRINT,"THE PROBABILITY DIFFERENCES ARE:"
137 PRINT 6,PZO0,PZ11
138 PRINT,"NUMBER OF ITERATIONS ="
139 PRINT, MID
140 MID=0
141 GO TO 26
142 875 D=DDT
143 GO TO 39
144 48 PZO0=1-PROBOT-PBO
145 PZ11=1-PROBIT-PB1
146 IF(PZO0-PZ11) 208,209,210
147 208 DDL=DDT
148 GO TO 888
149 210 DDU=DDT
150 GO TO 888
151 209 CONTINUE
152 PRINT,"PROBABILITY EQUALITY, K="
153 PRINT 7,DDT
154 PRINT,"NUMBER OF ITERATIONS ="
155 PRINT,MID
156 MID=0
157 PRINT,"THE PROBABILITY DIFFERENCES ARE:"
163 30 CONTINUE
164 35 CONTINUE
165 40 CONTINUE
166 PRINT "+++++
167 GO TO 4
168 END
169 FUNCTION GP(X)
170 GP=0
171 Y=X*X
172 IF(324-Y)2
173 Z=.3989422804*EXP(-Y/2)
174 IF(Y<9)3
175 Y=11/X
176 DO1C=1,10
177 1Y=(11-C)/(Y+X)
178 GP=-Z/(Y+X)
179 2IF(X)5
180 GP=1+GP
181 GO TO 5
182 3GP=Z
183 C=1
184 4C=C+2
185 Z=Z*X/C
186 GP=GP+Z
187 IF(GP-Z*1E8)4
188 GP=.5+X*GP
189 5
190 END