A CHARACTERIZATION OF TETRAHEDRAL GRAPHS

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Abstract

A tetrahedral graph may be defined as a graph G, whose vertices may be identified with the \(n(n-1)(n-2)/6\) unordered triplets on n symbols, such that two vertices are adjacent if and only if the corresponding triplets have two symbols in common. If \(d(x,y)\) denotes the distance between two vertices x and y and \(\Delta(x,y)\) denotes the number of vertices adjacent to both x and y, then a tetrahedral graph G has the following properties: (b_1) The number of vertices is \(n(n-1)(n-2)/6\). (b_2) G is connected and regular of valence 3(n-3). (b_3) For any two adjacent vertices x and y, \(\Delta(x,y)=n-2\). (b_4) \(\Delta(x,y)=4\) if \(d(x,y)=2\). We show that if \(n > 16\), then any graph G (without loops and with utmost one edge connecting two vertices) having the properties (b_1)-(b_4) must be a tetrahedral graph.

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I. Introduction

1. We shall consider only finite undirected graphs, with at most one edge joining a pair of vertices and no edge joining a vertex to itself.

The valence \( d(u) \) of the vertex \( u \) of a graph \( G \), is defined to be the number of vertices adjacent to \( u \). If all vertices of \( G \) have the same valence \( n \), the graph \( G \) is said to be a regular graph of valence \( n \).

A chain \( x_1, x_2, \ldots, x_n \) is a sequence of vertices of \( G \), not necessarily all different, such that any two consecutive vertices in the chain are adjacent. Thus the pairs \( (x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n) \) are edges of \( G \). The number of edges \( n-1 \) is said to be the length of the chain. The chain is said to begin at \( x_1 \) and terminate at \( x_n \), and is said to join \( x_1 \) and \( x_n \).

The graph \( G \) is said to be connected if for every pair of distinct vertices \( x \) and \( y \), there is a chain beginning at \( x \) and terminating at \( y \). For a connected graph the distance \( d(x,y) \) between two vertices \( x \) and \( y \) is defined to be the length of the shortest chain joining \( x \) and \( y \).

For any two vertices \( u \) and \( v \), \( \Delta(u,v) \) denotes the number of vertices \( w \), adjacent to both \( u \) and \( v \). If \( u \) and \( v \) are adjacent, i.e. \( d(x,y)=1 \), \( \Delta(u,v) \) is

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called the edge degree of the edge \((u, v)\). A regular graph \(G\) for which all edges have the same edge-degree \(\Delta\), is said to be edge-regular, with edge-degree \(\Delta\).

2. A graph \(G\) is said to be triangular if the vertices of \(G\) can be identified with unordered pairs on \(n\) symbols, so that two pairs are adjacent if and only if, the corresponding pairs have one symbol in common. A triangular graph \(G\) obviously possesses the following properties:

- \((a_1)\) The number of vertices in \(G\) is \(n(n-1)/2\).
- \((a_2)\) \(G\) is regular of valence \(2(n-2)\).
- \((a_3)\) \(G\) is edge regular with edge degree \(n-2\), i.e. \(\Delta(u,v) = n-2\) if \(u\) and \(v\) are adjacent.
- \((a_4)\) \(\Delta(u,v) = 4\), if \(u\) and \(v\) are non-adjacent.

Connor [4] showed (with a somewhat different terminology) that for \(n > 8\), the properties \((a_1)-(a_4)\), characterize a triangular graph, i.e. if \(G\) has the properties \((a_1)-(a_4)\), then \(G\) must be triangular. Shrikhande [9], Li-chien [7,8] and Hoffman [5,6] completed Conner's work by demonstrating that the same result holds for \(n < 8\), but if \(n=8\), then there exist other non-isomorphic graphs which are not triangular.

3. In this paper we consider the problem of characterization of tetrahedral graphs. A tetrahedral graph may be defined as a graph \(G\) whose vertices can be identified with unordered triplets on \(n\) symbols, such that two vertices are adjacent if and only if the corresponding triplets have two common symbols. It is readily seen that \(G\) has the following properties:

- \((b_1)\) The number of vertices in \(G\) is \(n(n-1)(n-2)/6\).
- \((b_2)\) \(G\) is connected and regular of valence \(3(n-3)\).
(b_3) G is edge-regular, with edge degree n-2, i.e. \( \Delta(x,y) = n-2 \)
if \( d(x,y) = 1 \).

(b_4) \( \Delta(x,y) = 1 \) if \( d(x,y) = 2 \).

In Section III we prove that for \( n > 16 \), the properties (b_1)-(b_4) characterize a tetrahedral graph, i.e., if G possesses properties (b_1)-(b_4) and \( n > 16 \), then it is possible to establish a (1,\alpha) correspondence between the vertices of G, and the unordered triplets on \( n \) symbols, such that two vertices of G are adjacent if and only if the corresponding triplets have a pair of symbols in common.

The proof is based on certain theorems regarding the existence or non-existence of cliques and claws in edge-regular graphs, which are proved in Section II. These theorems generalize the previous work of Bruck [2] and (one of the authors) Bose [1]. Other applications of these theorems will be given in subsequent communications.

II. Claws and Cliques in Edge-regular graphs.

1. In this section we shall consider a graph G which has the following properties:

\( (c_1) \) G is connected and regular of valence \( r(k-1) \).

\( (c_2) \) G is edge-regular with edge-degree \( (k-2) + \alpha \).

\( (c_3) \) \( \Delta(x,y) \leq 1 + \beta \), for all pairs of non-adjacent vertices, \( x \) and \( y \) of G.
In the above, \( r, k, \alpha, \beta, \) are fixed positive integers, such that \( r \geq 1, \) \( k \geq 2, \alpha \geq 0, \beta \geq 0 \) and \( r\beta - 2\alpha \geq 0. \) All the lemmas and theorems in paragraphs 2 and 3 of this section are about the edge regular graph \( G, \) with properties \((c_1), (c_2), (c_3).\)

We define here some functions of the parameters \( r, k, \alpha, \) and \( \beta, \) which play an important role in subsequent developments.

\[
\begin{align*}
(2.1.1) \quad \gamma(r, \alpha) &= 1 + (r-1)\alpha. \\
(2.1.2) \quad q(r, \alpha) &= 1 + (2r-1)\alpha. \\
(2.1.3) \quad \rho(r, \alpha, \beta) &= 1 + \beta + (2r-1)\alpha. \\
(2.1.4) \quad p(r, \alpha, \beta) &= 1 + \frac{1}{2}(r+1)(r\beta - 2\alpha).
\end{align*}
\]

We shall denote as usual the cardinality of a set \( S \) by \( |S|. \)

A **clique** \( K \) of a graph is a set of vertices adjacent to each other. A clique \( K \) will be called **complete** if we cannot find a vertex \( x, \) not contained in \( K \) such that \( xUK \) is a clique. Thus a complete clique cannot be extended to a larger clique by the adjunction of a new vertex belonging to the graph.

Now consider the graph \( G \) with the properties, \((c_1), (c_2), (c_3).\) A clique \( K \) of \( G \) will be called a **major clique** if

\[
(2.1.5) \quad |K| \geq 1 + k - \gamma(r, \alpha) = k - (r-1)\alpha.
\]

A clique \( K \) of \( G \) will be called a **grand clique** if it is both major and complete.

A claw \([p, S]\) of \( G, \) consists of a vertex \( p, \) the vertex of the claw and a non-empty set \( S \) of vertices of \( G, \) not containing \( p, \) such that \( p \) is adjacent to every vertex in \( S, \) but any two vertices in \( S \) are non-adjacent. The order of the claw is defined to be the number \( s = |S|. \)

In the next two paragraphs of this section, we obtain a number of theorems about claws and cliques in a graph \( G \) having the properties \((c_1), (c_2), (c_3).\)
These theorems are very similar to those obtained in an earlier paper [1], but are now proved under less restrictive conditions, thereby substantially increasing their range of applicability. This will be illustrated in Section III, where they will be used to obtain a geometric characterization of tetrahedral graphs (defined in Section I). Further applications will be given in subsequent communications.

2. Theorem (2.2.1). If \( k > p(r, \alpha, \beta) \), there cannot exist a claw of order \( r+1 \) in \( G \).

Suppose there exists in \( G \) a claw \([p, S]\) of order \( s \). Let \( T \) be the set of vertices of \( G \), not belonging to \([p, S]\), and adjacent to \( p \). Let \( f(x) \) denote the number of vertices \( q \) in \( T \), such that \( q \) is adjacent to exactly \( x \) vertices in \( S \). Counting the number of vertices in \( T \) we have from \((c_1)\),

\[
(2.2.1) \quad \sum_{x=0}^{s} f(x) = r(k-1)-s = rk-r-s,
\]

Counting the number of ordered pairs \((b, q)\) where \( b \) and \( q \) are adjacent, \( b \) belongs to \( S \), and \( q \) belongs to \( T \), we have from \((c_2)\)

\[
(2.2.2) \quad \sum_{x=0}^{s} x f(x) = s(k-2+\alpha).
\]

Again counting the triplets \((b_1, b_2, q)\), where \( b_1, b_2 \) is an ordered pair of vertices in \( S \), \( q \) is a vertex in \( T \), and \( b_1, b_2 \) are both adjacent to \( q \), we have from \((c_3)\)

\[
(2.2.3) \quad \sum_{x=0}^{s} x(x-1) f(x) \leq s(s-1)\beta.
\]
If a claw of order \( r+1 \) exists, putting \( s = r+1 \), we have from (2.2.1), (2.2.2) and (2.2.3)

\[
(2.2.4) \quad f(0) + \frac{1}{2} \sum_{x=1}^{r+1} (x-1)(x-2) f(x) \leq -k + 1 + \frac{1}{2} (r+1)(s-2\alpha) = -k+p(r,\alpha,\beta)
\]

Since the left hand side is essentially non-negative whereas \( k > p(r,\alpha,\beta) \) by hypothesis, we have a contradiction. This proves our theorem.

Theorem (2.2.2). If \( k > \gamma(r,\alpha) \), then any claw of \( G \) of order \( s < r \), can be extended to a claw of order \( r \).

Suppose there exists in \( G \) a claw \([p,S]\) of order \( s \). From (2.2.1) and (2.2.2)

\[
(2.2.5) \quad f(0) - \sum_{x=1}^{s} (x-1) f(x) = (k-1)(s-r-1)-\alpha s.
\]

If \( k > \gamma(r,\alpha) = 1+(r-1)\alpha \), and \( s < r \)

\[
f(0) > \alpha r(r-s-1) \geq 0
\]

Hence \( f(0) > 0 \), which shows that there exists a vertex \( q \) in \( T \), which is not adjacent to any vertex of \( S \). If \( S^* = SUq \) we can extend the claw \([p,S]\) to the claw \([p,S^*]\) of order \( s+1 \). If \( s+1 < r \) we can continue the process till we arrive at a claw of order \( r \).

Theorem (2.2.3). Given a claw \([p,S]\) of \( G \) of order \( r-1 \) there exist at least \( k-\gamma(r,\alpha) \) distinct vertices \( q \) of \( G \) such that \([p,SUq]\) is a claw of order \( r \).

Putting \( s = r-1 \), in (2.2.5) we have

\[
f(0) \geq k-1-\alpha(r-1) = k-\gamma(r,\alpha).
\]
Hence there exist at least $k - \gamma(r, \alpha)$ vertices $q$ in $T$ such that $[p, S \cup q]$ is a claw of order $r$.

3. Lemma (2.3.1). If $k > \gamma(r, \alpha)$ and if $G$ has no claw of order $r+1$, then any pair of adjacent vertices $p$ and $q$ is contained in at least one major clique.

From theorem (2.2.2) we can extend the claw $[p, q]$ to a claw $[p, S]$ of order $r$. Let $b_1, b_2, \ldots, b_r$ be vertices in $S$ other than $q$. Let $\Omega$ be the set of vertices $\omega$, which when adjoined to $S - q$ give a claw $[p, S^*]$ of order $r$, where $S^* = (S - q) \cup \omega$. Of course $q$ is contained in $\Omega$ and from theorem (2.2.3)

$$|\Omega| \geq k - \gamma(r, \alpha).$$

The vertices in $\Omega$ are all adjacent to one another. If any two were not adjacent they could be added to $b_1, b_2, \ldots, b_r$ to give a claw of order $r+1$. Let $K = pU\Omega$. Then $K$ is a major clique since

$$|K| \geq 1 + k - \gamma(r, \alpha).$$

Corollary (2.3.1). In Lemma (2.3.1), the hypothesis may be replaced by

$$k > \max \{\beta(r, \alpha), p(r, \alpha, \beta)\}.$$

This follows at once from theorem (2.2.1).

Corollary (2.3.2). When the conditions of Lemma (2.3.1), or corollary (2.3.1) are satisfied, then any pair of adjacent vertices $p$ and $q$, is contained in at least one grand clique.

We can extend the major clique $K$ by adding new vertices till it is complete.
and therefore a grand clique.

**Lemma (2.3.2).** If K and L are cliques of G, and K∪L is not a clique, then $|K∪L| \leq 1 + \beta$.

Since K∪L is not a clique, there exists in K∪L a pair of vertices c and d which are non-adjacent, such that c belongs to K and d to L. Any vertex belonging to K∪L must be adjacent to both c and d. Hence $\Delta(c, d) \geq |K∩L|$. The lemma follows from ($c_3^-$).

**Lemma (2.3.3).** If K and L are cliques of G, K∩L contains at least two vertices a and b, then $|K∪L| \leq k + \alpha$.

Every vertex in K∪L, other than a and b is adjacent to both a and b. Hence

$$\Delta(a, b) \geq |K∪L| - 2.$$  

It follows from ($c_1$), that $|K∪L| \leq k + \alpha$.

**Lemma (2.3.4).** If K and L are cliques of G, K∪L is not a clique and K∩L contains at least two vertices, then

$$|K| + |L| \leq 1 + k + \alpha + \beta.$$  

This follows at once from Lemmas (2.3.2) and (2.3.3) by noting that

$$|K| + |L| = |K∩L| + |K∪L|.$$  

**Theorem (2.3.1A).** If $k > \rho(r, \alpha, \beta)$ and G has no claw of order r+1, then any pair of adjacent vertices p and q is contained in one and only one grand clique.

The existence of at least one grand clique containing p and q follows at once from corollary (2.3.2) by noting that $\rho(r, \alpha, \beta) \geq \gamma(r, \alpha)$. 

8
Suppose there exist at least two distinct grand cliques $K$ and $L$ both containing the adjacent vertices $p$ and $q$. Since $K$ and $L$ are complete, $K \cup L$ is not a clique. Hence from Lemma (2.3.4)

$$|K| + |L| \leq 1 + k + \alpha + \beta.$$ 

But $K$ and $L$ are both major cliques. Hence

$$|K| + |L| \geq 2 \{ k-(r-1)\alpha \}.$$ 

which shows that

$$k \leq 1 + \beta + (2r-1)\alpha = \rho(r, \alpha, \beta),$$

contrary to the hypothesis.

The previous theorem can be written in the following alternative form:

**Theorem (2.3.1B).** If $k > \max \{ \rho(r, \alpha, \beta) \ p(r, \alpha, \beta) \}$, then any two adjacent vertices $p$ and $q$ of $G$ are contained in exactly one grand clique.

This follows at once from Theorems (2.2.1) and (2.3.1A).

**Theorem (2.3.2A).** If $k > q(r,\alpha)$, there exists no claw of order $r+1$ in $G$, and every pair of adjacent vertices of $G$ is contained in utmost one grand clique of $G$, then each vertex of $G$ is contained in exactly $r$ grand cliques.

If we note that $q(r,\alpha) \geq \gamma(r,\alpha)$ it follows from corollary (2.3.2), that any pair of adjacent vertices is contained in exactly one grand clique. Again from Theorem (2.2.2), $p$ is the vertex of at least one claw of order $r$. Let $[p,S]$ be a claw of order $r$, where $S = \{ b_1, b_2, \ldots, b_r \}$. As in Theorem (2.2.1), let $T$ be the set of vertices not belonging to $S$, which are adjacent to $p$. 

9
Let $H_j$ be the set consisting of $p_j$, $b_j$, and $q$ belonging to $T$, such that $q$ is adjacent to $b_j$ but not adjacent to $b_i$, $i \neq j$. As in Theorem (2.2.1) let $f(x)$ denote the number of vertices in $T$, which are adjacent to exactly $x$ vertices in $S$. Then $f(0) = 0$, otherwise there would exist a claw of order $r+1$. Putting $s=r$ in (2.2.1) and (2.2.2), we have

$$(2.3.1) \quad \sum_{x=1}^{r} f(x) = r(k-2).$$

$$(2.3.2) \quad \sum_{x=1}^{r} x f(x) = r(k-2+\alpha).$$

Hence

$$\sum_{x=2}^{r} (x-1) f(x) = r\alpha.$$ 

Since

$$-f(1) + \sum_{x=1}^{r} f(x) = \sum_{x=2}^{r} f(x) \leq \sum_{x=2}^{r} (x-1) f(x) = r\alpha,$$

it follows that

$$(2.3.3) \quad f(1) \geq r(k-2-\alpha).$$
Any two vertices of $H_j$ are adjacent to one another, otherwise there would exist a claw of order $r+1$. Thus $H_j$ is a clique.

Put $H_j^* = H_j - (b_j \cup p)$. Then $H_j^*$ consists of exactly those vertices of $T$ which are adjacent to $b_j$ but to no other vertex of $S$. Hence $H_1^*, H_2^*, \ldots, H_r^*$ are disjoint sets, and the total number of vertices in these sets in $f(1)$.

Now there is a unique grand clique $K_j$ containing $b_j$ and $p$. The number of vertices in $K_j$ cannot be less than the number of vertices in $H_j$. If possible let $|K_j| < |H_j|$. Since $K_j$ is a grand clique it follows that $H_j$ is a major clique and contained in some grand clique $K_j'$. Since $b_j$ and $p$ are contained in $K_j$ and $K_j'$, they must coincide. Hence $K_j$ contains $H_j$, which contradicts $|K_j| < |H_j|$. 

Now consider the $r$ grand cliques, $K_1, K_2, \ldots, K_r$. Then $K_1 - p, K_2 - p, \ldots, K_r - p$ are disjoint. For if $K_i - p$ and $K_j - p, i \neq j$, have a common vertex $q$, then $K_i$ and $K_j$ would coincide, and would contain both $b_i$ and $b_j$ which is impossible since $b_i$ is not adjacent to $b_j$. Remembering (2.3.3), we have

\[
(2.3.4) \quad \sum_{j=1}^{r} |K_j - p| \geq \sum_{j=1}^{r} |H_j - p|
\]

\[
= r + \sum_{j=1}^{r} |H_j^*|
\]

\[
= r + f(1)
\]

\[
\geq r(k-1-\alpha).
\]

If possible, suppose there is another grand clique $K_{r+1}$ containing $p$. The vertices in $K_{r+1} - p$ must be disjoint from the vertices in $K_1 - p, K_2 - p, \ldots, K_r - p$. Since $K_{r+1}$ is a grand and therefore a major clique, $|K_{r+1} - p| \geq k-1-(r-1)\alpha$. 

11
But from \((c_1)\), the number of vertices adjacent to \(p\) is exactly \(r(k-1)\). Hence from \((2.3.4)\)

\[
 r(k-1) \geq \sum_{j=1}^{r+1} |K_j - p| \geq r(k-1) + k - 1 - (2r-1) \alpha.
\]

\[
\therefore k \leq 1 + (2r-1)\alpha = q(r, \alpha)
\]

which is a contradiction. Thus \(p\) is contained in exactly \(r\) grand cliques.

**Theorem (2.3.2B).** If \(k > \rho(r, \alpha, \beta)\), and there exists no claw of order \(r+1\) in \(G\), then each vertex of \(G\) is contained in exactly \(r\) grand cliques.

This follows at once from the previous theorem and Theorem \((2.3.1A)\), remembering \(\rho(r, \alpha, \beta) \geq q(r, \alpha)\).

**Theorem (2.3.2C).** If \(k > \max \left[ \rho(r, \alpha, \beta), p(r, \alpha, \beta) \right] \), then each vertex in \(G\) is contained in exactly \(r\) grand cliques.

This follows from the previous theorem and theorem \((2.2.1)\).
III. Characterization of Tetrahedral Graphs

1. As mentioned earlier in the introduction a tetrahedral graph \( G \) is a graph whose vertices can be identified with the \( n(n-1)(n-2)/6 \) unordered triplets on \( n \) symbols, such that any two vertices are adjacent if and only if the corresponding triplets have a pair of common symbols. Then \( G \) clearly possesses the properties \((b_1)-(b_4)\) given in Section I, paragraph 3. We shall here prove that if \( n > 16 \), the converse also holds.

In the following lemmas \( G \) is a graph satisfying the conditions \((b_1)-(b_4)\), and such that \( n > 16 \).

If we set \( \gamma = 3 \), \( K = n-2 \), \( \alpha = 2 \), \( \beta = 3 \), then the conditions \((b_2)\), \((b_3)\), \((b_4)\) are the same as \((c_1),(c_2),(c_3)\) of Section II. Also from \((2.1.1)\), \((2.1.3)\), and \((2.1.4)\)

\[
\gamma(r,\alpha) = 5, \quad \rho(r,\alpha,\beta) = 14, \quad p(r,\alpha,\beta) = 11.
\]

Hence a clique \( K \) of \( G \) is a major clique if \( |K| \geq n-6 \), and if it is complete it is a grand clique. Since \( n > 16 \), the condition \( K > \max \{\rho(r,\alpha,\beta), p(r,\alpha,\beta)\} \) is satisfied. Hence from theorems \((2.3.1B)\) and \((2.3.2C)\) we have:

Lemma \((3.1.1)\). Any two adjacent vertices of \( G \) are contained in exactly one grand clique. Each vertex of \( G \) is contained in exactly 3 grand cliques.

The unique grand clique containing any two given adjacent vertices \( x \) and \( y \), may be denoted by \( K(x,y) \).

The null set will be denoted by \( \emptyset \).

The following six lemmas are directed towards proving that \( |K| = n-2 \), for any grand clique \( K \) in \( G \).

Lemma \((3.1.2)\). If \( K \) is a grand clique in \( G \), then

\[
n-4 \leq |K| \leq n.
\]
Let \( x \) and \( y \) be any two vertices in \( K \). There are \(|K| - 2\) vertices in \( K \) other than \( x \) and \( y \), and by definition each of these is adjacent to both \( x \) and \( y \). If \(|K| > n\), then \(|K| - 2 > n - 2\) which would contradict \((b_3)\). Hence \(|K| \leq n\).

Let \( A \) be the set of all vertices adjacent to both \( x \) and \( y \) but not contained in \( K \). Then from \((b_4)\)

\[ |K| - 2 + |A| = n - 2. \]

i.e.

\[ (3.1.1) \quad |K| + |A| = n. \]

If \( S_1 \) and \( S_2 \) are the two other cliques containing \( x \) and \( T_1, T_2 \) be those containing \( y \), then any vertex in \( A \) must be of the form

\[ Z_{ij} = S_i \cap T_j, \quad i, j = 1, 2, \]

if it exists at all. Since two distinct cliques can have at most one vertex in common, we have

\[ |A| \leq 4. \]

Hence from \((3.1.1)\)

\[ |K| \geq n - 4. \]

Lemma \((3.1.3)\). If \( K \) is a grand clique in \( G \), then

\[ |K| \neq n - 4. \]

Suppose \(|K| = n - 4\), and let \( x, y, A, S_i, T_j \) \((i, j = 1, 2)\) be as in lemma \((3.1.2)\). Since there are only \( n - 6 \) vertices in \( K \) adjacent to both \( x \) and \( y \), we must have \(|A| = 4\). Hence, \( S_i \cap T_j \neq \emptyset \) for \( i, j = 1, 2 \). It follows from \((b_2)\) that

\[ (3.1.2) \quad |K - x| + |S_1 - x| + |S_2 - x| = 3n - 9. \]

Since

\[ |K - x| = n - 5, \quad \text{we have} \]

\[ (3.1.3) \quad |S_1 - x| + |S_2 - x| = 2n - 4. \]
Thus, at least one $S_1$-x, say $S_1$-x, has at least n-2 vertices, and therefore

$$ (3.1.4) \quad |S_1| \geq n-1. $$

Consider the two vertices $x$ and $z_{11}$ in $S_1$. They are both adjacent to 
$|S_1|-2$ other vertices in $S_1$ as well as to $y$ and $z_{21}$ not in $S_1$. From 
(3.1.4) it follows that $\Delta(x, z_{11}) \geq n-1$, which contradicts $(b_3)$.

Lemma (3.1.4). If $K$ is a grand clique in $G$, then $|K| \neq n-3$.

Suppose $|K| = n-3$. Then from (3.1.1), $|A| = 3$, and one of the grand 
cliques, say $S_1$, containing $x$ must intersect both of the other two grand 
cliques containing $y$ and the other grand clique $S_2$, containing $x$, must 
intersect exactly one of $T_1, T_2$, say $T_2$.

Since $|K-x| = n-4$, from (3.1.2) we have

$$ (3.1.5) \quad |S_1-x| + |S_2-x| = 2n-5. $$

It follows then, that one of $S_1$-x and $S_2$-x has at least n-2 vertices. If 
$|S_1-x| \geq n-2$, then $|S_1| > n-1$ and considering the vertices $x$ and $z_{12}$ we have 
at most one vertex not in $S_1$ adjacent to both. This is contradicted since 
$z_{22}$ and $y$ are adjacent to both. If $|S_2| \geq n-1$, the same argument can be 
applied to $x$ and $z_{22}$.

Lemma (3.1.5). If $K$ is a grand clique in $G$, then $|K| \neq n-1$.

Suppose $|K| = n-1$, then from (3.1.1), $|A| = 1$. Hence, exactly one 
of $S_1, S_2$ intersects exactly one of $T_1, T_2$.

Suppose, $z_{11} = S_1 \cap T_1$ and $S_2 \cap T_2 = \emptyset$.

Since $|K-x| = n-2$, it follows from (3.1.2) that

$$ |S_1-x| + |S_2-x| = 2n-7. $$
Hence for one $i$, $|S_i - x| \leq n^{-4}$ and then $|S_i| \leq n^{-3}$. But by lemmas (3.1.2), (3.1.3), (3.1.4), we have $|K| \geq n^{-2}$ for every $K$ in $G$.

**Lemma (3.1.6).** If $K$ is a grand clique in $G$, then $|K| \neq n$.

Suppose $|K| = n$, then $|A| = 0$ and hence $S_i \cap T_j = \emptyset$, $(i = 1, 2,)$ $(j = 1, 2,).$ Since $|K - x| = n^{-1}$, from (3.1.2) we have

$$|S_1 - x| + |S_2 - x| = 2n^{-8}.$$ 

Hence at least one of $S_1 - x$ and $S_2 - x$, say $S_1 - x$, has at most $n^{-4}$ vertices. Thus $|S_1| \leq n^{-3}$ which contradicts at least one of the lemmas (3.1.2), (3.1.3), (3.1.4).

**Lemma (3.1.7).** If $K$ is a grand clique in $G$, then $|K| = n^{-2}$.

This follows immediately from lemmas (3.1.2)–(3.1.6).

**Lemma (3.1.8).** Let $x$ be a vertex in $G$ and let $L$ be a grand clique not containing $x$. Then the three grand cliques $K_1$, $K_2$, $K_3$ containing $x$ cannot all intersect $L$.

Suppose $K_1$, $K_2$, $K_3$ all meet $L$, and let $y_i = K_i \cap L$, $i = 1, 2, 3$. From Lemma (3.1.1) the vertices $y_i$, $i = 1, 2, 3$ are all distinct. Let $S_i$, $i = 1, 2, 3$ be the third grand clique containing $y_i$ in addition to $K_i$ and $L$.

Suppose $S_i \cap K_j \neq \emptyset$, for some pair $i, j$, $i \neq j$, and let $z = S_i \cap K_j$. Then, the vertices $z$, $y_j$ and $y_k$ are such that, each of these is adjacent to both $x$ and $y_i$ but none contained in the grand clique $K_i$ containing $x$ and $y_i$ $(i, j, k$ are different). From lemma (3.1.7) $|K_i| = n^{-2}$, and hence the $n^{-4}$ vertices in $K_i$, other than $x$ and $y_i$, together with the three vertices $z$, $y_j$ and $y_k$ constitute a set of $n^{-1}$ vertices which are adjacent to both.
and $y_i$. This contradicts $(b_2)$. Hence

$$(3.1.6) \quad S_i \cap K_j = \emptyset \quad \text{for } i \neq j, \ i, j = 1, 2, 3$$

By lemma $(3.1.7)$ there are $n-5$ vertices in $L$ other than $y_1$, $y_2$ and $y_3$. Each of these vertices must be non-adjacent to $x$, for otherwise we would have 4 grand cliques containing $x$ contradicting lemma $(3.1.1)$. Let $z$ be one of these $n-5$ vertices in $L-y_1-y_2-y_3$. Then, $d(z, x) = 2$ and by $(b_{1i})$ there are exactly 4 vertices adjacent to both $z$ and $x$. Clearly three of these are $y_1$, $y_2$, $y_3$ and the fourth vertex must be on some $K_i$, $(1 \leq i \leq 3)$ and be distinct from $x$ and $y_i$. Thus for each of the $n-5$ vertices in $L-y_1-y_2-y_3$, there is exactly one vertex in the set

$$T = \bigcup_{i=1}^{3} \{ K_i-x-y_i \}$$

which is adjacent to it. Let us define three sets $A_1$, $A_2$, $A_3$ where $A_1$ consists of all those vertices in $K_i-x-y_i$ which are adjacent to at least one vertex of $L$ other than $y_i$. Since from $(3.1.6)$ no vertex $z \in K_i-x-y_i$ can be adjacent to $y_j$, $j \neq i$, it is clear that $A_1$ consists of all those vertices in $K_i-x-y_i$ which are adjacent to a vertex of $L-y_1-y_2-y_3$.

Now, since each vertex in $L-y_1-y_2-y_3$ is adjacent to exactly one vertex $T$, and since

$$|L-y_1-y_2-y_3| = n-5,$$

we have

$$(3.1.7) \quad \sum_{i=1}^{3} |A_i| \leq n-5.$$  

Again, since $|K_i-x-y_i| = n-k$, and the $K_i-x-y_i$, $i = 1, 2, 3$ are disjoint, we have
$$|T| = \sum_{i=1}^{3} |K_i \cdot x - y_i| = 3n - 12.$$  

If we define $B_i$ to be the set of all those vertices in $K_i \cdot x - y_i$ which are adjacent to no vertex of $L$ other than $y_i$, then clearly $A_i$ is disjoint from $B_i$ and

$$A_i \cup B_i = K_i \cdot x - y_i.$$  

Hence, from (3.1.8),

$$\sum_{i=1}^{3} |A_i| + \sum_{i=1}^{3} |B_i| = (3n - 12).$$

Combining (3.1.7) and (3.1.9), we have

$$\sum_{i=1}^{3} |B_i| \geq 2n - 7.$$  

It follows from (3.1.10) that for at least one $i = i_0$, we have

$$|B_{i_0}| \geq \frac{2n - 7}{3}.$$  

Now, let $b \in B_{i_0}$ and consider the vertices $b$ and $y_{i_0}$. Since $|K_{i_0}| = n - 2$ from lemma (3.1.7), it follows from (b.3) that there are exactly two vertices adjacent to both $b$ and $y_{i_0}$ and not in $K_{i_0}$. From the definition of $B_{i_0}$, neither of these two vertices can be in $L$. Hence they must both lie in $S_{i_0} - y_{i_0}$.

Let $b$ and $b'$ be two distinct vertices in $B_{i_0}$ and let $s_1, s_2$ be the two vertices of $S_{i_0} - y_{i_0}$ adjacent to $b$ and $s_1', s_2'$ be the two vertices of $S_{i_0} - y_{i_0}$ adjacent to $b'$. Suppose
\[ s_i' = s_j, \text{ for some pair } i, j, \quad 1 \leq i, j \leq 2. \]

Without loss of generality, suppose \( s_1' = s_1 \). Let \( M \) be the grand clique containing the adjacent vertices \( b \) and \( s_1 \). Then, there are \( n-4 \) vertices in \( M \) adjacent to both \( b \) and \( s_1 \) as well as 3 others not in \( M \), namely, \( s_2', b' \) and \( y_{i_0} \). It follows that, there are at least \( n-1 \) vertices adjacent to both \( b \) and \( s_1 \) which contradicts \( (b_3) \). Hence the 4 vertices \( s_1, s_2, s_1', s_2' \) must all be distinct. Hence the number of distinct vertices in \( s_{i_0} - y_{i_0} \) adjacent to vertices in \( B_{i_0} \) is exactly \( 2 |B_{i_0}| \). Consequently

\[(3.1.12) \quad |S_{i_0} - y_{i_0}| \geq 2 |B_{i_0}| \]

or from \( (3.1.11) \) and lemma \( (3.1.7) \),

\[ n-3 \geq 2\left(\frac{2n-7}{3}\right) \]

which contradicts the assumption \( n > 16 \). This completes the proof.

Lemma (3.1.9). Let \( x \) and \( y \) be two adjacent vertices in \( G \) and let \( K \) be the grand clique containing both \( x \) and \( y \). Let \( S_1, S_2 \) be the other two grand cliques containing \( x \) and \( T_1, T_2 \) be the other two grand cliques containing \( y \). Then the grand cliques \( S_1 \) may be put in \( (1,1) \) correspondence with grand cliques \( T_j \) so that only corresponding cliques intersect,

\[ S_i \cap T_1 \neq \varnothing \]

\[ S_i \cap T_j = \varnothing, \quad i \neq j. \]

Since \( |K| = n-2 \), it is clear that exactly two of the 4 intersections

\[ S_i \cap T_j, \quad i, j = 1, 2 \]
are non-empty. If $S_1$ and $S_2$ both intersect one of the grand cliques containing $y$, other than $K$, say $T_1$, then all three grand cliques containing $x$ intersect $T_1$ and $x \notin T_1$. This contradicts lemma (3.1.8). Hence each of the grand cliques $S_i$ intersects one and only one of the grand cliques and vice versa.

Lemma (3.1.10). Let $x$ and $y$ be two vertices in $G$ such that, $d(x,y) = 2$. Then there is one grand clique $S_3$ containing $x$ which does not intersect any grand clique containing $y$, and one grand clique $T_3$ containing $y$ which does not intersect any grand clique containing $x$. The other two grand cliques $S_1, S_2$ containing $x$ and the other two grand cliques $T_1, T_2$ containing $y$ mutually intersect.

From (b_4), there are exactly 4 vertices adjacent to both $x$ and $y$. Let $z_{11}$ be one of these and let $S_1$ be the grand clique containing $x$ and $z_{11}$ and $T_1$ be the grand clique containing $y$ and $z_{11}$. Clearly $S_1 \neq T_1$, since $d(x,y) = 2$.

Since $d(x,z_{11}) = 1$ by lemma (3.1.9) there exists exactly one grand clique $S_2$ containing $x$, other than $S_1$, which intersects $T_1$. Let $z_{21} = S_2 \cap T_1$. Clearly $z_{11} \neq z_{21}$, $y \neq z_{21}$. Similarly since $d(z_{11}, y) = 1$, there exists exactly one grand clique $T_2$ containing $y$, which intersects $S_1$ in $z_{12}$ say. Clearly $x \neq z_{12}$, $z_{11} \neq z_{12}$. Suppose $T_2 \cap S_2 = \emptyset$. Then since $d(z_{12}, x) = 1$, from the previous lemma the third grand clique $S_3$ containing $x$ must intersect $T_2$ in $z_{32}$, say. Similarly, the third grand clique $T_3$ containing $y$ must intersect $S_2$ in $z_{23}$ say. Since $z_{23} \in S_2$ and $z_{32} \in T_2$ and $S_2 \cap T_2 = \emptyset$, we must have $z_{23} \neq z_{32}$. But then we have 5 vertices adjacent to both $x$ and $y$, namely, $z_{11}, z_{12}, z_{21}, z_{23}, z_{32}$. This contradicts (b_4) and hence $S_2 \cap T_2 \neq \emptyset$. Let $z_{22} = S_2 \cap T_2$, we then have 4 vertices adjacent to both $x$ and $y$, namely, $z_{ij}$, $i,j = 1,2$.

It follows from (b_4) that
$$S_j \cap T_j = \emptyset, \quad j = 1,2,3$$

$$S_i \cap T_3 = \emptyset, \quad i = 1,2,3.$$ 

Lemma (3.1.11). Given two distinct grand cliques $K_1$ and $K_2$ of $G$, with a common vertex $x$, there is a $(1,1)$ correspondence between the vertices of $K_1$ and $K_2$ such that the corresponding vertices are contained in a grand clique.

Let $y_1$ be a vertex of $K_1$, $y_1 \neq x$. If $z \neq x$ is a vertex of $K_2$, then $d(y_1, z) \leq 2$. It follows as in the previous lemma that every vertex of $K_2$ is not adjacent to $y_1$. Hence there exists a vertex $z$ in $K_2$ such that $d(y_1, z) = 2$. From lemma (3.1.10) there is another clique $K^*$, besides $K_1$, which contains $y_1$ and intersects $K_2$. Let $y_2 = K^* \cap K_2$. Then $y_1$ is adjacent to $y_2$. Also the third clique containing $y_1$ does not intersect $K_2$. Thus there is exactly one vertex $y_2$ in $K_2$ which is adjacent to $y_1$. In the same way we can start from a vertex $y_2$ in $K_2$ and show that there is exactly one vertex in $K_1$ adjacent to $y_1$.

Lemma (3.1.12). There are exactly $\frac{n(n-1)}{2}$ grand cliques in $G$.

Consider ordered pairs $(x, K)$ where $x$ is a vertex and $K$ is a grand clique of $G$ containing $x$. Since each vertex is contained in 3 grand cliques we get $3v$ such pairs, where $v = \frac{n(n-1)(n-2)}{6}$ is the number of vertices in $G$. But each grand clique accounts for $n-2$ pairs. Hence the number of grand cliques is $3v/(n-2) = \frac{n(n-1)}{2}$.

Lemma (3.1.13). Each grand clique in $G$ is intersected by exactly $2(n-2)$ other grand cliques.

This follows at once by noting that each vertex of a grand clique $K$ is contained in exactly two other grand cliques.
Lemma (3.1.14). If $K_1$ and $K_2$ are two intersecting grand cliques in
$G$, there exist exactly $n-2$ grand cliques which intersect both $K_1$ and $K_2$.

Let $x = K_1 \cap K_2$. Then through $x$ there passes another grand clique
$K_3$ (intersecting $K_1$ and $K_2$ in $x$). Again if $y_1$ is any one of the other
$n-3$ points on $K_1$, then from Lemma (3.1.11) there exists a corresponding point
$y_2$ on $K_2$ such that $y_1$ and $y_2$ are adjacent, $K(y_1, y_2)$ being a grand clique
intersecting both $K_1$ and $K_2$. Each pair of corresponding points gives one such
clique and vice versa.

The following three lemmas are directed towards proving that there exist
exactly 4 grand cliques, which intersect each of the two given non-intersecting
grand cliques $K_1$ and $K_2$.

Lemma (3.1.15). If $K_1$ and $K_2$ are two non-intersecting grand cliques in
$G$, $x_1$ and $x_2$ are two adjacent vertices belonging to $K_1$ and $K_2$ respectively,
then there exist vertices $y_1, y_2$ belonging to $K_1$ and $K_2$ respectively, such
that $x_1, x_2, y_1, y_2$ are mutually adjacent ($y_1 \neq x_1, y_2 \neq x_2$).

Since $K_1$ and $K(x_1, x_2)$ are intersecting cliques, by lemma (3.1.11), there
exists a vertex $y_1$ in $K_1$ which corresponds to $x_2$ and is therefore adjacent
to it. Similarly there exists a vertex $y_2$ in $K_2$ which is adjacent to $x_1$.

Clearly $d(y_1, y_2) \leq 2$. We want to show that $y_1$ and $y_2$ are adjacent.
If not, then $d(y_1, y_2) = 2$. Then by lemma (3.1.10) there exists a clique $K_1^*$
containing $y_1$, and a clique $K_2^*$ containing $y_2$ such that $K_1^*$ does not intersect
any clique containing $y_2$, $K_2^*$ does not intersect any clique containing $y_1$.
Hence $K_1^*$ must be distinct from $K_1$ and $K(y_1, x_2)$. Also $K_2^*$ must be distinct
from $K_2$ and $K(y_2, x_1)$. The cliques $K_1$ and $K(y_1, x_2)$ must therefore
intersect the cliques $K_2$ and $K(y_2, x_1)$. Since by hypothesis $K_1$ does not
intersect $K_2$, this is a contradiction. Hence $d(y_1, y_2) = 1$. 

22
Lemma (3.1.16). If $K_1$ and $K_2$ are two non-intersecting grand cliques in $G$, such that there exist vertices $x_1$ and $x_2$ belonging to $K_1$ and $K_2$ respectively with $d(x_1, x_2) = 2$, then there exists at least one pair of adjacent vertices one of which belongs to $K_1$ and the other to $K_2$.

From lemma (3.1.10), there exist two cliques (different from $K_1$) containing $x_1$, which intersect each of the two cliques (other than $K_2$) containing $x_2$. Let $z$ be the vertex of intersection of one of the cliques containing $x_1$ with one of the cliques containing $x_2$. Then $z$ is different from $x_1$ and $x_2$. Since $K_1$ and $K(x_1, z)$ are intersecting cliques, from lemma (3.1.11), there exists a vertex $y_1$ in $K_1$ ($y_1 \neq x_1$), such that $y_1$ is adjacent to $z$. Similarly there exists a vertex $y_2$ in $K_2$ adjacent to $z$. $x_1$ and $x_2$ are non-adjacent by hypothesis. If the lemma is false then the pairs $(x_1, y_2)$, $(y_1, x_2)$ and $(y_1, y_2)$ must also be non-adjacent. Hence $K(z, x_1)$, $K(z, x_2)$, $K(z, y_1)$ and $K(z, y_2)$ must all be different grand cliques, contradicting the lemma (3.1.1).

Lemma (3.1.7). If $K_1$ and $K_2$ are two non-intersecting grand cliques in $G$, there must exist a pair of adjacent vertices one of which belongs to $K_1$ and the other to $K_2$.

Let $x_1$ be a vertex in $K_1$ and $x_2$ a vertex in $K_2$. If $d(x_1, x_2) = 1$, the lemma is true. If $d(x_1, x_2) = 2$, the required result follows from the previous lemma. Suppose $d(x_1, x_2) = r > 2$. Then, there exists a chain $x_1, z_1, z_2, \cdots, z_{r-1}, x_2$ joining $x_1$ and $x_2$. Clearly $d(z_{r-2}, x_2) = 2$. Consider the grand cliques $K(z_{r-3}, z_{r-2})$ and $K_2$. These must be non-intersecting, otherwise $d(z_{r-3}, x_2) = 2$, and $x$ and $y$ would be at distance $r-1$. Hence from the previous lemma we can find a vertex $y_2$ in $K(z_{r-3}, z_{r-2})$ and $z_{r-2}^*$ in $K_2$ such that $y_2$ and $z_{r-2}^*$ are adjacent. Then $d(x_1, y_2) = r-1$. By successive
repetition of the same process we can find a vertex \( v_2 \) in \( K_2 \) such that \( d(x_1, v_2) = 2 \). The required result then follows from the previous lemma.

Lemma (3.1.18). If \( K_1 \) and \( K_2 \) are two non-intersecting grand cliques in \( G \), then there exist exactly 4 grand cliques, which intersect each of \( K_1 \) and \( K_2 \).

From the previous lemma we can find a vertex \( x_1 \) in \( K_1 \) and a vertex \( x_2 \) in \( K_2 \) such that \( x_1 \) is adjacent to \( x_2 \). Now from lemma (3.1.15) we can find in \( K_1 \) a vertex \( y_1 \neq x_1 \), and in \( K_2 \) a vertex \( y_2 \neq x_2 \), such that \( x_1, x_2, y_1, y_2 \) are mutually adjacent. Hence \( K(x_1, x_2), K(x_1, y_2), K(y_1, x_2) \) and \( K(y_1, y_2) \) are 4 grand cliques each of which intersects \( K_1 \) and \( K_2 \).

If possible let there be another grand clique which intersects \( K_1 \) in \( z_1 \) and \( K_2 \) in \( z_2 \). Then \( z_1 \) is distinct from \( x_1 \) and \( y_1 \) and \( z_2 \) is distinct from \( x_2 \) and \( y_2 \). Otherwise there would be 4 grand cliques containing some vertex. Again \( z_1 \) is non-adjacent to \( x_2 \). Otherwise there would be 4 grand cliques containing \( x_2 \). Hence \( d(z_1, x_2) = 2 \). The grand cliques \( K_1 \) and \( K_2 \) containing \( z_1 \) and \( x_2 \) respectively are non-intersecting. Since the grand cliques \( K(x_2, x_1) \) and \( K(x_2, y_1) \) intersect the grand clique \( K_1 \) containing \( z_1 \), it follows from lemma (3.1.10), that \( K_2 \) the third grand clique containing \( x_2 \), cannot intersect any grand clique containing \( z_1 \). This is a contradiction since \( K(z_1, z_2) \) intersects \( K_2 \) in \( z_2 \).

2. Theorem (3.2.1). For a tetrahedral graph \( G \), the conditions \((b_1)\)-(b_4) hold. Conversely if \( n > 16 \) and conditions \((b_1)-(b_4)\) are satisfied then \( G \) is tetrahedral.

Given \( n \) symbols, 1,2,\ldots, \( n \), a tetrahedral graph \( G \) is the graph whose vertices are the unordered triplets on these symbols, two triplets being adjacent when the corresponding triplets have two symbols in common. The
conditions \((b_1)-(b_4)\) are easily checked.

(i) The number of unordered triplets on \(n\) symbols is clearly \(n(n-1)(n-2)/6\) which is the number of vertices in \(G\).

(ii) Let \(x\) be the vertex corresponding to the symbol \((i,j,k)\), for a vertex adjacent to \(x\), the corresponding triplet must contain 2 of the symbols \(i,j,k\) and one of \(n-3\) symbols other than \(i,j,k\). Hence the number of vertices adjacent to \(x\) is \(3(n-3)\). Hence \(G\) is regular of valence \(3(n-3)\).

Given any two triplets we can readily construct a chain of triplets beginning with the first and ending with the second so that two consecutive triplets have two symbols in common. Hence \(G\) is connected.

(iii) Let \(x\) and \(y\) be two adjacent vertices of \(G\). Let \(x\) and \(y\) correspond to \((i,j,k_1)\) and \((i,j,k_2)\). Then the triplets which have two symbols in common with each of the two triplets are the triplets \((i,k_1,k_2)\), \((j,k_1,k_2)\) and the \(n-4\) triplets \((i,j,s)\) where \(s\) is any of the \(n\) symbols other than \(i,j,k_1,k_2\). Hence the number of vertices adjacent to both \(x\) and \(y\) is \(n-2\), i.e. \(\Delta(x,y) = n-2\).

(iv) Let \(x\) and \(y\) be two vertices such that \(d(x,y) = 2\). Then we may take \(x\) to correspond to \((i,j_1,k_1)\) and \(y\) to correspond to \((i,j_2,k_2)\). Then the only triplets which have two symbols in common with both these triplets are \((i,j_1,j_2)\), \((i,j_1,k_2)\), \((i,j_2,k_1)\) and \((i,k_1,k_2)\). Thus \(\Delta(x,y) = 4\) if \(d(x,y) = 2\).

Conversely, suppose \(n > 16\) and the conditions \((b_1)-(b_4)\) are satisfied for \(G\). Then lemmas (3.1.1) - (3.1.18) hold. Let \(H\) be a graph whose vertices are the grand cliques of \(G\), two vertices of \(H\) being adjacent if and only if the corresponding grand cliques of \(G\) have a non-empty intersection.
Then $H$ satisfies the following conditions:

1. The number of vertices in $H$ is $n(n-1)/2$.

2. $H$ is regular and of valence $2(n-2)$.

3. $H$ is edge regular with edge degree $n-2$, i.e. $\Delta(u,v) = n-2$ if $u$ and $v$ are adjacent.

4. $\Delta(u,v) = 4$ if $u$ and $v$ are non-adjacent.

Conditions (a$_1$), (a$_2$), (a$_3$), (a$_4$) are satisfied in virtue of lemmas (3.1.12), (3.1.13), (3.1.14) and (3.1.18). It now follows from Connor's theorem (see section I), that for $n > 8$, $H$ is triangular. Hence to each vertex of $H$ we can associate an unordered pair $(i,j)$ of two distinct symbols chosen out of $n$ symbols 1, 2, ..., $n$ so that two vertices of $H$ are adjacent if and only if the corresponding pair have a common symbol. From the correspondence between $H$ and $G$ it follows that to each grand clique in $G$ we can associate an unordered pair of symbols chosen out of $n$ symbols, such that the pairs corresponding to two grand cliques have a common symbol if and only if the two grand cliques intersect.

Let $K_1$ and $K_2$ be any two intersecting grand cliques of $G$, having the vertex $x$ in common and let $(i,j)$ and $(i,k)$ be the corresponding pairs. To the vertex $x$ we associate the unordered triplet $(i,j,k)$. The third grand clique $K_3$ containing $x$, intersects both $K_1$ and $K_2$ and hence must correspond to the pair $(j,k)$. Note that the triplet assigned to $x$ is unambiguously determined by any two of the three cliques intersecting in $x$. Thus to each of the $n(n-1)(n-2)/6$ vertices of $G$ we can associate a unique triplet.

If two vertices $x$ and $y$ of $G$ are adjacent then there is a unique grand clique $K$ containing $x$ and $y$. If $(i,j)$ is the pair corresponding to $K$, then
the triplets corresponding to \( x \) and \( y \) must contain the symbols \( i \) and \( j \).

Conversely if the triplets corresponding to the vertices \( x \) and \( y \) contain the symbols \( i \) and \( j \), then \( x \) and \( y \) are contained in the grand clique \( K \) corresponding to the pair \((i,j)\), and must therefore be adjacent. Thus two vertices of \( G \) are adjacent if and only if the corresponding triplets have a common pair of symbols. Hence \( G \) must be tetrahedral.

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References


