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A MARTINGALE DECOMPOSITION THEOREM

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Let \( Z \) be a random variable with \( \mathbb{E}|Z| < \infty \) and define recursively

\[
Z_0 = EZ, \quad Z_n = \mathbb{E}^{F_n} Z,
\]

where

\[
F_n = \mathcal{B}(Z_{n-1}, I(Z \geq Z_{n-1})) \text{ for } n = 1, 2, \ldots.
\]

The \( Z_n \) sequence constitutes a martingale decomposition of \( Z \) in the sense of the following

**THEOREM.**

(i) \( Z_0, Z_1, \ldots, Z_n, \ldots, Z \) is a martingale.

(ii) The conditional distribution of \( Z_n \) given \( Z_{n-1} \) is a one or two point distribution a.s. for \( n = 1, 2, \ldots \).

(iii) \( Z_n \to Z \) a.s. as \( n \to \infty \).

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1 We shall assume that everything is defined on a basic probability space \((\Omega, F, P)\). For an arbitrary event \( A \in F \) and arbitrary random vector \( W \), we denote \( \mathbb{I}(A) \) and \( \mathcal{B}(W) \) as the indicator function (taking the value 1 on \( A \) and 0 off \( A \)) and the \( \sigma \)-field generated by \( W \) respectively. \( \mathcal{B}(W) \) will refer to the smallest \( \sigma \)-field containing \( \mathcal{B}(W) \) and the null sets of \( F \).

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Proof. It is useful to define a closely related sequence by

\[ Y_0 = EZ, \quad Y_n = E_{G_n}^n Z, \]

where

\[ G_n = B(Y_i, I(Z \geq Y_i); \ i = 0, \ldots, n-1) \text{ for } n = 1, 2, \ldots. \]

We shall show that

\[ \bar{F}_n = \bar{G}_n \]

from which we may conclude (i) (cf., [1] pg 293) and

\[ Y_n = Z_n \text{ a.s. for } n = 0, 1, \ldots. \]

To show (5), it suffices to show for \( 0 \leq j < k \) that

\[ Z \geq Y_j \text{ if, and only if, } Y_k \geq Y_j \text{ a.s.} \]

and

\[ Y_j \text{ is measurable with respect to } \bar{B}(Y_k). \]

For then

\[ \bar{G}_n = \bar{B}(Y_i, I(Z \geq Y_i); \ i = 0, \ldots, n-1) \]

\[ = \bar{B}(Y_{n-1}, I(Z \geq Y_{n-1}), Y_i, I(Y_{n-1} \geq Y_i); \ i = 0, \ldots, n-2) \text{ (cf., (7))} \]

\[ = \bar{B}(Y_{n-1}, I(Z \geq Y_{n-1})) \text{ (cf., (8))} \]

\[ = \bar{B}(Z_{n-1}, I(Z \geq Z_{n-1})) \text{ (cf., (1), (2), (3), (4))} \]

\[ = \bar{F}_n \]

(7) follows from
\[(9) \quad I(Z \geq Y_j) (Y_k - Y_j) = E^G_k I(Z \geq Z_j) (Z - Y_j) \geq 0 \text{ a.s.} \]

and

\[(10) \quad I(Z < Y_j) (Y_k - Y_j) = E^G_k I(Z < Y_j) (Z - Y_j) < 0 \text{ a.s. on } [Z < Y_j].\]

(8) is true for \(j = 0\) and if true for \(j = 0, \ldots, \alpha - 1 < k - 1\), then

\[\bar{G}_\alpha = \bar{B}(Y_1, I(Y_k \geq Y_1); i = 0, \ldots, \alpha - 1) \subset \bar{B}(Y_k)\]

and, hence, (8) is true for \(j = \alpha\).

(ii) is immediate from (1) and (2). Preliminary to showing (iii), we observe that for \(0 \leq j < k,\)

\[E|Z - Y_j| = EE^G_k (I(Z \geq Y_j) - I(Z < Y_j))(Z - Y_j)\]

\[(11) = E(I(Z \geq Y_j) - I(Z < Y_j))(Y_k - Y_j)\]

\[= E(I(Y_k \geq Y_j) - I(Y_k < Y_j))(Y_k - Y_j) = E|Y_k - Y_j|.\]

It is easily seen that \(\sup E|Y_n| \leq E|Z| < \infty\) and that the \(Y_n\) are uniformly integrable. Hence, by the martingale convergence theorem, there is a random variable \(Y_\infty\) with \(Y_n \Rightarrow Y_\infty\) a.s. and

\[E|Y_\infty - Y_n| \rightarrow 0 \text{ as } n \rightarrow \infty.\]

In view of (6) and (11), (iii) clearly follows.

**REMARKS:**

(A) It is easy to see that if one of the \(Z_k\) is decomposed as we have \(Z\) into a sequence \(Z_{kn}\) \((n = 0, 1, \ldots)\), say, then \(Z_{kn} = Z_{k \land n}\) a.s. where \(k \land n\) is the smaller of \(k\) and \(n\).
(B) This decomposition leads to an obvious procedure for embedding $Z$ into Brownian motion when $EZ = 0$ and more generally for embedding zero mean martingales. (cf., Skorokhod [3] page 163, Strassen [4] page 318) The procedure, which does not require external randomization, becomes identical to the one suggested by Lester Dubins [2]. His paper, which hints at such a decomposition, has provided some of the motivation for this note. Primarily, we have found this type of decomposibility of interest in itself.
BIBLIOGRAPHY


