This research was partially supported by the National Science Foundation under Grant GU-2059.

ON EQUIVALENCE OF PROBABILITY MEASURES

CHARLES R. BAKER

Department of Statistics
University of North Carolina at Chapel Hill

Institute of Statistics Mimeo Series No. 701

July, 1970
ON EQUIVALENCE OF PROBABILITY MEASURES

Charles R. Baker
Department of Statistics
University of North Carolina at Chapel Hill

ABSTRACT

Let $H$ be a real and separable Hilbert space, $\Gamma$ the Borel $\sigma$-field of $H$ sets, and $\mu_1$ and $\mu_2$ two probability measures on $(H,\Gamma)$. $\mu_1$ and $\mu_2$ are equivalent (mutually absolutely continuous) if, for $A \in \Gamma$,

$$\mu_1(A) = 0 \iff \mu_2(A) = 0.$$  

Several sufficient conditions for equivalence are obtained in this paper. Some of these results do not require that $\mu_1$ and $\mu_2$ be Gaussian. The conditions obtained are applied to show equivalence for some specific measures when $H$ is $L_2[\mathbb{T}]$.

This research was partially supported by the National Science Foundation under Grant GU-2059.
ON EQUIVALENCE OF PROBABILITY MEASURES

Charles R. Baker
University of North Carolina at Chapel Hill

INTRODUCTION

Conditions for equivalence of probability measures on the Borel \( \sigma \)-field of a Hilbert space have been the subject of much research during recent years \([1] - [9]\). For two Gaussian measures \( \mu_1, \mu_2 \), either \( \mu_1 \) and \( \mu_2 \) are equivalent (mutually absolutely continuous, denoted by \( \mu_1 \sim \mu_2 \)) or else \( \mu_1 \) and \( \mu_2 \) are orthogonal \( (\mu_1 \perp \mu_2) \), and general necessary and sufficient conditions for equivalence have been obtained (e.g., \([5]\)).

When one or both of the measures is not Gaussian, few conditions for equivalence are known. Moreover, even when both measures are Gaussian the general conditions for equivalence are often difficult to verify, requiring one to prove existence of a Hilbert-Schmidt operator with prescribed spectral properties.

In this paper, several conditions for equivalence are given. Most of these conditions are stated in terms of sample function properties. Several results do not require that both measures be Gaussian; when the two measures are Gaussian, the sufficient conditions given here will often be easier to verify than those previously obtained.

This research was partially supported by the National Science Foundation under Grant GU-2059.
DEFINITIONS AND PROBLEM STATEMENT

Let $H$ be a real and separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and Borel $\sigma$-field $\Gamma$. Let $(\Omega, \mathcal{B}, P)$ be a probability space, and suppose that $S$ and $N$ are $\beta/\Gamma$ measurable mappings (e.g., $S^{-1}(A) \in \beta$ for all $A \in \Gamma$) of $\Omega$ into $H$. Following Mourier [10], a $\beta/\Gamma$ measurable mapping of $\Omega$ into $H$ will be called a "random element" in $H$. Let $(H \times H, \Gamma \times \Gamma)$ be the usual product measurable space; $\Gamma \times \Gamma$ is the smallest $\sigma$-field containing all measurable rectangles $A \times B, A, B \in \Gamma$. $H \times H$ is a separable Hilbert space under the inner product defined by $\langle (x_1, x_2), (y_1, y_2) \rangle_{H \times H} = \langle x_1, x_1 \rangle + \langle y_2, y_2 \rangle$ for $x_1, x_2, y_1, y_2 \in H$ [11].

Define the map $(S, N) : \Omega \to H \times H$ by $(S, N)(\omega) = (S(\omega), N(\omega))$; this map is $\beta/\Gamma \times \Gamma$ measurable, since for $A, B$ in $\Gamma$

$$\{ \omega : (S(\omega), N(\omega)) \in A \times B \} = \{ \omega : S(\omega) \in A \} \cap \{ \omega : N(\omega) \in B \}.$$ 

Hence $(S, N)$ induces from $P$ a measure $\mu_{S,N}$ on $(H \times H, \Gamma \times \Gamma)$ defined by $\mu_{S,N}(C) = P(\omega : (S(\omega), N(\omega)) \in C)$, $C$ in $\Gamma \times \Gamma$. Further, $S$ and $N$ induce measures $\mu_S$ and $\mu_N$, respectively, on $\Gamma$; e.g., for $A$ in $\Gamma$,

$$\mu_S(A) = P[S^{-1}(A)].$$

$S$ and $N$ are independent if and only if $\mu_{S,N} = \mu_S \otimes \mu_N$, where $\mu_S \otimes \mu_N(A \times B) = \mu_S(A)\mu_N(B)$, $A, B$ in $\Gamma$.

Consider the map $f : H \times H \to H$, $f(x, y) = x + y$. $f$ is continuous relative to the norm topologies of $H \times H$ and $H$, so that $f$ is $\Gamma \times \Gamma/\Gamma$ measurable. Hence $S+N$ is $\beta/\Gamma$ measurable, and induces a measure $\mu_{S+N}$ from $P$,

$$\mu_{S+N}(A) = \mu_{S,N}(f^{-1}(A)) = P(\omega : (S(\omega), N(\omega)) \in f^{-1}(A)).$$

Let $\chi$ denote any fixed element of $H$. Define $f_\chi : H \to H$ by $f_\chi(\chi) = \chi + \chi$; $f_\chi$ is the section of $f$ at $\chi$, therefore is $\Gamma/\Gamma$ measurable and for $A$ in $\Gamma$ we define the measure $\mu_{\chi+N}$ by $\mu_{\chi+N}(A) = \mu_N(f^{-1}_\chi(A))$. 


Two measures $\mu_1$ and $\mu_2$ on $(\mathcal{H}, \mathcal{F})$ are said to be equivalent $(\mu_1 \sim \mu_2)$ if for $A \in \mathcal{F}$ $\mu_1(A) = 0 \iff \mu_2(A) = 0$. They are orthogonal $(\mu_1 \perp \mu_2)$ if there exists $A$ in $\mathcal{F}$ such that $\mu_1(A) = 0$, $\mu_2(A) = 1$.

The problem considered in this paper is that of obtaining sufficient conditions for equivalence of $\mu^N$ and $\mu_{s+N}$. Several of the conditions obtained are stated in terms of the measures $\mu_{s+N}$, $s \in \mathcal{H}$. 
COVARIANCE OPERATORS: GAUSSIAN MEASURES

Suppose $\mathbb{E}| | S(\omega)| |^2 < \infty$; then there exists [10] an element $m_S$ of $H$ and an operator $R_S$ in $H$ such that $\langle m_S, y \rangle = \mathbb{E}\langle S(\omega), y \rangle$, $\langle R_S y, y \rangle = \mathbb{E}\langle S(\omega) - m_S, y \rangle \cdot \langle S(\omega) - m_S, y \rangle$, for all $y, y$ in $H$. The operator $R_S$ is a "covariance" operator; i.e., it is linear, bounded, non-negative, self-adjoint, and trace-class.

If $\mathbb{E}| | N(\omega)| |^2 < \infty$, then $S$ and $N$ have a "cross-covariance" operator $R_{SN}: H \to H$, defined by $\langle R_{SN} u, v \rangle = \mathbb{E}\langle S(\omega) - m_S, y \rangle \cdot \langle N(\omega) - m_N, y \rangle$ for all $y, y$ in $H$; moreover, $R_{SN} = R_S^{\frac{i}{2}}V_{N}^{\frac{i}{2}}$ for a bounded linear operator $V$, with $||V|| \leq 1$ [12]. $R_{SN}$ is thus trace class, and $R_{NS} = R_{SN}^*$, where $*$ denotes adjoint.

$S$ is said to be a Gaussian element, and $\mu_S$ a Gaussian measure, if $\langle S, y \rangle$ is a Gaussian random variable for all $y$ in $H$. $\mu_S$ then has a covariance operator $R_S$ and a mean element $m_S$, and $\mathbb{E}| | S(\omega)| |^2 < \infty$ [10].

If $\mu_N$ and $\mu_{S+N}$ are Gaussian measures, then they are either equivalent or orthogonal [2,3]. A number of conditions for equivalence have been given by various authors; the conditions most useful for our purposes are the following [5]:

**Lemma 1.** $\mu_N \sim \mu_{S+N}$ if and only if

(a) $m_S - m_N$ is in the range of $R_N^{\frac{i}{2}}$;

(b) $R_{S+N} = R_N^{\frac{i}{2}}W R_N^{\frac{i}{2}} + R_N^*$, where $W$ is a Hilbert-Schmidt operator that does not have $-1$ as an eigenvalue.

In dealing with the equivalence of Gaussian measures, one often needs the following results on the range of square roots of covariance operators [13], [14]:

**Lemma 2.** Suppose $R_1$ and $R_2$ are covariance operators. Then

(1) $\text{range } (R_1^{\frac{i}{2}}) \subset \text{range } (R_2^{\frac{i}{2}})$ if and only if the following (equivalent) conditions are satisfied:
(a) $R_1^{\frac{k}{2}} = R_2^{\frac{k}{2}} G$ for $G$ linear and bounded;

(b) $R_1 = R_2^{\frac{k}{2}} Q R_2^{\frac{k}{2}}$ for $Q$ linear and bounded;

(c) $\langle R_1 y, y \rangle \leq k \langle R_2 y, y \rangle$ for all $y$ in $H$ and some finite scalar $k$.

(2) range $(R_1^{\frac{k}{2}}) = \text{range } (R_2^{\frac{k}{2}})$ if and only if the following (equivalent) conditions are satisfied:

(a) $R_1^{\frac{k}{2}} = R_2^{\frac{k}{2}} G$, for $G$ linear and bounded with bounded inverse

(b) $R_1 = R_2^{\frac{k}{2}} Q R_2^{\frac{k}{2}}$ for $Q$ linear and bounded with bounded inverse.
APPLICATIONS

In most applications $H$ is $L_2^1[T]$ (Lebesgue measure) for some compact interval $T$ of the real line. In such cases, $S$ and $N$ are random functions corresponding to measurable stochastic processes $(S_t^1), (N_t^1)$, whose sample functions belong almost surely to $L_2[T]$. It is easy to verify that a measurable stochastic process $(S_t^1)$ with sample functions a.s. in $L_2[T]$ is a $\beta / \Gamma$ measurable function; one uses the facts that $\langle S, y^1 \rangle$ is $\beta / B[R]$ measurable ($B[R]$ $\equiv$ Borel sets of the real line) for all $y$ in $L_2[T]$, and that $\Gamma$ is the smallest $\sigma$-field such that all bounded linear functionals on $L_2[T]$ are $\Gamma / B[R]$ measurable.

In many signal detection problems (see, e.g., [6], Chapter 20), one observes a sample function $X(\omega)$ belonging to $L_2[T]$ and must determine whether the sample function is "noise" ($X(\omega) = N(\omega)$) or "signal-plus-noise" ($X(\omega) = S(\omega) + N(\omega)$), these two possibilities being mutually exclusive and exhaustive. If $\nu_{S+N} \perp \nu_N$, there exists a statistical test such that the probability of making an incorrect classification is zero. However, if $\nu_{S+N} \sim \nu_N$, then the probability of deciding correctly when signal is present can be unity only if the probability of deciding incorrectly is unity when signal is absent. Equivalence holds in virtually all practical signal detection problems. Thus conditions for equivalence are of interest to aid in constructing valid mathematical models of signal detection problems.
Equivalence Conditions for Independent S, N

In this section it is assumed that \( \mu_{S,N} = \mu_S \otimes \mu_N \).

**Theorem 1.** If \( \mu_{\mathcal{X}^N} \sim \mu_N \) a.e. \( d\mu_S(\mathcal{X}) \), then \( \mu_{S+N} \sim \mu_N \).

**Proof:** For \( A \) in \( \Gamma \),

\[
\mu_{S+N}(A) = \mu_{S,N}[f^{-1}(A)] = \mu_S \otimes \mu_N[f^{-1}(A)]
\]

\[
= \int_H \mu_N[f^{-1}(A)] d\mu_S(\mathcal{X})
\]

\[
= \int_H \mu_{\mathcal{X}^N}(A) d\mu_S(\mathcal{X}). \tag{*}
\]

Suppose \( \mu_{\mathcal{X}^N} \sim \mu_N \) a.e. \( d\mu_S(\mathcal{X}) \). Then, \( \mu_N(A) = 0 \Rightarrow \mu_{\mathcal{X}^N}(A) = 0 \) a.e. \( d\mu_S(\mathcal{X}) \)

\( \Rightarrow \mu_{S+N}(A) = 0 \), from (*). Also, \( \mu_{S+N}(A) = 0 \Rightarrow \mu_{\mathcal{X}^N}(A) = 0 \) a.e. \( d\mu_S(\mathcal{X}) \Rightarrow \mu_N(A) = 0 \). Hence \( \mu_{S+N} \sim \mu_N \). (Note that \( \mu_{\mathcal{X}^N} \perp \mu_N \) a.e. \( d\mu_S(\mathcal{X}) \) does not imply \( \mu_{S+N} \perp \mu_N \), since the set \( A_\mathcal{X} \) satisfying \( \mu_{\mathcal{X}^N}(A_\mathcal{X}) = 1 - \mu_N(A_\mathcal{X}) = 0 \) can vary with \( \mathcal{X} \).)

Although the above result assumes that \( \mu_{S,N} = \mu_S \otimes \mu_N \), it can be applied to yield conditions for equivalence when \( S \) and \( N \) are not independent. For example, suppose that \( N = N_1 + N_2 \), where \( N_1 \) and \( N_2 \) are \( \beta/\Gamma \) measurable transformations inducing measures \( \mu_{N_1} \) and \( \mu_{N_2} \). If \( \mu_{N_1,N_2} = \mu_{N_1} \otimes \mu_{N_2} \) and \( \mu_{S_1,N_2} = \mu_S \otimes \mu_{N_2} \), one can use the above result to determine if \( \mu_N \sim \mu_{N_2} \) and \( \mu_{S+N} \sim \mu_{N_2} \). If both these equivalences hold, then by the chain rule for Radon-Nikodym derivatives one must have \( \mu_{S+N} \sim \mu_N \). This simple modification should be useful in many applications, especially in practical signal detection problems, where the noise usually contains an additive Gaussian component that is independent of the signal and of the remainder of the noise.
In order to apply Theorem 1, one must first determine sufficient conditions for $\mu_{X+N} \sim \mu_N$, $x \in H$. Several such conditions are known when $\mu_N$ is Gaussian, and are utilized to obtain the following corollary.

**Corollary.** Suppose that $\mu_N$ is Gaussian with $||m_N|| = 0$, and $\mu_{S+N} = \mu_S \otimes \mu_N$. Then $\mu_{S+N} \sim \mu_N$ if any of the following conditions is satisfied:

1. $y \in \text{range}\ (R_N^{1/2})$ a.e. $du_S(y)$;
2. $E||S(\omega)||^2 < \infty$, $E_S \in \text{range}\ (R_N^{1/2})$, and $R_S = R_N^{1/2}WR_N^{1/2}$ with $W$ trace-class;
3. $H$ is $L_2[T]$ for a compact interval $T$, $\mu_N$ is the measure induced by a measurable mean-square-continuous stationary stochastic process with rational spectral density $\hat{\phi}_N$, $E||S(\omega)||^2 < \infty$, and
   a. there exists a rational spectral density function $\hat{\phi}_0$ such that
      \[
      \langle R_S u, u \rangle \leq k \int_{-\infty}^{\infty} \frac{\hat{\phi}_0(\lambda)}{\hat{\phi}_N(\lambda)}|\hat{\mu}(\lambda)|^2\ d\lambda
      \]
      for all $y$ in $H$ ($\hat{\mu}$ is the Fourier transform of $y$, $y(t) \equiv 0$ for $t \notin T$) and some finite scalar $k$, with
      \[
      \int_{-\infty}^{\infty} \frac{\hat{\phi}_0(\lambda)}{\hat{\phi}_N(\lambda)}\ d\lambda < \infty,
      \]
   b. $E_S \in \text{range}\ (R_N^{1/2})$.
4. $H$ is as in (3), $\mu_N$ is induced by a measurable mean-square continuous stationary stochastic process with a spectral density function, $\hat{\phi}_N$, and
   \[
   \int_{-\infty}^{\infty} \frac{|\hat{\mu}(\lambda)|^2}{\hat{\phi}_N(\lambda)}\ d\lambda < \infty \quad \text{a.e.} \quad du_S(y).
   \]

**Proof:** (1) From Lemma 1, $y$ in range $(R_N^{1/2})$ $\iff$ $\mu_{X+N} \sim \mu_N$.

(2) Let $\{\lambda_n\}, \{g_n\}$ be the non-zero eigenvalues and an associated set of orthonormal eigenvectors of $R_N$. Then
\[ \mathbb{E} \sum_{n=1}^{N} \frac{\langle S(\omega), e_n^2 \rangle^2}{\lambda_n} = \sum_{n=1}^{N} \left( \frac{\langle R, e_n^2 \rangle}{\lambda_n} + \frac{\langle m, e_n^2 \rangle^2}{\lambda_n} \right) \]

and (2) follows, since \( S(\omega) \in \text{range } (R_N^{\frac{k}{2}}) \) almost surely.

(3) According to Hajek [4],

\[ \int_{-\infty}^{\infty} \frac{\hat{R}_0(\lambda)}{\hat{\lambda}_n(\lambda)} d\lambda < \infty \]

implies that \( R_0 = R_N^{\frac{k}{2}} W R_N^{\frac{k}{2}}, \) W trace-class. By Lemma 2, the condition \( \langle R_S^H, H \rangle \leq k \int_{-\infty}^{\infty} \frac{\hat{R}_0(\lambda)}{\hat{\lambda}_n(\lambda)} |\hat{\lambda}(\lambda)|^2 d\lambda \) implies that range \( (R_S^{\frac{k}{2}}) = \text{range } (R_0^{\frac{k}{2}}), \) and this last condition implies (Lemma 2) that \( R_S^{\frac{k}{2}} = R_0^{\frac{k}{2}} G \) for \( G \) bounded; while the representation \( R_0 = R_N^{\frac{k}{2}} W R_N^{\frac{k}{2}} \) with \( W \) trace-class implies \( R_0^{\frac{k}{2}} = R_N^{\frac{k}{2}} Q, \) Q Hilbert-Schmidt. Hence

\( R_S = R_N^{\frac{k}{2}} Q G G^* Q R_N^{\frac{k}{2}} \), and \( Q G G^* Q \) is trace-class, so that (3) follows from (2).

(4) This result is due to Kelly, Reed, and Root [15], and can be proved quickly as follows. By Lemma 2, \( y \) is in range \( (R_N^{\frac{k}{2}}) \) if and only if there exists a finite scalar \( k \) such that \( \langle y, y \rangle^2 \leq k \langle R_N^H, y \rangle \), all \( y \) in \( H \). The condition given in (4) implies that

\[ \langle y, y \rangle^2 \leq \int_{-\infty}^{\infty} \frac{|\hat{\lambda}_n(\lambda)|^2}{\hat{R}_n(\lambda)} d\lambda \langle R_N^H, y \rangle \quad \text{a.e. } d\mu_S(y), \]

so that \( y \in \text{range } (R_N^{\frac{k}{2}}) \) a.e. \( d\mu_S(y) \).

The conditions cited above can be used to determine equivalence when \( N = N_1 + N_2 \), where \( \nu_{11} = \nu_{11}^1 \ast \nu_{11}^2 \), \( \nu_{12} = \nu_{12}^1 \ast \nu_{12}^2 \), \( \nu_{21} = \nu_{21}^2 \ast \nu_{21}^1 \), and \( \nu_{22}^2 \) is Gaussian, as discussed previously.
Dr. T.T. Kadota has kindly provided the author with a preprint of a forthcoming paper [16]. In that paper, part (1) of the above corollary is proved for $H = L_2[0,b]$, $b < \infty$, under the additional assumptions that $u_N$ has continuous covariance function and $R_N$ is strictly positive definite.
DEPENDENT S, N

When the assumption that \( \nu_{S,N} = \nu_S \otimes \nu_N \) is not valid, the equivalence of \( \nu_{S+N} \) and \( \nu_N \) is not implied by \( \mu_{S+N} \sim \mu_N \) a.e. \( d\mu_S(\gamma) \). As a counterexample, suppose that \( \mu_N \) is Gaussian and that \( S(\omega) = k R_N^{1/p} N(\omega) \), for \( p \) a positive integer and some scalar \( k \). \( S \) is then Gaussian, and for \( p = 1 \) or \( 2 \), \( S(\omega) \in \text{range}(R_N^{1/p}) \) a.e. \( dP(\omega) \), implying \( \mu_{S+N} \sim \mu_N \) a.e. \( d\mu_S(\gamma) \). One has

\[
R_{S+N} = R_N^{1/2} \left[ k^2 R_N^{-2/p} + 2 k R_N^{1/p} + 1 \right]^{1/2} R_N^{1/2}.
\]

From Lemma 1 and Lemma 2, \( \mu_{S+N} \parallel \mu_N \) if and only if \( k^2 R_N^{-2/p} + 2 k R_N^{1/p} + 1 \) has zero as an eigenvalue; this will occur if and only if \( k = -\lambda_1^{-1/p} \) for some non-zero eigenvalue \( \lambda_1 \) of \( R_N \). As a specific example, \( \mu_{S+N} \parallel \mu_N \) if \( H \) is \( L_2[0,1] \), \( \mu_N \) is Wiener measure, and

\[
S_t(\omega) = -((2n+1)^2/4) \pi^2 \int_0^1 \int_0^1 \nu_N(\omega) d\nu d\mu = -((2n+1)^2/4) \pi^2 [R_N]_t(\omega),
\]

for any integer \( n \).
DEPENDENT S, N: GAUSSIAN MEASURES

In this section, it is assumed that \( \mu_N, \mu_S, \) and \( \mu_{S+N} \) are Gaussian. As previously noted, this implies the existence of covariance operators \( R_N, R_S, \) and \( R_{S+N} \) and mean elements \( E_N, E_S, \) and \( E_{S+N} \); we assume \( ||E_N|| = 0 \). Let \( \mu_{(S+N)} \) denote the Gaussian measure defined by

\[
\mu_{(S+N)}(A) = \int_{f^{-1}(A)} d\mu_S \otimes \mu_N \quad \text{for} \quad A \in \Gamma
\]

\((f(y,x) = y+x)\). \( \mu_{(S+N)} \) has covariance operator \( R_S + R_N \) and mean element \( E_S \).

Finally, we assume that the range of \( R_N \) is dense in \( H \); in cases where this is not satisfied, one can obtain the results given in this section by defining \( H \) to be the closure of \( \text{range}(R_N) \). We proceed to obtain some new sufficient conditions for equivalence of \( \mu_{S+N} \) and \( \mu_N \).

**Lemma 3.** Suppose \( \mu_S(\text{range}(R_N^{-\frac{1}{2}})) = 1 \). Then \( R_S = R_N^{-\frac{1}{2}} WR_N^{-\frac{1}{2}} \) for a covariance operator \( W \), and \( E_S \in \text{range}(R_N^{-\frac{1}{2}}) \).

**Proof:** By the corollary to Theorem 1, \( \mu_S(\text{range}(R_N^{-\frac{1}{2}})) = 1 \Rightarrow \mu_{(S+N)} \sim \mu_N \); from Lemma 1 this implies \( R_S = R_N^{-\frac{1}{2}} WR_N^{-\frac{1}{2}} \), \( W \) Hilbert-Schmidt, and \( E_S \in \text{range}(R_N^{-\frac{1}{2}}) \). Let \( x = R_N^{-\frac{1}{2}} E_S \).

Since \( \mu_S \) is Gaussian, \( \langle S, y \rangle \) is a Gaussian random variable for all \( y \in H \).

Define \( Y: \Omega \to H \) by

\[
Y(\omega) = \begin{cases} 
R_N^{-\frac{1}{2}} S(\omega) & \text{for } S(\omega) \in \text{range}(R_N^{-\frac{1}{2}}) \\
\emptyset & \text{for } S(\omega) \notin \text{range}(R_N^{-\frac{1}{2}}) 
\end{cases}
\]

For any \( A \in \Gamma \), \( R_N^{-\frac{1}{2}}[A] = \{ x : x = R_N^{-\frac{1}{2}} y \text{ for } y \in A \} \) is an element of \( \Gamma \), since \( R_N^{-\frac{1}{2}} \) is a continuous map. Moreover, \( Y^{-1}(A) = \{ \omega : S(\omega) \in R_N^{-\frac{1}{2}}[A] \} \) if \( \emptyset \notin A \); if \( \emptyset \in A \), then \( Y^{-1}(A) = \{ \omega : S(\omega) \in R_N^{-\frac{1}{2}}[A] \} \cup \{ \omega : S(\omega) \notin \text{range}(R_N^{-\frac{1}{2}}) \} \). In either
case, \( Y^{-1}(A) \in \beta \), so that \( Y \) is \( \beta/\Gamma \) measurable and thus induces from \( P \) a measure \( \mu_Y \) on \((H, \Gamma)\). For any \( y \) in \( H \) we show that \( \langle Y, y \rangle \) is a Gaussian random variable; first, note that there exists \( \{u_n\} \) such that \( R_N^{\frac{1}{2}}u_n \to y \).

Using \( R_S = R_N^{\frac{1}{2}}WR_N^{\frac{1}{2}} \), \( W \) bounded, and \( E_S = R_N^{\frac{1}{2}}E \), one has that \( \langle S, u_n \rangle \to \langle Y, y \rangle \) almost surely and in \( L_2(\Omega, \mathbb{B}, P) \). Hence \( \langle Y, y \rangle \) is a Gaussian r.v. for all \( y \) in \( H \), \( \mu_Y \) is Gaussian with covariance operator \( R_Y \) and mean element \( E_Y \), \( R_S = R_N^{\frac{1}{2}}R_Y R_N^{\frac{1}{2}} \), and \( E_S = R_N^{\frac{1}{2}}E_Y \).

**Theorem 2.** If \( \mu_S(\text{range } (R_N^{\frac{1}{2}})) = 1 \), then \( \mu_{S+N} \sim \mu_N \) if and only if \( \text{range } (R_S^{\frac{1}{2}}) \supset \text{range } (R_N^{\frac{1}{2}}) \).

**Proof:** \( \mu_S(\text{range } (R_N^{\frac{1}{2}})) = 1 \) implies, by Lemma 3, that \( \mu_S \in \text{range } (R_N^{\frac{1}{2}}) \) and \( R_S = R_N^{\frac{1}{2}}WR_N^{\frac{1}{2}} \), \( W \) trace class. From Lemma 1, \( \mu_{S+N} \sim \mu_N \) if and only if \( R_{S+N} = R_N^{\frac{1}{2}}(I+Q)R_N^{\frac{1}{2}} \), where \( Q \) is Hilbert-Schmidt and does not have \(-1\) as an eigenvalue. \( R_S = R_N^{\frac{1}{2}}WR_N^{\frac{1}{2}} \) implies \( R_S^{\frac{1}{2}} = R_N^{\frac{1}{2}}G \), \( G \) Hilbert-Schmidt; hence

\[
R_{S+N} = R_S + R_{SN} + R_{NS} + R_N = R_N^{\frac{1}{2}}[\mathbb{W} + \mathbb{G}V + V^*G^* + I]R_N^{\frac{1}{2}},
\]

where \( V \) is an operator of norm \( \leq 1 \) satisfying \( R_{SN} = R_N^{\frac{1}{2}}VR_N^{\frac{1}{2}} \) [12]. Now \( Z = \mathbb{W} + \mathbb{G}V + V^*G^* \) is Hilbert-Schmidt, so that \( \mu_{S+N} \sim \mu_N \) if and only if \( Z \) does not have \(-1\) as an eigenvalue. From Lemma 2, this is precisely the condition for \( \text{range } (R_S^{\frac{1}{2}}) \supset \text{range } (R_N^{\frac{1}{2}}) \).

The results summarized in Lemma 2 can be used to determine whether \( \text{range } (R_N^{\frac{1}{2}}) \subset \text{range } (R_S^{\frac{1}{2}}) \), and thus \( \mu_{S+N} \sim \mu_N \) whenever \( \mu_{Y+N} \sim \mu_N \) a.e. \( du_S(y) \). The following corollary gives two additional conditions for \( \mu_{S+N} \sim \mu_N \).

**Corollary.** \( \mu_{S+N} \sim \mu_N \) if \( \mu_S(\text{range } (R_N^{\frac{1}{2}})) = 1 \) and either of the two following conditions is satisfied:

(a) There exists no non-null \( y \) in \( H \) satisfying \( G^*y = -Vy \) and
\[ V^*G^*u = -u, \text{ where } G, V \text{ are operators satisfying } R_S^{\frac{1}{2}} = R_N^{\frac{1}{2}} G, ||V|| \leq 1, \]
\[ R_{SN} = R_S^{\frac{1}{2}} V R_N^{\frac{1}{2}}. \text{ (This condition is also necessary for } \mu_{SN} \sim \mu_N) \]

(b) \( H \) is \( L_2[\mathbb{T}], \mathbb{T} \) a compact interval, \( \mu_N \) is induced by a measurable mean-square continuous stochastic process, \( (S_t) \) is measurable, and for all \( t \in \mathbb{T} \), \( S_t \) is independent of \( N_v \) for all \( v \geq t \).

**Proof:** (a) From the theorem, it is sufficient to show that the stated condition is necessary and sufficient for \( \text{range } (R_{SN+\frac{1}{2}}) \supseteq \text{range } (R_N^{\frac{1}{2}}) \), or, by Lemma 2, for \( (I+GG^*+GV+V^*G^*)y = 0 \) to imply \( ||y|| = 0 \). If \( (I+GG^*+GV+V^*G^*)y = 0 \), then \( ||y||^2 + |G^*y|^2 + 2\langle G^*y, V_y \rangle = 0 \). The LHS of this last equality is \( \geq ||y||^2 + ||G^*y||^2 - 2||V^*G^*y|| ||y|| \geq ||y||^2 + ||G^*y||^2 - 2||V^*G^*y|| ||y|| \geq ||y||^2 + ||G^*y||^2 - 2||G^*y|| ||y|| = (||y|| - ||G^*y||)^2 \geq 0 \), with equality throughout if and only if \( G^*y = -V_y \) and \( ||G^*y|| = ||y|| \). Further, \( ||y||^2 + ||G^*y||^2 + 2\langle G^*y, V_y \rangle \geq ||y||^2 + ||G^*y||^2 - 2||V^*G^*y|| ||y|| \geq ||y||^2 + ||G^*y||^2 - 2||G^*y|| ||y|| = (||y|| - ||G^*y||)^2 \geq 0 \), with equality throughout if and only if \( ||G^*y|| = ||y|| \) and \( V^*G^*y = -y \). Hence, if \( G^*y = -V_y \) and \( V^*G^*y = -y \) only for \( y = 0 \), \( \text{range } (R_{SN+\frac{1}{2}}) \supseteq \text{range } (R_N^{\frac{1}{2}}) \). It is clear that \( V^*G^*y = -y \) and \( G^*y = -V_y \) together imply \( (I+GG^*+GV+V^*G^*)y = 0 \), so that the condition is also necessary for \( \text{range } (R_{SN+\frac{1}{2}}) \supseteq \text{range } (R_N^{\frac{1}{2}}) \).

(b) We show that the assumptions imply \( \text{range } (R_{SN+\frac{1}{2}}) \supseteq \text{range } (R_N^{\frac{1}{2}}) \). Suppose not. Then by Lemma 2 there must exist \( \{y_n\} \) in \( H \) such that \( R_N^{\frac{1}{2}} y_n + y, ||y|| = 1, \) and \( R_{SN+\frac{1}{2}} y_n \sim 0 \). This implies the existence of a random variable \( X \) in \( L_2(\Omega, \beta, P) \) such that \( X = \ell.i.m. \langle y_n, N \rangle = \ell.i.m. \langle -y_n, S^{-\frac{1}{2}} S \rangle \) (\( \ell.i.m. \) for \( L_2(\Omega, \beta, P) \)). Since \( (N_t) \) is mean-square continuous, the subspace of \( L_2(\Omega, \beta, P) \) spanned by \( \{\langle N_t, y \rangle, y \in H\} \) is identical to the subspace spanned by \( \{N_t, t \in \mathbb{T}\} \).

Moreover, this subspace is separable since it is isometric to the closure of the range of \( R_N^{\frac{1}{2}} \) (under the map taking \( R_N^{\frac{1}{2}} y \) into \( \langle y, N \rangle \)). Hence, there are scalars \( \{a_k^M \}, k = 1, 2, \ldots, M; M = 1, 2, \ldots \) such that \( X = \ell.i.m. \sum_{k=1}^{M} a_k^M N_t^k \).
let $X_M = \sum_{k=1}^{M} \alpha_k N_{t_k}^M$. Note that the set $\{t_k\}$ can be assumed independent of $M$, since $\{N_{t_k}\}$, $t \in T$, spans a separable subspace of $L_2(\Omega, \beta, \mathbb{P})$.

We now show that $\text{EX}(\omega)[S_{t}(\omega) - \mathbb{E}_{S_t} + N_{t}(\omega)] = 0$ for all $t \in T$. To see this, one notes that

$$\text{EX}(\omega)[S_{t}(\omega) - \mathbb{E}_{S_t} + N_{t}(\omega)] = -\lim_{n} \mathbb{E}[\mathbb{E}_{S_{t_n}}(S_{t}(\omega) - \mathbb{E}_{S_t} + N_{t}(\omega))]$$

$$= -\lim_{n} [R_{S_{t_n}}^n + R_{N_{t_n}}^n](t).$$

In $L_2(T)$, one has $R_{S_{t_n}}^n + R_{N_{t_n}}^n = R_N^{\frac{1}{2}}(GG^* + V^*G^*)R_N^{\frac{1}{2}}u_n + R_N^{\frac{1}{2}}(GG^* + V^*G^*)u_n = 0$, from the assumptions on $y$ and part (a) of this corollary.

Since $R_N$ has continuous kernel $R(t,u)$, this implies that $[R_{S_{t_n}}^n + R_{N_{t_n}}^n](t) \to 0$ uniformly for all $t$ in $T$. To prove this, let $R_0(t,u)$ be the kernel of $R_N^{\frac{1}{2}}$.

Then

$$([R_{S_{t_n}}^n + R_{N_{t_n}}^n](t))^2 = \left[\int_0^T R_0(t,s)[(GG^* + V^*G^*)R_N^{\frac{1}{2}}u_n](s)ds\right]^2$$

$$\leq \int_0^T R_0^2(t,s)ds \|(GG^* + V^*G^*)R_N^{\frac{1}{2}}u_n\|^2$$

$$= R(t,t) \|(GG^* + V^*G^*)R_N^{\frac{1}{2}}u_n\|^2$$

$$\leq \sup_{t \in T} R(t,t) \|(GG^* + V^*G^*)R_N^{\frac{1}{2}}u_n\|^2 \to 0.$$

Hence $\text{EX}(\omega)[N_{t}(\omega) + S_{t}(\omega) - \mathbb{E}_{S_t}] = 0$ for all $t \in T$.

For $X_M$ defined as above,

$$E[S_{t_1}(\omega) - \mathbb{E}_{S_{t_1}}]X_M(\omega) = \sum_{l=1}^{M} \alpha_l^M R_{SN}(t_1, t_k) = 0,$$

since $R_{SN}(t_1, t_k) = 0$ for $k \geq 1$ (we are assuming that $t_k \geq t_1$ for $k \geq 1$). Hence

$$E[S_{t_1}(\omega) - \mathbb{E}_{S_{t_1}}]X(\omega) = 0 \Rightarrow EN_{t_1}(\omega)X(\omega) = 0$$

$X$ is contained in the subspace of $L_2(\Omega, \beta, \mathbb{P})$ spanned by $\{N_{t_k}\}, \ k = 2, 3, \ldots$

Suppose next that $X_M \to X$ in $L_2(\Omega, \beta, \mathbb{P})$ with $X_M = \sum_{k=m}^{M} \alpha_k^M N_{t_k}$, where $1 < m \leq M$. 


m fixed and independent of M. Then \( \text{EX}_M(\omega)[S_{t_m}(\omega) - \mathbb{E}S_{t_m}] = 0 \) for all M

\( \Rightarrow \text{EX}_M N_{t_m} = 0, \) all M \( \Rightarrow \) X is contained in the linear subspace spanned by \( \{N_{t_k}\}, \) \( k \geq m+1. \) Hence \( \text{EX}(\omega) \sum_{k=1}^{M} N_{t_k}(\omega) = 0, \) all M

\( \Rightarrow \text{EX}^2(\omega) = 0 \Rightarrow \lim E \left< N(\omega), u_n \right>^2 = 0 \)

\( \Rightarrow \lim_{n} \left\| R_N^{\frac{1}{2}} u_n \right\|^2 = 0 \Rightarrow \left\| y \right\| = 0 \)

\( \Rightarrow \text{range } (R_{s+N}^{\frac{1}{2}}) = \text{range } (R_N^{\frac{1}{2}}). \)
Examples

Suppose that $H$ is $L_2[0,b]$, $b < \infty$, and that $\mu_N$ is Gaussian with null
mean function and covariance function defined as follows:

1. $R_1(t,s) = \min(t,s)$
2. $R_2(t,s) = b - \max(t,s)$
3. $R_3(t,s) = b - |t-s|$  
4. $R_4(t,s) = e^{-\alpha|t-s|}$, $\alpha > 0$
5. $R_5(t,s) = \int_{-\infty}^{\infty} e^{i\lambda(t-s)} \hat{R}(\lambda) d\lambda$

where $\hat{R}$ is a rational spectral density function with denominator of
degree exactly two greater than the degree of the numerator.

Suppose that $\mu_S$ is induced by the stochastic process $(S_t)$ defined by

(a) $S_t(\omega) = \int_{0}^{t} Y_s(\omega) ds$
(b) $S_t(\omega) = \int_{t}^{b} Y_s(\omega) ds$

or

(c) $S_t(\omega) = c(\omega) + \int_{0}^{t} Y_s(\omega) ds$, all $t$ in $[0,b]$,

where in (a), (b) and (c) $(Y_t)$ is a measurable stochastic process with sample
functions almost surely in $L_2[0,b]$. $c$ is an a.s. finite random variable.

Assume that one of the following two conditions is satisfied: (A) $S$ and
$N$ are independent; (B) $S$ is Gaussian with $S_t$ independent of $N_v$ for $v \geq t$
and all $t$ in $[0,b]$. One then has the following results:

(i) For $S$ defined as in (a), $N$ defined by (1), (3), (4), or (5),

$\mu_{S+N} \sim \mu_N$.

(ii) For $S$ defined as in (b), $N$ defined by (2) – (5), $\mu_{S+N} \sim \mu_N$.

(iii) For $S$ defined as in (c), $N$ defined by (3) – (5), $\mu_{S+N} \sim \mu_N$.

These results are unchanged if any of the $N$ covariance functions are multiplied
by a positive real scalar. (The result given in (i) for the Wiener process (co-
variance function $R_1$) has been previously obtained under weaker assumptions [16].)
To obtain the preceding results we note the following [17]:

(1') The integral operator in \( L_2[0,b] \) with kernel \( R_1(t,s) \) has square root with range containing all absolutely continuous functions on \([0,b]\) that vanish at \(0\) and have \(L_2[0,b]\) derivative.

(2') The integral operator with kernel \( R_2(t,s) \) has square root with range containing all absolutely continuous functions on \([0,b]\) that vanish at \(b\) and have \(L_2[0,b]\) derivative.

(3') - (5') The integral operators with \( R_3(t,s), R_4(t,s) \) and \( R_5(t,s) \) as kernel have square roots with the same range; this range space contains all absolutely continuous functions on \([0,b]\) with derivatives belonging to \(L_2[0,b]\).

The results stated above now follow directly from the corollaries to the two theorems.
REFERENCES


