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CONSTRUCTION OF RULED SURFACES IN 5-DIMENSIONAL FINITE PROJECTIVE GEOMETRIES*

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1. Introduction. I want to describe, as briefly as possible, the methods for constructing one of my ruled surfaces in PG(5,q). Here q is any prime-power. However, the theory becomes trivial (and exceptional) for q=2. So assume

(1.1) \( q \geq 3 \) \quad (q \text{ a prime-power}).

2. Preliminary. Nothing is said in my paper on ruled surfaces (the Luxembourg paper) about reguli. I need the following as background:

Lemma 2.1. Let \( S \) be a surface in \( \Sigma = PG(5,q) \) ruled by 3 systems of planes, belonging, say, to classes I, II and III. Let \( L \) be a line lying in a ruling plane of (say) class I. Let \( R \) be the set of \( q+1 \) ruling planes of (say) class II each of which contains exactly one point of \( L \). Then:

(i) \( R \) is a regulus of planes of \( \Sigma \).

(ii) Each of the \( q^2 \) and \( q+1 \) ruling planes of class I meets the \( q+1 \) planes of \( R \) in the \( q+1 \) points of a line -- a transversal line to \( R \).

(iii) Each of the \( q^2 \) and \( q+1 \) ruling planes of class III meets the \( q+1 \) planes of \( R \) in the \( q+1 \) points of a conic.

Proof. It is to be understood that \( S \) is constructed as in my Luxembourg paper. Set

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\[ F = GF(q), \quad K = GF(q^3), \]

and let \( V = K \times K \) be the 6-dimensional vector space over \( F \) consisting of ordered pairs \((x,y), x,y \in K\), with

\[
\begin{align*}
(x,y) + (x',y') &= (x+x', y+y') \quad \forall \ x,y,x',y' \in K \\
f(x,y) &= (fx, fy) \quad \forall \ f \in F.
\end{align*}
\]

Then \( S \) consists of all points \( \langle x,y \rangle \) such that

\[
N(x) = N(y);
\]
equivalently

\[
x^{q^2+q+1} = y^{q^2+q+1}.
\]

The classes (I), (II), (III) are defined as the set of planes of the following sorts:

(I) \( y = kx \)

(II) \( y = kx^q \)

(III) \( y = kx^{q^2} \)

For any fixed \( k \) such that

\[
N(k) = k^{q^2+q+1} = 1.
\]

For the proof, we may assume that \( L \) is the 2-dimensional vector space

\[
L = \langle (1,1), \quad (a,a) \rangle
\]

for some fixed \( a \),

\[
a \in K-F.
\]

Then the \( q+1 \) points of \( L \) are the following: the point

\[
\langle (1, 1) \rangle
\]
and \( q \) points of form

\[
(f(1,1) + (a,a)) = (f+a, f+a), \quad f \in F.
\]

The plane of class (II) through \((1,1)\) is

\[
\Pi_\infty: \ y = x^q.
\]

The plane of class (II) through \((f+a, f+a)\) is

\[
\Pi_f: \ y = \frac{f+a}{f+aq}x^q.
\]

We may check that, for each \( k \in K, \ k \neq 0 \), the line

\[
L_k = (k, k^q), \ (ka, k^qa)
\]

meets \( \Pi_\infty \) in \((k, k^q)\) and \( \Pi_f \) in \((k(a+f), k^q(a+q+f))\), and hence is a transversal to

\[
R = \{\Pi_\infty\} \cup \{\Pi_f | f \in F\}.
\]

Furthermore, \( L_k \) lies in the following plane of class I,

\[
y = k^q-1x.
\]

We also note that

\[
L_k = L_k', \iff \frac{k'}{k} \in F,
\]

so that the total number of distinct lines \( L_k \) is

\[
\frac{q^3-1}{q-1} = q^2 + q + 1.
\]

Thus: \( R \) is a regulus of (skew) planes of \( \Sigma:PG(5,q) \), with the \( q^2+q+1 \) lines \( L_k \) as its transversal lines.
Next consider a typical ruling plane of class III, say
\[ \alpha: y = kx^2, \]
where \( k \) is some fixed element of \( K \) with
\[ k^{q^2+q+1} = 1. \]
We can choose \( b \) (in \( q-1 \) ways) so that
\[ k = b^{q-1}. \]
Thus the equation of \( \alpha \) becomes
\[ \alpha: y = b^{q-1}x^2. \]
Then we may check that
\[ \Pi_\infty \cap \alpha = \langle b^{-q^2}, b^{-1} \rangle \]
\[ \Pi_f \cap \alpha = \langle b^{-q^2} (f+a)^{-q^2}, b^{-1} (f+a)^{-1} \rangle, \quad \forall f \in F. \]
To complete the proof, we must show that the \( q+1 \) points so obtained lie on a conic (in \( \alpha \)). If \( q \) is odd, it is enough to show that no three of the points lie on a line. Consider the special case of these points
\[ \Pi_{f_i} \cap \alpha, \quad i = 1, 2, 3, \]
where \( f_1, f_2, f_3 \) are distinct elements of \( F \). If these points lie on a line, we must have
\[ u(b^{-2}(f_1+a)^{-2}, b^{-1}(f_1+a)^{-1}) + v(b^{-2}(f_2+a)^{-2}, b^{-1}(f_2+a)^{-1}) \\
+ w(b^{-2}(f_3+a)^{-2}, b^{-1}(f_3+a)^{-1}) = 0 \]

for elements \( u, v, w \) of \( F \), not all zero. Considering components, we get two equations for \( u, v, w \) which (after cancelling some non-zero factors) become

\[ (f_1+a)^{-2}u + (f_2+a)^{-2}v + (f_3+a)^{-2}w = 0 \]

\[ (f_1+a)^{-1}u + (f_2+a)^{-1}v + (f_3+a)^{-1}w = 0. \]

We note that (2) implies (1). However, (2) (with \( u, v, w \) in \( F \), not all zero) means that \( a \) satisfies a quadratic equation with coefficients in \( F \). Since \( a \in K-F \), and \( K \) is three dimensional over \( F \), this is false.

Similarly, three points \( \Pi_\infty \cap \alpha, \Pi_{f_1} \cap \alpha, \Pi_{f_2} \cap \alpha \), cannot be collinear.

We leave the proof of Lemma 2.1 at this point.

3. Construction of surfaces \( S \). (Some algebraic details.)

We suppose given a regulus \( R \) of planes of \( \Sigma = PG(5,q) \), consisting of \( q+1 \) skew planes \( \Pi_i \);

\[ R = \{ \Pi_i | i = 1, 2, \ldots, q+1 \} \]

such that each of the \( q^2+q+1 \) transversal lines to \( \Pi_1, \Pi_2, \Pi_3 \) meets every \( \Pi_i \) in a point. We also suppose given one more plane, \( \Pi \), disjoint from the \( q+1 \) planes \( \Pi_i \).
We consider the problem of constructing a triply-ruled surface \( S \) of \( \Sigma \) such that the set
\[ R \cup \{ \Pi \} \]
of \( q+2 \) skew planes forms part of one of the three systems of \( q^2+q+1 \) ruling planes of \( S \).

In the light of Lemma 2.1, we may assume that the planes of \( R \cup \{ \Pi \} \) belong to Class II, and that every plane of Class I contains a (unique) transversal line to \( R \).

We may suppose that \( \Sigma = \text{PG}(5,q) \) is given by a six-dimensional vector space \( V \) over \( F = \text{GF}(q) \) with a basis \( \ell_1, \ell_2, \ell_3, \ell'_1, \ell'_2, \ell'_3 \) chosen so that
\[ \Pi_1 = J(\infty) \text{ has basis } \ell_1, \ell_2, \ell_3 \]
\[ \Pi_2 = J(0) \text{ has basis } \ell'_1, \ell'_2, \ell'_3, \]
\[ \Pi_3 = J(1) \text{ has basis } \ell_1+\ell'_1, \ell_2+\ell'_2, \ell_3+\ell'_3. \]

Then every plane skew to \( \Pi_1 = J(\infty) \) has form \( J(X) \) for a unique \( 3 \times 3 \) matrix \( X = (x_{ij}) \) over \( F = \text{GF}(q) \), where

\[ J(X) \text{ has basis } x_{11}\ell_1 + x_{12}\ell_2 + x_{13}\ell_3 + \ell'_1, \]
\[ x_{21}\ell_1 + x_{22}\ell_2 + x_{23}\ell_3 + \ell'_2, \]
\[ x_{31}\ell_1 + x_{32}\ell_2 + x_{33}\ell_3 + \ell'_3. \]

In particular, \( R \) consists of \( J(\infty) \) and the \( J(fI) \), \( f \in F \), where
\[
 fI = \begin{pmatrix}
 f & 0 & 0 \\
 0 & f & 0 \\
 0 & 0 & f 
\end{pmatrix}.
\]
Also
\[ \Pi = J(U) \]
for some irreducible 3x3 matrix \( U \).

Every vector \( a \) in \( J(\infty) \) has form
\[ a = a_1\ell_1 + a_2\ell_2 + a_3\ell_3 \]
for unique elements \( a_1, a_2, a_3 \) in \( F \). We define
\[ a^X = (a_1x_{11} + a_2x_{21} + a_3x_{31})\ell_1 + (a_1x_{12} + a_2x_{22} + a_3x_{32})\ell_2 + (a_1x_{13} + a_2x_{23} + a_3x_{33})\ell_3 \]
for every 3x3 matrix \( X \). In this notation
\[ J(X) \] has basis \( \ell_1^X + \ell_1', \ell_2^X + \ell_2', \ell_3^X + \ell_3' \).

Let \( a \) be a non-zero vector in \( J(\infty) \), so that \( \langle a \rangle \) is a point of the plane \( J(\infty) \). The transversal line, \( L_a \), to \( \mathcal{R} \) through \( \langle a \rangle \) is the two-dimensional vector space
\[ L_a = \langle a, a' \rangle \]
where we define
\[ a' = a_1\ell_1' + a_2\ell_2' + a_3\ell_3'. \]

A ruling plane, of our proposed surface \( S \), which has Class (I) and contains \( L_a \) must (in particular) meet \( \Pi = J(U) \) in a point, say in
\[ \langle b^U + b' \rangle, \]
where \( b \) is some non-zero vector in \( J(\infty) \). Then this ruling plane is (say)
\[ \alpha = \langle a, a', b^U + b' \rangle. \]
The ruling plane $\alpha$ should meet each plane of $R$ in a point, namely in a point of $L_a$. This puts some restrictions on the point $<b^U+b'>$ of $N$ or, equivalently, the point $<b>$ of $J(\infty)$. To see this, we note that each point of $\alpha$ has form $<v>$ where $v$ is a non-zero vector of form

$$v = fa + ga' + h(b^U+b'),$$

where $f, g, h$ are elements of $F$, not all zero. Equivalently,

$$v = (fa+hb^U) + (ga+hb)' .$$

The point $<v>$ will be in $J(\infty)$ if and only if

$$ga + hb = 0.$$

We wish this condition to imply $h = 0$. Thus we want

(1) $<b> \neq <0> .$

If $t \in F$, so that $J(tI)$ is in $R$, the point $<v>$ will be in $J(tI)$ if and only if

$$fa + hb^U = t(ga+hb)$$
or

$$h(b^U-tb) = (-f+tg)a$$
or

$$h \cdot b^U-tI = (-f+tg)a .$$

We want this equation to imply $h=0$. Thus we want

$$<b^U-tI> \neq <a>$$
or, equivalently,

(2) $<b> \neq <a(U-tI)^{-1}>, \ \forall \ t \in F .\ \ $
It may be shown that the point \( <a> \) of \( J(\infty) \) together with the \( q \) points
\[
<a(U-tI)^{-1}>, \quad t \in F
\]
of \( J(\infty) \) form a conic of \( J(\infty) \). Hence, also, the corresponding points of \( \Pi \), namely
\[
<a U+a'> \quad \text{and} \quad <a(U-tI)^{-1}U + (a(U-tI)^{-1})'>, \quad t \in F,
\]
form a conic of \( \Pi \). Also the point \( <b U+b'> \) must avoid the \( q+1 \) points of the latter conic. This gives
\[
(q^2+q+1) - (q+1) = q^2
\]
choices of the point \( <b U+b'> \) so that
\[
\alpha = <a, a', b U+b'>
\]
meets each of the \( q+2 \) planes in \( \mathbb{R} \cup \{ \Pi \} \) in (exactly) a point.

It is easy to check that the conditions on \( b \) are equivalent to the following:

(3) \( a, b, b^U \) form a basis of \( J(\infty) \) over \( F = GF(q) \).

However, since \( U \) is irreducible and \( b \neq 0 \), the vectors \( b, b^U, b^U^2 \) form a basis of \( J(\infty) \) over \( F = GF(q) \). Hence (3) means that
\[
\alpha = fb + gb^U + hb^U^2
\]
for some \( f, g, h \) in \( F \) with \( h \neq 0 \). Equivalently (the case \( h=1 \))

(3') \( <a> = <f_0 b + g_0 b^U + b^U^2> \)
for some $f_0', g_0$ in $F$. (Note that the ordered pair $f_0', g_0$ can be chosen in $q^2$ ways.)

Assuming $(3')$, we can write the plane $\alpha$ as the plane $\alpha(b)$ where

$$\alpha(b) = \langle f_0 b + g_0 b^U + b^U, f_0 b' + g_0 (b^U)' + (b^U)', b^U + b' \rangle.$$  

Here $b$ can be any non-zero vector in $J(\infty)$. If, now, for fixed $f_0', g_0$ in $F$, we consider all the planes $\alpha(b)$, we check easily that they form a set of $q^2 + q + 1$

distinct, mutually skew, planes, each of which meets every plane of $R \cup \Pi$ in exactly a point, and each of which contains a unique transversal line to $R$, namely the line

$$\langle f_0 b + g_0 b^U + b^U, f_0 b' + g_0 (b^U)' + (b^U)', b^U + b' \rangle.$$  

Although we are not supplying a proof here, these $q^2 + q + 1$ planes $\alpha(b)$ constitute the desired (complete) collection of planes of Class I.

We still have to supply the

$$q^2 + q + 1 - (q+2) = q^2 - 1$$

missing planes of Class II, and the $q^2 + q + 1$ planes of Class III.

4. Geometric approach. The material of Section 3 may be summarized as follows:

Suppose given a regulus, $R$, of $q+1$ skew planes of $\Sigma = PG(5,q)$ and a plane $\Pi$ disjoint from each of the planes in $R$. Then, in precisely $q^2$ distinct ways, we can set up a one-to-one correspondence

$$L \leftrightarrow P$$

(4.1)
between the $q^2+q+1$ transversal lines $L$ to $R$ are the $q^2+q+1$ points $P$ of $\Pi$ such that:

(i) If $L, P$ are a corresponding pair, the plane $L+P$ intersects $\Pi$ in $P$ and intersects each plane $\Pi_i$ of $R$ in the point $L \cap \Pi_i$.

(ii) The $q^2+q+1$ planes $L+P$ obtained by letting $L, P$ range over all corresponding pairs are structurally skew.

This is essentially all we need to complete the construction of the surface $S$.

We assume that some fixed correspondence (4.1) has been chosen.

Consider any corresponding pair $L, P$. We note first that the projective space

(4.2) \[ H_P = L + \Pi \]

is four-dimensional and hence is a hyperplane of $\Sigma = \text{PG}(5,q)$. Next we construct a projective 3-space,

\[ T_P' \]

in the following manner: Let $\Pi_1, \Pi_2$ be two distinct planes of $R$ and let $M$ be the unique transversal line through $P$ to $\Pi_1, \Pi_2$, meeting $\Pi_1, \Pi_2$ in points $P_1, P_2$ respectively. Let $L_1, L_2$ be the transversal lines to $R$ through $P_1, P_2$ respectively. Note that $L_1 \neq L_2$, else the point $P$ of $\Pi$ would lie on a transversal to $R$ and hence on one of the planes of $R$, a contradiction. Since $L_1 \neq L_2$, then the space

(4.3) \[ T_P = L_1 + L_2 \]

is a projective 3-space. Since $M$ contains $P, P_1, P_2$ and since $P_1$ is on $L_1$, $P$, is on $L_1$, then $T_P$ contains $P$ and $M$ is the transversal through $P$ to $L_1, L_2$ in $T_P$. 

We need to note the following:

(a) The 3-space $T_p$ depends only on $R$ and $P$, not on $\Pi$ or on the choice of the planes $\Pi_1, \Pi_2$ of $R$.

(b) $T_p$ meets each plane of $R$ in a line, giving $q+1$ such lines, and contains precisely $q+1$ transversals to $R$, namely the transversals to the first set of lines. These two sets of $q+1$ lines form the two sets of rulings of a doubly-ruled quadric $Q_p$ of $T_p$; and $Q_p$ depends only on $P$ and $R$.

(c) $P$ lies on exactly $q+1$ planes to $Q_p$ in $T_p$. Each of these contains exactly one transversal line to $R$ and meets exactly one plane of $R$ in a line. Each of the remaining $q^2$ planes of $T_p$ through $P$ meets $Q_p$ in a conic, contains no transversal line to $R$, and meets no plane of $R$ in a line.

(d) $T_p \cap \Pi = P$.

Next we need the following:

(4.4) $T_p \cap L$ is the empty point-set.

(4.5) $\Pi_p = T_p \cap H_p$ is a plane (disjoint from $L$).

To prove (4.4), first suppose that $L$ is contained in $T_p$. Then $P+L$ is a tangent plane to $Q_p$ and hence meets some plane of $R$ in a line. This is a contradiction. Hence $L$ is not in $T_p$. Next suppose that $L$ meets $T_p$ in a point $P'$. (Since $P$ is not on $L$, necessarily $P' \neq P$.) Since $P'$ is on $L$, then $P'$ is on some plane, say $\Pi_1$, of $R$. Since $P'$ is in $\Pi_1 \cap T_p$, the $P'$ is on $Q_p$. In particular, the transversal line through $P'$ to $R$ is a ruling of $Q_p$, and is in $T_p$. But this transversal, being the unique transversal line through $P'$ to
$R$, must be $L$. Hence $L$ is in $T_p$, a contradiction. Therefore, (4.4) must be true.

In view of (4.4), the projective space $T_p + L$ has dimension $3 + 1 - (-1) = 5$. Therefore,

$$T_p + L = \mathbb{P}.$$  \hspace{1cm} (4.6)

Since $H_p$ contains $L$, we see from (4.6) that

$$T_p + H_p = \mathbb{P}.$$  \hspace{1cm} (4.7)

Hence

$$\dim(III_p) = \dim T_p + \dim H_p - \dim \mathbb{P} = 3 + 4 - 5 = 2.$$  \hspace{1cm} (4.8)

Thus $III_p$ is a plane. Moreover

$$III_p \cap L = T_p \cap H_p \cap L$$

is empty by (4.4). This proves (4.5). From (4.3) we get

$$III_p + L = H_p.$$  \hspace{1cm} (4.9)

Indeed, the left-hand side of (4.7) is contained in the right-hand side, and both sides have projective dimension 4. From (4.7) and the fact that $P$ is in $III_p$ we get

$$III_p + (P + L) = H_p$$

and hence

$$III_p \cap (P + L) = P.$$  \hspace{1cm} (4.10)
Next we need

\[(4.9)\] If \( E_4 \) is a projective 4-space of \( E \) containing \( H \), then \( E_4 \) contains one and only one transversal line to \( R \).

To see this, we note that there are precisely \( q^2+q+1 \) distinct transversal lines to \( R \) and precisely \( q^2+q+1 \) distinct projective 4-spaces of \( E \) containing \( H \). If \((4.9)\) is false, there must be a \( E_4 \) which contains \( H \) but contains no transversal line to \( R \). Let \( E_4 \) be such a 4-space, and let \( H_1 \) be one of the planes of \( R \). Then \( E_4 \) intersects \( H_1 \) is a line, say \( M \). The \( q+1 \) transversal lines to \( R \) through the points of \( M \), each meet \( E_4 \) in a single point, namely a point of \( M_1 \). The same process, carried out for the \( q+1 \) distinct planes of \( R \), shows that there must be at least

\[(q+1)^2 > q^2+q+1\]

distinct transversal lines to \( R \). This is a contradiction. Hence \((4.9)\) is true. As a special case of \((4.9)\),

\[(4.10)\] \( L \) is the only transversal line to \( R \) contained in \( H = L + H \).

By \((4.10)\) and \((4.4), (4.5), \)

\[(4.11)\] The plane \( III_p \) contains no transversal line to \( R \).

In view of \((4.11), III_p \) meets the \( q+1 \) planes of \( R \) in the points of a conic. (The conic lies on \( Q_p \).

The \( q^2+q+1 \) planes \( III_p \), one for each point \( P \) of \( H \), are our candidates for the ruling planes of Class III. (Compare Lemma 2.1.) We omit the proof that every two of these are disjoint.
Of course, the planes of class I are the planes

\[ \frac{I}{P} = P + L \]

where \( L, P \) is a corresponding pair in the sense of (4.1). It has to be shown that every plane of class I meets every plane of class III in a point.

In addition, we have in class II only the set

\[ R \cup \{\Pi\} \]

of \( q+2 \) planes. But it should be clear at this point that the missing \( q^2 - 1 \) planes are uniquely determined by the planes of Classes I and III. We have only to consider Lemma 2.1 with classes I, II, III replaced (for example) by classes III, I, II respectively.

I will stop here. Note that one must prove that the construction can actually be completed consistently. (This is, in fact, true.)