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ON THE DISTRIBUTION OF A TRACE OF A MULTIVARIATE QUADRATIC FORM IN THE MULTIVARIATE NORMAL SAMPLES

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ABSTRACT

This paper considers the derivation of the p.d.f. of the trace of a non-central multivariate quadratic form using a polynomial $P_\kappa(T,A)$ defined by the author and compares with the results of Kotz et al and Ruben. The complex case is also discussed.

KEY WORDS

Non-central multivariate quadratic form
generating function
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representation
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1. INTRODUCTION.

Recently the probability density function (p.d.f.) of latent roots of a multivariate non-central quadratic form and of a trace of it were obtained by the use of a polynomial $P_\kappa(T,A)$ introduced by Hayakawa [3]. However, it only shows that if we use the polynomial $P_\kappa(T,A)$, we can represent the p.d.f.'s in terms of a power series representation. In this paper, we discuss another type of representation, that is, Γ-type representation of the p.d.f. and give a more concrete form than Hayakawa [3]. By using this representation, we can compare with the results of Ruben [8] and Kotz et al [7]. We also give a p.d.f. for the case of the complex variables.

2. NOTATIONS AND SOME USEFUL RESULTS.

Let T and U be $m\times n$ $(m\leq n)$ real arbitrary matrices each of rank m, and let A be an $n\times n$ positive definite symmetric (p.d.s.) matrix. Hayakawa [3] defined a new polynomial $P_\kappa(T,A)$ as follows;

$$(1)\quad etr(-TT')P_\kappa(T,A) = \frac{(-1)^k}{\pi^{mn/2}}\int_U etr(-2iTU')etr(-UU')C_\kappa(UAU')dU,$$

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where κ is a partition of k into not more than m parts, i.e., $\kappa = (k_1,k_2,\ldots,k_m)$, $k = k_1+\ldots+k_m$, $k_1 \geq k_2 \geq \ldots \geq k_m \geq 0$, and $C_\kappa(UAU')$ is a zonal polynomial of UAU' corresponding to a partition κ of k, James [5].

$P_\kappa(T,A)$ has the following properties.

$$(2) \qquad P_\kappa(0,A) = (\tfrac{n}{2})_\kappa\, C_\kappa(A)/C_\kappa(I_n),$$

$$(3) \qquad P_\kappa(T,I_n) = H_\kappa(T),$$

$$(4) \qquad |P_\kappa(T,A)| \leq \mathit{etr}(TT')(\tfrac{n}{2})_\kappa\, C_\kappa(A)/C_\kappa(I_n),$$

where

$$(a)_\kappa = \prod_{\alpha=1}^{m} (a - \tfrac{\alpha-1}{2})_{k_\alpha}, \quad (a)_n = a(a+1)\ldots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)},$$

and $H_\kappa(T)$ is a generalized Hermite polynomial of matrix argument T.

The generating function of $P_\kappa(T,A)$ is given by

$$(5) \qquad \int_{O(m)} \int_{O(n)} \mathit{etr}(-UH_2AH_2'U' + 2H_1UH_2A^{\frac{1}{2}}T')\, d(H_1)\, d(H_2)$$

$$= \sum_{k=0}^{\infty} \sum_{\kappa} \frac{P_\kappa(T,A)C_\kappa(UU')}{k!(\frac{n}{2})_\kappa C_\kappa(I_m)}$$

and the right hand side (R.H.S.) of (5) converges absolutely with respect to U. $d(H_1)$ and $d(H_2)$ are the orthogonal invariant measures on the orthogonal groups $O(m)$ and $O(n)$, respectively. A detailed discussion of $H_\kappa(T)$ and $P_\kappa(T,A)$ may be found in Hayakawa [3].

We give here some useful lemmas which will be applied to the representation of the p.d.f. of a trace of a non-central quadratic form.

Lemma 1.

(6)
$$\sum_\kappa P_\kappa(T,A) = (-1)^k [A_k + \frac{1}{2}\sum_{\ell=0}^{k} A_{k-\ell}A_\ell + \frac{1}{3!}\sum_{\ell_1=0}^{k}\sum_{\ell_2=0}^{\ell_1} A_{k-\ell_1}A_{\ell_1-\ell_2}A_{\ell_2} + \cdots + \frac{1}{k!}\sum_{\ell_1=0}^{k}\sum_{\ell_2=0}^{\ell_1}\cdots\sum_{\ell_{k-1}=0}^{\ell_{k-2}} A_{k-\ell_1}A_{\ell_1-\ell_2}\cdots A_{\ell_{k-2}-\ell_{k-1}}A_{\ell_{k-1}}],$$

where

$$A_\ell = \frac{m}{2\ell}\,tr A^\ell - tr TA^\ell T', \qquad \ell = 1,2,\ldots,k$$

and

$$A_0 = 0, \quad \text{for convenience.}$$

Proof: From the definition of $P_\kappa(T,A)$, we construct the generating function of $(-1)^k \sum_\kappa P_\kappa(T,A)$.

(7)
$$\sum_{k=0}^{\infty}\frac{(-x)^k}{k!}\sum_\kappa P_\kappa(T,A)$$

$$= \frac{etr(TT')}{\pi^{\frac{1}{2}mn}}\int_U etr(-2iTU')\,etr(-UU')\,etr(xUAU')\,dU$$

$$= det(I-xA)^{-(m/2)}\,etr(TT').$$

$$\cdot \frac{1}{\pi^{mn/2}} \int_U etr(-UU'-2iT(I-xA)^{-\frac{1}{2}}U')\, dU$$

$$= det(I-xA)^{-(m/2)}\, etr(T(I-(I-xA)^{-1})T'),$$

$$||xA|| < 1,$$

where $||A||$ means the maximum value of the absolute value of the characteristic roots of A.

We expand the R.H.S. of (7) with respect to x by noting that $||xA|| < 1$ using

$$\log det(I-xA) = x\,trA + \frac{x^2}{2} trA^2 + \ldots + \frac{x^k}{k} trA^k + \ldots$$

and

$$(I-xA)^{-1} = I + xA + x^2A^2 + \ldots + x^kA^k + \ldots \quad .$$

Hence

$$\text{R.H.S.} = exp[-\frac{m}{2} \log det(I-xA)]\, etr(-xTA(I-xA)^{-1}T')$$

$$= exp[x\,tr(\tfrac{m}{2}A-TAT') + x^2 tr(\tfrac{m}{4}A^2-TA^2T') + \ldots$$

$$\ldots + x^k tr(\tfrac{m}{2k}A^k-TA^kT') + \ldots] \, .$$

Here we set

$$A_\ell = tr(\tfrac{m}{2\ell}A^\ell-TA^\ell T'), \quad \ell = 1,2,\ldots$$

and

$$A_0 = 0, \quad \text{for convenience.}$$

We can obtain the value of $(-1)^k \Sigma_\kappa P_\kappa(T,A)$ by comparing the coefficients of x^k on the two sides of (7). By differentiating the left hand side with respect to x and by setting $x = 0$, we have $(-1)^k \Sigma_\kappa P_\kappa(T,A)$. Let us divide the R.H.S. into two parts such that

$$\text{R.H.S.} = \mathit{exp}(g(x))\ \mathit{exp}(f(x))$$

$$= \{1 + g(x) + \frac{1}{2!} g^2(x) + \ldots + \frac{1}{k!} g^k(x) + \ldots\}$$

$$\cdot \{1 + f(x) + \frac{1}{2!} f^2(x) + \ldots\} ,$$

where

$$g(x) = \sum_{j=0}^{k} A_j x^j, \quad f(x) = \sum_{j=k+1}^{\infty} A_j x^j .$$

The degrees of x in the second factor are all greater than or equal to $k+1$, and so there is no contribution to the coefficient of x^k in the R.H.S. Hence we need only consider the first factor. We have

$$g'(0) = A_1, \quad g''(0) = 2A_2, \ldots, g^{(k)}(0) = k!A_k ,$$

and

$$g(0) = 0 \ (\equiv A_0, \text{ by definition}).$$

Now we differentiate k times the power series of $g(x)$ and set $x = 0$.

$$g^{(k)}(0) = k!A_k ,$$

$$(g^2(x))^{(k)}\Big|_{x=0} = \sum_{\ell=0}^{k} \binom{k}{\ell} g^{(\ell)}(x) g^{(k-\ell)}(x)\Big|_{x=0} = k! \sum_{\ell=0}^{k} A_{k-\ell} A_\ell ,$$

$$(g^3(x))^{(k)}\Big|_{x=0} = \sum_{\ell_1=0}^{k} \binom{k}{\ell_1} (g^2(x))^{(\ell_1)} g^{(k-\ell_1)}(x)\Big|_{x=0}$$

$$= \sum_{\ell_1=0}^{k} \sum_{\ell_2=0}^{\ell_1} \binom{k}{\ell_1}\binom{\ell_1}{\ell_2} g^{(\ell_2)}(x) g^{(\ell_1-\ell_2)}(x) g^{(k-\ell_1)}(x)\Big|_{x=0}$$

$$= k! \sum_{\ell_1=0}^{k} \sum_{\ell_2=0}^{\ell_1} A_{k-\ell_1} A_{\ell_1-\ell_2} A_{\ell_2}.$$

In the same way we have

$$(g^k(x))^{(k)}\Big|_{x=0} = k! \sum_{\ell_1=0}^{k} \sum_{\ell_2=0}^{\ell_1} \cdots \sum_{\ell_{k-1}=0}^{\ell_{k-2}} A_{k-\ell_1} A_{\ell_1-\ell_2} \cdots A_{\ell_{k-2}-\ell_{k-1}} A_{\ell_{k-1}},$$

and

$$(g^p(x))^{(k)}\Big|_{x=0} = 0, \quad \text{for} \quad p \geq k+1.$$

Hence combining the results, we have

$$\left\{exp(g(x))\right\}^{(k)}\Big|_{x=0} = k![A_k + \frac{1}{2} \sum_{\ell=0}^{k} A_{k-\ell} A_\ell + \frac{1}{3!} \sum_{\ell_1=0}^{k} \sum_{\ell_2=0}^{\ell_1} A_{k-\ell_1} A_{\ell_1-\ell_2} A_{\ell_2}$$

$$+ \cdots + \frac{1}{k!} \sum_{\ell_1=0}^{k_1} \cdots \sum_{\ell_{k-1}=0}^{\ell_{k-2}} A_{k-\ell_1} A_{\ell_1-\ell_2} \cdots A_{\ell_{k-2}-\ell_{k-1}} A_{\ell_{k-1}}],$$

which completes the proof.

EXAMPLES:

$k = 1$: $P_1(T,A) = (-1)A_1 = -\frac{m}{2}\, tr A + tr TAT'$,

$k = 2$: $\sum_{(2)} P_{(2)}(T,A) = 2![A_2 + \frac{1}{2} A_1]$

$$= 2![tr(\frac{m}{4} A^2 - TA^2T') + \frac{1}{2}\{tr(\frac{m}{2} A - TAT')\}^2],$$

$$k = 3: \quad \sum_{(3)} P_{(3)}(T,A) = -3![A_3 + A_1A_2 + \frac{1}{3!} A_1^3]$$

$$= -3![tr(\frac{m}{6} A^3 - TA^3T') + tr(\frac{m}{2} A - TAT') \cdot$$

$$\cdot\, tr(\frac{m}{4} A^2 - TA^2T')$$

$$+ \frac{1}{3!} \{tr(\frac{m}{2} A - TAT')\}^3],$$

etc.

REMARK. If we set $A = I_n$, then we have immediately from Hayakawa [4, (18)]

$$\sum_{\kappa} P_{\kappa}(T,I_n) = \sum_{\kappa} H_{\kappa}(T) = (-1)^k L_k^{(mn/2)-1} (trTT') .$$

3. THE P.D.F. OF $tr\, \Sigma^{-1}XAX'$.

Let X be an $m\times n$ $(m \le n)$ matrix whose density function is given by

$$\text{(8)} \qquad \frac{1}{(2\pi)^{mn/2}(det\Sigma)^{n/2}(detB)^{m/2}}\, etr[-\frac{1}{2}\Sigma^{-1}(X-M)B^{-1}(X-M)'] ,$$

Where Σ is an $m\times m$ p.d.s. matrix, B is an $n\times n$ p.d.s. matrix and M is an $m\times n$ $(m \le n)$ matrix such that $E(X) = M$ and rank $M = m$. Let A be an $n\times n$ p.d.s. matrix.

LEMMA 2. *(Power series representation.)* Let X be distributed with p.d.f. (8), then the p.d.f. of the latent roots $\Lambda = diag(\lambda_1,\ldots,\lambda_m)$ of $\Sigma^{-\frac{1}{2}}XAX'\Sigma^{-\frac{1}{2}}$ is given by

$$(9) \qquad \frac{\pi^{(m^2/2)} etr(-\frac{1}{2}MB^{-1}M'\Sigma^{-1})}{\Gamma_m(\frac{n}{2})\Gamma_m(\frac{m}{2})(det 2AB)^{(m/2)}} (det\Lambda)^{\frac{1}{2}(n-m-1)} \prod_{i<j} (\lambda_i-\lambda_j)$$

$$\cdot \sum_{k=0}^{\infty} \sum_{\kappa} \frac{P\ (\frac{1}{\sqrt{2}}\ \Sigma^{-\frac{1}{2}}MB^{-1}A^{-\frac{1}{2}}C^{\frac{1}{2}}, C^{-1})C_\kappa(\frac{1}{2}\Lambda)}{k!(\frac{n}{2})_\kappa C_\kappa(I_m)},$$

where $\Gamma_m(a) = \pi^{\frac{1}{4}m(m-1)} \prod_{\alpha=1}^{m} \Gamma(a-\frac{\alpha-1}{2})$ and $C = A^{\frac{1}{2}}BA^{\frac{1}{2}}$.

PROOF: See Hayakawa [3].

Next we give another type (say, Γ-type) expression for this p.d.f.

THEOREM 1. *(Γ-type representation.)* Let X be distributed with p.d.f. (8), then the p.d.f. of the latent roots $\Lambda = diag(\lambda_1,\dots,\lambda_m)$ of $\Sigma^{-\frac{1}{2}}XAX'\Sigma^{-\frac{1}{2}}$ is given by, for $||AB|| < p$,

$$(10) \qquad \frac{\pi^{(m^2/2)} etr(-\frac{1}{2}\Sigma^{-1}MB^{-1}M')}{\Gamma_m(\frac{n}{2})\Gamma_m(\frac{m}{2})(det 2AB)^{m/2}}\, etr(-\frac{1}{2p}\Lambda)(det\Lambda)^{\frac{1}{2}(n-m-1)} \prod_{i<j} (\lambda_i-\lambda_j)$$

$$\cdot \sum_{k=0}^{\infty} \sum_{\kappa} \frac{P_\kappa(\frac{1}{\sqrt{2}}\ \Sigma^{-\frac{1}{2}}MB^{-1}A^{-\frac{1}{2}}(C^{-1}-\frac{I}{p})^{-\frac{1}{2}}, C^{-1}-\frac{I}{p})C_\kappa(\frac{1}{2}\Lambda)}{k!(\frac{n}{2})_\kappa C_\kappa(I_m)}$$

where $C = A^{\frac{1}{2}}BA^{\frac{1}{2}}$.

The power series converges absolutely for $\Lambda > 0$.

PROOF: We decompose $Y = \Sigma^{-\frac{1}{2}}XA^{\frac{1}{2}}$ as

$$Y = \Sigma^{-\frac{1}{2}} XA^{\frac{1}{2}} = H_1 \Lambda^{\frac{1}{2}} L,$$

where H_1 is an orthogonal matrix of order m whose elements of the first column are positive and $\Lambda^{\frac{1}{2}} = diag(\lambda_1^{\frac{1}{2}},\dots,\lambda_m^{\frac{1}{2}})$ and $\lambda_1,\dots,\lambda_m$ are latent roots of $YY' = \Sigma^{-\frac{1}{2}}XAX'\Sigma^{-\frac{1}{2}}$, and L is an $m\times n$ Stiefel matrix such that $LL' = I_m$. By inserting this decomposition into (8), we have the joint p.d.f. of Λ, H_1 and L:

$$(11)\qquad \frac{\pi^{(m^2/2}etr(-\frac{1}{2}\Sigma^{-1}MB^{-1}M')}{\Gamma_m(\frac{n}{2})\Gamma_m(\frac{m}{2})(det2AB)^{(m/2)}}\, etr(-\frac{1}{2}\Lambda^{\frac{1}{2}}LA^{-\frac{1}{2}}B^{-1}A^{-\frac{1}{2}}L'\Lambda^{\frac{1}{2}}$$

$$+ H_1\Lambda^{\frac{1}{2}}LA^{-\frac{1}{2}}B^{-1}M'\Sigma^{-\frac{1}{2}})(det\Lambda)^{\frac{1}{2}(n-m-1)} \prod_{i<j}(\lambda_i-\lambda_j)d\Lambda d(H_1)d(L)\ .$$

If we set $L \to LH_2$, $H_2\epsilon O(n)$, then $LH_2(LH_2)' = I_m$ and $d(L)$ remains invariant with respect to H_2. Then the integral with respect to $O(m)$ and $O(n)$ is the same form as (5) with $U = \frac{1}{\sqrt{2}}\Lambda^{\frac{1}{2}}L$ and $T = \frac{1}{\sqrt{2}}\Sigma^{-\frac{1}{2}}MB^{-1}A^{-\frac{1}{2}}(C^{-1} - \frac{I}{p})^{\frac{1}{2}}$, where $C = A^{\frac{1}{2}}BA^{\frac{1}{2}}$. Hence

$$\int_{LL'=I_m} d(L)etr(-\frac{1}{2p}\Lambda)\frac{1}{2^m}\int_{O(m)}\int_{O(n)} etr[-\frac{1}{2}\Lambda^{\frac{1}{2}}LH_2(C^{-1}-\frac{I}{p})H_2'L'\Lambda^{\frac{1}{2}}$$

$$+ H_1\Lambda^{\frac{1}{2}}LH_2(C^{-1}-\frac{I}{p})^{\frac{1}{2}}(C^{-1}-\frac{I}{p})^{-\frac{1}{2}}A^{-\frac{1}{2}}B^{-1}M'\Sigma^{-\frac{1}{2}})\ d(H_1)\ d(H_2)$$

$$= etr(-\frac{1}{2p}\Lambda)\sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(\frac{1}{2}\Lambda)}{k!(\frac{n}{2})_\kappa C_\kappa(I_m)} P_\kappa(\frac{1}{\sqrt{2}}\Sigma^{-\frac{1}{2}}MB^{-1}A^{-\frac{1}{2}}(C^{-1}-\frac{I}{p})^{-\frac{1}{2}}, C^{-1}-\frac{I}{p})\ .$$

The proof of the absolutely convergence is easily achieved by using (4), which completes the proof.

NOTE. Since

$$\int_{\lambda_1>\dots>\lambda_m>0} etr(-\frac{1}{2p}\Lambda)(det\Lambda)^{\frac{1}{2}(n-m-1)}\prod_{i<j}(\lambda_i-\lambda_j)C_\kappa(\frac{1}{2}\Lambda)d\Lambda$$

$$= \frac{\Gamma_m(\frac{n}{2})\Gamma_m(\frac{m}{2})}{\pi^{m^2/2}} (2p)^{mn/2} (\tfrac{n}{2})_\kappa p^k C_\kappa(I) ,$$

we have the following relations.

(12) $$\sum_{k=0}^{\infty} \sum_\kappa \frac{p^k}{k!} P_\kappa(\tfrac{1}{\sqrt{2}}\Sigma^{-\frac{1}{2}}MB^{-1}A^{-\frac{1}{2}}(C^{-1} - \tfrac{I}{p})^{-\frac{1}{2}}, C^{-1} - \tfrac{I}{p})$$

$$= etr(-\tfrac{1}{2}\Sigma^{-1}MB^{-1}M')(detAB/p)^{mn/2}.$$

This formula can be obtained from (7) directly if we replace x with $-p$, T with $\frac{1}{\sqrt{2}}\Sigma^{-\frac{1}{2}}MB^{-1}A^{-\frac{1}{2}}(C^{-1} - \frac{I}{p})^{-\frac{1}{2}}$, and A with $C^{-1} - \frac{I}{p}$, respectively.

Hayakawa [3] obtained the p.d.f. of $tr\Sigma^{-\frac{1}{2}}XAX'\Sigma^{-\frac{1}{2}}$ as a power series representation. We give this as Lemma 3 for comparison with Theorem 2.

LEMMA 3. *(Power Series representation.)* Let Λ be distributed with p.d.f. (9), then the p.d.f. of $T = tr\Lambda$ is given by

(13)

$$\frac{etr(-\frac{1}{2}\Sigma^{-1}MB^{-1}M')}{\Gamma(\frac{mn}{2})(det2AB)^{m/2}} T^{(mn/2)-1} \sum_{k=0}^{\infty} \frac{1}{k!(\frac{mn}{2})_k} (\tfrac{T}{2})^k \sum_\kappa P_\kappa(\tfrac{1}{\sqrt{2}}\Sigma^{-\frac{1}{2}}MB^{-1}A^{-\frac{1}{2}}C^{\frac{1}{2}}, C^{-1}),$$

where $C = A^{\frac{1}{2}}BA^{\frac{1}{2}}$.

PROOF: See [3].

THEOREM 2. *(Γ-type representation.)* Let Λ be distributed with p.d.f. (10), then the p.d.f. of $T = tr\Lambda$ is given by

$$(14)\qquad \frac{etr(-\frac{1}{2}\Sigma^{-1}MB^{-1}M')}{\Gamma(\frac{mn}{2})(det2AB)^{m/2}}\, exp(-\frac{T}{2p})\cdot$$

$$\cdot T^{(mn/2)-1}\sum_{k=0}^{\infty}\frac{1}{k!(\frac{mn}{2})_k}(\frac{T}{2})^k\sum_{\kappa}P_{\kappa}(\frac{1}{\sqrt{2}}\Sigma^{-\frac{1}{2}}MB^{-1}A^{-\frac{1}{2}}(C^{-1}-\frac{I}{p})^{-\frac{1}{2}},$$

$$C^{-1}-\frac{I}{p}),$$

where $C = A^{\frac{1}{2}}BA^{\frac{1}{2}}$.

The series converges absolutely for $T > 0$.

PROOF: By applying a Fourier transform to $T = tr\Lambda$ and inverting it, we obtain (14) easily. Q.E.D.

4. THE RELATION WITH A UNIVARIATE QUADRATIC FORM.

We can derive the p.d.f. of $tr\Sigma^{-1}XAX'$ by another way. We denote X and M as

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}, \qquad M = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_m \end{bmatrix},$$

where $x_\alpha = (x_{\alpha 1},\ldots,x_{\alpha n})$ and $\mu_\alpha = (\mu_{\alpha 1},\ldots,\mu_{\alpha n})$, $\alpha = 1,2,\ldots,m$. Let $x = (x_1,x_2,\ldots,x_m)$ and $\mu = (\mu_1,\mu_2,\ldots,\mu_m)$, then x is distributed with mean μ and a covariance matrix $\Sigma\otimes B$, where $\otimes$ denotes a Kronecker product of Σ and B. On the other hand, $trXAX' = \Sigma_{\alpha=1}^{n}x_\alpha Ax_\alpha' = x[I_m\otimes A]x'$. Hence the problem is reduced to the one of the univariate non-central quadratic form. To compare with the results of Kotz et al [7] and Ruben [8], we can assume that A is a diagonal matrix whose diagonal elements are $a_1,a_2,\ldots,a_n$ and $a_1 \geq a_2 \geq \ldots \geq a_n > 0$.

The p.d.f. of $tr XAX'$ is derived as follows.

(15) $$\frac{1}{(2\pi)^{\frac{mn}{2}}(det\Sigma)^{\frac{n}{2}}(detB)^{\frac{m}{2}}}\int_{T=x(I_m\otimes A)x'} exp[-\tfrac{1}{2}(x-\mu)(\Sigma^{-1}\otimes B^{-1})(x-\mu)']\,dX$$

$$=\frac{exp[-\frac{1}{2}\mu(\Sigma^{-1}\otimes B^{-1})\mu']}{(2\pi)^{\frac{mn}{2}}(det\Sigma)^{\frac{n}{2}}(detAB)^{\frac{m}{2}}}\int_{T=xx'} exp[-\tfrac{1}{2}(\Sigma^{-1}\otimes C^{-1})x' + x(\Sigma^{-1}\otimes A^{-\frac{1}{2}}B^{-1})\mu']dX$$

where $C = A^{\frac{1}{2}}BA^{\frac{1}{2}}$.

From the above integral, we derive two types of the representation.

THEOREM 3. *(Power series representation.)* Let X be distributed with p.d.f. (8), then the p.d.f. of $T = tr XAX'$ is given by (16)

$$\frac{exp[-\frac{1}{2}\mu(\Sigma^{-1}\otimes B^{-1})\mu']}{2^{\frac{mn}{2}}\Gamma(\frac{mn}{2})(det\Sigma)^{\frac{n}{2}}(detAB)^{\frac{m}{2}}}\,T^{(mn/2)-1}$$

$$\sum_{k=0}^{\infty}\frac{1}{k!(\frac{mn}{2})_k}(\tfrac{T}{2})^k\,P_k(\tfrac{1}{\sqrt{2}}\mu(\Sigma^{-\frac{1}{2}}\otimes B^{-1}A^{-\frac{1}{2}}C^{\frac{1}{2}}),\ \Sigma^{-1}\otimes C^{-1}),$$

where $P_k(\cdot,\cdot)$ is a polynomial for $m = 1$ in the definition of $P_K(T,A)$ and $C = A^{\frac{1}{2}}BA^{\frac{1}{2}}$.

PROOF: As for Lemma 2.

COROLLARY. The p.d.f. of $T = tr\Sigma^{-1}XAX'$ is given by

(17) $$\frac{exp[-\frac{1}{2}\mu(\Sigma^{-1}\otimes B^{-1})\mu']}{2^{\frac{mn}{2}}\Gamma(\frac{mn}{2})(detAB)^{\frac{m}{2}}}\,T^{(mn/2)-1}$$

$$\sum_{k=0}^{\infty} \frac{1}{k!(\frac{mn}{2})_k} (\tfrac{T}{2})^k P_k(\tfrac{1}{\sqrt{2}}\mu(\Sigma^{-\frac{1}{2}} \otimes B^{-1}A^{-\frac{1}{2}}C^{\frac{1}{2}}),\ I \otimes C^{-1}).$$

We can easily show that (17) is the same form as (13).

Here we compare with the results of Kotz et al [7]. Kotz et al showed the following Lemma.

LEMMA. *(Kotz et al.)* Let $x = (x_1,\ldots,x_n)$ be normally distributed with mean 0 and covariance matrix I_n and A be a diagonal matrix, i.e. $diag(a_1,\ldots,a_n)$, $a_1 \geq a_2 \geq \ldots \geq a_n > 0$, and $b = (b_1,\ldots,b_n)$, then the p.d.f. of $T = (x+b)A(x+b)'$ is given by

$$(18) \qquad \sum_{k=0}^{\infty} \alpha_k^p (-1)^k (\tfrac{T}{2})^{(n/2)+k-1} \frac{1}{2\Gamma(\frac{n}{2}+k)} .$$

The α_k^p's are determined by

$$(19) \qquad \sum_{k=0}^{\infty} \alpha_k^p \theta^k = (det A)^{-\frac{1}{2}} exp[-\tfrac{1}{2}\sum_{i=1}^{n} b_i^2/(1-\theta/a_i)] \prod_{i=1}^{n} (1-\theta/a_i)^{-\frac{1}{2}}$$

where the recurrence relation

$$(20) \qquad \alpha_0^p = (det A)^{-\frac{1}{2}} exp(-\tfrac{1}{2}\sum_{i=1}^{n} b_i^2)$$

$$\alpha_k^p = \sum_{r=0}^{k-1} b_{k-r}\, \alpha_r^p/k, \quad k \geq 1,$$

with $b_k^p = \frac{1}{2}\sum_{i=1}^{n}(1-kb_i^2)(a_i)^{-k}$ can be obtained.
(α_0^p and b_k^p of Kotz et al should be changed to above form.)

To compare with Theorem 3 and Lemma (Kotz et al), we set $\Sigma = I_m$ and $B = I_n$ in (16) and we have

(16)'
$$\frac{exp[-\frac{1}{2}\mu\mu']}{\Gamma(\frac{mn}{2})(det2A)^{\frac{m}{2}}} T^{(mn/2)-1} \sum_{k=0}^{\infty} \frac{1}{k!(\frac{mn}{2})_k} (\frac{T}{2})^k P_k(\frac{1}{\sqrt{2}}\mu, I\otimes A^{-1})$$

$$= \frac{etr(-\frac{1}{2}MM')}{\Gamma(\frac{mn}{2})(det2A)^{\frac{m}{2}}} T^{(mn/2)-1} \sum_{k=0}^{\infty} \frac{1}{k!(\frac{mn}{2})_k} (\frac{T}{2})^k \sum_{\kappa} P_{\kappa}(\frac{1}{\sqrt{2}}M, A^{-1}).$$

Now replacing A by $I_m \otimes A$ and b by μ in Lemma (Kotz et al), we have the following form.

(18)'
$$\sum_{k=0}^{\infty} \alpha_k^p (-1)^k (\frac{T}{2})^{(mn/2)+k-1} \frac{1}{2\Gamma(\frac{mn}{2}+k)},$$

(19)'
$$\sum_{k=0}^{\infty} \alpha_k^p \theta^k = (detA)^{-(m/2)} exp[-\frac{1}{2}\sum_{j=1}^{n} \frac{1}{1-\theta/a_j} \sum_{i=1}^{m} \mu_{ij}^2] \prod_{j=1}^{n} (1-\theta/a_j)^{-(m/2)}$$

(20)'
$$\alpha_0^p = (detA)^{-(m/2)} etr(-\frac{1}{2} MM')$$

(21)'
$$b_k^p = \sum_{j=1}^{n} (\frac{1}{a_j})^k \sum_{i=1}^{m} (1-k\mu_{ij}^2).$$

Therefore, by comparing with (16)' and (18)', we have the following relation.

(22)
$$\alpha_k^p = \frac{(-1)^k}{k!} \frac{etr(-\frac{1}{2}MM')}{(detA)^{m/2}} \sum_{\kappa} P_{\kappa}(\frac{1}{\sqrt{2}}M, A^{-1}),$$

where $(-1)^k\Sigma_{\kappa}P_{\kappa}(\frac{1}{\sqrt{2}}M,A^{-1})$ is given by Lemma 1. Hence (22) gives an explicit form for α_k^p not involving a recurrence relation. We can also easily check that if we insert (22) into the left hand side of (19)', then we have (7).

Next we compare with the Γ-type representation.

THEOREM 4. *(Γ-type representation.)* Let X be distributed with the p.d.f. (8), then the p.d.f. of $T = tr XAX'$ is given by

$$(23)\qquad \frac{exp(-\frac{1}{2}\mu(\Sigma^{-1}\otimes B^{-1})\mu')}{\Gamma(\frac{mn}{2})(det\Sigma)^{\frac{n}{2}}(det 2AB)^{\frac{m}{2}}}\ exp(-\frac{1}{2p}T)\ T^{(mn/2)-1}$$

$$\sum_{k=0}^{\infty}\frac{1}{k!(\frac{mn}{2})_k}(\frac{T}{2})^k\ P_k(\frac{1}{\sqrt{2}}\ \mu(\Sigma^{-1}\otimes B^{-1}A^{-\frac{1}{2}})\ D^{-\frac{1}{2}},\ D),$$

where $D = \Sigma^{-1}\otimes C^{-1} - I_m\otimes I_n/p$.

PROOF: From (15) and the proof of Theorem 3, we can easily show (23).

COROLLARY. The p.d.f. of the $T = tr\Sigma^{-1}XAX'$ is given by

$$(24)\qquad \frac{exp[-\frac{1}{2}\mu(\Sigma^{-1}\otimes B^{-1})\mu']}{\Gamma(\frac{mn}{2})(det 2AB)^{\frac{m}{2}}}\ exp(-\frac{1}{2p}T)\ T^{(mn/2)-1}$$

$$\sum_{k=0}^{\infty}\frac{1}{k!(\frac{mn}{2})_k}(\frac{T}{2})^k P_k(\frac{1}{\sqrt{2}}\mu(\Sigma^{-\frac{1}{2}}\otimes B^{-1}A^{-\frac{1}{2}})(I\otimes(C^{-1}-I/p))^{\frac{1}{2}}\ ,\ \ I\otimes(C^{-1}-I/p)).$$

We can show easily that (24) is the same form as (14).

Ruben [8] gave a Γ-type representation of a quadratic form which is obtained in the following lemma.

LEMMA. *(Ruben)* Under the same condition of Lemma (Kotz et al), the p.d.f. of $T = (x+b)A(x+b)'$ is given by

$$(25)\qquad \sum_{k=0}^{\infty}\alpha_k^c\ \frac{e^{-(T/2p)}T^{(n/2)+k-1}}{2^{(n/2)+k}\ \Gamma(\frac{n}{2}+k)}\ (\frac{1}{p})^{(n/2)+k}\ .$$

The α_k^c are determined by

$$(26)\qquad \sum_{k=0}^{\infty} \alpha_k^c \theta^k = (\det A/p)^{-\frac{1}{2}} \exp\left[-\frac{1}{2}\sum_{k=1}^{n} b_k^2 \frac{1-\theta}{1-(1-p/a_k)\theta}\right] \prod_{j=1}^{n} \frac{1}{\{1-(1-p/a_j)\theta\}^{\frac{1}{2}}} .$$

Hence the recurrence relation

$$\alpha_0^c = (\det A/p)^{-\frac{1}{2}} \exp\left[-\frac{1}{2} \sum_{i=1}^{n} b_i^2\right]$$

$$\alpha_k^c = \sum_{r=0}^{k-1} b_{k-r}^c \alpha_r^c/2k, \quad k \geq 1,$$

with

$$b_k^c = \sum_{j=1}^{n} \left(1 - \frac{p}{a_j}\right)^k + kp \sum_{j=1}^{n} \frac{b_j^2}{a_j} \left(1 - \frac{p}{a_j}\right)^{k-1}$$

can be obtained.

To compare with Theorem 4 and Lemma (Ruben), we set $\Sigma = I_m$ and $B = I_n$ in (23) and we replace A with $I_m \otimes A$ and b with μ in (26). Then we have

$$(23)'\qquad \frac{\exp(-\frac{1}{2}\mu\mu')}{2^{\frac{mn}{2}}\Gamma(\frac{mn}{2})(\det A)^{\frac{m}{2}}} \exp\left(-\frac{T}{2p}\right) T^{(mn/2)-1}$$

$$\sum_{k=0}^{\infty} \frac{1}{k!(\frac{mn}{2})_k} \left(\frac{T}{2}\right)^k P_k\left(\frac{1}{\sqrt{2}}\mu(I\otimes A^{-\frac{1}{2}}(A^{-1} - \frac{I}{p})^{\frac{1}{2}}, (I\otimes A^{-1} - I/p)\right)$$

$$= \frac{etr(-\frac{1}{2}MM')}{2^{\frac{mn}{2}}\Gamma(\frac{mn}{2})(\det A)^{\frac{m}{2}}} \exp\left(-\frac{T}{2p}\right) T^{(mn/2)-1}$$

$$\sum_{k=0}^{\infty} \frac{1}{k!(\frac{mn}{2})_k} \left(\frac{T}{2}\right)^k \sum_{\kappa} P_{\kappa}\left(\frac{1}{\sqrt{2}}MA^{-\frac{1}{2}}(A^{-1}-I/p)^{\frac{1}{2}}, A^{-1} - I/p\right)$$

and

$$(25)' \qquad \sum_{k=0}^{\infty} \alpha_k^c \frac{e^{-(T/2p)}T^{(mn/2)+k-1}}{2^{(mn/2)+k}\Gamma(\frac{mn}{2}+k)} (\frac{1}{p})^{(mn/2)+k}$$

$$(26)' \qquad \sum_{k=0}^{\infty} \alpha_k^c\theta^k = (detA/p)^{-(m/2)} \prod_{j=1}^{n} \{1-(1-P/a_j)\theta\}^{-(m/2)}$$

$$exp[-\frac{1}{2}\sum_{j=1}^{n}\frac{1-\theta}{1-(1-P/a_j)\theta}\sum_{i=1}^{m}\mu_{ij}^2] .$$

Therefore, by comparing with (23)' and (25)', we have the following relation.

$$(27) \qquad \alpha_k^c = \frac{1}{k!}(detA/p)^{-(m/2)}etr(-\frac{1}{2}MM')\sum_{\kappa} P_{\kappa}(\frac{1}{\sqrt{2}}MA^{-\frac{1}{2}}(A^{-1}-I/p)^{\frac{1}{2}},A^{-1}-I/p)$$

and $\Sigma_{\kappa}P_{\kappa}(\frac{1}{\sqrt{2}}MA^{-\frac{1}{2}}(A^{-1}-I/p)^{\frac{1}{2}},A^{-1}-I/p)$ is given by Lemma 1. Hence (27) gives an explicit form for α_k^c, not involving a recurrence relation. We can easily check that if we insert (27) into the left hand side of (26)', then we obtain (7) by changing T and A into the appropriate variables.

From (27), we have the relation

$$(28) \qquad \frac{E(Q^k H_{2k}(L/Q^{\frac{1}{2}}))}{2^k(2k-1)!!} = \sum_{\kappa} P_{\kappa}(\frac{1}{\sqrt{2}}MA^{-\frac{1}{2}}(A^{-1}-I/p)^{\frac{1}{2}}, A^{-1}-I/p),$$

where

$$L = \sum_{j=1}^{n}\frac{1}{\sqrt{a_j}}\sum_{i=1}^{m}\mu_{ij}x_{ij}, \quad Q = \sum_{i=1}^{m}\sum_{j=1}^{n}(\frac{1}{a_j}-\frac{1}{p})x_{ij}^2,$$

x_{ij} are independent normal variables with zero mean and unit variances, and $H_{2k}(y)$ is an Hermite polynomial of order $2k$, and $(2k-1)!! = 1\cdot 3\ldots(2k-1)$.

NOTE. If we set $M = 0$ and $B = I_n$ in (16), then we have the p.d.f. of a central case, given in Hayakawa [2] by using the zonal polynomials.

5. THE COMPLEX MULTIVARIATE QUADRATIC FORM.

In this section, we shall state the above results for the complex Gaussian distribution studied by Goodman [1], James [5] and Khatri [6].

Let T and U be $m\times n$ $(m \le n)$ complex arbitrary matrices whose rank are m, respectively, and A be an $n\times n$ positive definite Hermitian matrix. We define $\tilde{P}_\kappa(T,A)$ as follows:

$$(29)\qquad etr(-T\bar{T}')\tilde{P}_\kappa(T,A) = \frac{(-1)^k}{\pi^{mn}}\int_U etr(-i(T\bar{U}'+U\bar{T}'))etr(-U\bar{U}')\tilde{C}_\kappa(UA\bar{U}')dU,$$

where $\tilde{C}_\kappa(UA\bar{U}')$ is a zonal polynomial of a Hermitian matrix $UA\bar{U}'$. Then we can show that

$$(30)\qquad \tilde{P}_\kappa(0,A) = [n]_\kappa\, \tilde{C}_\kappa(A)/\tilde{C}_\kappa(I_n),$$

$$(31)\qquad \tilde{P}_\kappa(T,I_n) = \tilde{H}_\kappa(T),$$

$$(32)\qquad |\tilde{P}_\kappa(T,A)| \le etr(T\bar{T}')[n]_\kappa\, \tilde{C}_\kappa(A)/\tilde{C}_\kappa(I_n),$$

where

$$[n]_\kappa = \prod_{\alpha=1}^{m} (a-\alpha+1)_{k_\alpha}$$

and $\tilde{H}_\kappa(T)$ is a generalized complex Hermite polynomial of a matrix argument T.

The generating function of $\tilde{P}_\kappa(T,A)$ is given by

$$(33)\qquad \int_{U(m)}\int_{U(n)} etr(-SU_2A\bar{U}_2'\bar{S}'+U_1SU_2A^{\frac{1}{2}}\bar{T}'+TA^{\frac{1}{2}}\bar{U}_2'\bar{S}'\bar{U}_1')d(U_1)d(U_2)$$

$$= \sum_{k=0}^{\infty}\sum_{\kappa}\frac{\tilde{P}_{\kappa}(T,A)\tilde{C}_{\kappa}(S\bar{S}')}{k![n]_{\kappa}\tilde{C}_{\kappa}(I_m)} .$$

The R.H.S. of (33) converges absolutely with respect to S, where S is an $m\times n$ $(m \le n)$ complex arbitrary matrix, U_1 and U_2 are unitary matrix of order m and n, respectively, and $d(U_1)$ and $d(U_2)$ are the unitary invariant measures over the unitary groups $U(m)$ and $U(n)$, respectively.

A detailed discussion of $\tilde{H}_{\kappa}(T)$ may be found in Hayakawa [4].

LEMMA 2.

$$(34)\quad \sum_{\kappa}\tilde{P}_{\kappa}(T,A) = (-1)^k k![\tilde{A}_k+\frac{1}{2}\sum_{\ell=0}^{k}\tilde{A}_{k-\ell}\tilde{A}_{\ell}+\frac{1}{3!}\sum_{\ell_1=0}^{k}\sum_{\ell_2=0}^{\ell_1}\tilde{A}_{k-\ell_1}\tilde{A}_{\ell_1-\ell_2}\tilde{A}_{\ell_2}$$

$$+\dots+\frac{1}{k!}\sum_{\ell_1=0}^{k}\sum_{\ell_2=0}^{\ell_1}\dots\sum_{\ell_{n-1}=0}^{\ell_{k-2}}\tilde{A}_{k-\ell_1}\dots\tilde{A}_{\ell_{k-2}-\ell_{k-1}}\tilde{A}_{\ell_{k-1}}],$$

where

$$\tilde{A}_{\ell} = \frac{m}{k}\,tr A^{\ell} - tr TA^{\ell}\bar{T}', \quad \ell = 1,2,\dots,k$$

and

$$\tilde{A}_0 = 0, \quad \text{for convenience.}$$

PROOF: Similar to Lemma 1.

Let X be an $m\times n$ $(m\le n)$ complex matrix whose denisty function is given by

$$\frac{1}{\pi^{mn}(det\Sigma)^n(detB)^m} etr[-\Sigma^{-1}(X-M)B^{-1}(\bar{X}'-\bar{M}')], \tag{35}$$

where Σ is an $m\times m$ p.d. Hermite matrix, M is an $m\times n$ complex matrix whose rank is m, and B is an $n\times n$ p.d. Hermitian matrix. Let A be an $n\times n$ p.d. Hermitian matrix. We denote X and M as

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}, \quad M = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_m \end{bmatrix},$$

where $x_\alpha = (x_{\alpha1}, x_{\alpha2}, \ldots, x_{\alpha n})$ and $\mu_\alpha = (\mu_{\alpha1}, \ldots, \mu_{\alpha n})$, $\alpha = 1,2,\ldots,m$. By setting $x = (x_1, x_2, \ldots, x_m)$ and $\mu = (\mu_1\ \mu_2\ ..\mu_m)$, we rewrite $trXA\bar{X}'$ $= x(I_m\otimes A)\bar{x}'$ and x has an mn dimensional complex Gaussian distribution with mean μ and covariance $\Sigma\otimes B$. Then by applying the same method as in the real variate case, we have the following theorem.

THEOREM 5. *(Power series representation.)* Let X be distributed with the p.d.f. (35), then the p.d.f. of $T = trXA\bar{X}'$ is given by

$$\frac{exp(-\mu(\Sigma^{-1}\otimes B^{-1})\bar{\mu}')}{\Gamma(mn)(det\Sigma)^n(detAB)^m} T^{mn-1} \tag{36}$$

$$\sum_{k=0}^{\infty} \frac{1}{k!(mn)_k} T^k \tilde{P}_\kappa(\mu(\Sigma^{-\frac{1}{2}}\otimes B^{-1}A^{-\frac{1}{2}}C^{\frac{1}{2}}, \Sigma^{-1}\otimes C^{-1}),$$

where $\tilde{P}_k(\cdot,\cdot)$ is a polynomial for $m = 1$ in the definition of $\tilde{P}_\kappa(T,A)$ and $C = A^{\frac{1}{2}}BA^{\frac{1}{2}}$.

THEOREM 6. *(Γ-type representation.)* Let X be distributed with the p.s.f. (35), then the p.d.f. of $T = trXA\bar{X}'$ is given by

$$(37)\qquad \frac{exp(-\mu(\Sigma^{-1}\otimes B^{-1})\mu')}{\Gamma(mn)(det AB)^m(det\Sigma)^n}\, exp(-\frac{1}{2p}T)\, T^{mn-1}$$

$$\sum_{k=0}^{\infty} \frac{1}{k!(mn)_k}\, T^k\, P_k(\mu(\Sigma^{-1}\otimes B^{-1}A^{-\frac{1}{2}})D^{-\frac{1}{2}}, D),$$

where $D = \Sigma^{-1}\otimes C^{-1} - I_m \otimes I_n/p$ and $||AB|| < p$.

Theorem 5 and 6 can be proved in the same way as the real variate case. However, since the procedure is exactly the same, we will omit it.

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