MATCHING THEOREMS FOR COMBINATORIAL GEOMETRIES

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1. INTRODUCTION

Let \( G(S) \) and \( G(T) \) be combinatorial geometries [2] on sets \( S \) and \( T \), respectively, and let \( R \subseteq S \times T \) be a binary relation between the points of \( G(S) \) and \( G(T) \). A matching from \( G(S) \) into \( G(T) \) is a triple \( (A, B, f) \), where \( A \) and \( B \) are independent sets in \( G(S) \) and \( G(T) \), respectively, and \( f \) is a one-one function from \( A \) onto \( B \) such that \( (a, f(a)) \in R \) for all \( a \in A \).

The present paper presents a characterization of matchings of maximum cardinality, a max-min theorem, and a number of related results. In the case where both \( G(S) \) and \( G(T) \) are free geometries, Theorem 4 reduces to a characterization of maximum matchings associated with the "Hungarian method" as introduced by Egerváry and Kuhn (see [1]), Theorem 5 to the König-Egerváry Theorem, Theorem 6 to a theorem of Ore [5], and Corollary 6.1 to the classical Marriage Theorem. The latter corollary for the case when \( G(S) \) is free and \( G(T) \) arbitrary was first obtained by Rado [6] (see also Crapo-Rota [2]). Theorem 7 is known (see [2], [3], [4]), but the proof, based on earlier results in the paper, is new.

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2. DEFINITIONS, NOTATION, AND TERMINOLOGY

For completeness, several definitions and results on combinatorial geometries from Crapo-Rota [2] are included in this section.

A closure relation on a set \( S \) is a function \( A \rightarrow \overline{A} \) defined for all subsets \( A \subseteq S \), satisfying

\[
(2.1) \quad A \subseteq \overline{A},
\]

\[
(2.2) \quad A \subseteq \overline{B} \implies \overline{A} \subseteq \overline{B},
\]

for all subsets \( A, B \) of \( S \). A set endowed with a closure relation is a closure space. A subset \( A \subseteq S \) is closed if and only if \( A = \overline{A} \).

A closure relation on a set \( S \) has finite basis if and only if

\[
(2.3) \quad \text{Any subset } A \subseteq S \text{ has a finite subset } A_0 \subseteq A \text{ such that } \overline{A_0} = \overline{A}.
\]

A closure relation satisfies the exchange property if and only if

For any elements \( a, b \in S \), and for any subset \( A \subseteq S \),

\[
(2.4) \quad a \in A \cup b, \quad a \notin \overline{A} \implies b \in \overline{A} \cup a.
\]

A pregeometry (matroid) \( G(S) \) is any closure space consisting of a set \( S \) and a closure relation with finite basis and the exchange property. A pregeometry \( G(S) \) is a combinatorial geometry if and only if

\[
(2.5) \quad \overline{\emptyset} = \emptyset, \quad \overline{a} = a \text{ for all } a \in S.
\]
Associated with every pregeometry $G(S)$ on a set $S$ is a unique geometry $G(S_0)$ on the set $S_0$ of equivalence classes of $S$ under the equivalence relation

$$a \sim b \quad \text{if and only if} \quad \bar{a} = \bar{b}.$$ 

We shall confine our attention to geometries with no loss of generality; the results may easily be extended to pregeometries.

The cardinality of a set $S$ will be denoted $\nu(S)$. In a geometry $G(S)$, all minimal subsets $A_0$ of any subset $A \subseteq S$ satisfying $\overline{A_0} = \overline{A}$ have the same cardinality, which is defined as the rank $r(A)$ of the set $A$. A set $A$ is independent if and only if $r(A) = \nu(A)$, i.e., if and only if no proper subset of $A$ has closure $\overline{A}$. The rank function $r$ of $G(S)$ satisfies the semi-modular inequality

$$r(A \cup B) + r(A \cap B) \leq r(A) + r(B). \quad (2.6)$$

A combinatorial geometry $G(S)$ is free if $\overline{A} = A$ for all subsets $A \subseteq S$. In this case, every subset of $S$ is both independent and closed. The concept of independence is thus non-trivial only with regard to "multisets" on "lists", in which an element may occur more than once. A multiset in a free geometry is independent if and only if all its elements are distinct, i.e., if and only if it is a set.

The following definitions and notation are introduced in the present paper. For a further theory of combinatorial geometries, the reader is referred to [2].

Let $G(S), G(T)$ be combinatorial geometries on sets $S, T$, respectively, and let $R \subseteq S \times T$ be a binary relation between the points of $S$ and $T$. The system consisting of $G(S), G(T), \text{and } R$ will be denoted $(G(S), G(T), R)$. We shall denote the rank functions of both $G(S), G(T)$
by $r$ and the closure relations by $A \to \overline{A}$, $B \to \overline{B}$, where $A \subseteq S$, $B \subseteq T$. For $A \subseteq S$, $R(A)$ denotes the set of points $b \in T$ such that $(a, b) \in R$ for some $a \in A$.

A matching in $(G(S), G(T), R)$ is a triple $(A, B, f)$, where $f$ is a one-one function from $A$ onto $B$ such that $(a, f(a)) \in R$ for all $a \in A$, and $A, B$ are independent sets in $G(S), G(T)$, respectively. A matching $(A, B, f)$ is characterized by its edge set

$$M = \{(a, f(a)) : a \in A\},$$

and we formally identify these two concepts by writing $M = (A, B, f)$. The common cardinality of $A, B, M$ is called the size $\nu(M)$ of the matching $M$. We shall be interested in matchings of maximum size in $(G(S), G(T), R)$.

A support of $(G(S), G(T), R)$ is a pair $(C, D)$ of closed sets in $G(S), G(T)$, respectively, such that $(c, d) \in R$ implies at least one of $c \in C$, $d \in D$ holds. The order of a support $(C, D)$ is the number $\lambda(C, D) = r(C) + r(D)$.

Note that if both $G(S)$ and $G(T)$ are free geometries, then the system $(G(S), G(T), R)$ is, apart from the orientation of the edges from $S$ to $T$, a bipartite graph, and the above definitions of a matching and a support reduce to the usual ones for this case. The following definition uses the exchange property to generalize a notion associated with the "Hungarian method" for finding a maximum matching in a bipartite graph.

Let $M = (A, B, f)$ be a matching in $(G(S), G(T), R)$. A sequence

$$(a_0', b_1'), (b_1', a_1), (a_1', b_2'), \ldots, (b_n', a_n), (a_n', b_{n+1}')$$

(2.7)
of $2n + 1$ distinct pairs $(n \geq 0)$ is an augmenting chain with respect to $M$ if and only if

$$ (a_i, b_i) \in M, \quad (a'_i, b'_{i+1}) \in R-M, $$

$$ a'_0 \in S-A, \quad b'_{n+1} \in T-B, $$

$$ a'_i \in \overline{A}, \quad a'_i \notin (A - \bigcup_{j=1}^{i-1} a_j) \cup \bigcup_{j=1}^{i} a'_j, $$

$$ b'_i \in \overline{B}, \quad b'_i \notin (B - \bigcup_{j=1}^{i} b_j) \cup \bigcup_{j=1}^{i-1} b'_j, $$

for $1 \leq i \leq n$.

Note that if both $G(S)$, $G(T)$ are free geometries, (2.10) implies $a'_i = a_i$, $b'_i = b_i$ for $1 \leq i \leq n$, so that the sequence represents an ordinary augmenting chain in the bipartite graph.

3. MATCHING THEOREMS

**Theorem 1.** If there exists an augmenting chain with respect to a matching $M = (A, B, f)$ in $(G(S), G(T), R)$, then $M$ is not of maximum size.

**Proof.** Let the augmenting chain be given by (2.7) and define

$$ P = \{(a_i, b_i): 1 \leq i \leq n\}, $$

$$ P' = \{(a'_i, b'_{i+1}): 0 \leq i \leq n\}. $$

A straightforward inductive argument using (2.10) and the exchange property shows that
\[ (A - \bigcup_{j=1}^{i} a_j) \cup \bigcup_{j=1}^{i} a'_j \]

and

\[ (B - \bigcup_{j=1}^{i} b_j) \cup \bigcup_{j=1}^{i} b'_j \]

are independent sets with closures \( \bar{A}, \bar{B} \), respectively, for all \( i \), \( 1 \leq i \leq n \). Thus by (2.9),

\[ (3.1) \quad (A - \bigcup_{j=1}^{n} a_j) \cup \bigcup_{j=0}^{n} a'_j \]

and

\[ (3.2) \quad (B - \bigcup_{j=1}^{n} b_j) \cup \bigcup_{j=1}^{n+1} b'_j \]

are independent sets of cardinality \( \nu(M) + 1 \). The edges of

\[ (3.3) \quad M' = (M - P) \cup P' \]

define a one-one function \( f' \) of (3.1) onto (3.2), so \( M' \) is a matching in \( (G(S), G(T), R) \), and \( \nu(M') = \nu(M) + 1 \). Thus \( M \) is not of maximum size.

**Theorem 2.** If \( M = (A, B, f) \) is a matching and \((C, D)\) is a support in \( (G(S), G(T), R) \), then

\[ \nu(M) \leq \lambda(C, D). \]

**Proof.** By definition, \( R(S-C) \leq D \). Therefore

\[ \nu(M) = \nu(A) \]

\[ = \nu(A \cap C) + \nu(A \cap (S-C)) \]

\[ = \nu(A \cap C) + \nu(f(A \cap (S-C))) \]
\[ = r(A \cap C) + r(f(A \cap (S-C))) \]
\[ \leq r(A \cap C) + r(R(A \cap (S-C))) \]
\[ \leq r(C) + r(R(S-C)) \]
\[ \leq r(C) + r(D) \]
\[ = \lambda(C, D). \]

**Theorem 3.** If \( M = A, B, f \) is a matching in \((G(S), G(T), R)\) and there does not exist an augmenting chain with respect to \( M \), then there exists a support \((\overline{A - A_m}, f(A_m))\), where \( A_m \subseteq A \).

The proof of Theorem 3 is constructive. We require several lemmas before proceeding with the main proof.

**Lemma 1.** If \( B_1, B_2 \) are independent sets in \( G(T) \), and \( B_1 \cup B_2 \) is independent, then

\[ \overline{B_1} \cap \overline{B_2} = \overline{B_1 \cap B_2}. \]

**Proof.** Clearly \( B_1 \cap B_2 \subseteq \overline{B_1 \cap B_2} \), and since the latter is a closed set, \( \overline{B_1 \cap B_2} \subseteq B_1 \cap B_2 \). By the semimodular inequality (2.6) for the rank function \( r \) of \( G(T) \),

\[ r(\overline{B_1 \cap B_2}) \leq r(\overline{B_1}) + r(\overline{B_2}) - r(\overline{B_1 \cup B_2}) \]
\[ = r(\overline{B_1}) + r(\overline{B_2}) - r(B_1 \cup B_2) \]
\[ = \nu(B_1) + \nu(B_2) - \nu(B_1 \cup B_2) \]
\[ = \nu(B_1 \cap B_2) \]
\[ = r(B_1 \cap B_2), \]

and the lemma follows.
LEMMA 2. Let \( B \) be an independent set in \( G(T) \) and suppose \( D \subseteq B \). Then the set

\[
B_1 = \{ b \in B : D \nsubseteq B - b \}
\]

is the unique minimal subset of \( B \) whose closure contains \( D \).

**Proof.** Let \( B_2 \) be any subset of \( B \) such that \( D \subseteq B_2 \). If \( B_1 \nsubseteq B_2 \), then there exists \( b \in B_1 \) such that \( B_2 \subseteq B - b \). But then \( D \subseteq B - b \), a contradiction.

Using Lemma 1 and the definition of \( B_1 \), we have

\[
D \subseteq b' \in B - B_1 \cap \overline{B - b'}
\]

\[
= \cap_{b' \in B - B_1} (B - b')
\]

\[
= \overline{B_1}.
\]

LEMMA 3. Suppose \( B \) is an independent set in \( G(T) \) and

\[
B_0 \subseteq B_1 \subseteq B_2 \subseteq \ldots \subseteq B_n
\]

is an increasing sequence of subsets of \( B \). Let \( b_i, b'_i, 1 \leq i \leq n \), be points satisfying

(i) \( b_i \in B_i - B_{i-1} \),

(ii) \( b'_i \in \overline{B_i - B - b_i} \).

Then

\[
b'_i \notin (B - \bigcup_{j=1}^{i-1} B_j) \cup \bigcup_{j=1}^{i} \bigcup_{j=1}^{i-1} B_j
\]

for \( 1 \leq i \leq n \).
Proof. We first show that

\[
B'_i = (B_i - U b'_j) \cup U b'_j \quad j=1 \to i
\]

is an independent set with closure \( \overline{B'_i} \). Now by (ii) \( b'_1 \in B_i - b'_1 \), so \( b'_1 \in \overline{B'_i} \) implies \( \overline{B'_i} = B'_i \) by the exchange property. Assuming the result true for \( i-1 \), let

\[
C_i = (B_i - U b'_j) \cup U b'_j \quad j=1 \to i-1
\]

\[
= (B_i - B_{i-1}) \cup B'_{i-1}.
\]

Then

\[
\overline{C_i} = (\overline{B_i - B_{i-1}}) \cup \overline{B'_{i-1}}
\]

\[
= (B_i - B_{i-1}) \cup B'_{i-1}
\]

\[
= (B_i - B_{i-1}) \cup B_{i-1}
\]

\[
= \overline{B_i},
\]

and by a similar argument

\[
\overline{C_i} - b_i = (\overline{B_i - B_{i-1}} - b_i) \cup \overline{B'_{i-1}}
\]

\[
= B_i - b_i.
\]
It follows now from (ii) and the exchange property that

\[ \overline{B_i} = \overline{C_i} \]

\[ = (C_i - b_i) \cup b_i' \]

\[ = \overline{B_i'}, \]

so by induction \( B_i' \) has closure \( \overline{B_i} \) for \( 1 \leq i \leq n \). Therefore

\[ \overline{B_i} = \overline{B_i' - B_{i-1} - b_i} \]

and the lemma follows by (ii).

**Lemma 4.** Suppose \( A \) is an independent set in \( G(S) \) and

\[ A_0 \subset A_1 \subset A_2 \subset \ldots \subset A_n \]

is an increasing sequence of subsets of \( A \). Let \( a_i, a_i', 1 \leq i \leq n \), be points satisfying

(i) \[ a_i \in A_i - A_{i-1}, \]

(ii) \[ a_i' \in A - A_{i-1} - A - a_i. \]

Then

\[ a_i' \notin (A - \bigcup_{j=1}^{i} a_j) \cup \bigcup_{j=1}^{i-1} a_j' \]

for \( 1 \leq i \leq n \).
PROOF. By (ii), $a'_1 \notin A-a_1$, so assume inductively that the lemma holds for $i-1$. Then

$$A^*_{i-1} = (A - \bigcup_{j=1}^{i-1} a_j) \cup \bigcup_{j=1}^{i-1} a'_j$$

is an independent set, and thus so also is $A^*_{i-1} - a_1$. Since

$$(A^*_{i-1} - a_1) \cup (A - A_{i-1}) = A^*_{i-1},$$

it follows from Lemma 1 that

$$A^*_{i-1} - a_1 \cap \overline{A - A_{i-1}} = \overline{A - A_{i-1} - a_1}.$$

Hence if $a'_1 \in A^*_{i-1} - a_1$, then by (ii),

$$a'_1 \in A - A_{i-1} - a_1 \subseteq A - a_1,$$

a contradiction.

PROOF OF THEOREM 3. Let $C_0 = S - \overline{A}$. Then $R(C_0) \subseteq \overline{B}$ since there is no augmenting chain with respect to $M = (A, B, f)$. Let $B_1$ be the minimal subset of $B$, defined according to Lemma 2, such that $R(C_0) \subseteq B_1$. Then let $A_1 = f^{-1}(B_1)$, $C_1 = S - A - A_1$. In general, having constructing $C_{i-1}$, we define $B_{i-1}$ as the minimal subset of $B$ such that $R(C_{i-1}) \cap \overline{B} \subseteq B_{i-1}$, and set $A_{i-1} = f^{-1}(A_{i-1})$, $C_{i-1} = S - A_{i-1}$. Since $A_{i-1} \subseteq C_{i-1}$, $f(A_{i-1}) \subseteq R(C_{i-1}) \cap \overline{B}$, but $f(A_{i-1}) \nsubseteq \overline{B} - b$ for any $b \in B_{i-1}$. Thus by Lemma 2, $B_{i-1} \subseteq B_i$ and so $A_{i-1} \subseteq A_i$, $C_{i-1} \subseteq C_i$. It is, moreover, clear that each of the sequences $A_i, B_i, C_i$ is strictly increasing up to and including some index $m$ after which the process terminates. Thus $R(C_m) \cap \overline{B} \subseteq B_m$, but $R(C_i) \cap \overline{B} \nsubseteq B_i$ for $0 \leq i \leq m$, where $B_0 = \emptyset$. 
We shall show that $R(C_i) \subseteq \overline{B}$ for all $i$, $0 \leq i \leq m$. Assuming otherwise, let $n$ be the smallest index, $0 \leq n \leq m$, such that $R(C_n) \ni \overline{B}$. We obtain a contradiction by showing that this assumption implies the existence of an augmenting chain with respect to $M$.

Since $R(C_n) \ni \overline{B}$, but $R(C_i) \subseteq \overline{B}$ for $0 \leq i < n$, there exists an edge $(a'_n, b'_{n+1})$ such that

$$b'_{n+1} \in T-\overline{B},$$

$$a'_n \in C_n - C_{n-1} = \overline{A-A_{n-1}} - \overline{A-A_n}.$$

If $a'_n \in \overline{A-a_n}$ for all $a_n \in A_n$, then $a'_n \in \overline{A-A_n}$ by Lemma 1, so there exists $a_n \in A_n$ such that $a'_n \notin \overline{A-a_n}$. Since $a'_n \in \overline{A-A_{n-1}}$, $a'_n \in \overline{A-a_{n-1}}$ for all $a_{n-1} \in A_{n-1}$, and hence $a_n \in A_n - A_{n-1}$. Let $b_n = f(a_n)$, then $b_n \in B-\overline{B}$, and hence $a_n \in A_n - A_{n-1}$. Let $b_n = f(a_n)$, then $b_n \in B-\overline{B}$, and hence $a_n \in A_n - A_{n-1}$.

By definition of $B_n$ and Lemma 2, there exists an edge $(a'_{n-1}, b'_n)$ such that $a'_{n-1} \in C_{n-1}$, $b'_n \in \overline{B-\overline{B}}$. Thus $b'_n \notin \overline{B_{n-1}}$, and so $b'_n \in \overline{B_{n-1}}$. Since $R(C_{n-2}) \subseteq \overline{B_{n-1}}$ by hypothesis, it follows that $a'_{n-1} \in C_{n-1} - C_{n-2}$ and $(a'_{n-1}, b'_n) \notin M$.

We may then continue this argument starting with $a'_{n-1} \in C_{n-1} - C_{n-2}$, and arrive finally at a sequence

$$\text{(3.4)} \quad (b'_{n+1}, a'_n), (a'_n, b'_n), (b'_n, a'_{n-1}), \ldots, (a'_1, b'_1), (b'_1, a'_0),$$

where

$$\text{(3.5)} \quad (a'_1, b'_1) \in M, \quad (a'_1, b'_{i+1}) \in R-M,$$

$$\text{(3.6)} \quad a'_0 \in S-\overline{A}, \quad b'_{n+1} \in T-\overline{B},$$

$$\text{(3.7)} \quad a'_1 \in A_{1} - A_{1-1}, \quad a'_i \in \overline{A-A_{i-1}} - \overline{A-A_i},$$

$$b'_i \in B_{i} - B_{i-1}, \quad b'_i \in \overline{B_{i} - B_{i-1}}.$$


for $1 \leq i \leq n$. Now (3.5) and (3.6) are simply (2.8) and (2.9), and by Lemmas 3 and 4, (3.7) implies (2.10). It follows that the sequence (3.4) in reverse order represents an augmenting chain, which contradicts the hypothesis of Theorem 3.

Thus $R(C_{i-1}) \subseteq \overline{B_i}$ for $1 \leq i \leq m$, and $R(C_m) \subseteq \overline{B_m}$. Since $C_m = S - \overline{A-A_m}$, the pair $(\overline{A-A_m}, \overline{B_m})$ is a support, and the proof of Theorem 3 is complete.

**Theorem 4.** A matching $M = (A,B,f)$ in $(G(S), G(T), R)$ is of maximum size if and only if there does not exist an augmenting chain with respect to $M$.

**Proof.** The necessity of the condition follows by Theorem 1. If there does not exist an augmenting chain, then the support given by Theorem 3 has order equal to the size of $M$, which together with Theorem 2 implies that $M$ is of maximum size.

**Corollary 4.1.** If $M = (A,B,f)$ is a matching not of maximum size, there exists a matching $M' = (A'\cup a, B'\cup b, f')$ such that $\overline{A'} = \overline{A}$, $\overline{B'} = \overline{B}$.

**Proof.** By Theorem 4, there exists an augmenting chain with respect to $M$, and the required matching $M'$ is constructed as in the proof of Theorem 1.

**Theorem 5.** The maximum size of a matching in $(G(S), G(T), R)$ is equal to the minimum order of a support.
PROOF. If $M = (A, B, f)$ is a matching of maximum size, then by Theorem 4 there does not exist an augmenting chain with respect to $M$. A support of order $\nu(M)$ therefore exists by Theorem 3, and this support is necessarily of minimum order by Theorem 2.

Following Ore [5] for the case of a bipartite graph, we define the deficiency $\delta_S(A)$ of a subset $A \subseteq S$ by

$$\delta_S(A) = r(S) - r(S-A) - r(R(A)),$$

and let

$$\delta_S = \max_{A \subseteq S} \delta_S(A).$$

Note that $\delta_S \geq 0$ since $\delta_S(\emptyset) = 0$.

THEOREM 6. In the system $(G(S), G(T), R)$,

$$\max_{M \text{ matching}} \nu(M) = \min_{(C, D) \text{ support}} \lambda(C, D) = r(S) - \delta_S.$$

PROOF. Note that every support of minimum order is necessarily of the form $(C, D)$, where $D = R(S-C)$. Now

$$r(S) - \delta_S = r(S) - \max_{A \subseteq S} [r(S) - r(S-A) - r(R(A))]$$

$$= \min_{A \subseteq S} [r(S-A) + r(R(A))]$$

$$= \min_{A \subseteq S} [r(A) + r(R(S-A))],$$

and the latter minimum is clearly attained when $A$ is a closed set.
**COROLLARY 6.1.** There exists a matching of size \( r(S) \) in \( (G(S), G(T), R) \) if and only if

\[
r(S) - r(S - A) \leq r(R(A))
\]

for every subset \( A \subseteq S \).

**THEOREM 7.** (See also [2], [3], [4].) If the geometry \( G(S) \) is free in the system \( (G(S), G(T), R) \), then the subsets \( S' \) of \( S \) for which there exists a matching \( (S', T', f) \) for some \( T' \) and \( f \), are the independent sets of a pregeometry, the **transversal pregeometry** on \( S \).

**PROOF.** Let \( I \) be the family of subsets \( S' \) of \( S \) for which there exists a matching \( (S', T', f) \) for some \( T' \) and \( f \). Given any subset \( S' \subseteq S \), let \( G(S') \) be the free subgeometry on \( S' \). Applying Theorem 6 to the system \( (G(S'), G(T), R \cap (S' \times T)) \), we have \( S' \in I \) if and only if \( \delta_{S'} = 0 \). Equivalently, \( S' \in I \) if and only if \( \nu(S') \leq \nu(S') - \delta_{S'} \).

The theorem will therefore follow from Proposition 7.3 of [2] if the function

\[
r^*(S') = \nu(S') - \delta_{S'}
\]

is increasing and semimodular. We proceed to establish these properties for \( r^* \).

Let \( S' \subseteq S \) and \( A \subseteq S' \). Since \( G(S') \) is free, the deficiency

\[
\delta_{S'}(A) = \nu(A) - r(R(A))
\]

is independent of \( S' \), so we may omit the subscript. Then

\[
\delta_{S'} = \max_{A \subseteq S'} \delta(A).
\]
Given \( A_1, A_2 \subseteq S' \), it follows from the relations

\[
R(A_1 \cup A_2) = R(A_1) \cup R(A_2)
\]

\[
R(A_1 \cap A_2) \subseteq R(A_1) \cap R(A_2)
\]

and the semimodular inequality (2.6) for the rank function \( r \) of \( G(T) \) that

\[
\delta(A_1 \cup A_2) \cup \delta(A_1 \cap A_2) \geq \delta(A_1) + \delta(A_2).
\]

Let \( F(S') \) be the family of subsets \( A \) of \( S' \) for which \( \delta(A) = \delta_S' \).

Then by (3.8), \( F(S') \) is closed under the operation of intersection, and therefore contains a minimal set, which we denote by \( A_S' \).

Clearly if \( a \in S'-A_S' \), then \( A_{S'-a} = A_S' \) and \( \delta_{S'-a} = \delta_S' \).

Suppose that \( a \in A_S' \). Since \( A_S' \) is minimal, \( \delta_{S'-a} \leq \delta_S' - 1 \). But

\[
\delta(A_S', -a) = \vee(A_S', -a) - r(R(A_S', -a))
\]

\[
\geq \vee(A_{S'}') - r(R(A_{S'})) - 1
\]

\[
= \delta(A_{S'}') - 1,
\]

so \( \delta_{S'-a} = \delta_{S'} - 1 \). We have therefore that

\[
r^*(S'-a) = \begin{cases} 
r^*(S') - 1, & a \in S'-A_S', \\
r^*(S'), & a \in A_S'. \end{cases}
\]

Thus the function \( r^* \) is not only increasing, but unit-increasing.

Note from the argument above that if \( a \in S' \), then \( A_{S'-a} \in F(S'-a) \) and \( A_{S'-a} \subseteq A_{S'-a} \). We conclude that if \( S_1 \subseteq S_2 \subseteq S \), then \( A_{S_1} \subseteq A_{S_2} \cap S_1 \) and \( \delta(A_{S_1}) = \delta(A_{S_2} \cap S_1) \).
Now let $S_1, S_2$ be any two subsets of $S$, and let

$$A_1 = A_{S_1 \cup S_2} \cap (S_1 - S_2),$$

$$A_2 = A_{S_1 \cup S_2} \cap (S_2 - S_1),$$

$$A_3 = A_{S_1 \cup S_2} \cap (S_1 \cap S_2).$$

Then by (3.8) and the above remark we have

$$\delta_{S_1 \cup S_2} + \delta_{S_1 \cap S_2} = \delta(A_{S_1 \cup S_2}) + \delta(A_{S_1 \cap S_2})$$

$$= \delta(A_1 \cup A_2 \cup A_3) + \delta(A_3)$$

$$\geq \delta(A_1 \cup A_3) + \delta(A_2 \cup A_3)$$

$$= \delta(A_{S_1}) + \delta(A_{S_2})$$

$$= \delta_{S_1} + \delta_{S_2}.$$

Thus

$$r^*(S_1 \cup S_2) + r^*(S_1 \cap S_2) \leq r^*(S_1) + r^*(S_2)$$

and the proof is complete.

It should be noted that Theorem 7 is false if the geometry $G(S)$ is arbitrary. The function

$$r^*(S') = r(S') - \delta_S,$$

is unit-increasing, but not semimodular in general, so that Proposition 5.7 of [2] cannot be applied. For the same reason, Theorem 6 cannot be
proved by extending Ore's inductive argument [5] for the case of a bi-
partite graph to the general case, although this approach works when
G(S) is free and G(T) arbitrary.

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