# A Class of Robust Adaptive Controllers for Infinite Dimensional Dynamical Systems *


M. A. Demetriou K. Ito
Center for Research in Scientific Computation
Department of Mathematics
North Carolina State University
Raleigh, NC 27695-8205


August 25, 1995


**Abstract**

An adaptive controller for a perturbed infinite dimensional plant is developed to force the state of the plant track the state of a reference model. The reference model is based on the nominal plant that has a physical similarity with the plant. Using a Lyapunov stability argument, which is based on the $H^\infty$-Riccati equation of the nominal plant, an adaptive law is developed for the adjustment of the feedback gain. It is proved that the closed-loop system is stable with the tracking error remaining bounded, and converging to zero provided that the norm of the structured perturbation is less than a specified attenuation bound. Results of numerical studies regarding a heat equation and a beam equation are presented to demonstrate the applicability of the proposed control algorithm.


## 1 Introduction

We consider an infinite dimensional plant whose dynamics are governed by a linear evolution equation of the form

$$\frac{d}{dt}x_p(t) = A_p x_p(t) + Bu(t), \quad x_p(0) = x_0 \in X. \tag{1.1}$$

The linear operator $A_p$ is an unknown infinitesimal generator of a $C_0-$semigroup on a Hilbert space $X$ with inner product and norm $\langle\cdot,\cdot\rangle$ and $|\cdot|_X$, respectively. The input operator $B$ is known and it is assumed that $B \in \mathcal{L}(\mathbb{R}^m, X)$ has finite rank. The objective is to find a robust dynamic feedback control law of the form

$$u(t) = -K(t)x_p(t) + r(t) \tag{1.2}$$

such that the state of the plant $x_p(t)$ tracks the state of a reference model $x_m(t)$. This reference model is based on the nominal closed-loop dynamics and is given by

$$\frac{d}{dt}x_m(t) = (A_0 - BK_m)x_m(t) + Br(t) \tag{1.3}$$


*Research supported in part by the Air Force Office of Scientific Research under grant AFOSR F49620-93-1-0198, and in part by NASA under grant NAG-1-1600.

where $r(\cdot) \in L^1_{loc}(0,\infty;\mathbb{R}^m)$. The open loop operator $A_0$ is the *nominal* plant generator in the sense that the unknown plant generator $A_p$ has the following perturbation form

$$A_p = A_0 + \Delta A, \quad \Delta A \in \mathcal{L}(X),$$

and $K_m \in \mathcal{L}(X,\mathbb{R}^m)$ is an appropriately chosen stabilizing feedback gain operator for the nominal plant that will be defined below. The reference signal $r(t)$ may be determined by designing an optimal control for the reference dynamics (1.3) with certain performance index. For example, it might be chosen to minimize the performance index

$$\int_0^T \left(|x_m(t) - \bar{x}|^2 + \rho\,|r(t)|^2\right)\,dt + |x_m(T) - \bar{x}|^2$$

associated with the problem of regulating the state $x_m(t)$ to a desired state $\bar{x} \in X$. Following the model reference adaptive feedback law [3, 9], the dynamic feedback gain operator $K(t)$ is given by

$$\beta\,\frac{d}{dt}K(t) = B^*\Pi_0\,(x_p(t) - x_m(t)) \otimes x_p(t) \tag{1.4}$$

where $\beta > 0$ is the acceleration parameter. The derivation of the adaptation rule (1.4) is based on the Lyapunov synthesis approach, [3, 9]. In our approach we choose the nonnegative definite self-adjoint operator $\Pi_0$ to be a solution to the $H^\infty$ Riccati equation (1.8), discussed below, and

$$K_m = B^*\Pi_0. \tag{1.5}$$

It is shown in Section 3 that, under certain assumptions on the norm of $\Delta A$ and the reference trajectory $x_m(t)$, the closed-loop system (1.1) - (1.5) has a unique global solution $(x_p(t), K(t))$ and that the asymptotic tracking property

$$\int_0^\infty |x_p(t) - x_m(t)|_X^2\,dt < \infty, \quad \text{and} \quad e(t) = x_p(t) - x_m(t) \to 0 \quad \text{as} \quad t \to \infty \tag{1.6}$$

is satisfied (see Theorem 3.2). The control law (1.2) provides a sub-optimal control law for the uncertain plant (1.1) in the sense of finding an optimal control law for the reference model (1.3) and an asymptotic regulator via the adaptive feedback law (1.4). Thus, the problem of constructing a feedback law for the uncertain plant (1.1) can be decomposed into (i) the optimal control problem of $r(t)$ for the reference model (1.3) and (ii) the asymptotic tracking problem in the sense of (1.6).

The following *matching condition* is normally assumed in the adaptive control literature [9]. There exists a feedback operator, $K : \mathbb{R}^m \to X$ such that

$$A_p + BK = A_m = A_0 - BK_m, \tag{1.7}$$

which is equivalent to assume that

$$\Delta A \subset \mathcal{R}(B).$$

The range condition (1.7) is very restrictive for the infinite dimensional plant (1.1). Our objective here is to remove this assumption and allow general uncertainty in the plant. Our choice for $\Pi_0$ in the adaptive law (1.4) is based on the fact that the $H^\infty$ feedback control is developed to construct a feedback law that stabilizes all perturbed plants with a specified uncertainty bound for $\|\Delta A\|$. That is, if for given $\gamma > 0$, there exists a nonnegative, self-adjoint operator $\Pi_0$ on $X$ satisfying the game-theoretic Riccati equation

$$\left[A_0^*\Pi_0 + \Pi_0 A_0 - \Pi_0\left(BR^{-1}B^* - \frac{1}{\gamma^2}DD^*\right)\Pi_0 + C^*C\right]x = 0, \tag{1.8}$$

for all $x \in \mathcal{D}(A_0)$, then for $K = R^{-1}B^*\Pi_0$, the closed-loop plant operator

$$A_p - BK = A_0 + D\Gamma C - BK$$

generates an exponentially stable semigroup on $X$ provided that

$$\|\Gamma\| < \frac{1}{\gamma}.$$

Here we assume the structured perturbation

$$\Delta A = D\Gamma C, \quad \Gamma \in \mathcal{L}(X)$$

with known operators $D \in \mathcal{L}(X)$ and $C \in \mathcal{L}(X)$. It is shown (e.g. [4, 12, 13, 14, 16]) that the least $\gamma^* > 0$, such that (1.8) has a nonnegative solution, gives the best stability bound in the robust linear feedback control, i.e.

$$\gamma^* = \inf\left\{ \gamma > 0 \;\middle|\; \begin{array}{l} \exists \text{ a linear feedback } K \in \mathcal{L}(\mathbb{R}^m, X) \text{ such that} \\ A_0 - BK + D\Gamma C \text{ generates an exponentially} \\ \text{stable semigroup, provided that } \|\Gamma\| < \dfrac{1}{\gamma} \end{array} \right\}.$$

Moreover, it can be seen that this robust stabilization problem is equivalent to the standard $H^\infty$-attenuation problem for the state feedback case (see [6, 7]). Throughout this paper we consider the case when $D$, $C$ are chosen to be the identity operators in order to account for unstructured perturbation, and the control weight is taken, for simplicity, to be $R = I$. All discussions can be simply extended to the structured perturbation case.

For the sake of completeness of our discussions, the asymptotic tracking property of the constant feedback synthesis

$$u(t) = -K_m\, x_p(t) + r(t) \quad \text{with} \quad K_m = B^*\Pi_0 \tag{1.9}$$

is given in Section 2. Comparison of the performance between the constant feedback synthesis (1.9) and the adaptive control law (1.2), (1.4) is of our interest and will be analyzed in Section 3 in terms of the Lyapunov stability argument, and in Section 5 through numerical testings.

An outline of the paper is as follows. In Section 2 we summarize the results for the robust model reference control of infinite dimensional systems. In Section 3 we present our main results on the model reference adaptive control based on the Lyapunov stability argument. The well-posedness of the proposed control law (1.2) with the adaptation rule (1.4) is given in Section 4. Numerical studies of a heat equation with spatially varying parameter and of an Euler-Bernoulli beam with piezoceramic actuators are shown in Section 5 and conclusions and discussion of possible extensions of this work are given in Section 6.

## 2 Robust Model Reference Control

When the control law (1.9) is applied to the plant equation (1.1), it yields the closed loop system

$$\frac{d}{dt}x_p(t) = (A_p - BK_m)x_p(t) + Br(t). \tag{2.1}$$

The closed loop system (2.1) combined with the reference model (1.3) give, for $e(t) = x_p(t) - x_m(t)$, the following state error equation

$$\begin{aligned} \frac{d}{dt}e(t) &= (A_p - BB^*\Pi_0)x_p(t) + Br(t) - (A_0 - BB^*\Pi_0)x_m(t) - Br(t) \\ &= (A_p - BB^*\Pi_0)e(t) + (A_p - A_0)x_m(t) = \overline{A}_p e(t) + \Delta A x_m(t) \\ &= (A_0 - BB^*\Pi_0)e(t) + (A_p - A_0)x_p(t) = \overline{A}_0 e(t) + \Delta A x_p(t), \end{aligned} \tag{2.2}$$

where $\overline{A}_p = A_p - BB^*\Pi_0$, $\overline{A}_0 = A_0 - BB^*\Pi_0$ and $\Delta A = A_p - A_0 = \overline{A}_p - \overline{A}_0$. Here, solutions to (1.3), (2.1), (2.2) are understood in the sense of the mild solution, see [10]. In addition, if $x_p(0), x_m(0) \in \mathcal{D}(A_0)$ and $r(t)$ is continuously differentiable, then equation (1.3) as well as (2.1) have a classical solution $x(t) \in C^1(0,\tau;X) \cap C(0,\tau;\mathcal{D}(A_0))$. Hence, we have that (2.2) holds in the strong sense. The corresponding Lyapunov function is given by

$$V(t) = \langle \Pi_0 e(t), e(t) \rangle. \tag{2.3}$$

We assume here that $\Pi_0 \geq 0$ is the solution to the Riccati equation

$$\left[A_0^*\Pi_0 + \Pi_0 A_0 - \Pi_0 \left(BB^* - \frac{1}{\gamma^2}I\right)\Pi_0 + Q\right] x = 0 \tag{2.4}$$

for all $x \in \mathcal{D}(A_0)$ and $Q \geq I$. Or, equivalently

$$\left(\overline{A}_0^*\Pi_0 + \Pi_0\overline{A}_0\right) x = -\left[\Pi_0 \left(BB^* + \frac{1}{\gamma^2}I\right)\Pi_0 + Q\right] x. \tag{2.5}$$

We assume that

$$\|\Delta A\| < \frac{1}{\gamma} \tag{2.6}$$

and thus we have that

$$\gamma \, \|\Delta A\| = 1 - \delta, \qquad 0 < \delta \leq 1. \tag{2.7}$$

The following calculations are understood in the sense of Theorem 3.1 presented in § 3.

Taking the derivative of (2.3) along the solutions of (2.2) and using (2.5) - (2.7), we have that

$$\frac{d}{dt}V(t) = -\langle(\Pi_0 BB^*\Pi_0 + \frac{1}{\gamma^2}\Pi_0\Pi_0 + Q)e(t), e(t)\rangle + \langle(\overline{A}_p - \overline{A}_0)e(t), \Pi_0 e(t)\rangle$$

$$
\begin{aligned}
&+\langle \Pi_0 e(t), (\overline{A}_p - \overline{A}_0)e(t)\rangle + \langle (A_0 - A_p)x_m(t), \Pi_0 e(t)\rangle \\
&+\langle \Pi_0 e(t), (A_0 - A_p)x_m(t)\rangle \\
\leq\; & -\frac{1}{\gamma^2}|\Pi_0 e(t)|_X^2 - |e(t)|_X^2 + 2\,|\Pi_0 e(t)|_X \left(\|\Delta A\||e(t)|_X + \|\Delta A\||x_m(t)|_X\right) \\
\leq\; & -(1-\delta)\left(|e(t)|_X - \frac{1}{\gamma}|\Pi_0 e(t)|_X\right)^2 - \delta|e(t)|_X^2 - \frac{\delta}{\gamma^2}|\Pi_0 e(t)|_X^2 \\
&+2\,|\Pi_0 e(t)|_X |x_m(t)|_X \|\Delta A\| \\
\leq\; & -\delta|e(t)|_X^2 - \frac{\delta}{2\gamma^2}|\Pi_0 e(t)|_X^2 + \frac{2\gamma^2}{\delta}\|\Delta A\|^2 |x_m(t)|_X^2 \\
\leq\; & -\delta \min\left\{1, \frac{1}{2\gamma^2}\right\}\left(|e(t)|_X^2 + |\Pi_0 e(t)|_X^2\right) + \frac{2\gamma^2}{\delta}\|\Delta A\|^2 |x_m(t)|_X^2 \\
\leq\; & -2\delta \min\left\{1, \frac{1}{2\gamma^2}\right\}\frac{1}{2}\left(|e(t)|_X^2 + |\Pi_0 e(t)|_X^2\right) + \frac{2\gamma^2}{\delta}\frac{(1-\delta)^2}{\gamma^2}|x_m(t)|_X^2 \\
\leq\; & -c_1\left(|e(t)|_X^2 + \langle \Pi_0 e(t), e(t)\rangle\right) + \frac{2(1-\delta)^2}{\delta}\,|x_m(t)|_X^2, \\
=\; & -c_1\left(|e(t)|_X^2 + V(t)\right) + c_2\,|x_m(t)|_X^2,
\end{aligned}
\tag{2.8}
$$

for some positive constants $c_1$, $c_2$, where we used the completion of squares and the fact that

$$
V(t) = \langle \Pi_0 e(t), e(t)\rangle \leq |\Pi_0 e(t)|_X |e(t)|_X \leq \frac{1}{2}\left(|\Pi_0 e(t)|_X^2 + |e(t)|_X^2\right). \tag{2.9}
$$

This implies that for all $t \geq 0$, the tracking error $e(t)$ satisfies

$$
V(t) \leq e^{-c_1 t}\, V(0) + c_2 \int_0^t e^{-c_1(t-s)}\, |x_m(s)|_X^2\, ds \tag{2.10}
$$

and

$$
\int_0^t |e(t)|_X^2\, dt \leq \frac{1}{c_1}\left(V(0) + c_2 \int_0^t |x_m(s)|_X^2\, ds\right). \tag{2.11}
$$

By imposing additional conditions on the state of the reference model, we can show that the function $|e(t)|_X^2 \in L^1(0,\infty)$ with $\left|\frac{d}{dt}\langle \Pi_0 e(t), e(t)\rangle_X\right| < \infty$. If $|e(t)|_X^2$ is uniformly continuous and integrable, then it follows from Barbălat's lemma ([3, 9, 11]) that $|e(t)|_X \to 0$ as $t \to \infty$. If the operator $\Pi_0$ is coercive, i.e. $\langle \Pi_0 e(t), e(t)\rangle \geq \alpha_0 |e(t)|_X^2$, $\alpha > 0$, then we have that $|e(t)|_X$ is indeed uniformly continuous and integrable and thus we can apply Barbălat's lemma to conclude convergence of $e(t)$ to zero. While this coercivity condition is true in finite dimensional systems, it is not necessarilly satisfied by infinite dimensional systems. Since no such information of coercivity of $\Pi_0$ can be extracted from the game-theoretic Riccati equation (2.4), we must find an alternate route to show convergence of the state error to zero. Thus, assuming

$$
\int_0^\infty |x_m(t)|_X^2\, dt < \infty, \tag{2.12}
$$

i.e. $x_m(\cdot) \in L^2(0,\infty;X)$, we can show that $V(t) \to 0$ as $t \to \infty$. Moreover, we can prove that $\sqrt{\langle \Pi_0 x, x\rangle}$ defines a norm in $X$. Thus, we conclude that $e(t) \to 0$ as $t \to \infty$.

We present this in the form of the theorem below, but first we prove that $\sqrt{\langle \Pi_0 x, x\rangle}$ indeed defines a norm on $X$.

**Lemma 2.1** *The seminorm $\sqrt{\langle \Pi_0 x, x\rangle}$ is a norm on $X$.*

**Proof.** We first note that

$$\langle \Pi_0 x, x\rangle \geq \int_0^\infty |S(t)x|_X^2 \, dt \tag{2.13}$$

for $x \in X$, provided that $Q \geq I$, where $S(t)$ is the semigroup generated by $\overline{A}_0$ on $X$. Suppose there exists an element $x \in X$ such that $\langle \Pi_0 x, x\rangle = 0$. Then, from (2.13) we have that

$$\int_0^1 |S(t)x|^2 \, dt = 0$$

which implies that $S(t)x = 0$, $0 \leq t \leq 1$ since $t \to |S(t)x|$ is continuous in $[0, 1]$. Hence, $S(0)x = 0$ and therefore $x = 0$. □

**Theorem 2.1** *Given $\gamma > 0$, assume that the Riccati equation* (2.4) *has a nonnegative solution* $\Pi_0$. *Then, if the control* (1.9) *is applied to the plant* (2.1), *the tracking error $e(t)$ satisfies*

$$\int_0^\infty |e(t)|_X^2 dt \leq \frac{1}{c_1}\left(V(0) + c_2 \int_0^\infty |x_m(s)|_X^2 ds\right)$$

*and $\lim_{t\to\infty} V(t) = 0$, provided that $x_m(\cdot) \in L^2(0, \infty; X)$. Moreover,*

$$\lim_{t\to\infty} e(t) = 0$$

*in the sense that the norm $\sqrt{\langle \Pi_0 e(t), e(t)\rangle} \to 0$ as $t \to \infty$.*

**Proof.** We first show that the norm $V(t) \to 0$ as $t \to \infty$ using the estimates (2.8) - (2.11). Then, an application of Lemma 2.1 would give the desired results.

In addition to the estimate (2.11), using (2.8), we get that

$$\int_0^t |\Pi_0 e(t)|_X^2 \, dt \leq \frac{1}{c_1}\left(V(0) + c_2 \int_0^t |x_m(s)|_X^2 \, ds\right). \tag{2.14}$$

which implies that $\int_0^\infty V(t)\, dt < \infty$. Using the above (and implicitly (2.12)), we get that

$$\int_0^\infty \left|\frac{d}{dt}V(t)\right| \, dt \leq M\left[\int_0^\infty \left(|e(t)|_X^2 + |\Pi_0 e(t)|_X^2\right) \, dt + \int_0^\infty |x_m(t)|_X^2 \, dt\right], \tag{2.15}$$

for some positive constant $M > 0$. It thus follows from (2.9), (2.11), (2.12), (2.14) and (2.15) that

$$\int_0^\infty \left|\frac{d}{dt}V(t)\right| \, dt < \infty, \tag{2.16}$$

and

$$\int_0^\infty |V(t)| \, dt < \infty. \tag{2.17}$$

Equation (2.17) implies that $\liminf V(t) = 0$. But, (2.16) implies that $\lim V(t)$ exists since $V(t) = V(0) + \int_0^t \frac{d}{dt}V(s)\, ds$. Hence, $\lim V(t) = \liminf V(t) = 0$.

Using Lemma 2.1 we have that

$$\lim_{t\to\infty} e(t) = 0,$$

which concludes our proof. □

## 3 Model Reference Adaptive Control

We now proceed, as in the finite dimensional case, see [3, 9], to introduce the adaptive control law for the plant given by equation (1.1). Let us identify the space $\mathcal{L}(X,\mathbb{R}^m)$ with $X^m$. We consider the adaptive control law of the form

$$u(t) = -K(t)x_p(t) + r(t), \tag{3.1}$$

where we aim to adjust $K(t) \in X^m$ to the optimal feedback operator of the uncertain plant (1.1) and not to the optimal feedback operator for the nominal plant, (i.e. $K_m = B^*\Pi_0$). The rationale behind such a choice is the desire to improve the performance of the nominal controller $K_m$. The update law for $K(t)$ is given by

$$\beta \frac{d}{dt}K(t) = B^*\Pi_0 e(t) \otimes x_p(t) \tag{3.2}$$

where $\beta > 0$; in this case $\beta$ can be viewed as the *weighting* on the $X^m$-inner product, or as the reciprocal of the *adaptive gain*, see [3, 9]. In the adaptive law (3.1), (3.2), the parameter $\gamma$ is a design parameter that is not necessarily the *optimal* one as in the non-adaptive case, i.e. (2.4), because it is desired to study the combined effect of robust feedback and adaptation.

The Lyapunov stability argument used below requires regularity of solutions. This is guranteed under the following theorem.

**Theorem 3.1 (Ito-Powers, [5])** *Consider the abstract evolution equation in $X$*

$$\frac{d}{dt}x(t) = (A - BK(t))\,x(t) + f(t) \tag{3.3}$$

*with $x_0 \in \mathcal{D}(A)$, $f \in C^1(0,T;X)$, where we assume that $A$ is the infinitesimal generator of $C_0-$semigroup $S(t)$ on $X$ and $t \to BK(t)x$, $x \in X$ is continuously differentiable for each $x \in X$. Then, the mild solution defined by*

$$x(t) = S(t)x + \int_0^t S(t-s)(-BK(s)x(s) + f(s))\,ds \tag{3.4}$$

*is a strong (classical) solution [10], in the sense that $x(t) \in C(0,T;\mathcal{D}(A))$, $x(t) \in AC(0,T;X)$, $\frac{d}{dt}x(t) \in C(0,T;X)$.*

From Theorem 3.1 all functions that appear in the following stability arguments are strongly differentiable, assuming $x_0 \in \mathcal{D}(A)$, $r \in C^1(0,T;\mathbb{R}^m)$ and thus the calculations make sense. Then, we use the continuity of solutions $x(t) \in C(0,T;X)$ to (1.1) with respect to $(x_0, f(\cdot))$ in $X \times L^2(0,T;X)$ and the Lyapunov functions to generalize the stability results to the general initial condition $x_0 \in X$ and the reference signal $r(\cdot) \in L^2(0,T;\mathbb{R}^m)$. It follows from (1.1), (1.2) and (3.1) that

$$\frac{d}{dt}e(t) \quad = \quad (A_p - BK(t))x_p(t) + Br(t) - (A_0 - BK_m)x_m(t) - Br(t)$$

$$
\begin{aligned}
&= (A_p - BK(t) + BK_m - BK_m)x_p(t) - (A_0 - BK_m)x_m(t) \\
&= (A_p - BK_m)x_p(t) - (A_0 - BK_m)x_m(t) - B(K(t) - K_m)x_p(t) \\
&= \overline{A}_p x_p(t) - \overline{A}_p x_m(t) - B(K(t) - K_m)x_p(t) + \Delta A x_m(t) \\
&= \overline{A}_p e(t) - B\Phi(t)x_p(t) + \Delta A x_m(t) \\
&= \overline{A}_0 e(t) - B\Phi(t)x_p(t) + \Delta A e(t) + \Delta A x_m(t),
\end{aligned} \tag{3.5}
$$

where $\Phi(t) = K(t) - K_m \in X^m$ and $K_m = B^*\Pi_0$. Let us define the Lyapunov function

$$
V(t) = \langle \Pi_0 e(t), e(t)\rangle + \beta\,|\Phi(t)|^2_{X^m}. \tag{3.6}
$$

Then, we have

$$
\begin{aligned}
\frac{d}{dt}V(t) &= -\langle(\Pi_0 BB^*\Pi_0 + \frac{1}{\gamma^2}\Pi_0\Pi_0 + Q)e(t), e(t)\rangle - 2\langle \Pi_0 e(t), \Delta A e(t)\rangle \\
&\quad +2\langle \Pi_0 e(t), \Delta A x_m(t)\rangle - 2\langle \Pi_0 e(t), B\Phi(t)x_p(t)\rangle + 2\beta\langle \frac{d}{dt}\Phi(t), \Phi(t)\rangle_{X^m} \\
&\leq -\delta \min\left\{1, \frac{1}{2\gamma^2}\right\}\left(|e(t)|^2_X + |\Pi_0 e(t)|^2_X\right) + \frac{2(1-\delta)^2}{\delta}|x_m(t)|^2_X \\
&= -\kappa_1\left(|e(t)|^2_X + |\Pi_0 e(t)|^2_X\right) + c_2|x_m(t)|^2_X,
\end{aligned} \tag{3.7}
$$

for some positive constants $\kappa_1$, $c_2$, where we used the fact that

$$
\beta\,\langle \frac{d}{dt}\Phi(t), \Phi(t)\rangle_{X^m} = \langle \Pi_0 e(t), B\Phi(t)x_p(t)\rangle. \tag{3.8}
$$

This implies that

$$
V(t) + \kappa_1\int_0^t \left(|e(s)|^2_X + |\Pi_0 e(s)|^2_X\right)\,ds \leq V(0) + c_2\int_0^t |x_m(t)|^2_X\,ds \tag{3.9}
$$

for all $t > 0$. If we assume

$$
\int_0^\infty |x_m(t)|^2_X\,ds < \infty \tag{3.10}
$$

then $|\Phi(t)|_{X^m} \leq$ const. for $t \geq 0$. Moreover, we can show state error convergence to zero using the same arguments as in the proof of Theorem 2.1.

**Theorem 3.2** *Assume that for given $\gamma > 0$ and $Q \geq I$, the Riccati equation* (2.4) *has a nonnegative self-adjoint solution $\Pi_0$, that the uncertainty bound $\|\Delta A\| < \frac{1}{\gamma}$ is satisfied, and that the state of the reference model has finite energy (i.e. equation* (3.10)*), then the closed loop system for the adaptive law* (3.1) - (3.2) *satisfies*

$$
\lim_{t\to\infty} e(t) = 0,
$$

*in the sense that the norm $\sqrt{\langle \Pi_0 e(t), e(t)\rangle} \to 0$ as $t \to \infty$, and $u(\cdot) \in L^2(0,\infty;\mathbb{R}^m)$.*

**Proof.** Define

$$
W(t) = \langle \Pi_0 e(t), e(t)\rangle.
$$

Using (3.7), (3.9) and (3.10) we have that

$$\int_0^\infty \left|\frac{d}{dt}W(t)\right| \, dt \le M \left[\int_0^\infty \left(|e(t)|_X^2 + |\Pi_0 e(t)|_X^2\right) \, dt + \int_0^\infty |x_m(t)|_X^2 \, dt\right],$$

for some positive constant $M$, where we used the fact that $|\Phi(t)|_{X^m} \le \text{const}$. It thus follows from (3.9), (3.10) that

$$\int_0^\infty \left|\frac{d}{dt}W(t)\right| \, dt < \infty, \tag{3.11}$$

and that

$$\int_0^\infty |W(t)| \, dt < \infty. \tag{3.12}$$

Using arguments similar to those for Theorem 2.1, we conclude that

$$\lim_{t\to\infty} W(t) = \lim_{t\to\infty} \langle \Pi_0 e(t), e(t)\rangle = 0. \tag{3.13}$$

It thus follows from Lemma 2.1 that

$$\lim_{t\to\infty} e(t) = 0. \tag{3.14}$$

Finally, it follows from (3.9) and (3.10) that $x_p(\cdot) \in L^2(0,\infty;X)$ and $|\Phi(t)|_{X^m} < \infty$ and thus $u(\cdot) \in L^2(0,\infty;\mathbb{R}^m)$. □

It is not easy to analyze the performance of the adaptive feedback synthesis against the one of the constant feedback law $K_m = B^*\Pi_0$ in the quantitative manner. However, we can argue the qualitative improvement of the adaptive law (3.1) over the constant feedback (1.9) as follows. Assume that there exists a feedback law $K_p \in X^m$ such that

$$\|\Delta A - B\Delta K\| \le \alpha \, \|\Delta A\|, \quad 0 \le \alpha < 1. \tag{3.15}$$

where $\Delta K = K_p - K_m$. Then, as above we have

$$\begin{aligned} \frac{d}{dt}e(t) &= (A_p - BK_p)e(t) - B\widehat{\Phi}(t)x_p(t) + (\Delta A - B(K_p - K_m))x_m(t) \\ &= (A_0 - BK_m)e(t) - B\widehat{\Phi}(t)x_p(t) + (\Delta A - B\Delta K)(e(t) + x_m(t)), \end{aligned} \tag{3.16}$$

where $\widehat{\Phi}(t) = K(t) - K_p$. Let

$$\widehat{V}(t) = \langle \Pi_0 e(t), e(t)\rangle + \beta \, |\widehat{\Phi}(t)|_{X^m}^2 \tag{3.17}$$

Using the same calculation as above, we obtain

$$\begin{aligned} \frac{d}{dt}\widehat{V}(t) &= -\langle (\Pi_0 BB^*\Pi_0 + \frac{1}{\gamma^2}\Pi_0\Pi_0 + Q)e(t), e(t)\rangle \\ &\quad -2\langle \Pi_0 e(t), (\Delta A - B\Delta K)e(t)\rangle + 2\langle \Pi_0 e(t), (\Delta A - B\Delta K)x_m(t)\rangle \\ &\le -\frac{1}{\gamma^2}|\Pi_0 e(t)|_X^2 - |e(t)|_X^2 + 2|\Pi_0 e(t)|_X \left(\|\Delta A - B\Delta K\||e(t)|_X + \|\Delta A - B\Delta K\||x_m(t)|_X\right) \\ &\le -\frac{1}{\gamma^2}|\Pi_0 e(t)|_X^2 - |e(t)|_X^2 + 2\alpha|\Pi_0 e(t)|_X \left(\|\Delta A\||e(t)|_X + \|\Delta A\||x_m(t)|_X\right) \end{aligned}$$

$$
\begin{aligned}
&\le -\frac{1}{\gamma^2}|\Pi_0 e(t)|_X^2 - |e(t)|_X^2 + 2\alpha\frac{(1-\delta)}{\gamma}|\Pi_0 e(t)|_X|e(t)|_X \\
&\quad +2\alpha|\Pi_0 e(t)|_X\|\Delta A\||x_m(t)|_X \\
&\le -\alpha(1-\delta)\left(|e(t)|_X - \frac{1}{\gamma}|\Pi_0 e(t)|_X\right)^2 - (1-\alpha(1-\delta))|e(t)|_X^2 \\
&\quad -(1-\alpha(1-\delta))\frac{1}{\gamma^2}|\Pi_0 e(t)|_X^2 + 2\alpha|\Pi_0 e(t)|_X\|\Delta A\||x_m(t)|_X \\
&\le -(1-\alpha(1-\delta))|e(t)|_X^2 - (1-\alpha(1-\delta))\frac{1}{2\gamma^2}|\Pi_0 e(t)|_X^2 \\
&\quad +\frac{2\gamma^2}{(1-\alpha(1-\delta))}\alpha^2\|\Delta A\|^2|x_m(t)|_X^2 \\
&\le -(1-\alpha(1-\delta))\min\left\{1,\frac{1}{2\gamma^2}\right\}\left(|e(t)|_X^2 + |\Pi_0 e(t)|_X^2\right) \\
&\quad +\frac{2\gamma^2}{(1-\alpha(1-\delta))}\frac{(1-\delta)^2}{\gamma^2}\alpha^2|x_m(t)|_X^2 \\
&= -(1-\alpha(1-\delta))\min\left\{1,\frac{1}{2\gamma^2}\right\}\left(|e(t)|_X^2 + |\Pi_0 e(t)|_X^2\right) \\
&\quad +\frac{2(1-\delta)^2}{(1-\alpha(1-\delta))}\alpha^2|x_m(t)|_X^2 \\
&= -\tilde{\kappa}_1\left(|e(t)|_X^2 + |\Pi_0 e(t)|_X^2\right) + \tilde{c}_2|x_m(t)|_X^2, \qquad (3.18)
\end{aligned}
$$

where we used (3.15) and (2.6), (2.7). The positive constants $\tilde{\kappa}_1$ and $\tilde{c}_2$ are given by

$$\tilde{\kappa}_1 = (1-\alpha(1-\delta))\min\left\{1,\frac{1}{2\gamma^2}\right\}, \quad \text{and} \quad \tilde{c}_2 = \frac{2(1-\delta)^2}{(1-\alpha(1-\delta))}\alpha^2.$$

The equivalent constants from equation (2.8) are given (as they were explicitly found in equation (3.7), or simply using the above with $\alpha \equiv 1$) by

$$\kappa_1 = \delta\min\left\{1,\frac{1}{2\gamma^2}\right\}, \quad \text{and} \quad c_2 = \frac{2(1-\delta)^2}{\delta}.$$

Using the fact that $0 < \delta \le 1$ (from (2.7)) and that $0 \le \alpha < 1$ (from (3.15)) along with the inequality $\delta \le 1-\alpha(1-\delta)$, we can show that $\tilde{\kappa}_1 \ge \kappa_1$ and $\tilde{c}_2 < c_2$. Hence, it would mean better tracking and asymptotic behavior of the adaptive law (3.1) comparing to the constant feedback synthesis control law (1.9).

**Remark 3.1** *In regard to* Theorem 3.2, *the existence of the nonnegative solution for the Riccati equation* (2.4) *implies that the pair* $(A_0, B)$ *is stabilizable.*

**Remark 3.2** *Concerning possible limit of* $K(t)$, *a more precise condition than* (3.15) *can be assumed. According to our energy estimate* (3.18), *the best* $K_p$ *can be chosen such that*

$$(\Delta A - B\Delta K)^*\Pi_0 + \Pi_0(\Delta A - B\Delta K) \le \rho I$$

*is satisfied for the least* $\rho \ge 0$. *In the event that the matching condition* (1.7) *is satisfied, we then have* $\rho = 0$.

**Remark 3.3** *For the finite dimensional case with the matching condition* (1.7) *satisfied and with persistence of excitation, [3, 9, 15], we have $K(t) \to K$. But,* Theorem 3.2 *only shows that $K(t)$ is bounded, which implies that $K(t)$ converges to some $K_\infty$ subsequentially, i.e. there exists a sequence $\{t_n\}$ such that $K(t_n) \to K_\infty$ as $n \to \infty$ and $t_n \to \infty$.*

*So, a natural question to ask is what could be a set of subsequential limit of $K(t)$ as $t \to \infty$? According to the energy estimate* (3.18)*, $\frac{d}{dt}V(t)$ is more negative if $K(t) = K_p$ provides the least upper bound for the following inequality*

$$(\Delta A - B\Delta K)^* \Pi + \Pi_0 (\Delta A - B\Delta K) \leq \rho I, \tag{3.19}$$

*for $\rho \geq 0$. These heuristic observations make sense in the following argument. Suppose the matching condition* (1.7) *is satisfied and* (3.19) *is satisfied with $\rho = 0$ and $K = \overline{K}$ where $A_p - B\overline{K} = A_m$ in* (1.7)*. This matches with the fact that the asymptotic limit of $K(t)$ is $\overline{K}$ if it converges.*

# 4 Well-Posedness

In this section we establish the well-posedness of the adaptive law (3.1), i.e., the existence of solutions to the closed loop (2.1), (3.1), (3.2) and (3.5).

**Theorem 4.1** *For arbitrary initial condition $(x_0, K(0)) \in X \times X^m$, there exists a $\tau > 0$ such that the system of equations* (2.1), (3.1) (3.2) *and* (3.5) *has a solution $(x(t), K(t))$ in $C(0, \tau, X) \times AC(0, \tau; X^m)$ with $t \to K(t)x$ being continuous for each $x \in X$. Moreover, if* (2.4) *is satisfied then there exists a unique global solution.*

**Proof:** The proof is based on the Banach fixed-point theorem. Let $Y = C(0, \tau; X^m)$ and define the mapping $\Psi$ on $Y$ by $\widehat{K} = \Psi(K)$ where

$$x(t) = S(t)x_0 - \int_0^t S(t-s)\,(BK(s)x(s) - Br(s))\,ds \tag{4.1}$$

and

$$\beta\,\widehat{K}(t) = \beta\,K(0) + \int_0^t (B^*\Pi_0 e(s) \otimes x(s))\,ds, \tag{4.2}$$

with $e(t) = x(t) - x_m(t)$. Let $\mathcal{S}$ be the closed subset in $Y$ defined by

$$\mathcal{S} = \left\{ \max_{t\in[0,\tau]} |K(t)|_{X^m} \leq \alpha \right\}$$

Then, we prove

(i) that for $\alpha \geq |K(0)|_{X^m}$ there exists a $\tau > 0$ such that $\Psi : \mathcal{S} \to \mathcal{S}$

(ii) and that $\Psi$ is a contraction,

$$|\Psi(K_1) - \Psi(K_2)|_Y \leq \rho |K_1 - K_2|_Y, \quad K_1, K_2 \in \mathcal{S}, \quad 0 < \rho < 1. \tag{4.3}$$

Proof of (i): Let $\|S(t)\| \le Me^{\omega t}$ where $S(t)$ is the semigroup generated by $A_p$. Then, from (4.1)

$$\begin{aligned}
|x(t)|_X &\le Me^{\omega t}|x_0|_X + \int_0^t Me^{\omega(t-s)}|B|\left(\alpha\,|x(s)|_X + |r(s)|_{R^m}\right)\,ds \\
&\le \tilde{M}\left(|x_0|_X + |B|\int_0^t |r(s)|_{R^m}\,ds\right) + \alpha\tilde{M}|B|\int_0^t |x(s)|_X ds \\
&\le c_1 + \alpha c_2 \int_0^t |x(s)|_X\,ds.
\end{aligned}$$

where $\tilde{M} = Me^{\omega\tau}$ and thus $Me^{\omega t} \le \tilde{M}$ for $t \in [0,\tau]$,

$$c_1 = \tilde{M}\left(|x_0|_X + |B|\int_0^\tau |r(s)|_{R^m}\,ds\right), \quad c_2 = \tilde{M}|B|.$$

By Gronwall's inequality, we have

$$|x(t)|_X \le c_1 e^{\alpha c_2 \tau}, \qquad 0 \le t \le \tau. \tag{4.4}$$

From (4.2)

$$\begin{aligned}
\beta\,|\widehat{K}(t)|_{X^m} &\le \beta\,|K(0)|_{X^m} + c\int_0^t |e(s)|_X |x(s)|_X\,ds \\
&\le \beta\,|K(0)|_{X^m} + c\tau\left(c_3 + c_1 e^{\alpha c_2\tau}\right)c_1 e^{\alpha c_2 \tau} \equiv \mathbf{F}(\tau).
\end{aligned}$$

where $c = |B^*\Pi_0|_{X^m}$ and we assumed that $|x_m(t)|_X \le c_3$ on $[0,\tau]$. Since $\tau \to \mathbf{F}(\tau)$ is continuous and monotonically increasing, it follows that for $\alpha \ge |K(0)|_{X^m}$, there exists a $\tau_0 > 0$ such that

$$\mathbf{F}(\tau_0) \le \alpha\beta,$$

and thus $\Psi : \mathcal{S} \to \mathcal{S}$ with $\tau = \tau_0$.

Proof of (ii): Let $\widehat{K}_i = \Psi(K_i)$ for $K_i \in \mathcal{S}$, $i = 1,2$. It follows from (4.1) that if we define $\xi(t) = x_1(t) - x_2(t)$, then

$$\xi(t) = -\int_0^t S(t-s)B\left[\left(K_1(s) - K_2(s)\right)x_1(s) + K_2(s)\xi(s)\right]\,ds$$

Thus, we have as above

$$|\xi(t)|_X \le c_2 e^{\alpha c_2 \tau}\int_0^t |K_1(s) - K_2(s)|_{X^m}|x_1(s)|_X\,ds \le c_2 c_4 \tau e^{\alpha c_2\tau}\,|K_1 - K_2|_Y,$$

where we assumed that $|x_1(t)|_X \le c_4$ on $[0,\tau]$. Hence, using (4.2) and the above we have

$$\begin{aligned}
\beta\,|\widehat{K}_1(t) - \widehat{K}_2(t)|_{X^m} &\le c\int_0^t \left(|x_1(s) - x_2(s)|_X|x_2(s)|_X + |e_1(s)|_X|x_1(s) - x_2(s)|_X\right)ds \\
&\le cc_2c_4\tau^2 e^{\alpha c_2\tau}[c_4 + (c_4 + c_3)]|K_1 - K_2|_Y.
\end{aligned}$$

We may choose $0 < \tau \le \tau_0$ such that $\rho = \frac{1}{\beta}cc_2c_4\tau^2 e^{\alpha c_2\tau}[c_4 + (c_4 + c_3)] < 1$ and thus (4.3) is satisfied. By the Banach fixed point theorem, there exits a unique fixed point $K(\cdot)$ in $\mathcal{S}$ which defines a local solution to (2.1), (3.1), (3.2) and (3.5).

Suppose (2.6) is satisfied, then it follows from (3.9) that

$$|K(t)|_{X^m} \le |K(0)|_{X^m} + \text{const}\left(|x_0|_X + |K(0) - K_m|_{X^m} + \int_0^t |x_m(s)|_X^2\, ds\right), \quad t \in [0, \tau]$$

which implies that a locally defined solution $K(\cdot)$ is a unique global solution. □

# 5 Examples and Numerical Results

Here we present two examples to demonstrate the feasibility of our proposed adaptive scheme.

**Remark 5.1** *The examples presented below cannot be treated within our theoretical framework in Sections 3 and 4, because $\Delta A \notin \mathcal{L}(X, X)$. However, the plant generator in these two examples generates an analytic semigroup in $X$. Thus, we can restate* Theorem 3.2 *for the case of analytic semigroups as follows.*

**Theorem 5.1** *We assume that $A_0$ and $A_p$ generate an analytic semigroup on $X$. Furthermore we assume that*

*(A1)* $(\Delta A)^*\Pi_0 + \Pi_0\Delta A + Q > \mu I$, *for some* $\mu > 0$

*(A2)* $\Delta A x_m(\cdot) \in L^2_{loc}(0, \infty; X)$.

*Then the closed loop system is well posed. Moreover, if $\Delta A x_m(\cdot) \in L^2(0, \infty; X)$, then*

$$\lim_{t\to\infty} e(t) = 0,$$

*and $u(\cdot) \in L^2(0, \infty; \mathbb{R}^m)$.*

**Proof:** It follows from (3.5) - (3.9)

$$\begin{aligned}\frac{d}{dt}V(t) &= -2\langle \Delta A e(t), \Pi_0 e(t)\rangle - \langle Q e(t), e(t)\rangle - \langle (\Pi_0 B B^* \Pi_0 + \frac{1}{\gamma^2}\Pi_0\Pi_0)e(t), e(t)\rangle \\ &\quad +2\langle \Pi_0 e(t), \Delta A x_m(t)\rangle.\end{aligned}$$

Thus, from $(A1)$

$$\frac{d}{dt}V(t) \le -\min\left\{\mu, \frac{1}{2\gamma^2}\right\}\left(|e(t)|_X^2 + |\Pi_0 e(t)|^2|_X\right) + 2\gamma^2|\Delta A x_m(t)|_X^2,$$

which implies that

$$V(t) + \min\left\{\mu, \frac{1}{2\gamma^2}\right\}\int_0^t \left(|e(\tau)|_X^2 + |\Pi_0 e(\tau)|_X^2\right)\, d\tau \le V(0) + 2\gamma^2\int_0^t |\Delta A x_m(\tau)|_X^2\, d\tau.$$

Using $(A2)$ and similar arguments as in the proof of Theorem 3.2, we can show that

$$\int_0^\infty |e(\tau)|_X^2\, d\tau < \infty$$

and $\sqrt{\langle \Pi_0 e(t), e(t)\rangle} \to 0$ as $t \to \infty$.

**Remark 5.2** *A detailed discussion concerning assumption* $(A1)$ *will appear in a forthcoming paper. In finite dimensional systems it resembles the definition of quadratic stability, [8]. The second assumption of* Theorem 5.1 *is satisfied if* $x_m(0) \in \mathcal{D}(A_p)$ *and* $r(\cdot) \in L^2_{loc}(0,\infty;\mathbb{R}^m)$.

**Example 1:** We consider the heat diffusion equation in the interval $[0,1]$ given by

$$\frac{\partial x_p(t,\xi)}{\partial t} = a_p \frac{\partial^2 x_p(t,\xi)}{\partial \xi^2} + b_p \frac{\partial x_p(t,\xi)}{\partial \xi} + c_p x_p(t,\xi) + B(\xi)u(t),$$

with boundary and initial conditions

$$x_p(t,0) = 0 = x_p(t,1), \qquad x_p(0,\cdot) = x_0 \in L^2(0,1).$$

In this case we have that the operator $A_p$ is given by

$$A_p\varphi(\xi) = \frac{d}{d\xi}\left(a_p q(\xi)\frac{d\varphi(\xi)}{d\xi}\right) + b_p \frac{d\varphi(\xi)}{d\xi} + c_p\varphi(\xi),$$

with $\mathcal{D}(A_p) = H^2(0,1) \cap H^1_0(0,1)$, where we choose

$$q(\xi) = 1.0 - \frac{1}{2}\sin\left[2\pi(\xi - 0.25)\right], \quad 0 \le \xi \le 1.$$

The nominal plant operator $A_0$ is chosen to be

$$A_0\varphi(\xi) = a_0 \frac{d^2\varphi(\xi)}{d\xi^2} + c_0\varphi(\xi).$$

In our calculations we have chosen the following numerical values for the coefficients of $A_p$ and $A_0$ to be

$$\begin{aligned} a_p &= 1.5\times 10^{-2} \\ b_p &= 1.0\times 10^{-2} \\ c_p &= 2.0\times 10^{-2} \end{aligned} \qquad \text{and} \qquad \begin{aligned} a_0 &= 0.5\times 10^{-2} \\ c_0 &= 2.0\times 10^{-3}. \end{aligned}$$

The input operator $B$ is taken to be

$$B(\xi) = \chi_{[0.3,0.7]}(\xi), \quad 0 \le \xi \le 1,$$

and the reference signal is given by

$$r(t) = \left[3.0 + 0.1\sin\left(\frac{\pi t}{50}\right)\right] e^{-0.01t}.$$

All computations are carried out by a numerical approximation method using finite element methods with the linear spline elements and the Fehlberg fourth-fifth order Runge-Kutta method for time integration. The numerical implementation of the proposed adaptive law and its finite dimensional approximation and convergence proofs will appear in greater detail in a forthcoming paper.

In Figure 1 we plot the state error versus time and we see that the error converges to zero. For comparison, two cases are presented which represent the robust nonadaptive case (dashed line) and the adaptive case (solid line). We can observe that the state error using the adaptive law (3.1)

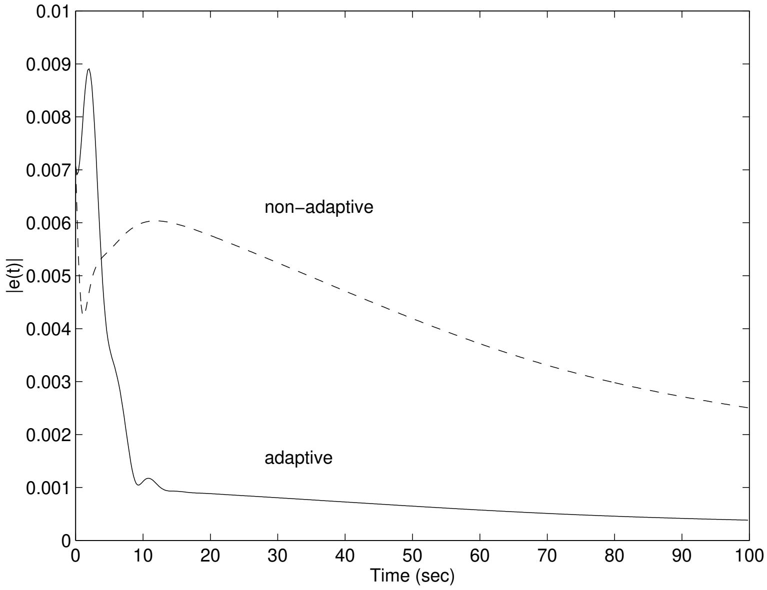


Figure 1: State error, $e(t) = x_p(t) - x_m(t)$: adaptive and non-adaptive case.

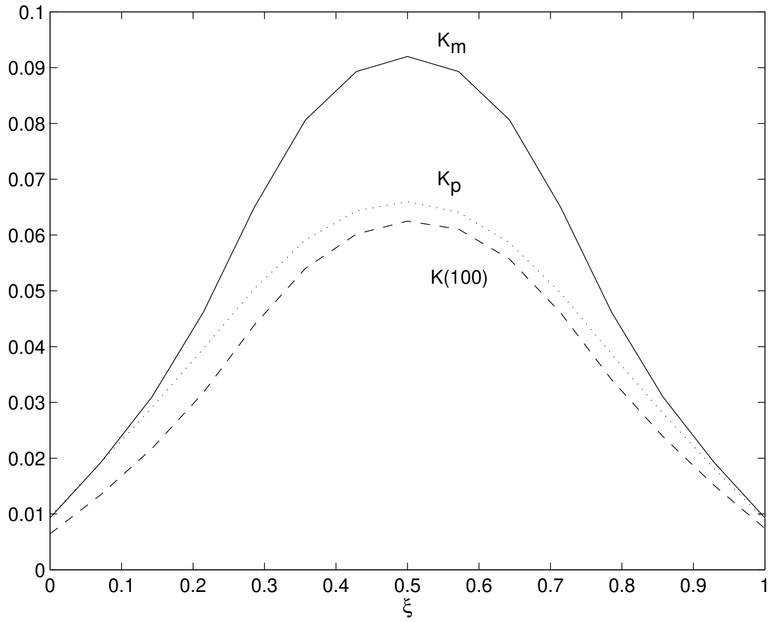


Figure 2: Nominal gain $K_m$, plant gain $K_p$ and estimated gain $K(100)$.

converges to zero faster that the non-adaptive case where the constant feedback law (1.9) was used. This agrees with the theoretical estimates of §2 and §3 concerning the constants $\kappa_1$ and $\tilde{\kappa}_1$ that are related to the tracking of the state error. Our calculations were carried out by choosing the initial values of the plant and reference model states and the initial guess of the estimate $K(t)$ at $t = 0$ as $x_p(0, \xi) = 1.0 \times 10^{-2} \sin(\pi\xi)$, $x_m(0, \xi) = 0$ and $K(0) \equiv 0$, respectively.

In Figure 2, we plot the nominal feedback gain $K_m$ (solid line), and the final value of the estimate $K(100)$ (dashed line). In addition, we plot the feedback gain $K_p$ (dotted line) which is the feedback gain that corresponds to the "optimal" gain based on the actual plant operator $A_p$, i.e. $K_p = B^* \Pi_p$ where $\Pi_p$ solves

$$\left( A_p^* \Pi_p + \Pi_p A_p - \Pi_p \left( BB^* - \frac{1}{\gamma^2} I \right) \Pi_p + Q \right) x = 0, \quad \forall x \in \mathcal{D}(A_p).$$

This shows that the adaptive law $K(t)$ is adjusted to *some* optimal feedback gain for the uncertain plant *but* not to the nominal gain $K_m$ or the optimal gain $K_p$.

**Example 2:** As a second example, we consider a one dimensional Euler-Bernoulli beam with Kelvin-Voigt viscoelastic damping. It is assumed that a single piezoceramic patch located in the center of the beam is used for actuation, see [2]. The underlying equation is given by

$$\frac{\partial^2 x_p(t,\xi)}{\partial t^2} + \frac{\partial^2}{\partial \xi^2}\left(EI_p \frac{\partial^2 x_p(t,\xi)}{\partial \xi^2} + c_D I_p \frac{\partial^3 x_p(t,\xi)}{\partial \xi^2 \partial t}\right) = \frac{\partial^2}{\partial \xi^2}\left(\mathcal{K}^B \chi_{[\xi_1,\xi_2]}(\xi) u(t)\right) + f(t,\xi) \qquad (5.1)$$

with boundary and initial conditions

$$x_p(t,0) = \frac{\partial x_p(t,0)}{\partial \xi} = x_p(t,l) = \frac{\partial x_p(t,l)}{\partial \xi} = 0,$$

$$x_p(0,\xi) = w_0(\xi) \quad \text{and} \quad \frac{\partial x_p(0,\xi)}{\partial t} = w_1(\xi), \quad 0 \le \xi \le l.$$

If we write (5.1) in a first order form (see [1]) on the state space $X = H_0^2(0,l) \times L^2(0,l)$, then the plant generator is given by

$$A_p = \begin{bmatrix} 0 & I \\ K_p & D_p \end{bmatrix},$$

with $\mathcal{D}(A_p) = \{(\phi,\psi) \in X \,:\, \psi \in H_0^2(0,l) \text{ and } K_p\phi + D_p\psi \in L^2(0,l)\}$, where the plant *stiffness* and *damping* operators are given by

$$K_p\varphi(\xi) = \frac{d^2}{d\xi^2}\left(EI_p \frac{d^2\varphi(\xi)}{d\xi^2}\right) \qquad D_p\varphi(\xi) = \frac{d^2}{d\xi^2}\left(c_D I_p \frac{d^3\varphi(\xi)}{d\xi^2 \partial t}\right).$$

Similarly, the nominal plant generator is given by

$$A_0 = \begin{bmatrix} 0 & I \\ K_0 & D_0 \end{bmatrix},$$

with the corresponding stiffness and damping operators given by

$$K_0\varphi(\xi) = \frac{d^2}{d\xi^2}\left(EI_0 \frac{d^2\varphi(\xi)}{d\xi^2}\right) \qquad D_0\varphi(\xi) = \frac{d^2}{d\xi^2}\left(c_D I_0 \frac{d^3\varphi(\xi)}{d\xi^2 \partial t}\right).$$

and $\mathcal{D}(A_0) = \{(\phi,\psi) \in X \,:\, \psi \in H_0^2(0,l) \text{ and } K_0\phi + D_0\psi \in L^2(0,l)\}$. We have chosen the following numerical values for the coefficients of $A_p$ and $A_0$ to be

$$\begin{array}{lcl} EI_p = 25 & & EI_0 = 20 \\ c_D I_p = 1.0 \times 10^{-3} & \text{and} & c_D I_0 = 2.0 \times 10^{-3}. \end{array}$$

The (unbounded) input operator $B \in \mathcal{L}(\mathbb{R}, H^{-2}(0,l))$ is given by

$$Bu(t) = \frac{\partial^2}{\partial \xi^2}\left(\mathcal{K}^B \chi_{[\xi_1,\xi_2]}(\xi) u(t)\right)$$

where $\chi_{[\xi_1,\xi_2]}(\xi)$ is the characteristic function on the interval $[\xi_1,\xi_2]$ for $0 \le \xi \le l = 0.6$ with $[\xi_1,\xi_2] = [0.15, 0.45]$, and the constant $\mathcal{K}^B$ is a piezoceramic constant that is given by $\mathcal{K}^B = 2.331655 \times 10^{-3}$. The reference signal $r(t)$ is given by

$$r(t) = 2000\left[5 + \sin\left(\frac{\pi t}{50}\right) + \cos\left(\frac{\pi t}{10}\right)\right].$$

In Figure 3 we plot the norm of the displacement error for the adaptive (solid line) and non-adaptive cases (dashed line). Once again we observe that the state error using the adaptive law (3.1) converges to zero faster than the non-adaptive case with the constant feedback law (1.9). The displacement feedback gains $K_m$ (solid line), $K_p$ (dotted line) and the final value $K(1)$ (dashed line) are plotted in Figure 4. It is observed, like the previous example, that the displacement feedback gain $K(t)$ converges to *some* optimal feedback gain but not to the nominal displacement gain $K_m$ or the optimal displacement gain for the uncertain plant $K_p$. The 3-D plots of the displacement errors for the adaptive and non-adaptive cases are plotted in Figures 5 & 6 respectively. Inspection of the last two figures reveals that the state error $e(t, \xi)$ converges to zero faster in the adaptive case than that of the non-adaptive case.

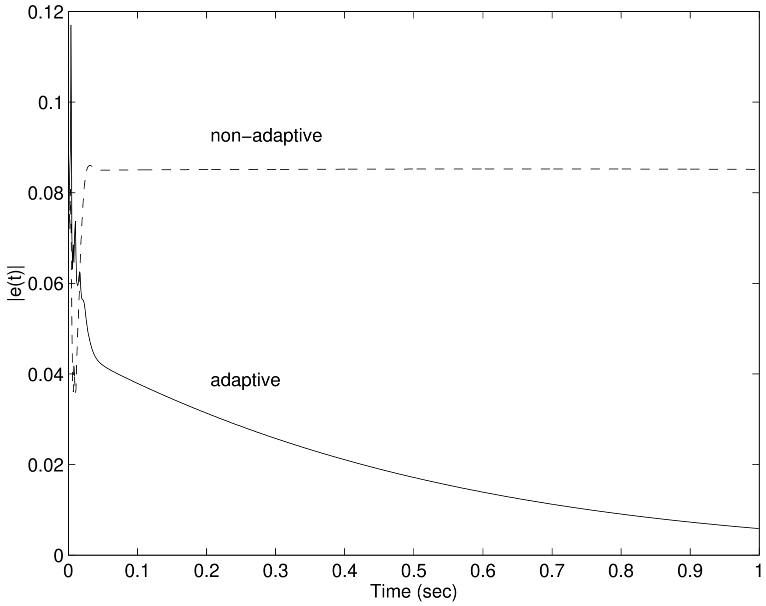


Figure 3: State error, $e(t) = x_p(t) - x_m(t)$: adaptive and non-adaptive case.

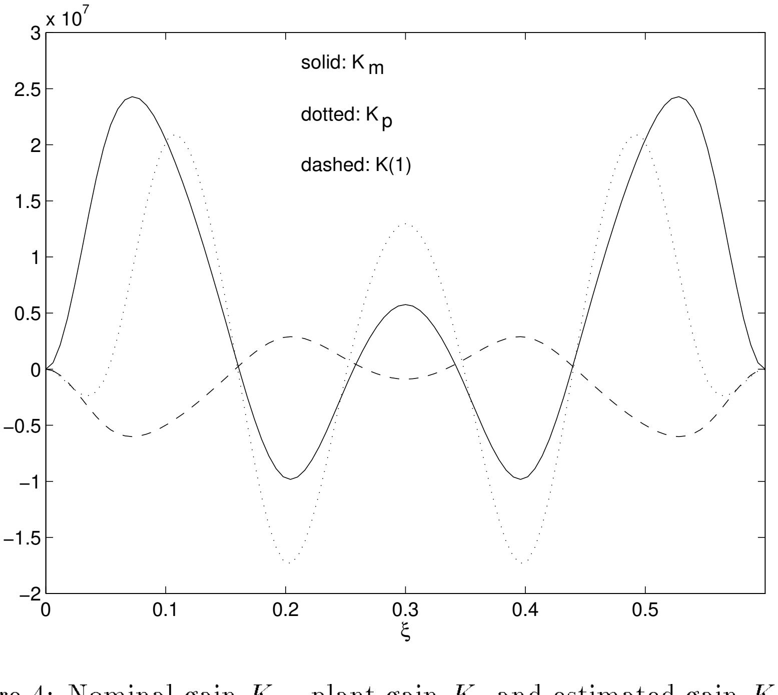


Figure 4: Nominal gain $K_m$, plant gain $K_p$ and estimated gain $K(1)$.

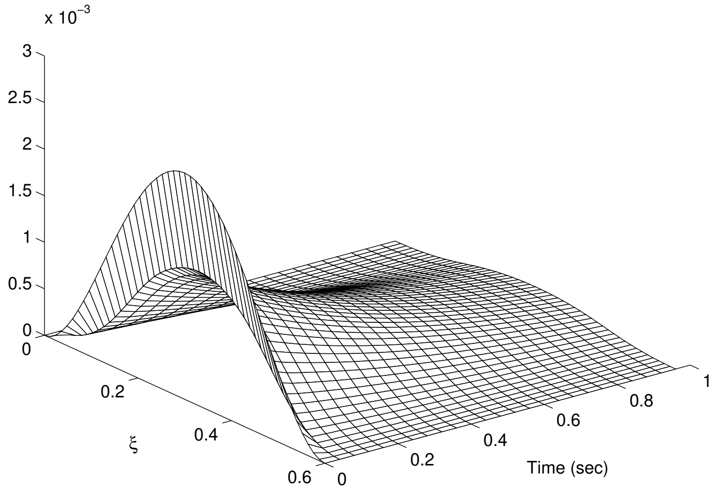


Figure 5: State error, $e(t,\xi) = x_p(t,\xi) - x_m(t,\xi)$: adaptive case.

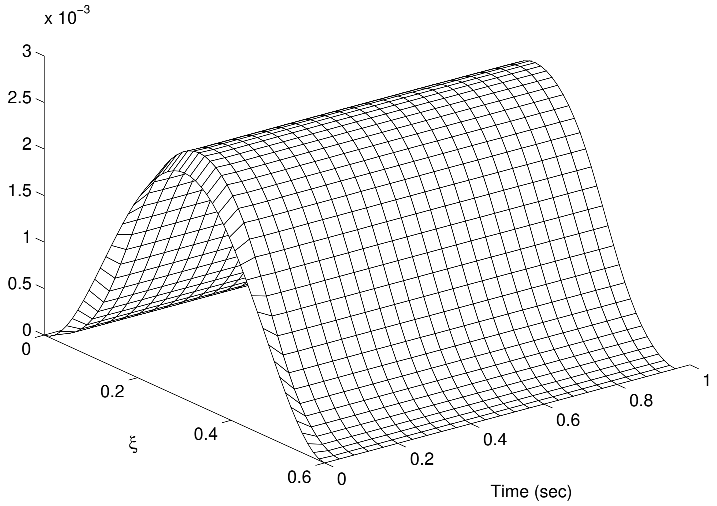


Figure 6: State error, $e(t,\xi) = x_p(t,\xi) - x_m(t,\xi)$: non-adaptive case.

## 6 Conclusions and Future Research

An adaptive control law was proposed and shown to actually enhance the robustness properties of the closed loop system. Its performance exceeds the performance of a non-adaptive robust controller. Through numerical findings, the above claim was justified and shown that even with unknown state initial conditions the adaptive controller yields better tracking. Possible extensions of the above adaptive scheme would involve systems with unbounded input operators and uncertainties in input operators as well as a more general class of system operators in order to include a wider class of infinite dimensional systems. This is the subject of our current research. The proposed adaptive control law (1.2), (1.4) assumes full state observation. We will extend our study to the construction of adaptive compensator dynamics based on a partial state observation. Moreover, the convergence properties of the approximation of the adaptive law that uses a finite dimensional approximation of the Riccati solution $\Pi_0$ and an approximated reference model $x_m(t)$ will be studied.

## References


[1] H. T. Banks and K. Ito, *A unified framework for approximation and inverse problems for distributed parameter systems*, Control Theory and Advanced Technology, 4 (1988), pp. 73–90.

[2] ——, *Approximation in LQR problems for infinite dimensional systems with unbounded input operators*, CRSC Report 94-22, Center for Research in Scientific Computation, North Carolina State University, Raleigh, NC 27695-8205, November, 1994; Journal of Mathematical Systems, Estimaton and Control, to appear.

[3] P. A. Ioannou and J. Sun, *Robust Adaptive Control*, Prentice Hall, Englewood Cliffs, NJ, 1995.

[4] K. Ito, *Well-posedness and Stability of Perturbed Linear Evolution Equations*, in Proceedings of Conference on Sensing, Identification, and Control of Flexible Structures, K. A. Morris, ed., Fields Institute Communications, The Fields Institute for Research in Mathematical Sciences,Waterloo, Canada, 1993, AMS, pp. 179–196.

[5] K. Ito and R. K. Powers, *Chandrasekhar equations for infinite dimensional systems*, SIAM J. Control and Optimization, 25 (1987), pp. 596–611.

[6] B. V. Keulen, *$H_\infty$-Control for Distributed Parameter Systems: A State-Space Approach*, Birkhäuser, Boston-Basel-Berlin, 1993.

[7] B. V. Keulen, M. Peters, and R. Curtain, *$H_\infty$ control with state feedback: The infinite-dimensional case*, Journal of Mathematical Systems, Estimation, and Control, 3 (1993), pp. 1–39.

[8] P. Khargonekar, I. R. Petersen, and K. Zhou, *Robust stabililization of uncertain linear systems: quadratic stabilizability and $H_\infty$ theory*, IEEE Trans. on Automat. Control, 35 (1991), pp. 356–361.

[9] K. S. Narendra and A. M. Annaswamy, *Stable Adaptive Systems*, Prentice Hall, Englewood Cliffs, NJ, 1989.

[10] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[11] V. M. Popov, *Hyperstability of Control Systems*, Springer-Verlag, Berlin, 1973.

[12] A. J. Pritchard and S. Townley, *Robustness of linear systems*, J. of Diff. Eqns, 77 (1989), pp. 254–286.

[13] ——, *Robustness optimization for abstract, uncertain control systems: unbounded inputs and perturbations*, in Proc. of IFAC Symposium on Distributed Parameter Systems, E. Jai, ed., Perpignan, France, July 1989, Pergamon Press, pp. 117–121.

[14] ——, *Robustness optimization for uncertain infinite-dimensional systems with unbounded inputs*, IMA J. of Math. Control and Information, 8 (1991), pp. 121–134.

[15] S. Sastry and M. Bodson, *Adaptive Control: Stability, Convergence and Robustness*, Prentice-Hall, Englewood Cliffs, NJ, 1989.

[16] S. Townley, *Robust stability radii for distributed parameter systems: A survey*, in Analysis and Optimization of Systems: State and Frequency Domain Approaches for Inifnite Dimensional Systems, R. F. Curtain, ed., Proceedings of the 10th International Conference Sophia-Antipolis, France, June 9-12 1992, Springer-Verlag, pp. 302–313.