LECTURES ON RANDOM MEASURES

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PREFACE

Random measure theory is a new and rapidly growing branch of probability of increasing interest both in theory and applications. Loosely speaking, it is concerned with random quantities which can only take non-negative values, such as e.g. the number of random variables in a given sequence possessing a certain property, the time spent by a random process in a certain region etc. In the special case when these quantities are integer valued, our random measures reduce to point processes, which constitute an important subclass of random measures. However, I am convinced that general random measures are potentially just as important from the point of view of applications. (See e.g. Kallenberg (1973c) for some results supporting this opinion.)

The justification of random measure theory as a separate topic lies in the overwhelming amount of important results which are specific to random measures in the sense that they do not carry over to general random processes. As a contrast, comparatively few results are specific in this sense to the subclass of point processes. Indeed, it will be clear from the development on the subsequent pages that a substantial part of point process theory, as presented e.g. in the monograph of Kerstan, Matthes and Mecke (1974), carries over to the class of general random measures. The latter may therefore be considered as a natural scope of a theory.

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The origin of these notes is a series of lectures delivered at the Statistics Department at Chapel Hill during the spring semester of 1974, where I intended to collect the (in my opinion) central facts about random measures from the scattered literature in the field. During the course of my lectures, however, I often elaborated new ideas and approaches, and as a result I found many improvements of known theorems and proofs, and also a large number of new results. The present notes should therefore be regarded both as a survey of known results and as an original contribution to the research in the area.

Here follows a summary of the contents and an indication about the nature of the most important novelties. Section 1 is introductory. After having set up the general framework, we demonstrate how the basic point processes may be defined by a simple mixing procedure. Here and in subsequent sections, the Laplace transform plays the role of a powerful general tool. In Section 2, the main purpose is to extend the classical simplicity criteria for point processes into a general theory of atomic properties for arbitrary random measures. Section 3 is devoted to the question of uniqueness of random measure distributions. In particular, it is shown that the distribution of a simple point process or diffuse random measure $\xi$ is uniquely determined by the set function $\phi_t(B) = e^{-t\xi B}$ for bounded Borel sets $B$, where $t > 0$ is arbitrary but fixed. Note that, as $t \to \infty$, $\phi_t(B) \to P(\xi B = 0)$ and our result turns formally into a well-known one for point processes. In Section 4, we derive the
corresponding convergence criteria. A novelty here is a relationship between the notions of convergence based on the weak and vague topologies respectively. The idea of tightness leads in Section 5 to a simple approach to the existence theorems of Harris and others.

Sections 6 and 7 are devoted to infinite divisibility and the limit theorems for null-arrays. From a simple representation of an infinitely divisible random measure \( \xi \), we are able in Section 6 to draw interesting conclusions about the relationship between the properties of \( \xi \) itself and its canonical spectral measure; in Section 7 we show among other things how the above-mentioned uniqueness theorem in Section 3 leads in particular cases to improved convergence criteria for null-arrays. In Section 8 on compounding and thinning of point processes, the known results for the case when the compounding variables \( \beta_n \) tend to zero in distribution are accompanied by an analogous theory for the complementary case when no subsequence tends to zero.

Symmetrically distributed random measures are examined in Section 9, and in particular we dwell on the case when the random measures are at the same time infinitely divisible. The peak of our treatment is the characterization of the class of canonical measures which can arise in the corresponding spectral representations. After a general discussion of Palm distributions in Section 10, containing improvements of known uniqueness and continuity results, we finally enter in Section 11 into the fascinating but difficult subject of characterizations by factorization and invariance properties of the Palm distributions. As for the factorization
properties, we are able to give a very simple proof of the Kerstan-
Kummer-Matthes theorem, while a modification of the original proof
yields a radically strengthened version of this result.

I hope that the technicalities of the above presentation, mainly
intended for specialists, will not discourage newcomers to the field.
Great efforts have been made to provide a smooth approach to the subject,
and wherever I have relied on "known" facts, detailed references are given.
Still some requisites may prove helpful, including some basic knowledge of
measures on locally compact spaces (to be found in Bauer (1972)), and of
convergence of probability measures (as presented in Billingsley (1968),
Chapter 1).

Finally, I would like to express my thanks to my audience in Chapel
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Göteborg in August 1974

O.K.
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1. BASIC CONCEPTS

We could confine ourselves to random measures defined on the real line $\mathbb{R}$ or on some Euclidean space $\mathbb{R}^k$, $k \in \mathbb{N} = \{1, 2, \ldots\}$, but it turns out that we can choose a much more general framework without any essential changes in notation or proofs. Hence we consider instead an abstract topological space $S$ which is assumed to be locally compact, second countable and Hausdorff, (see e.g. Simmons (1963)). By choosing a suitable metric $\rho$, we can convert $S$ into a separable and complete metric space. This possibility is often useful, but it is important to realize that our definitions or results never depend on the actual choice of $\rho$.

Let $\mathcal{B}$ be the Borel $\sigma$-algebra in $S$, i.e. the $\sigma$-algebra generated by the class of open sets, and let $\mathcal{B}$ be the ring of all bounded $\mathcal{B}$-sets. (By definition, a set $B$ is bounded whenever its closure $\overline{B}$ is compact. In general, this is stronger than metric boundedness.) Write $F = F(S)$ for the class of $\mathcal{B}$-measurable functions $f: S \to \mathbb{R}_+ = [0, \infty)$, and let $F_c = F_c(S)$ denote its subclass of continuous functions with compact support.

To simplify statements, we introduce three types of subclasses of $\mathcal{B}$.

By a DC-semiring (D for dissecting, C for covering) we mean a semi-ring $I \subset \mathcal{B}$ (see e.g. Kolmogorov and Fomin (1970) for the definition) with the property that, given any $B \in \mathcal{B}$ and $\varepsilon > 0$, there exists some finite covering of $B$ by $I$-sets of maximal diameter $< \varepsilon$ in some fixed metric.

A DC-ring is a ring with the same property. Note that such classes always exist and may be chosen countable. (We may e.g. take the ring generated
by some countable base in $\mathcal{B}$. On $\mathbb{R}$, typical DC-semirings and -rings consist of finite intervals and interval unions respectively, hence our notation. Finally, we say that a class $\mathcal{C} \subseteq \mathcal{B}$ is covering if any set $B \in \mathcal{B}$ can be covered by finitely many sets in $\mathcal{C}$.

Note that the notions of DC-semiring and -ring are purely topological and do not depend on the choice of metric $\rho$. In fact, suppose that $\rho$ and $\rho'$ are two metrizations of $S$, and let $B \in \mathcal{B}$ and $\varepsilon > 0$ be arbitrary. Choose a bounded open set $G \supset \overline{B}$. (Note that any countable base in $\mathcal{B}$ covers $B$, and that a finite subcovering exists since $\overline{B}$ is compact.) Since $G$ is compact, the identity mapping of $G$ onto itself is uniformly continuous with respect to $\rho'$ and $\rho$, and in particular we may choose some $\varepsilon' > 0$ such that $\rho(s, t) < \varepsilon$ whenever $s, t \in G$ with $\rho'(s, t) < \varepsilon'$. Thus any covering of $B$ with $\rho'$-diameters $< \varepsilon'' = \varepsilon' \wedge \rho(B, G^c)$ ($G^c = S \setminus G$ denoting the complement of $G$) is included in $G$ and hence has $\rho$-diameters $< \varepsilon$. It remains to verify that $\varepsilon'' > 0$, which follows easily from the facts that $\overline{B}$ is compact while $G$ is open.

We now introduce the class $M = M(S)$ of Borel measures $\mu$ on $(S, \mathcal{B})$ which are locally finite, i.e. take finite values on $\mathcal{B}$, and further the subclass $N = N(S)$ of $\mathbb{Z}_+^*$-valued measures ($\mathbb{Z}_+^* = \{0, 1, \ldots\}$). Let $M$ and $N$ be the smallest $\sigma$-algebras in $M$ and $N$ respectively making the mapping $\mu \mapsto \mu B$ of $M$ or $N$ into $\mathbb{R}_+$ measurable for each $B \in \mathcal{B}$. Note that, by monotone convergence, the integral $\mu f = \int f d\mu = \int_S f(s) \mu(ds)$.

1 Here and below, $\overline{B}$ denotes the closure of $B$. 

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is then automatically measurable for any \( f \in F \). More generally, assuming \( f \in F \) to be bounded with compact support and defining the measure \( f\mu \) by
\[
(f\mu)B = \int_B f(s)\mu(ds) , \ B \in \mathcal{B} ,
\]
it is seen that the mapping \( \mu \mapsto f\mu \) is measurable. In particular, the restriction \( B\mu = 1_B\mu \) of \( \mu \) to \( B \) is measurable for every \( B \in \mathcal{B} \). (As usual, \( 1_B \) denotes the indicator of the set \( B \), being equal to 1 on \( B \) and 0 elsewhere.)

**Lemma 1.1.** The \( \sigma \)-algebras \( M \) in \( M \) and \( N \) in \( N \) are both generated by the mappings \( \mu \mapsto \mu f \), \( f \in F_C \), and also by the mappings \( \mu \mapsto \mu I \), \( I \in I \), for any fixed \( DG \)-semiring \( I \subseteq \mathcal{B} \).

**Proof.** The same argument applies to both \( M \) and \( N \), so let us e.g. consider \( M \). Let \( M' \) be the \( \sigma \)-algebra in \( M \) generated by the mappings \( \mu \mapsto \mu f \), \( f \in F_C \). By the definition of \( M \), we have to prove that the mapping \( \mu \mapsto \mu B \) is \( M' \)-measurable for every \( B \in \mathcal{B} \). If \( B \) is compact, we may choose open bounded sets \( G_1, G_2, \ldots \) such that \( G_n \subseteq B \). By Urysohn's lemma (cf. Simmons (1963)) there exist functions \( f_1, f_2, \ldots \in F_C \) such that
\[
1_B \leq f_n \leq 1_{G_n} , \ n \in N ,
\]
so for any \( \mu \in M \),
\[
\mu B = \mu 1_B \leq \mu f_n \leq \mu 1_{G_n} = \mu G_n + \mu B .
\]

Hence \( \mu f_n + \mu B \), and the \( M' \)-measurability of \( \mu B \) follows from that of \( \mu f_n \) for \( n \in N \).

We next consider an arbitrary fixed compact set \( C \) and define
\[
\mathcal{D} = \{ B \in \mathcal{B} : \mu \mapsto \mu( B \cap C ) \text{ is } M' \text{-measurable} \} .
\]
Then \( \mathcal{D} \) is clearly closed under monotone limits and proper differences and it contains \( \mathcal{S} \). On the other hand, \( \mathcal{D} \) contains the class \( \mathcal{C} \) of compact sets, which is closed under finite intersections. Hence by Dynkin's monotone class theorem\(^2\) (cf. Bauer (1974)), \( \mathcal{D} \) contains the \( \sigma \)-algebra \( \sigma(\mathcal{C}) \) generated by \( \mathcal{C} \). But \( \sigma(\mathcal{C}) = \overline{\mathcal{B}} \), and so \( \mu + \mu(B \cap C) \) is \( \mathcal{M}' \)-measurable for every \( B \in \mathcal{B} \). Since \( \mathcal{C} \) was an arbitrary compact set, this proves that \( \mathcal{M}' = \mathcal{M} \).

The assertion involving \( I \) is proved by a similar argument, but instead of \( f_n \) we consider a finite covering of \( B \) by \( I \)-sets \( I_{n1}, \ldots, I_{nk} \) contained in \( G_n \). Since \( I \) is a semiring, we may take the \( I_{nj} \) to be disjoint, and then

\[
\mu B \leq \mu \cup \bigcup_{j} I_{nj} = \sum_{j} \mu I_{nj} \leq \mu G_n \downarrow \mu B,
\]

proving that \( \sum_{j} \mu I_{nj} \to \mu B \). This completes the proof.

**Lemma 1.2.** \( N \in M \).

**Proof.** Let \( I \in \mathcal{B} \) be a countable DC-semiring, and let \( \mu \) be a fixed measure in the set

\[
M = \{ \mu \in M : \mu I \in Z_+, I \in I \}.
\]

For any fixed finite union \( U \) of \( I \)-sets, we further define

\[
\mathcal{D} = \{ B \in \mathcal{B} : \mu(B \cap U) \in Z_+ \}.
\]

Then \( \mathcal{D} \) is closed under monotone limits and proper differences, and it

\(^2\) Though commonly ascribed to Dynkin, this theorem is actually due to Sierpinski, *Fund. Math.* 12 (1928).
contains $S$ since $U$ may be written as a disjoint union of $I$-sets. On the other hand, it is easily seen that $\mathcal{D}$ contains $I$, and $I$ being closed under finite intersections, it follows by Dynkin's theorem that $\mathcal{D} = \sigma(I) = \mathcal{B}$, and we obtain $\mu(B \cap U) \in Z_+$ for every $B \in \mathcal{B}$. But for given $B \in \mathcal{B}$, we may always choose $U$ such that $U \supseteq B$, so we get $\mu B \in Z_+$ for all $B \in \mathcal{B}$, proving that $\mu \in N$. Hence $M \subseteq N$, and the converse relation being obvious, we have in fact $N = M$. Since clearly $M \in M$, this completes the proof.

Let us now consider any fixed probability space $(\Omega, \mathcal{A}, \mathbb{P})$. By a random measure or point process $\xi$ on $S$ we mean any measurable mapping of $(\Omega, \mathcal{A}, \mathbb{P})$ into $(M, M)$ or $(N, N)$ respectively. By Lemma 1.2, a point process may alternatively be defined as an $N$-valued random measure. Conversely, any a.s. (almost surely w.r.t. $\mathbb{P}$) $N$-valued random measure coincides with a point process except on a $\mathbb{P}$-null set (i.e. on an $\Omega$-set of $\mathbb{P}$-measure 0). For these reasons, we shall make no difference in the sequel between point processes and a.s. $N$-valued random measures. Similarly, we shall allow a random measure to take values outside $M$ on a $\mathbb{P}$-null set.

The class of random measures is clearly closed under addition and multiplication by non-negative random variables. For a proof, it suffices by the definition of $M$ to observe that, if $\xi$ and $\eta$ are random measures while $\alpha$ and $\beta$ are $R_+$-valued random variables, the quantity $\alpha \xi B + \beta \eta B$ is a random variable for every $B \in \mathcal{B}$. Slightly less obvious is the following
Lemma 1.3. A series $\sum_j \xi_j$ of random measures on $S$ is itself a random measure iff $\sum_j \xi_j B < \infty$ a.s. for each $B \in \mathcal{B}$.

Proof. First note that, by monotone convergence, $\xi = \sum_j \xi_j$ is (everywhere on $\Omega$) $\sigma$-additive on $\mathcal{B}$ and is therefore measure valued. Next suppose that $C \subseteq B$ is a countable base in $S$. Then $\xi C < \infty$ for all $C \in C$ outside some fixed $P$-null set $A \in \mathcal{A}$, and since $C$ is covering, it follows that $\xi B < \infty$, $B \in \mathcal{B}$, except on $A$. Hence $\xi \in M$ a.s. It remains to prove that $\xi$ is measurable, and by the definition of $M$, this follows from the elementary fact that $\xi B = \sum_j \xi_j B$ is a random variable for every $B \in \mathcal{B}$.

The distribution of a random measure or point process $\xi$ is the probability measure $P\xi^{-1}$ on $(M,M)$ or $(N,N)$ defined by

$$(P\xi^{-1})M = P(\xi^{-1}M) = P(\xi \in M), \quad M \in M \text{ or } N.$$  

We further define the intensity $E\xi$ of $\xi$ by

$$(E\xi)B = E(\xi B), \quad B \in \mathcal{B},$$

where $E$ denotes the "expectation" or integral w.r.t. $P$. By Fubini's theorem, $E\xi$ is always a measure, though it need not be locally finite.

We finally define the $L$-transform $L_{\xi}$ of $\xi$ ($L$ for Laplace) by

$$L_{\xi}(f) = Ee^{-\xi f}, \quad f \in F.$$

Note in particular that, for any $k \in \mathbb{N}$ and $B_1, \ldots, B_k \in \mathcal{B}$, the function
is the elementary L-transform of the random vector \((\xi B_1, \ldots, \xi B_k)\).

One of the most important ways to define new random measures (or rather their distributions) is by mixing. For a definition, let \((\Theta, T, Q)\) be an arbitrary probability space and let \(\{\xi_\theta, \theta \in \Theta\}\) be a family of random measures which may or may not be defined on the same probability space. If \(P\{\xi_\theta \in M\}\) is a \(T\)-measurable function of \(\theta\) for every \(M \in \mathcal{M},\) we may then consider \(\theta\) as a random element in \(\Theta\) with distribution \(Q\) and form the mixture of \(P_{\xi_\theta}^{-1}\) with respect to \(Q\), thus obtaining the joint distribution \(R\) of the mixed random measure \(\xi\) and the mixing element \(\theta\). Formally, \(R\) is defined by the relation

\[
R(M \times A) = \int_A P\{\xi_\theta \in M\} \, Q(d\theta), \quad M \in \mathcal{M}, \quad A \in T.
\]

Note that \(P_{\xi_\theta}^{-1}\) is then a version of the conditional distribution of \(\xi\), given \(\theta\), and that

\[
E'f(\xi, \theta) = E'E[f(\xi_\theta, \theta) | \theta]
\]

for any measurable function \(f: M \times \Theta \to \mathbb{R}_+\). (See Loève (1963) page 359 for a complete discussion.)

The measurability requirement above is not easy to verify. Here is a convenient criterion:

**Lemma 1.4.** Let \((\Theta, T, Q)\) be an arbitrary probability space and let \(\{\xi_\theta, \theta \in \Theta\}\) be a family of random measures on \(S\). Then there exists

\[
E'[\cdot] \quad \text{denotes expectation w.r.t. } R.
\]
a unique probability measure \( R \) on \( M \times T \) satisfying (1) if \( L_{\xi_\theta}(f) \) is \( T \)-measurable in \( \theta \) for every fixed \( f \in F_c \).

**Proof.** The necessity of our condition follows by monotone convergence from simple functions. Conversely, suppose that \( L_{\xi_\theta}(f) \) is measurable in \( \theta \) for every \( f \in F_c \). Then

\[
L_{\xi_\theta}(\sum_j t_j f_j) = E \exp\left(\sum_j t_j \xi_\theta f_j\right)
\]

is measurable in \( \theta \) for any fixed \( k \in \mathbb{N}, t_1, \ldots, t_k \in R_+ \) and \( f_1, \ldots, f_k \in F_c \). Now it follows by the extended Stone-Weierstrass theorem (see Simmons (1963)) that any continuous function \( g(x_1, \ldots, x_k) \) on \( R_+^k \) which vanishes at infinity can be uniformly approximated by linear combinations of the functions \( \exp(-\sum_j t_j x_j) \) with \( t_1, \ldots, t_k \in R_+ \). Therefore \( E g(\xi_\theta f_1, \ldots, \xi_\theta f_k) \) is measurable in \( \theta \) for any such \( g \), and hence, by an easy application of Urysohn's lemma, it follows that the probability

\[
P\{\xi_\theta f_1 \leq t_1, \ldots, \xi_\theta f_k \leq t_k\}
\]

is measurable in \( \theta \) for any \( t_1, \ldots, t_k \in R_+ \).

Let us now define \( D \) as the class of all sets \( M \in M \) such that \( P\{\xi_\theta \in M\} \) is measurable. Then \( D \) is closed under proper differences and monotone limits, and it contains \( M \). On the other hand, it was shown above that \( D \) contains the class \( C \) of all sets

\[
\{u \in M: u f_1 \leq t_1, \ldots, u f_k \leq t_k\}, \quad k \in \mathbb{N}, f_1, \ldots, f_k \in F_c, \quad t_1, \ldots, t_k \in R_+, \quad
\]
and since \( C \) is closed under finite intersections, it follows by Dynkin's theorem that

\[
\mathcal{D} \supset \sigma(C) = \sigma\left\{ \{ \mu \in M : \mu f \leq t \} , \ f \in F_c , \ t \in \mathbb{R}_+ \right\} .
\]

But by Lemma 1.1, the right-hand side of this relation equals \( M \), and so \( P(\xi_\Theta \in M) \) is indeed measurable in \( \Theta \) for all \( M \in M \), as required.

We shall now introduce some important point processes and at the same time illustrate the use of L-transforms and mixing.

For any \( s \in S \), let \( \delta_s \in N \) be defined by \( \delta_s B = 1_B(s) , \ B \in \mathcal{B} \), and note that the mapping \( s \to \delta_s \) is measurable \( \mathcal{B} \to M \). Hence \( \delta_\tau \) is a point process on \( S \) whenever \( \tau \) is a random element in \( (S, \mathcal{B}) \). Writing \( \omega = P\tau^{-1} \) for the distribution of \( \tau \), it is easily seen that \( \delta_\tau \) has intensity \( E\delta_\tau = P\tau^{-1} = \omega \) and L-transform

\[
E \exp(-\delta_\tau f) = E e^{-f(\tau)} = \omega e^{-f} , \ f \in F .
\]

Next suppose that \( n \in \mathbb{Z}_+ \) and that \( \tau_1 , \ldots , \tau_n \) are independent random elements in \( S \) with common distribution \( \omega \in M \). Then any point process \( \xi \), which is distributed as \( \sum_{j=1}^{n} \delta_{\tau_j} \), is called a sample process with intensity \( n\omega \). By independence, \( \xi \) has L-transform

\[
E e^{-\xi f} = \prod_{j=1}^{n} E \exp(-\delta_{\tau_j} f) = (\omega e^{-f})^n , \ f \in F .
\]

Here we may mix with respect to \( n = \nu \), regarded as a random variable, to obtain a mixed sample process with intensity \( E\nu \cdot \omega \) and L-transform

\[
E e^{-\xi f} = E(\omega e^{-f})^\nu = \psi(\omega e^{-f}) , \ f \in F ,
\]
where
\[ \psi(s) = E s^v, \quad s \in [0,1], \]
is the (probability) generating function of \( v \).

In the particular case when \( v \) is Poissonian with mean \( a \), we get
\[ \psi(s) = e^{-a(1-s)} \quad s \in [0,1], \]
and the L-transform of \( \xi \) becomes
\[ (2) \quad E e^{-\xi f} = \exp\left(-a(1 - w e^{-f})\right) = \exp\left(-\lambda(1 - e^{-f})\right) \quad f \in F, \]
where we have put \( \lambda = aw = E\xi \). This measure \( \lambda \) is totally bounded, but we may also define point processes \( \xi \) satisfying (2) when \( \lambda \in M \) is arbitrary unbounded. For this purpose, let \( S_1, S_2, \ldots \in B \) be a disjoint partition of \( S \). Then the restrictions \( S_j \lambda \) of \( \lambda \) to \( S_j \) are totally bounded, and so there exist independent point processes \( \xi_1, \xi_2, \ldots \) on \( S \) with L-transforms
\[ (3) \quad E \exp(-\xi_j f) = \exp\left(-(S_j \lambda)(1 - e^{-f})\right) \quad f \in F. \]
Since their sum \( \xi = \sum_j \xi_j \) satisfies
\[ E\xi B = \sum_j E\xi_j B = \sum_j (S_j \lambda) B = \left(\sum_j S_j \lambda\right) B = \lambda B < \infty, \quad B \in B, \]
it follows by Lemma 1.3 that \( \xi \) is a point process. Furthermore, it is seen from (3) and the independence of \( \xi_1, \xi_2, \ldots \) that, for any \( f \in F \),
\[ E e^{-\xi f} = E \prod_j e^{-\xi_j f} = \prod_j E e^{-\xi_j f} = \prod_j \exp\left(-(S_j \lambda)(1 - e^{-f})\right) \]
\[ = \exp\left(-\sum_j (S_j \lambda)(1 - e^{-f})\right) = e^{-\lambda(1 - e^{-f})}, \]
which shows that \( \xi \) satisfies (2). (The formal calculations here are easily justified, using the fact that \( \xi_{j} f \) and \( (S_{j} \lambda)(1 - e^{-f}) \) are non-negative.)

Any point process \( \xi \) satisfying (2) will be called a Poisson process with intensity \( \lambda \). If \( \xi \) is a process of this type, it is easily seen from (2) that, for any \( k \in \mathbb{N} \) and disjoint \( B_{1}, \ldots, B_{k} \in \mathcal{B} \), the random variables \( \xi_{B_{1}}, \ldots, \xi_{B_{k}} \) are independent and Poissonian with means \( \lambda B_{1}, \ldots, \lambda B_{k} \).

Let us now consider a Poisson process \( \xi^{\alpha} \) with intensity \( \alpha \lambda \), where \( \alpha \in \mathbb{R}_{+} \) while \( \lambda \in \mathbb{M} \). By Lemma 1.4, we may then consider \( \alpha \) as a random variable and mix with respect to its distribution to obtain a mixed Poisson process \( \xi \) with L-transform

\[
E e^{-\xi f} = E \exp(-\alpha \lambda (1 - e^{-f})) = \phi(\lambda (1 - e^{-f})) , \quad f \in \mathcal{F},
\]

where

\[
\phi(t) = E e^{-\alpha t} , \quad t \geq 0 ,
\]

is the L-transform of \( \alpha \). More generally, it is possible by Lemma 1.4 to consider \( \lambda = \eta \) in (2) as a random measure, thus obtaining a Cox process directed by \( \eta \), possessing the L-transform

\[
L_{\eta}(f) = E e^{-\eta f} = E \exp(-\eta (1 - e^{-f})) = L_{\eta}(1 - e^{-f}) , \quad f \in \mathcal{F}.
\]

Let us finally consider some fixed measure \( \mu \in \mathbb{N} \), say \( \mu = \sum_{j=1}^{k} \delta_{t_{j}} \), where \( k \in \mathbb{Z}_{+} = \mathbb{Z}_{+} \cup \{ \infty \} \) and \( t_{j} \in \mathcal{S} \), \( j \leq k \). Further suppose that \( \beta, \beta_{1}, \beta_{2}, \ldots \) are independent and identically distributed \( \mathbb{R}_{+} \)-valued random variables, say with \( L_{\beta} = \phi \). Then it follows from Lemma 1.3 that
\[ \xi = \sum_j \beta_j \delta_{t_j} \] is a random measure on \( S \), and we get for \( f \in F \)

\[
E e^{-\xi f} = E \exp\left\{ - \sum_j \beta_j f(t_j) \right\} = \prod_j E \exp\left\{ -\beta_j f(t_j) \right\} = \prod_j \phi \circ f(t_j)
\]

\[
= \exp\left\{ \sum_j \log \phi \circ f(t_j) \right\} = \exp(\mu \log \phi \circ f).
\]

By Lemma 1.4, it is permissible to mix with respect to \( \mu = \eta \) when regarded as a point process, thus obtaining

\[
(5) \quad L_\xi(f) = E e^{-\xi f} = E \exp(\eta \log \phi \circ f) = L_\eta(-\log \phi \circ f), \quad f \in F.
\]

Any random measure with this mixed distribution will be called a \( \beta \)-compound of \( \eta \). In the particular case when \( \beta \) attains the values 0 and 1 only with probabilities \( 1 - p \) and \( p \) respectively, \( \xi \) will be called a \( p \)-thinning of \( \eta \). In this case, (5) takes the form

\[
(6) \quad L_\xi(f) = L_\eta\left(-\log\left(1 - p(1 - e^{-f})\right)\right), \quad f \in F.
\]

Scrutinizing the arguments used in the above constructions, it is seen that they do not depend on the nature of \( S \). Thus we may e.g. proceed as above to construct Poisson processes with arbitrary \( \sigma \)-finite intensities on any measurable space \( (S, \mathcal{B}) \).

**NOTES.** Here and below, our main source is Kallenberg (1973a). However, Lemma 1.4 is new and is essentially a strengthening of proposition 1.6.2 in Kerstan, Matthes and Mecke (1974). The present method on constructing
Poisson and Cox processes is simpler and more general than methods in the current literature.

**PROBLEMS.**

1.1. By a $\sigma$-ring $R \subseteq B$ we mean a ring which is closed under non-decreasing bounded limits. Show that, if $I$ is a DC-semiring, then $B$ is the smallest $\sigma$-ring generated by $I$. In symbols: $B = \sigma(I)$.

1.2. Show that Lemma 1.1 remains true for any semiring $I \subseteq B$ with $\sigma(I) = B$.

1.3. Let $I \subseteq B$ be an arbitrary DC-semiring. Show that Lemma 1.4 remains true with $F_c$ replaced by the class of all functions of the form $\sum_{j=1}^{k} t_j I_j$ for arbitrary $k \in \mathbb{N}$, $t_1, \ldots, t_k \in R^+$ and $I_1, \ldots, I_k \in I$.

1.4. Let $\xi$ be a Cox process on $S$ directed by $\eta$. Show that $\xi$ has independent increments, in the sense that $\xi_{B_1}, \ldots, \xi_{B_k}$ are independent for any $k \in \mathbb{N}$ and disjoint $B_1, \ldots, B_k \in B$, iff this is true for $\eta$. Prove the corresponding fact for compound point processes with $\beta \not\propto 0$. (Hint: use analytic continuation.)

2. **ATOMIC PROPERTIES**

Let $M_d$ denote the class of diffuse (or non-atomic) measures in $M$.

Every measure $\mu \in M$ may clearly be written in the form

$$\mu = \mu_d + \sum_{j=1}^{k} b_j \delta_{t_j}$$

(1)
for some $\mu_d \in M_d$, $k \in \mathbb{Z}_+ = \mathbb{Z}_+ \cup \{\infty\}$, $b_1, b_2, \ldots \in \mathbb{R}_+ = (0, \infty)$ and distinct $t_1, t_2, \ldots \in S$, and this decomposition is clearly unique apart from the order of terms. Moreover, $\mu \in \mathbb{N}$ iff $\mu_d = 0$ and $b_1, b_2, \ldots \in \mathbb{N}$.

When considering random measures, we often need a measurable decomposition.

**Lemma 2.1.** There exists a decomposition (1) of every $\mu \in M$ such that the quantities $\mu_d \in M_d$, $k \in \mathbb{Z}_+$, $b_1, b_2, \ldots \in \mathbb{R}_+$ and $t_1, t_2, \ldots \in S$ are measurable functions of $\mu$.

The measurable functions occurring in Lemma 2.1 are by no means unique, not even apart from the order of terms. However, when considering random measures $\xi$ on $S$ with $\xi_S < \infty$ a.s. (which holds automatically when $S$ is compact), we may require the atom sizes $\beta_1, \beta_2, \ldots$ to be taken in order $\beta_1 \geq \beta_2 \geq \ldots$ and the corresponding atom positions $\tau_1, \tau_2, \ldots$ to be chosen in random order within sets of equal $\beta_j$. (We shall always assume the basic probability space $(\Omega, \mathcal{F}, P)$ to be rich enough to support any such randomization.) Though the decomposition

$$\xi = \xi_d + \sum_{j=1}^\infty \beta_j \delta_{\tau_j}$$

is still not unique unless all the atom sizes are different with probability 1, we obtain in this way a unique joint distribution of the random elements occurring in (2).

For the proof of Lemma 2.1, we shall need an auxiliary result, the formulation and proof of which requires some further terminology and notation.
Given any set $B \in \mathcal{B}$, we shall say that $\{B_{nj}\} \subset \mathcal{B}$ is a null-array of partitions of $B$, if the $B_{nj}$ for fixed $n \in \mathbb{N}$ form a finite disjoint partition of $B$ and if $\max_j |B_{nj}| \to 0$, where $|\cdot|$ denotes the diameter in any fixed metrization $\rho$ of $S$. (Note that the last condition is independent of the choice of $\rho$.) For any $\mu \in \mathcal{M}$ and $\varepsilon > 0$, we define $\mu_e^* \in \mathcal{N}$ and $\mu_e' \in \mathcal{M}$ by

$$\mu_e^* B = \sum_{s \in B} 1_{\{\mu(s) \geq \varepsilon\}} (\mu), \quad B \in \mathcal{B},$$

$$\mu_e' B = \mu B - \sum_{s \in B} \mu(s) 1_{\{\mu(s) \geq \varepsilon\}} (\mu), \quad B \in \mathcal{B}.$$ 

**Lemma 2.2.** Let $\mu \in \mathcal{M}$, $\varepsilon > 0$ and $B \in \mathcal{B}$, and let $\{B_{nj}\} \subset \mathcal{B}$ be a null-array of partitions of $B$. Then

$$\lim_{n \to \infty} \sum_j 1_{\{\mu B_{nj} \geq \varepsilon\}} (\mu) = \mu_e^* B.$$ 

**Proof.** For sufficiently large $n \in \mathbb{N}$, all $\mu$-atoms in $B$ of size $\geq \varepsilon$ will lie in different partitioning sets $B_{nj}$, and we get

$$\sum_j 1_{\{\mu B_{nj} \geq \varepsilon\}} (\mu) \geq \mu_e^* B.$$

Thus it remains to prove that

$$\limsup_{n \to \infty} \sum_j 1_{\{\mu B_{nj} \geq \varepsilon\}} \leq \mu_e^* B.$$

If (2) were false, there would exist some subsequence $N' \subset \mathbb{N}$ and some numbers $j_n$, $n \in N'$, such that
Choosing arbitrary \( s_n \in B_{nj_n} \), \( n \in N' \), there exists by compactness some further subsequence \( N'' \subset N' \) and some \( s \in \overline{B} \) such that \( s_n \to s \), \( n \in N'' \), and since \( |B_{nj_n}| \to 0 \), it follows from (3) that \( \mu'_\varepsilon G \geq \varepsilon \) for every open set \( G \in \mathcal{B} \) containing \( s \). But then \( \mu'_\varepsilon(s) \geq \varepsilon \), which contradicts the definition of \( \mu'_\varepsilon \). Hence (2) must be true, and the lemma is proved.

**Proof of Lemma 2.1.** Define the set function \( \xi = \xi(\mu) \) by

\[
(4) \quad \xi(B \times [\varepsilon, \infty)) = \mu^*_\varepsilon B, \quad B \in \mathcal{B}, \quad \varepsilon > 0.
\]

By the Caratheodory extension theorem, (4) defines for each \( \mu \in \mathcal{M} \) a unique measure \( \xi \) on \( S \times R'_+ \), and it is easily seen that \( \xi \) is purely atomic with all its atoms of unit size. Furthermore, it follows from Lemmas 1.1 and 2.2 that \( \xi \) is a measurable function of \( \mu \). Suppose that the lemma is true for \( \xi \), i.e. that \( \xi = \sum_{j=1}^{k} \delta_{\sigma_j} \) for some measurable functions \( k \in Z_+ \) and \( \sigma_1, \sigma_2, \ldots \in S \times R'_+ \). Then (1) holds with \( \sigma_j \equiv (t_j, b_j) \), and since \( t_j \) and \( b_j \) are measurable functions of the pair \( (t_j, b_j) \), the measurability in (1) will follow by Lemma 1.3. We may therefore assume from now on that \( \mu_d = 0 \) and \( b_1 \equiv b_2 \equiv \ldots \equiv 1 \).

Let \( s_1, s_2, \ldots \) be an arbitrary dense sequence in \( S \), and note that \( \rho(s, s_n) = \rho(t, s_n) \), \( n \in N \), implies \( s = t \). (To see this, let \( N' \subset N \) be such that \( s_n \to s \), \( n \in N' \), and note that then
\[ \rho(s, t) \leq \rho(s, s_n) + \rho(s_n, t) = 2 \rho(s, s_n) \to 0 \quad n \in \mathbb{N} \]

We may therefore introduce a linear ordering of \( S \) by writing \( s < t \) whenever, for some \( k \in \mathbb{N} \),

\[ \rho(s, s_j) = \rho(t, s_j) \quad j = 1, \ldots, k - 1 ; \quad \rho(s, s_k) < \rho(t, s_k) . \]

For any fixed disjoint partition \( B_1, B_2, \ldots \in \mathcal{B} \) of \( S \), we now order the atom positions \( t_j \) of \( \mu \), first according to their occurrence in \( B_1, B_2, \ldots \), and then, within each \( B_n \), according to their linear order.

We have to show that \( t_1, t_2, \ldots \) are measurable when ordered in this way, and by induction it suffices to consider \( t_1 \), i.e. to prove that the set \( \{ t_1 \in B \} \) is measurable for every fixed \( B \in \mathcal{B} \). Now this set is clearly expressible by means of countable set operations in terms of the sets \( \left\{ \mu(C \cap B_1 \cap \{ s : \rho(s, s_j) < r \}) = k \right\} \) with \( C = B \) or \( B^c \) and with arbitrary \( i, j \in \mathbb{N} \), \( k \in \mathbb{Z}_+ \) and rational \( r > 0 \). Since the latter sets are measurable, this completes the proof.

We shall now establish a simple relationship between the one-dimensional distributions of a random measure \( \xi \) and the intensities of the corresponding point processes \( \xi^*_\varepsilon \), \( \varepsilon > 0 \), or more generally of \( \xi^*_\varepsilon \), \( 0 < a < b \leq \infty \).

**Theorem 2.3.** Let \( \xi \) be a random measure on \( S \), let \( B \in \mathcal{B} \) and let \( \{ B_{nj} \} \subset B \) be a null-array of partitions of \( B \). Further suppose that \( 0 < a < b \leq \infty \). Then
(5) \[ E_{\xi}^* \{ a, b \} B = \lim_{n \to \infty} \sum_j P \{ \xi B_{nj} \in [a, b) \} \]

holds provided \( E_{\xi}^* B < \infty \), while in general

(6) \[ E_{\xi}^* \{ a, b \} B \leq \liminf_{n \to \infty} \sum_j P \{ \xi B_{nj} \in [a, b) \} . \]

**Proof.** By taking differences in Lemma 2.2, we get

(7) \[ \lim_{n \to \infty} \sum_j 1 \{ \xi B_{nj} \in [a, b) \} = \xi_{\{a, b\}}^* B, \]

and hence, by Fatou's lemma,

\[ E_{\xi}^* \{ a, b \} B \leq \liminf_{n \to \infty} E \sum_j 1 \{ \xi B_{nj} \in [a, b) \} = \liminf_{n \to \infty} \sum_j P \{ \xi B_{nj} \in [a, b) \} , \]

proving (6). Since (6) implies (5) in the case \( E_{\xi}^* \{ a, b \} = \infty \), it remains to prove (5) in the case when \( E_{\xi}^* \{ a, b \} B \) and \( E_{\xi}^* a \) are both finite. But then (5) follows from (7) by dominated convergence, since we have

\[ \{ \xi B_{nj} \in [a, b) \} \subseteq \{ \xi_{\{a, b\}}^* B_{nj} \geq 1 \} \cup \{ \xi_{\{a\}}^* B_{nj} \geq a \} , \]

and hence

\[ \sum_j 1 \{ \xi B_{nj} \in [a, b) \} \leq \sum_j 1 \{ \xi_{\{a, b\}}^* B_{nj} \geq 1 \} + \sum_j 1 \{ \xi_{\{a\}}^* B_{nj} \geq a \} \]

\[ \leq \sum_j \xi_{\{a, b\}}^* B_{nj} + \sum_j a^{-1} \xi_{\{a\}}^* B_{nj} = \xi_{\{a, b\}}^* B + a^{-1} \xi_{\{a\}}^* B . \]

We next give a simple criterion ensuring \( \xi \) to have no atoms of size \( \geq \varepsilon \). Given any covering class \( C \subseteq B \) and any \( a > 0 \), we shall
say that $\xi$ is a-regular w.r.t. $C$, if there exists for every fixed $C \in C$ some array $\{C_{nj}\} \subset C$ of finite coverings of $C$ (one for each $n \in \mathbb{N}$), such that

\begin{equation}
\lim_{n \to \infty} \sum_{j} P\{\xi_{C_{nj}} \geq a\} = 0 .
\end{equation}

**Theorem 2.4.** Let $\xi$ be a random measure on $S$, let $C \subset B$ be a covering class and let $a > 0$. Then $\xi^*_a = 0$ a.s. if $\xi$ is a-regular w.r.t. $C$.

The converse is also true provided $C$ is a DC-semiring and $E\xi \in M$.

**Proof.** Suppose that $\xi$ is a-regular w.r.t. $C$, let $C \in C$ be arbitrary and let $\{C_{nj}\} \subset C$ be an array of coverings of $C$ such that (8) holds. Then

\[ P\left( \max_{s \in C} \xi(s) \geq a \right) \leq P\left( \max_{j} \xi_{C_{nj}} \geq a \right) = \bigcup_{j} P\{\xi_{C_{nj}} \geq a\} \leq \sum_{j} P\{\xi_{C_{nj}} \geq a\} , \]

and it follows from (8) that $\xi^*_a C = 0$ a.s. Since $C$ is covering, this implies $\xi^*_a = 0$ a.s.

Conversely, let $E\xi \in M$ and $\xi^*_a = 0$ a.s., and suppose that $C$ is a DC-semiring. For every $C \in C$, we may then choose a null-array $\{C_{nj}\} \subset C$ of partitions of $C$, and we get by Theorem 2.3

\[ \lim_{n \to \infty} \sum_{j} P\{\xi_{C_{nj}} \geq a\} \to E\xi^*_a C = 0 , \]

proving a-regularity of $\xi$ w.r.t. $C$. 
It is often useful to replace the global regularity condition (8) by a local one. For a first step, note that if $C$ is a DC-semiring and if $\lambda \in M$ is arbitrary, then (8) is implied by

$$\lim_{\epsilon \to 0} \sup \left\{ \frac{1}{\lambda C} P\{\xi C \geq a\}: C \in C, C \subseteq C_0, |C| < \epsilon \right\} = 0, C_0 \in C.$$  

We shall show that the uniform convergence in (9) may essentially be replaced by pointwise convergence.

**THEOREM 2.5.** Let $\xi$ be a random measure on $S$, let $\lambda \in M$ and let $a > 0$. Then $\xi_a^* = 0$ a.s. if

$$\lim_{\epsilon \to 0} \sup \left\{ \frac{1}{\lambda I} P\{\xi I \geq a\}: I \in I, s \in \overline{I}, |I| < \epsilon \right\} = 0, s \in S,$$

for some DC-semiring $I \subseteq B$, and also if

$$\lim_{\epsilon \to 0} \sup \left\{ \frac{1}{\lambda C} P\{\xi C \geq a\}: C \in C, s \in C, |C| < \epsilon \right\} = 0, s \in S,$$

where $C$ is the class of closed or the class of open $B$-sets.

**PROOF.** Suppose that $P(\xi_a^* \neq 0) > 0$, and let $I \subseteq B$ be a DC-semiring. Since $I$ is covering, we may choose some $I \subseteq I$ such that $P(\xi_a^* I > 0) > 0$.

We may further choose some null-array $\{I_{nj}\} \subseteq I$ of partitions of $I$, and we may assume $\{I_{nj}\}$ to be nested in the sense that it proceeds by successive refinements. Then

$$P(\xi_a^* I > 0) = P \bigcup_{j} \{ \xi_{aL_j}^* I_j > 0 \} \leq \sum_{j} P(\xi_{aL_j}^* I_j > 0),$$

and it follows that
(13) \[ \max_{j} \frac{1}{\lambda I_{1j}} P(\xi^*_a I_{1j} > 0) \geq \frac{1}{\lambda I} P(\xi^*_a I > 0), \]

provided 0/0 is interpreted as 0. In fact, this follows trivially from (12) if \( \lambda I = 0 \), while if \( \lambda I > 0 \), insertion of the converse of (13) into (12) yields the contradiction

\[ P(\xi^*_a I > 0) < \frac{1}{\lambda I} P(\xi^*_a I > 0) \sum_j \lambda I_{1j} = P(\xi^*_a I > 0). \]

By (13) there exists an index \( j_1 \) such that \( I_1 = I_{1j_1} \) satisfies

\[ \frac{1}{\lambda I_1} P(\xi^*_a I_1 > 0) \geq \frac{1}{\lambda I} P(\xi^*_a I > 0), \]

and proceeding inductively, we may construct a sequence \( I_1 \supset I_2 \supset \ldots \)

in \( I \) with \( |I_n| \to 0 \) and such that

\[ \frac{1}{\lambda I_n} P(\xi I_n \geq a) \geq \frac{1}{\lambda I_n} P(\xi^*_a I_n > 0) \geq \frac{1}{\lambda I} P(\xi^*_a I > 0) > 0, \quad n \in \mathbb{N}. \]

Now \( \{I_n\} \) has a non-empty intersection \( \{s\} \), and the last relation shows that (10) is violated at \( s \).

Let us now introduce the class

\[ I = \{ B \in \mathcal{B} : \lambda \emptyset B = 0 \}, \]

where \( \emptyset B = \overline{B} \cap B^C \) denotes the boundary of \( B \). If we can show that \( I \) is a DC-ring, it follows that we may apply the first assertion to \( I \).

Now the ring property of \( I \) follows from the elementary relations

\[ \emptyset (B \cup C) = \emptyset B \cup \emptyset C, \quad \emptyset (B \cap C) = \emptyset B \cup \emptyset C, \quad \emptyset (B^C) = \emptyset B. \]

To see that \( I \) has the DC-property, it suffices to show that, for any fixed \( t \in S \), the ball
\[ S(t,r) = \{ s \in S : \rho(s,t) < r \} \]

belongs to \( I \) for arbitrarily small \( r > 0 \). For a proof, note first that \( S(t,r) \in \mathcal{B} \) for small \( r \), say for \( r < \varepsilon \), since \( S \) is locally compact. Next verify that

\[ \exists S(t,r) \subset \{ s \in S : \rho(s,t) = r \} , \]

and conclude that \( \lambda \exists S(t,r) = 0 \) and hence \( S(t,r) \in I \) for all but countably many \( r < \varepsilon \).

With the above choice of \( I \) we have \( \lambda I = \lambda \overline{I} \), \( I \in I \), and hence

\[ \frac{1}{\lambda \overline{I}} \mathbb{P}(\xi I \geq a) \geq \frac{1}{\lambda I} \mathbb{P}(\xi I \geq a) , \ I \in I , \]

which proves that (11) implies (10) when \( \mathcal{C} \) is the class of closed sets. In the case of open sets, let \( \varepsilon > 0 \) be arbitrary and approximate each \( I \in I \) by an open bounded set \( G \supset I \) such that \( |G| < 2|I| + \varepsilon \) and

\[ \lambda G \leq \begin{cases} 2 \lambda I , \lambda I > 0 , \\ \mathbb{P}(\xi I \geq a) , \lambda I = 0 . \end{cases} \]

Then

\[ \frac{1}{\lambda G} \mathbb{P}(\xi G \geq a) \geq \begin{cases} \frac{1}{2\lambda I} \mathbb{P}(\xi I \geq a) , \lambda I > 0 , \\ 1 , \lambda I = 0 , \end{cases} \]

and again (11) implies (10).
From Theorem 2.4 it follows in particular that, given any covering class \( C \), a random measure \( \xi \) is a.s. diffuse provided \( \xi \) is \( \alpha \)-regular w.r.t. \( C \) for every \( \alpha > 0 \). In this case we shall say, simply, that \( \xi \) is regular w.r.t. \( C \). Other diffuseness criteria are implicit in Theorem 2.5.

Another case of particular interest is that of point processes. If \( \xi \) is a point process, we write \( \xi^* = \xi^* \) and say that \( \xi \) is simple if \( \xi^* = \xi \) a.s. Since for point processes \( \xi^*_I = 0 \) a.s., (5) holds generally with \( \alpha = 1 \). Furthermore, we obtain convenient simplicity criteria by taking \( \alpha = 2 \) in Theorems 2.4 and 2.5.

We conclude this section with a simple but useful lemma.

**Lemma 2.6.** Let \( \xi \) be a point process on \( S \) and let \( I \subset B \) be a DC-semiring. Then \( \xi \) is simple iff

\[
P(\xi I > 1) \leq P(\xi^* I > 1), \quad I \in I.
\]

**Proof.** The necessity of (14) is obvious. Converesely, suppose that (14) holds, and note that then

\[
0 \leq P(\xi I > \xi^* I = 1) = P(\xi I > 1) - P(\xi^* I > 1) \leq 0, \quad I \in I.
\]

Letting \( I \in I \) be fixed and choosing a null-array \( \{I_{nj}\} \in I \) of partitions of \( I \), we obtain by (15) and Lemma 2.2
\[ P(\xi I > \xi^* I) = \bigcup_j \{\xi I_{nj} > \xi^* I_{nj}\} = \bigcup_j \{\xi I_{nj} > \xi^* I_{nj} > 1\} \leq P \bigcup_j \{\xi^* I_{nj} > 1\} = P \left( \sum_j 1_{\{\xi^* I_{nj} > 1\}} > 0 \right) \to P(\xi^* I > 0) = 0, \]

proving that \((\xi - \xi^*)I = 0\) a.s. Since \(I\) is covering, it follows that \(\xi - \xi^* = 0\) a.s., as asserted.

**NOTES.** In the point process case, Theorem 2.3 with \(a = 1\) and \(b = \infty\) is known as Korolyuk's theorem, while the converse part of Theorem 2.4 with \(a = 2\) is known as Dobrushin's lemma, see Leadbetter (1972) and Belyaev (1969). The present generalizations to random measures are new, although a first step towards Theorem 2.4 was given in Kallenberg (1975) while the ideas behind Theorem 2.5 are implicit in Kallenberg (1973d). See also Gikhman and Skorokhod (1969) for a version for random processes in \(D[0,1]\) of the direct part of Theorem 2.4. We further point out that our conditions (10) and (11) generalize the notion of analytic orderliness in Daley (1974), while (9) generalizes his notion of uniform analytic orderliness. Finally, we mention that Lemma 2.1 was given without proof in Kallenberg (1973b) while Lemma 2.6 is taken from Kallenberg (1973a).

**PROBLEMS.**

2.1. The measurability of \(\mu + \mu^*_a\), \(a > 0\), is an immediate consequence of Lemma 2.1. Give a direct argument, proving the measurability of \(\mu + \mu'_a\), \(a > 0\), and conclude that \(\mu + \mu_d\) is measurable. (All these facts are of course contained in Lemma 2.1.)

2.2. Prove that the events \(\{\xi = \xi_d\}\) and (in the point process case) \(\{\xi = \xi^*\}\) are measurable. (Hint: consider the differences.)
2.3. Prove that the notion of null-array of partitions and the conditions (10) and (11) are independent of the choice of metric $\rho$.

2.4. Let $\xi$ be a Cox process directed by $\eta$. Show that $\xi$ is simple iff $\eta$ is diffuse. Prove also that, if $\xi$ is a $p$-thinning of $\eta$ ($p>0$), then $\xi$ and $\eta$ are simultaneously simple. (Hint: consider first the case of non-random $\eta$.)

2.5. Show that Lemma 2.6 follows easily from Theorem 2.3 in the particular case when $E\xi_2 \in M$.

2.6. Let $\xi$ be a random measure on $S$, let $I \in \mathcal{B}$ be a DC-semiring and let $a > 0$. Prove that $\xi_a^* = 0$ a.s. iff $P\{\xi I \geq a\} \leq P\{\xi'_a I \geq a\}$, $I \in I$.

2.7. Show by an example that the decomposition in Lemma 2.1 is not unique in general, not even apart from the order of terms. (Hint: take e.g. $S = [0,1]$, $\nu \equiv 2$, $\beta_1 \equiv \beta_2 \equiv 1$, and choose $(\tau_1, \tau_2)$ in two different ways.)

2.8. Show that the converse of Theorem 2.4 may be false when $E\xi \notin M$.
(Hint: consider a suitable mixture w.r.t. $\eta \in N$ of the non-random simple point processes on $R$ with atoms at $j2^{-n}$, $j \in N$.)

3. UNIQUENESS

The uniqueness results of this section will play a basic role in the subsequent discussion of convergence in distribution. Let us start with a theorem which applies to arbitrary random measures. For any $\mu \in M$ and $V = (B_1, \ldots, B_k) \in \mathcal{B}^k$, $k \in N$, we abbreviate $(\mu B_1, \ldots, \mu B_k) = \mu V$. 
Write $\xi \overset{d}{=} \eta$ for equality in distribution, i.e. $\xi \overset{d}{=} \eta$ iff $P_{\xi}^{-1} = P_{\eta}^{-1}$.

**Theorem 3.1.** Let $\xi$ be a random measure on $S$. Then $P_{\xi}^{-1}$ is uniquely determined by $P(\xi f)^{-1}$, $f \in F_c$, and even by $L_\xi(f)$, $f \in F_c$. It is further determined by $P(\xi V)^{-1}$, $V \in I^k$, $k \in N$, for any fixed DC-semiring $I \subset B$.

**Proof.** Let $\xi$ and $\eta$ be two random measures on $S$ satisfying $\xi V \overset{d}{=} \eta V$, $V \in I^k$, $k \in N$, for some fixed DC-semiring $I \subset B$. Letting $\mathcal{D}$ denote the class of all sets $M \in \mathcal{M}$ such that $P_{\xi}^{-1}M = P_{\eta}^{-1}M$, it is seen that $\mathcal{D}$ contains $M$ and is closed under proper differences and monotone limits. Furthermore, $\mathcal{D}$ contains by assumption the class $C$ of all sets of the form

$$\{m \in M : \mu I_1 \leq t_1, \ldots, \mu I_k \leq t_k\}$$

$k \in N$, $I_1, \ldots, I_k \in I$, $t_1, \ldots, t_k \in R_+$.

Since this class is closed under finite intersections, it follows by Dynkin's theorem that $\mathcal{D} \supseteq \sigma(C)$. But $\sigma(C) = M$ by Lemma 1.1, and we reach the desired conclusion $\xi \overset{d}{=} \eta$.

A similar argument shows that $P_{\xi}^{-1}$ is determined by the class of all distributions $P(\xi f_1, \ldots, \xi f_k)^{-1}$, $k \in N$, $f_1, \ldots, f_k \in F_c$.

But by the uniqueness theorem for multi-dimensional L-transforms, $P(\xi f_1, \ldots, \xi f_k)^{-1}$ is in turn determined by

$$L_{\xi f_1, \ldots, \xi f_k}(t_1, \ldots, t_k) = L_\xi(\sum_{j=1}^{k} t_j f_j) = L_\xi(f)$$

$t_1, \ldots, t_k \in R_+$,

where $f = \sum_{j} t_j f_j$. Since $f \in F_c$, this completes the proof of the first assertion.
Note in particular that the L-transforms which were calculated in Section 1 can be used to define the corresponding basic point processes.

**Corollary 3.2.** Let $\eta$ be a random measure on $S$ and let $\xi$ be a Cox process directed by $\eta$. Then $P_{\eta}^{-1}$ is determined by $P_{\xi}^{-1}$. This statement is also true when $\eta$ is a point process on $S$ and $\xi$ is a $\beta$-compound of $\eta$, provided $P_{\beta}^{-1}$ is known and such that $\beta \not= 0$.

**Proof.** Suppose that $\xi$ is a Cox process directed by $\eta$. Solving for $f$ in $1 - e^{-f}$, we obtain from (1.4) for any fixed $f \in F_c$

$$L_{\eta f}(t) = L_{\eta}(tf) = L_{\xi}(-\log(1 - tf)),$$

$$0 \leq t < ||f||^{-1},$$

where $||f|| = \sup f(s)$. Now $L_{\eta f}$ is analytic in the right half-plane, and so it is completely determined by its values on any finite interval. Hence by (1), $P(\eta f)^{-1}$ is determined by $P_{\xi}^{-1}$, and since $f \in F_c$ was arbitrary, the assertion follows from Theorem 3.1. A similar argument applies to the case of compound point processes, except that (1) is then replaced by

$$L_{\eta f}(t) = L_{\eta}(tf) = L_{\xi}(\phi^{-1} \circ e^{-tf}),$$

$$0 \leq t < -||f||^{-1} \log P(\beta = 0),$$

where $\phi^{-1}$ is the unique inverse of $\phi$ on the interval $(P(\beta = 0), 1]$.

Stronger results than in Theorem 3.1 are attainable if we confine ourselves to simple point processes or diffuse random measures.
THEOREM 3.3. Let $z$ and $\eta$ be two point processes on $\mathbb{S}$, and suppose that $z$ is simple. Further suppose that $U \subset B$ is a DC-ring and $I \subset B$ a DC-semiring. Then $z \overset{d}{=} \eta$ iff

\begin{equation}
P(zU = 0) = P(\eta U = 0), \quad U \in U,
\end{equation}

and we have even $z \overset{d}{=} \eta$ provided in addition

\begin{equation}
P(zI > 1) \geq P(\eta I > 1), \quad I \in I.
\end{equation}

PROOF. The conditions (2) and (3) are clearly necessary. Conversely, suppose that (2) holds. As before, the class $\mathcal{D}$ of all sets $M \in N$ satisfying $z^{-1}M = \eta^{-1}M$ is closed under proper differences and monotone limits, and it contains $N$. By assumption, it also contains the class $C$ of all sets of the form $\{\mu \in N: \mu U = 0\}, \quad U \in U$. Now it is easily seen that

$$\{\mu U = 0\} \cap \{\mu V = 0\} = \{\mu(U \cup V) = 0\}, \quad U, V \in U,$$

and since $U$ is closed under finite unions, it follows that $C$ is closed under finite intersections. Hence, by Dynkin's theorem, $\mathcal{D} \supset \sigma(C)$. But from Lemma 2.2 with $\varepsilon = 1$ it is seen that the mapping $\mu \mapsto \mu^*$ is measurable w.r.t. $\sigma(C)$, so we get $z = z^* \overset{d}{=} \eta^*$, proving the first assertion. If in addition (3) holds, then

$$P(\eta^* I > 1) = P(zI > 1) \geq P(\eta I > 1), \quad I \in I,$$

and it follows by Lemma 2.6 that $\eta = \eta^* \overset{d}{=} z$. This completes the proof.
Note that a variety of related results may be obtained by using the simplicity criteria of Section 2 rather than condition (3). A corresponding remark applies to the next theorem.

**THEOREM 3.4.** Let \( \xi \) and \( \eta \) be two point processes (random measures) on \( S \), and suppose that \( \xi \) is simple (diffuse). Further suppose that \( U \subset B \) is a \( D \)-class and \( C \subset B \) a covering class, and that \( s \) and \( t \) are fixed real numbers with \( 0 < s < t \). Then \( \xi \equiv \eta \) iff \( \eta \) is simple (a.s. diffuse) and

\[
E \exp \{-t\xi_U\} = E \exp \{-t\eta_U\}, \quad U \in U,
\]

and also iff (4) holds and in addition

\[
E \exp \{-s\xi_C\} \leq E \exp \{-s\eta_C\}, \quad C \in C.
\]

In the case \( E \xi \in \mathcal{M} \), we may replace (5) by

\[
E \xi_C \geq E \eta_C, \quad C \in C.
\]

It is interesting to observe that (2) is a limiting case of (4) obtained by letting \( t \to \infty \).

**PROOF.** Let us first consider the point process case, put \( p = 1 - e^{-t} \), and let \( \tilde{\xi} \) and \( \tilde{\eta} \) be \( p \)-thinnings of \( \xi \) and \( \eta \) respectively. By (1.6) we get

\[
E \exp \{-X\tilde{\xi}_B\} = E \exp \{\xi_B \log \left( 1 - p(1 - e^{-X}) \right) \}, \quad B \subset B \subseteq \mathbb{R}_+.
\]
and letting $x \to \infty$, we obtain by monotone convergence

$$P(\bar{\eta}B = 0) = E \exp(\xi B \log(1 - p)) = E e^{-t\xi B}, \quad B \in B.$$  \hspace{1cm} (8)

By (4), (8) and the corresponding relation for $\eta$, we obtain

$$P(\bar{\eta}U = 0) = E e^{-t\xi U} = E e^{-t\eta U} = P(\eta U = 0), \quad U \in U,$$

and it follows from Theorem 3.3 that $\bar{\xi} \overset{d}{=} \bar{\eta}$. Now $\xi$ and $\tilde{\xi}$ are simultaneously simple and correspondingly for $\eta$ and $\bar{\eta}$ (cf. Problem 2.4), so we get $\bar{\xi} \overset{d}{=} \bar{\eta} \overset{d}{=} \bar{\eta}$ in general, and if $\eta$ is simple even $\bar{\xi} \overset{d}{=} \bar{\eta}$.

Thus the first assertion follows by Corollary 3.2.

To prove the second assertion, we may assume that $\bar{\xi} \overset{d}{=} \bar{\eta}$ and that (5) holds. Choosing

$$x = -\log(1 - p^{-1}(1 - e^{-S}))$$

in (7) and in the corresponding relation for $\bar{\eta}$, which is possible since $s < t$, we obtain from (5)

$$E e^{-x\bar{\eta} C} = E e^{-x\tilde{\xi} C} \leq E e^{-x\eta C}, \quad C \in C,$$

or equivalently

$$E \left\{ e^{-x\bar{\eta} C} - e^{-x\eta C} \right\} \leq 0, \quad C \in C.$$ 

But here the random variable within brackets is non-negative, so it must in fact be a.s. 0, and it follows that $\bar{\eta}C = \bar{\eta}C$ a.s., $C \in C$, since
\( x > 0 \). Writing this result in the form \( \tilde{\eta} - \tilde{\eta}^* C = 0 \) a.s., \( C \in C \), and using the fact that \( C \) is covering, we obtain \( \tilde{\eta} = \tilde{\eta}^* \) a.s., so again \( \tilde{\xi} \overset{d}{=} \tilde{\eta} \), as required. The last assertion follows by a similar argument, based on the relations \( E\tilde{\xi} = pE\xi \) and \( E\tilde{\eta} = pE\eta \), which may e.g. be obtained from (7) by differentiation.

The random measure case may be proved from Theorem 3.3 by a similar argument where we consider Cox processes \( \tilde{\xi} \) and \( \tilde{\eta} \) directed by \( t\xi \) and \( t\eta \) respectively. Alternatively, we may choose any fixed \( x > t \), let \( \tilde{\xi} \) and \( \tilde{\eta} \) be Cox processes directed by \( x\xi \) and \( x\eta \) respectively, and apply the point process case of the present theorem.

As an application, we prove the following partial strengthening of Corollary 3.2.

**Corollary 3.5.** Let \( \xi \) be a \( \beta \)-compound of \( \eta \) and suppose that
\[ p = P(\beta > 0) > 0. \]
Further suppose that \( C_\eta \) is known to be simple for some \( C \in B \) with \( nC \overset{d}{=} 0 \). Then \( P_{n^{-1}} \) and \( P_{\beta^{-1}} \) are uniquely determined by \( P_{\xi^{-1}} \) and \( p \).

**Proof.** By (1.5),
\[ E e^{-t\xi B} = E \exp(nB \log \phi(t)) , \quad B \in B , \quad t \in R_+ , \]
and letting \( t \to \infty \), we obtain by monotone convergence
\[ P\{\xi B = 0\} = E \exp(nB \log(1 - p)) , \quad B \in B . \]
Considering sets $B$ contained in $C$, it follows by Theorem 3.4 that $P(Cn)^{-1}$ is uniquely determined by $P_{\xi}^{-1}$. Let us now write (9) for $B = C$ in the form

\begin{equation}
L_{\xi C} = L_{\eta C} \circ (-\log \phi).
\end{equation}

Since $\eta C \not\equiv 0$ by assumption, the function $L_{\eta C}$ is strictly increasing, so it must have a unique inverse $L_{\eta C}^{-1}$ on its range $\{P(\eta C = 0), 1\}$, and we obtain from (10)

\[ \phi = \exp(- L_{\eta C}^{-1} \circ L_{\xi C}), \]

proving that $\phi$ is determined by $P_{\xi}^{-1}$. We may now apply Corollary 3.2 to complete the proof.

**NOTES.** In Theorem 3.1, the first assertion is due to Prokhorov (1961), (see also von Waldenfels (1968) and Harris (1971)), while the second assertion is essentially a standard result for random processes. Theorem 3.3 was first proved in the particular case of Poisson processes by Rényi (1967), and then in the general case, independently by Mönch (1971) (in a weaker formulation) and Kallenberg (1973a). As for Theorem 3.4, the first assertion was proved for diffuse random measures (again in a weaker formulation) by Mönch (1970), and independently by Kallenberg (1973e) and Grandell (1973). The point process case of that theorem is new as are the last two assertions. Finally, Corollary 3.2 is due in the thinning case to Mecke (1968) and in the case of Cox processes to Kummer and Matthes (1970), while the assertions for compound processes in Corollaries 3.2 and 3.5 are new.
PROBLEMS.

3.1. Show that, for any random measure $\xi$, the distribution of $\xi$ is determined by $L_\xi(f)$ for all linear combinations $f$ of indicators $1_I$, $I \in I$, for any fixed DC-semiring $I$.

3.2. Theorem 3.1 clearly extends to arbitrary semirings generating $\mathcal{B}$. Prove that the first assertion in Theorem 3.3 extends to arbitrary rings $\mathcal{U}$ generating $\mathcal{B}$. (Hint: Show that the class of sets $\mathcal{U}$ satisfying (1) is closed under monotone bounded limits, and hence, by the classical monotone class theorem in Loève (1963), must contain $\mathcal{B} = \sigma(\mathcal{U})$. Then apply Theorem 3.3 with $\mathcal{U} = \mathcal{B}$.) Even the second part of Theorem 3.3 admits a similar strengthening, cf. Kerstan, Matthes and Mecke (1974), page 43.

3.3. Prove an alternative to the second part of Theorem 3.4 by applying the same method involving Cox processes to the second part of Theorem 3.3.

3.4. Prove that the distribution of an arbitrary point process $\xi$ is not even determined by $P(\xi_B)^{-1}$, $B \in \mathcal{B}$. (Hint: consider two random vectors $\alpha$ and $\beta$ on $\{0,1,2\}^2$ such that $\alpha \overset{d}{=} \beta$, and still $\alpha_1 \overset{d}{=} \beta_1$, $\alpha_2 \overset{d}{=} \beta_2$ and $\alpha_1 + \alpha_2 \overset{d}{=} \beta_1 + \beta_2$.)

3.5. Show that the ring property of $\mathcal{U}$ in Theorem 3.3 is essential. (Hint: we may e.g. introduce a suitable dependence in a stationary renewal process.) Cf. Kerstan, Matthes and Mecke (1974), page 213, for a stronger result and further references.

3.6. Prove that, for an arbitrary random measure $\xi$, the intensity $E\xi$ determines $P\xi^{-1}$ iff $E\xi = 0$. 
3.7. Show that \( t^{-1}(1 - Ee^{-t\xi B}) \rightarrow E\xi B \) as \( t \rightarrow 0 \), \( B \in \mathcal{B} \). This is interesting since, for simple point processes and diffuse random measures \( \xi \), \( P_{\xi}^{-1} \) is determined by the left-hand sides for arbitrarily small \( t > 0 \), and still it is not determined by the limit.

3.8. Assume that \( S \) is countable. Show that there exists some fixed function \( f \in F \) such that \( P_{\xi}^{-1} \) is determined by \( P_{f}^{-1} \) for every point process \( \xi \) on \( S \) with \( \xi S < \infty \) a.s. (Hint: let the numbers \( f(s) \), \( s \in S \), be rationally independent and bounded above and below by positive constants.)

3.9. Show that, if the class \( C \) in Theorem 3.4 is a DC-semiring, then \( \xi \) may be allowed to have atoms of fixed size and location. (Cf. the proof of Theorem 7.8 below.)

4. CONVERGENCE

In view of Theorem 3.1, it appears natural to say that the random measures \( \xi_n \), \( n \in \mathbb{N} \), converge in distribution to \( \xi \) (written \( \xi_n \overset{d}{\rightarrow} \xi \)) whenever

\[
\xi_n \overset{d}{\rightarrow} \xi V \quad , \quad V \in \mathcal{I}^k \quad , \quad k \in \mathbb{N} ,
\]

for some fixed DC-semiring \( \mathcal{I} \), and this is also the mode of convergence used in much classical work (with \( \mathcal{I} \) the class of real intervals). However, (1) requires too much in general since \( \tau_n \overset{d}{\rightarrow} \tau \) for random elements in \( S \) does not necessarily imply \( \delta_{\tau_n} \overset{d}{\rightarrow} \delta_{\tau} \) in the sense of (1), (cf. the trouble with pointwise convergence of distribution functions). Moreover, the above
mode of convergence depends strongly on the choice of \( I \). As will be seen below, we can avoid these disadvantages by assuming that \( I \) satisfies \( \xi \mathbb{1} = 0 \) a.s., \( I \in I \). We prove that DC-semirings \( I \) with this property do exist. Let us write \( B_\xi = \{ B \in B : \xi \mathbb{1} B = 0 \) a.s.\}.

**Lemma 4.1.** \( B_\xi \) is a DC-ring for every random measure \( \xi \) on \( S \).

**Proof.** Proceeding as in the proof of Theorem 2.5, we need only show that

\[(2) \quad \xi \{ s \in S : \rho(s,t) = r \} = 0 \text{ a.s.} \]

for any fixed \( t \in S \) and arbitrarily small \( r > 0 \). For this purpose, let \( \varepsilon > 0 \) be such that

\[ S(t, \varepsilon) = \{ s \in S : \rho(s,t) \leq \varepsilon \} \in B, \]

and define the random process \( X \) in \( D[0,\varepsilon] \) by

\[ X(r) = \xi S(t, r), \quad r \in [0,\varepsilon]. \]

Then

\[ \xi \{ s \in S : \rho(s,t) = r \} = X(r) - X(r^-), \quad r \in (0,\varepsilon], \]

and since we can have \( P\{X(r) \neq X(r^-)\} > 0 \) for at most countably many \( r \in (0,\varepsilon] \) (cf. Billingsley (1968), page 124), it follows that (2) holds almost everywhere on \( [0,\varepsilon] \). This completes the proof.
To be able to apply the powerful theory of weak convergence of probability measures, we shall now introduce a metrizable topology in the measure space such that (1) subject to the above restriction on \( T \) is equivalent to convergence in distribution in the sense of this theory.

For any \( \mu, \mu_1, \mu_2, \ldots \) in \( M \) or \( N \), we shall say that \( \mu_n \) tends vaguely to \( \mu \) and write \( \mu_n \rightharpoonup \mu \) whenever \( \mu_n f \to \mu f \) for all \( f \in F_c \). The so induced vague topology in \( M \) or \( N \) is known to be metrizable and separable (see Bauer (1972); it is indeed even Polish, see Lemma 10.1 below). It is often useful to relate the notion of vague convergence to the better known notion of weak convergence. For any \( \mu, \mu_1, \mu_2, \ldots \in M \) with \( \mu S < \infty \), we say that \( \mu_n \) tends weakly to \( \mu \) and write \( \mu_n \rightharpoonup \mu \) whenever \( \mu_n f \to \mu f \) for every bounded continuous function \( f \in F \). The reader should verify that \( \mu_n \rightharpoonup \mu \) iff \( \mu_n \rightharpoonup \mu \) and \( \mu_n S \to \mu S < \infty \).

The next lemma shows that the random elements in the topological Borel spaces \( M \) and \( N \) coincide with our random measures and point processes respectively.

**Lemma 4.2.** \( M \) and \( N \) coincide with the \( \sigma \)-algebras generated by the vague topologies in \( M \) and \( N \) respectively.

**Proof.** The vague topology in \( M \) (or \( N \)) is by definition the product topology induced by the mappings \( \mu \to \mu f \), \( f \in F_c \), so the class \( C \) of all sets \( \{ \mu f_j \leq t_j , j = 1, \ldots, k \} \), \( f_1, \ldots, f_k \in F_c \), \( t_1, \ldots, t_k \in \mathbb{R}^+ \), \( k \in \mathbb{N} \), is a topological base. Since \( M \) (or \( N \)) is separable, every vaguely open set is contained in \( \sigma(C) \), and so \( \sigma(C) \)
is the \( \sigma \)-algebra generated by the topology. On the other hand we have
\( \sigma(C) = M \) (or \( N \)) be Lemma 1.1. This completes the proof.

Unless otherwise stated, convergence in distribution of random measures
and point processes (denoted \( \overset{d}{\to} \), or sometimes \( \overset{v}{\to} \)) will henceforth mean
weak convergence of the corresponding distributions with respect to the
vague topology, i.e. \( \xi_n \overset{d}{\to} \xi \iff Ef(\xi_n) \to Ef(\xi) \) for every bounded and
vaguely continuous functional \( f: M \to \mathbb{R}_+ \) or \( N \to \mathbb{R}_+ \) respectively. Continuity in distribution holds for a large class of linear functionals.
Let \( D_f \) denote the set of discontinuity points of the function \( f \).
Let us further denote the mappings \( \mu \to \mu f \) and \( \mu \to \mu B \) by \( \pi_f \) and
\( \pi_B \) respectively.

**Lemma 4.3.** Let \( \xi, \xi_1, \xi_2, \ldots \) be random measures on \( S \) such that
\( \xi_n \overset{d}{\to} \xi \). Then \( \xi_n f \overset{d}{\to} \xi f \) for every bounded function \( f: S \to \mathbb{R}_+ \) with
compact support satisfying \( \xi D_f = 0 \) a.s. Furthermore, \( \xi_n V \overset{d}{\to} \xi V \) for
every \( V \in B_{\xi}^k \), \( k \in \mathbb{N} \).

**Proof.** Write \( C \in B \) for the support of \( f \) and choose some open set
\( G \in B \) with \( G \supset C \). By Urysohn's lemma, there exists some function
\( g \in F_C \) such that \( 1_C \leq g \leq 1_G \). Assuming \( \mu_n \overset{v}{\to} \mu \), we get for any bounded
continuous function \( h \in F \)

\[ (g \mu_n)h = \mu_n(gh) + \mu(gh) = (g \mu)h, \]

since \( gh \in F_C \), which proves that \( g \mu_n \overset{w}{\to} g \mu \). If we further assume that
\( \mu_{D_f} = 0 \), it follows by Theorem 5.2 in Billingsley (1968), (which clearly remains true for non-normalized measures), that

\[
\mu_n f = \mu_n (gf) = (g \mu_n) f \rightarrow (g \mu) f = \mu(f) = \mu f .
\]

This proves that

\[(3) \quad \{ \mu : \mu \in D_{nf} \} \subseteq \{ \mu : \mu D_f \neq 0 \} . \]

The first assertion now follows by Theorem 5.1 in Billingsley (1968).

Specializing to indicators \( f = 1_B , \ B \in \mathcal{B} \), it is seen from (3) that

\[
\{ \mu : \mu \in D_{nB} \} \subseteq \{ \mu : \mu \Delta B \neq 0 \} , \ B \in \mathcal{B} .
\]

Hence the second assertion follows by applying Theorem 5.1 in Billingsley (1968) to the mapping \( \mu \rightarrow \mu V , \ V \in \mathcal{B}_k^\mathcal{N} , \ k \in \mathbb{N} . \)

For point processes, convergence in distribution means the same thing in \( \mathbb{M} \) and \( \mathbb{N} \). In fact,

**Lemma 4.4.** \( \mathbb{N} \) is a vaguely closed subset of \( \mathbb{M} \). The class of point processes on \( \mathcal{S} \) is closed under convergence in distribution w.r.t. the vague topology on \( \mathbb{M} \).

**Proof.** Suppose that \( \mu_1, \mu_2, \ldots \in \mathbb{N} \) with \( \mu_n \Rightarrow \mu \in \mathbb{M} \). By Lemma 4.3 we have \( \mu_n B \rightarrow \mu B \) for any \( B \in \mathcal{B} \) with \( \mu \Delta B = 0 \), so \( \mu B \in \mathbb{Z}_+ \) for any such \( B \). Applying Dynkin's theorem as in the proof of Lemma 1.2, it follows that \( \mu B \in \mathbb{Z}_+ \) for all \( B \in \mathcal{B} \), and so \( \mu \in \mathbb{N} \). This proves that
N is a vaguely closed subset of M. Now suppose that \( \xi_1, \xi_2, \ldots \) are point processes on S converging in distribution to some random measure \( \xi \). By Theorem 2.1 in Billingsley (1968), we obtain

\[
P(\xi \in N) \geq \limsup_{n \to \infty} P(\xi_n \in N) = 1,
\]

proving that \( \xi \in N \) a.s., i.e. that \( \xi \) is a point process.

We shall now prove the basic fact that the classical notion of convergence introduced above is essentially equivalent to convergence in distribution with respect to the vague topologies.

**Theorem 4.5.** Let \( \xi, \xi_1, \xi_2, \ldots \) be random measures on S and let \( I \subset B_\xi \) be a DC-semiring. Then the following statements are equivalent.

\[(i) \quad \xi_n \xrightarrow{d} \xi, \]

\[(ii) \quad \xi_n f \xrightarrow{d} \xi f, \quad f \in F_c, \]

\[(iii) \quad L_{\xi_n}(f) \to L_\xi(f), \quad f \in F_c, \]

\[(iv) \quad \xi_n \mathbb{1} \xrightarrow{d} \xi \mathbb{1}, \quad \mathbb{1} \in T^k, \quad k \in \mathbb{N}. \]

Two lemmas are needed for the proof.

**Lemma 4.6.** Let \( \xi_1, \xi_2, \ldots \) be random measures on S. Then \( \{\xi_n\} \) is tight iff \( \{\xi_n B\} \) is tight for each \( B \in \mathcal{B} \).
PROOF. We shall make use of the fact (cf. Bauer (1972)) that a set $K$ in $M$ or $N$ is relatively compact iff

$$\sup_{\mu \in K} \mu B < \infty, \ B \in B.$$  

Let us first suppose that $\{\xi_n\}$ is tight. By definition, there exists for each $\epsilon > 0$ some compact set $K \subset M$ such that

$$P\{\xi_n \in K\} \geq 1 - \epsilon, \ n \in N.$$  

For fixed $B \in B$ we get by (4) $m = \sup_{\mu \in K} \mu B < \infty$, and hence by (5)

$$P\{\xi_n B > m\} \leq P\{\xi_n \notin K\} \leq \epsilon, \ n \in N,$$

proving tightness of $\{\xi_n B\}$.

Conversely, suppose that $\{\xi_n B\}$ is tight for each $B \in B$. Let $G_1, G_2, \ldots$ be open $B$-sets with $G_k \uparrow S$, and choose for fixed $\epsilon > 0$ the constants $m_1, m_2, \ldots > 0$ such that

$$P\{\xi_n G_k > m_k\} \leq \epsilon 2^{-k}, \ k, n \in N.$$  

Define

$$K = \bigcap_{k} \{\mu \in M : \mu G_k \leq m_k\},$$

and note that $K$ is relatively compact, since there exists for any $B \in B$ some $k \in N$ with $G_k \supseteq B$, and therefore

$$\sup_{\mu \in K} \mu B \leq \sup_{\mu \in K} \mu G_k \leq m_k \leq \infty.$$  

Furthermore,

\[ P(\xi_n \not\rightarrow K) = P \bigcup_k \{ \xi_n G_k > m_k \} \leq \sum_k P(\xi_n G_k > m_k) \leq \varepsilon \sum_k 2^{-k} = 2\varepsilon \quad n \in \mathbb{N}, \]

proving tightness of \( \{\xi_n\} \).

**Lemma 4.7.** Let \( \xi, \xi_1, \xi_2, \ldots \) be random measures on \( S \), let \( U \in \mathcal{B} \) be a \( DC \)-ring and let \( t > 0 \) be fixed. Further suppose that \( \xi_n \overset{d}{\rightarrow} \eta \) and that either

\[ \liminf_{n \to \infty} P(\xi_n U > \varepsilon) \leq P(\xi U > \varepsilon) \quad U \in \mathcal{B}, \quad \varepsilon \in (0,1), \]

or

\[ \limsup_{n \to \infty} E \exp(-t\xi_n U) \geq E \exp(-t\xi U) \quad U \in \mathcal{B}. \]

Then \( \eta F = 0 \) a.s. for every closed set \( F \in \mathcal{B} \) with \( \xi F = 0 \) a.s.

**Proof.** Suppose that (6) holds and let \( F \in \mathcal{B} \) be closed with \( \xi F = 0 \) a.s. By Lemma 4.1 there exists for every \( \varepsilon > 0 \) some closed set \( C \in \mathcal{B}_n \) such that \( C \supset F \) and \( P(\xi C \geq \varepsilon/2) \leq \varepsilon/2 \). We may next choose some \( U \in \mathcal{B} \) such that \( U \supset C \) and \( P(\xi (U \setminus C) \geq \varepsilon/2) \leq \varepsilon/2 \). Then

\[ P(\xi U \geq \varepsilon) \leq P(\xi C \geq \varepsilon/2) + P(\xi(U \setminus C) \geq \varepsilon/2) \leq \varepsilon. \]

Now \( \xi_n C \overset{d}{\rightarrow} \eta C \) by Lemma 4.3, so we obtain

\[ P(\eta F > \varepsilon) \leq P(\eta C > \varepsilon) \leq \liminf_{n \to \infty} P(\xi_n C > \varepsilon) \]

\[ \leq \liminf_{n \to \infty} P(\xi_n U > \varepsilon) \leq P(\xi U > \varepsilon) \leq \varepsilon, \]
and since \( \varepsilon \) was arbitrary, it follows that \( \eta F = 0 \) a.s. as asserted.

A similar proof applies to the case when (7) holds.

**Proof of Theorem 4.5.** The fact that (i) implies (ii) and (iv) was proved in Lemma 4.3. Since (iii) follows trivially from (ii), it remains to prove that (iii) and (iv) both imply (i).

Let us first suppose that (iii) holds. Then

\[ L_{\xi_n}(tf) \to L_{\xi}(tf), \quad t \in R^+ \quad f \in F_c, \]

so (ii) holds by the continuity theorem for L-transforms. But from (ii) and Lemma 4.6 it is seen that \( \{\xi_n\} \) is tight, and so it follows by Prohorov's theorem (Theorem 6.1 in Billingsley (1968)) that \( \{\xi_n\} \) is relatively compact. This means that every sequence \( N' \subset N \) contains a subsequence \( N'' \) such that \( \xi_n \overset{d}{\to} \eta \) (\( n \in N'' \)). From the implication (i) \( \Rightarrow \) (ii) we obtain \( \xi_n f \overset{d}{\to} \eta f \) (\( n \in N'' \)), \( f \in F_c \), and since also \( \xi_n \overset{d}{\to} \xi \), \( f \in F_c \), we must have \( \xi f \overset{d}{\to} \eta f \), \( f \in F_c \). By Theorem 3.1, this implies \( \xi \overset{d}{\to} \eta \), and since \( N' \) was arbitrary, we get \( \xi_n \overset{d}{\to} \xi \) (\( n \in N \)) by Theorem 2.3 in Billingsley (1968).

To prove that (iv) implies (i), the same argument goes through, except for the point where we conclude from \( \xi_n \overset{d}{\to} \eta \) (\( n \in N'' \)) that \( \xi_n V \overset{d}{\to} \eta V \) (\( n \in N'' \)), \( V \in I^k \), \( k \in N \). Now this step is permitted by Lemma 4.3 provided \( I \in \mathcal{B}_\eta \), and we shall show that this relation is indeed true. For this purpose, let \( \mathcal{U} \) be the ring of finite unions of \( I \)-sets, and note that these unions can always be taken disjoint. By (iv) and the fact that addition is a continuous operation from \( R^k_+ \) to \( R^+_+ \), we obtain
$\xi_n \overset{d}{\rightarrow} \xi$, $U \subset U$, and so

$$\liminf_{n \to \infty} P\{\xi_n U > \varepsilon\} \leq \limsup_{n \to \infty} P\{\xi_n U \geq \varepsilon\} \leq P\{\xi U \geq \varepsilon\}, \quad U \subset U, \quad \varepsilon > 0.$$ 

Hence it follows by Lemma 4.7 that $\eta F = 0$ a.s. for every closed set $F \subset B$ with $\xi F = 0$ a.s., and in particular $\eta \theta I = 0$ a.s., $I \subset I$, as desired.

We shall now improve Theorem 4.5 in the particular cases of convergence towards simple point processes and diffuse random measures. Given any covering class $C \subset B$, and any $a > 0$, we shall say that the sequence $\{\xi_n\}$ of random measures on $S$ is a-regular w.r.t. $C$ if there exists for every $C \subset C$ some array $\{C_{mk}\} \subset C$ of finite coverings of $C$ (one for each $m \in N$) such that

$$(8) \quad \lim_{m \to \infty} \limsup_{n \to \infty} \sum_k P\{\xi_n C_{mk} \geq a\} = 0.$$ 

Note that, in the particular case when $\xi_1 = \xi_2 = \ldots = \xi$, this notion reduces to a-regularity of $\xi$, as defined in Section 2. As before, we shall further say that $\{\xi_n\}$ is regular, if it is a-regular for every $a > 0$.

**Theorem 4.8.** Let $\xi, \xi_1, \xi_2, \ldots$ be point processes on $S$ and suppose that $\xi$ is simple. Further suppose that $U \subset B_\xi$ is a DC-ring and $1 \subset B_\xi$ a DC-semiring. Then $\xi_n \overset{d}{\rightarrow} \xi$ iff
(i) \( \lim_{n \to \infty} P(\xi_n U = 0) = P(\xi U = 0) \), \( U \in U \),

(ii) \( \limsup_{n \to \infty} P(\xi_n I > 1) \leq P(\xi I > 1) \), \( I \in I \),

(iii) \( \{\xi_n B\} \) is tight for every \( B \in \mathcal{B} \).

Moreover, \( \xi_n \xrightarrow{d} \xi \) follows from (i) and (ii) if \( \xi \) is 2-regular w.r.t. \( I \) and from (i) alone if \( \{\xi_n\} \) is 2-regular w.r.t. \( I \) or if

\[
\limsup_{n \to \infty} E\xi_n I \leq E\xi I < \infty, \quad I \in I.
\]

PROOF. The necessity of (i) - (iii) follows by Lemmas 4.3 and 4.6. Conversely, assuming (i) - (iii), it follows by Lemma 4.6 that \( \{\xi_n\} \) is tight and hence relatively compact, so any sequence \( N' \subset N \) must contain a sub-sequence \( N'' \) such that \( \xi_n \xrightarrow{d} \eta \) for some \( \eta \) \( (n \in N'') \). Here \( \eta \) is a point process by Lemma 4.4. By (i) and Lemma 4.7 we get \( \eta \# B = 0 \) a.s., \( B \in U \cup I \), and so by Lemma 4.3

\[
\xi_n B \xrightarrow{d} \eta B \quad (n \in N''), \quad B \in U \cup I.
\]

Hence

\[
P(\eta U = 0) = \lim_{n \to \infty} P(\xi_n U = 0) = P(\xi U = 0), \quad U \in U,
\]

\[
P(\eta I > 1) = \lim_{n \to \infty} P(\xi_n I > 1) \leq \limsup_{n \to \infty} P(\xi_n I > 1) \leq P(\xi I > 1) \}, \quad I \in I,
\]

and it follows by Theorem 3.3 that \( \xi \xrightarrow{d} \eta \). Since \( N' \) was arbitrary, this implies \( \xi_n \xrightarrow{d} \xi \).
Now suppose that $\xi$ is 2-regular w.r.t. $I$ and that (ii) holds.

Let $I \in I$ be fixed, and let $\{I_{nj}\} \subset I$ be an array of finite coverings of $I$ such that (2.8) holds with $a = 2$ and $C_{nj} \equiv I_{nj}$. If $r_m$ is the number of sets $I_{mk}$ for fixed $m \in N$, we get by (ii)

$$\limsup_{n \to \infty} P(\xi_n I > r_m) \leq \limsup_{n \to \infty} P \cup \{\xi_n I_{mk} > 1\} \leq \limsup_{n \to \infty} \sum_k P(\xi_n I_{mk} > 1) \leq \sum_k \limsup_{n \to \infty} P(\xi_n I_{mk} > 1) \leq \sum_k P(\xi I_{mk} > 1).$$

Hence by (2.8)

$$\lim_{m \to \infty} \limsup_{n \to \infty} P(\xi_n I > r_m) \leq \lim_{m \to \infty} \sum_k P(\xi I_{mk} > 1) = 0,$$

proving tightness of $\{\xi_n I\}$. Therefore, (iii) follows from (ii) in this case.

Next suppose that $\{\xi_n\}$ is 2-regular w.r.t. $I$ and that (i) holds. Then tightness of $\{\xi_n\}$ follows as above, so any sequence $N' \subset N$ must contain a sub-sequence $N''$ such that $\xi_n \overset{d}{\to} \eta$ (for some $n \in N''$). As before we may conclude from Lemma 4.7 that $\eta \in B = 0$ a.s., $B \in U \cup I$, so in particular (10) remains true and $\xi \overset{d}{\to} \eta \overset{*}{\to}$ follows by Theorem 3.3. Furthermore, it follows from (8) with $a = 2$ and $C_{mk} \equiv I_{mk} \in I$ that

$$\lim_{m \to \infty} \sum_k P(\eta I_{mk} > 1) = \lim_{m \to \infty} \sum_{n \in N''} \lim_{n \to \infty} P(\xi_n I_{mk} > 1) \leq \lim_{m \to \infty} \limsup_{n \to \infty} \sum_k P(\xi_n I_{mk} > 1) = 0,$$

so $\eta$ is 2-regular w.r.t. $I$, and hence simple by Theorem 2.4. Therefore $\xi \overset{d}{=} \eta$, and again $\xi_n \overset{d}{\to} \xi$ follows since $N'$ was arbitrary.
Finally suppose that (i) and (9) hold. By Cebyshev's inequality, (9) implies tightness of \( \{ \xi_n I \} \) for every \( I \in I \), so \( \{ \xi_n \} \) is tight by Lemma 4.6. If \( \xi_n \overset{d}{\rightarrow} \eta \) as \( n \rightarrow \infty \) through some \( N'' \subset N \), then it follows as above that \( \xi \overset{d}{\rightarrow} \eta \) and \( \xi_n I \overset{d}{\rightarrow} \eta I \) \( (n \in N'') \), \( I \in I \), so by (9) and Fatou's lemma (in the formulation for convergence in distribution, see Billingsley (1968), Theorem 5.3),

\[
E \xi I = E_n \xi_n I = \liminf_{n \in N''} E_n \xi_n I = \limsup_{n \rightarrow \infty} E_n \xi_n I \leq E \xi I, \quad I \in I.
\]

Hence \( E_n I = E_n \xi_n I \) for all \( I \in I \), and \( \eta \) must be simple. As before, we may conclude that \( \xi_n \overset{d}{\rightarrow} \xi \). The proof is now complete.

From Theorem 3.4 we obtain the following analogous result which may be proved in the same way.

**Theorem 4.9.** Let \( \xi, \xi_1, \xi_2, \ldots \) be point processes (random measures) on \( S \) and suppose that \( \xi \) is simple (diffuse). Further suppose that \( s \) and \( t \) are fixed real numbers with \( 0 < s < t \) and that \( U = B_{\xi} \) is a DC-ring and \( C \subset B_{\xi} \) a covering class. Then \( \xi_n \overset{d}{\rightarrow} \xi \) iff

\[
(i) \quad \lim_{n \rightarrow \infty} E_n e^{-t \xi U} = E e^{-t \xi U}, \quad U \in U,
\]

\[
(ii) \quad \liminf_{n \rightarrow \infty} E_n e^{-s \xi C} \geq E e^{-s \xi C}, \quad C \in C,
\]

(iii) \( \{ \xi_n C \} \) is tight for every \( C \in C \).

Moreover, \( \xi_n \overset{d}{\rightarrow} \xi \) follows from (i) alone if \( \{ \xi_n \} \) is 2-regular (regular) w.r.t. \( C \) or if (9) holds with \( C \) in place of \( I \).
In many applications we need to consider convergence in distribution with respect to the weak rather than the vague topology on \( M \). When wishing to emphasize the underlying topology, we write \( \xrightarrow{wd} \) or \( \xrightarrow{vd} \) instead of \( \xrightarrow{d} \). No new theory is needed for the case of the weak topology, since this case may easily be reduced to that of the vague topology:

**Theorem 4.10.** Let \( \xi, \xi_1, \xi_2, \ldots \) be a.s. bounded random measures. Then the following statements are equivalent.

(i) \( \xi_n \xrightarrow{wd} \xi \),

(ii) \( \xi_n \xrightarrow{vd} \xi \) and \( \xi_n \xrightarrow{d} \xi_S \),

(iii) \( \xi_n \xrightarrow{vd} \xi \) and \( \inf_{B \in \mathcal{B}} \limsup_{n \to \infty} P(\xi_n \cap B^c > \varepsilon) = 0 \), \( \varepsilon > 0 \).

Note that (i) is trivially equivalent to \( (\xi_n, \xi_n \xi_S) \xrightarrow{vd} (\xi, \xi_S) \).

The point is that the latter condition may be replaced by the seemingly much weaker conditions (ii) or (iii).

**Proof.** Since \( \mu \xrightarrow{w} \mu \) iff \( \mu \xrightarrow{y} \mu \) and \( \mu_n \xrightarrow{S} \mu S \), the weak topology is metrizable, so the theory of convergence in distribution in metric spaces applies to it. Write \( M(w) \) and \( M(v) \) for the space of bounded measures in \( M \) endowed with the weak and vague topology respectively. Let us first suppose that (i) holds. Since the mapping \( \mu \rightarrow (\mu, \mu S) \) from \( M(w) \) into \( M(v) \times \mathbb{R}_+ \) is continuous, (ii) follows by Theorem 5.1 in Billingsley (1968).
Next suppose that (ii) holds, and let $\epsilon, \delta > 0$ be arbitrary. Since 
\{n \in \mathbb{N}\} is tight, we may choose some $a > 0$ such that 

\[ P(\xi_n^* > a) < \delta, \quad n \in \mathbb{N}. \]

By dominated convergence, there further exists some $B \in \mathcal{B}$ with 

\[ E(1 - e^{-\xi B^C}) < \epsilon \delta e^{-a}, \]

and by Lemma 4.1 we may assume that $\xi \alpha B = 0$ a.s. Using (11), we obtain 

\[ P(\xi_n^* B^C > \epsilon) \leq P(\xi_n^* B^C > \epsilon, \xi_n^* S \leq a) + P(\xi_n^* S > a) \]

\[ \leq E[\epsilon e^{-\xi_n^* B^C}; \xi_n^* B^C > \epsilon, \xi_n^* S \leq a] + \delta \leq \epsilon e^{-a} E\xi_n^* B^C e^{-\xi_n^* S} + \delta \]

\[ \leq \epsilon e^{-a} E(e^{\xi_n^* B^C} - 1)e^{-\xi_n^* S} + \delta = \epsilon e^{-a} e^{-\xi_n^* B^C} + \delta \]

so by (ii), (12) and Lemma 4.3, 

\[ \limsup_{n \to \infty} P(\xi_n^* B^C > \epsilon) \leq \epsilon e^{-a} \lim_{n \to \infty} E(e^{\xi_n^* B^C} - e^{\xi_n^* S}) + \delta \]

\[ = \epsilon e^{-a} E(e^{-\xi B^C} - e^{-\xi S}) + \delta = \epsilon e^{-a} E[e^{-\xi B^C}(1 - e^{-\xi B^C})] + \delta \]

\[ \leq \epsilon e^{-a} E(1 - e^{-\xi B^C}) + \delta \leq \delta + \delta = 2\delta. \]

Since $\delta$ was arbitrary, this proves (iii).

Let us finally suppose that (iii) holds. Choose for each $k \in \mathbb{N}$ some $B_k \in \mathcal{B}_\xi$ such that $B_k + S$ and 

\[ \limsup_{n \to \infty} P(\xi_n^* B_k^C > k^{-1}) < k^{-1}, \quad n \in \mathbb{N}. \]
Then

(13) \[ \lim_{k \to \infty} \limsup_{n \to \infty} P(\xi_n B^c_k > \varepsilon) = 0, \quad \varepsilon > 0. \]

We now define the metric \( \rho \) in the topological product space \( M(v) \times R_+ \) by

\[ \rho([\nu_1, t_1], [\nu_2, t_2]) = \rho_v(\nu_1, \nu_2) \vee |t_1 - t_2|, \quad \nu_1, \nu_2 \in M, \quad t_1, t_2 \in R_+, \]

where \( \rho_v \) is any metrization of \( M(v) \). Then

\[ \rho((\xi_n, \xi_n B_k), (\xi_n, \xi_n S)) = |\xi_n B_k - \xi_n S| = \xi_n B^c_k, \quad n, k \in N, \]

so (13) may be written

(14) \[ \lim_{k \to \infty} \limsup_{n \to \infty} \left\{ \rho((\xi_n, \xi_n B_k), (\xi_n, \xi_n S)) > \varepsilon \right\} = 0, \quad \varepsilon > 0. \]

Furthermore, we have in \( M(v) \times R_+ \)

(15) \[ (\xi, \xi B_k) \to (\xi, \xi S) \text{ a.s. as } k \to \infty, \]

and finally, since the mappings \( \nu \to (\nu, \nu B_k) \) are continuous

\( M(v) \to M(v) \times R_+ \) at all points \( \nu \) with \( \nu B_k = 0 \),

(16) \[ (\xi_n, \xi_n B_k) \overset{d}{\to} (\xi, \xi B_k) \text{ as } n \to \infty, \quad k \in N. \]

Using Theorem 4.2 in Billingsley (1968), we may conclude from (14), (15) and (16) that

(17) \[ (\xi_n, \xi_n S) \overset{d}{\to} (\xi, \xi S) \text{ as } n \to \infty. \]
Now the mapping \((\mu, \mu S) \to \mu\) is continuous \(M(\nu) \times R^+ \to M(\nu)\), and therefore (i) follows from (17) by Theorem 5.1 in Billingsley (1968). This completes the proof.

NOTES. The equivalence of (i) - (iii) in Theorem 4.5 was proved by Prohorov (1961) for compact spaces and in general by von Waldenfels (1968), (see also Harris (1971)). The remaining equivalence was given in Kallenberg (1973a). This is also the source for Theorem 4.8, except for the assertion involving (9) which was added by Kurtz (1973), while Theorem 4.9 improves and extends a result by Kallenberg (1973e) and Grandell (1973). Theorem 4.10 is new, although it was used implicitly in Kallenberg (1973c) and (1974b). We finally refer to Jagers (1974) for Lemmas 4.2 and 4.4.

PROBLEMS.

4.1. Let \(\mu, \mu_1, \mu_2, \ldots \in M\) and let \(I \in B_\mu\) be a DC-semiring. Prove that \(\mu_n \overset{\mu}{\to} \mu\) iff \(\mu_n I \to \mu I, \ I \in I\).

4.2. Prove that \(\mu_n \overset{\mu}{\to} \mu\) iff \(\limsup \mu_n F \leq \mu F, \ F \in B\) closed, and 
\(\liminf \mu_n G \geq \mu G, \ G \in B\) open. Note that one of these conditions is not enough.

4.3. Let \(F \in B\) be closed and \(G \in B\) open, and let \(t \in R^+\). Show that the \(\mu\)-sets \(\{\mu F < t\}\) and \(\{\mu G > t\}\) are open in \(M\) and \(N\). (Hint: use Problem 4.2.)

4.4. Prove that, for bounded \(\mu, \mu_1, \mu_2, \ldots \in M\), \(\mu_n \overset{\mu}{\to} \mu\) iff \(\mu_n \overset{\mu}{\to} \mu\) and \(\mu_n S \to \mu S\).
4.5. Let \( \xi, \xi_1, \xi_2, \ldots \) be random measures on \( S \) and let \( C_1, C_2, \ldots \in B_{\xi} \) with \( C_k \uparrow S \). Prove that \( \xi_n \xrightarrow{\text{vd}} \xi \) iff \( C_k \xi_n \xrightarrow{\text{vd}} C_k \xi \), \( k \in N \). More generally, \( \xi \xrightarrow{\text{vd}} \xi \) implies \( f \xi_n \xrightarrow{\text{vd}} f \xi \) for any bounded function \( f \in F \) with bounded support satisfying \( \xi D_f = 0 \) a.s.

4.6. Prove that, if \( \tau, \tau_1, \tau_2, \ldots \) are random elements in \( (S, \mathcal{B}) \), then \( \tau_n \xrightarrow{\text{d}} \tau \) iff \( \delta_{\tau_n} \xrightarrow{\text{d}} \delta_\tau \).

4.7. Prove Theorem 4.9.

4.8. Extend Theorem 4.8 to the case when \( \xi_1, \xi_2, \ldots \) are general random measures.

4.9. Let \( \xi, \xi_1, \xi_2, \ldots \) be random measures on \( S \) such that \( \xi_n \xrightarrow{\text{d}} \xi \), and let \( \varepsilon \geq 0 \) be fixed. Show that \( \xi^*_\varepsilon \equiv \xi^*_{(\varepsilon, \infty)} = 0 \) a.s. if \( \{\xi_n\} \) is a-regular w.r.t. \( B_\xi \) for every \( a > \varepsilon \), and that the converse is also true whenever \( E\xi \in M \). (Cf. Kallenberg (1975).)

4.10. Show that, if the class \( C \) in Theorem 4.9 is a DC-semiring, then \( \xi \) may be allowed to have atoms of any fixed size and position.
   (Cf. Problem 3.9.)

4.11. Show that the equivalence of (i) and (iii) in Theorem 4.10 may essentially be restated as a tightness criterion for random measures subject to the weak topology in \( M \). Give also a direct proof of this criterion.

4.12. Let \( \xi_1, \xi_2, \ldots \) be Cox processes directed by \( \eta_1, \eta_2, \ldots \) respectively. Show that \( \xi_n \xrightarrow{\text{d}} \) some \( \xi \) iff \( \eta_n \xrightarrow{\text{d}} \) some \( \eta \), and that in this case \( \xi \) is a Cox process directed by \( \eta \). Prove the corresponding result for \( p \)-thinings with fixed \( p > 0 \).
4.13. Let \( \eta, \eta_1, \eta_2, \ldots \) be random measures on \( S \) with \( \eta_n \overset{d}{\rightarrow} \eta \) and suppose that the functions \( f, f_1, f_2, \ldots \in F \) are uniformly bounded with uniformly bounded support and satisfy

\[
\eta(s \in S : f_n(s_n) \rightarrow f(s) \text{ for some } s_1, s_2, \ldots \rightarrow s) = 0 \text{ a.s.}
\]

Show that in this case \( \eta_n f_n \overset{d}{\rightarrow} \eta f \). (Cf. the proof of Theorem 8.1 below.)

5. EXISTENCE

In the limit theorems of the preceding section, the limiting random measure \( \xi \) was always assumed to exist. We shall now show that, under mild additional conditions, the existence of \( \xi \) will hold automatically by compactness. This provides a new approach to the general existence problems in random measure theory.

We begin with the counterpart to Theorem 4.5.

**Lemma 5.1.** Let \( \xi_1, \xi_2, \ldots \) be random measures on \( S \) and let \( U \subset B \) be a DC-ring. Suppose that either

(i) \( \xi_n f \overset{d}{\rightarrow} \xi f \), \( f \in F_c \), or 

(ii) \( \xi_n V \overset{d}{\rightarrow} \xi V \), \( V \in U^k \), \( k \in \mathbb{N} \), where the random vectors \( \xi V \) are such that \( \xi U_k \overset{d}{\rightarrow} 0 \) whenever \( U, U_1, U_2, \ldots \in U \) with \( U_k \vdash \exists U \).
Then \( \xi_n \overset{d}{\rightarrow} \xi \) for some \( \xi \) satisfying \( \xi_f \overset{d}{=} \xi_f \), \( f \in F_c \), or \( \xi_V \overset{d}{=} \xi_V \), \( V \in \mathcal{U}_k \), \( k \in \mathbb{N} \), respectively.

**Proof.** In both cases, \( \{\xi_n\} \) is tight by Lemma 4.6, so by Prohorov's theorem, \( \xi_n \overset{d}{\rightarrow} \xi \) as \( n \to \infty \) through some sequence \( N' \subseteq \mathbb{N} \). But then \( \xi_n f \overset{d}{\rightarrow} \xi_f \) (\( n \in N' \)), \( f \in F_c \), by Theorem 4.5, so assuming (i), we obtain \( \xi_f \overset{d}{=} \xi_f \), and the assertion follows by Theorem 4.5.

In case of (ii), we may proceed as in the proof of Lemma 4.7 to conclude that \( \xi \mathbb{E} U = 0 \) a.s., \( U \in \mathcal{U} \). But then \( \xi_n V \overset{d}{\rightarrow} \xi V \) (\( n \in N' \)), \( V \in \mathcal{U}_k \), \( k \in \mathbb{N} \), by Theorem 4.5, and the assertion follows as before.

Arguing in the same way, we can easily derive analogous results for convergence towards simple point processes and diffuse random measures.

For example,

**Lemma 5.2.** Let \( \xi_1, \xi_2, \ldots \) be point processes on \( S \), and let \( U \subseteq \mathcal{B} \) be a DC-ring and \( I \subseteq \mathcal{B} \) a DC-semiring. Suppose that \( \{\xi_n\} \) is 2-regular w.r.t. \( I \) and that

\[
P(\xi_n U = 0) \rightarrow \text{some } \phi(U), \ U \in \mathcal{U}.
\]

Further suppose that \( \phi(U_n) \to 1 \) whenever \( U_1, U_2, \ldots \in \mathcal{U} \) and \( B \in \mathcal{U} \cup I \) with \( U_n \overset{d}{\rightarrow} \mathbb{E} B \). Then there exists some simple point process \( \xi \) on \( S \) satisfying \( \xi_n \overset{d}{\rightarrow} \xi \) and

\[
P(\xi U = 0) = \phi(U), \ U \in \mathcal{U}.
\]
We shall now use Lemma 5.1 to derive two important special cases of a general existence theorem corresponding to the second part of Theorem 3.1. For notational convenience, we consider random vectors in \( \mathbb{R}_+^k \) as random measures on the space \( \{1, \ldots, k\} \), \( k \in \mathbb{N} \).

**THEOREM 5.3.** Let \( U \subset B \) be a DC-ring and for any \( V \in U^k \), \( k \in \mathbb{N} \), let \( \xi_V \) be a random vector in \( \mathbb{R}_+^k \). Then there exists some random measure \( \xi \) on \( S \) satisfying \( U \subset B^\xi \) and \( \xi_V \overset{d}{=} \xi_V \), \( V \in U^k \), \( k \in \mathbb{N} \), iff

(i) the distributions \( P_{\xi_V}^{-1} \) are consistent,

(ii) \( \xi_V\{1\} + \xi_V\{2\} = \xi_V\{3\} \) a.s. whenever \( V = (U_1, U_2, U_1 \cup U_2) \in U^3 \) with disjoint \( U_1 \) and \( U_2 \),

(iii) \( \xi \overset{d}{=} 0 \) whenever \( U, U_1, U_2, \ldots \in U \) with \( U_n \rightarrow 3U \).

**PROOF.** The necessity of (i) - (iii) is obvious. Conversely, suppose that (i) - (iii) hold. For fixed \( U \in U \), let \( \{U_{nj}\} \subset U \) be a null-array of nested partitions of \( U \), choose arbitrary \( s_{nj} \in U_{nj} \) for all \( n \) and \( j \), and define the random measures \( \xi_1, \xi_2, \ldots \) on \( S \) by

\[
\xi_n = \sum_j \xi_V\{j\} \delta_{s_{nj}}, \quad \text{where } V = (U_{n1}, U_{n2}, \ldots), \quad n \in \mathbb{N}.
\]

Writing \( U_n \) for the ring generated by \( U_{n1}, U_{n2}, \ldots \), we obtain by (i) and (ii)

\[
\xi_n V \overset{d}{=} \xi_V, \quad V \in U_n^k, \quad k \in \mathbb{N},
\]
and so
\[
\xi_n V \xrightarrow{d} \xi_V, \quad V \in U^k, \quad k \in \mathbb{N},
\]
where \( U' = \bigcup_{n=0}^\infty U_n \). Now \( U' \) is a DC-ring in the subspace \( U \), so it follows by (iii) and Lemma 5.1 that \( \xi_n \xrightarrow{d} \) some \( \xi' \) satisfying

(3)
\[
\xi' V \xrightarrow{d} \xi_V, \quad V \in U^k, \quad k \in \mathbb{N}.
\]

To see that (3) extends to \( U \cap U \), let \( m \in \mathbb{N} \) and \( U_1, \ldots, U_m \in U \cap U \) be arbitrary, and repeat the above argument with a refined null-array such that \( U' \) is replaced by some \( U'' \supset U' \cup \{U_1, \ldots, U_m\} \). This yields the existence of some random measure \( \xi'' \) such that

(4)
\[
\xi'' V \xrightarrow{d} \xi_V, \quad V \in U^k, \quad k \in \mathbb{N}.
\]

Comparing (3) and (4), we obtain
\[
\xi' V \xrightarrow{d} \xi'' V, \quad V \in U^k, \quad k \in \mathbb{N},
\]
and so \( \xi' \xrightarrow{d} \xi'' \) by Theorem 3.1. Inserting this into (4) yields the desired extension of (3).

To complete the proof, let \( U_1, U_2, \ldots \in U \) with \( U_k^0 \uparrow S \), and let \( \xi'_1, \xi'_2, \ldots \) be random measures \( \xi' \) obtained as above with \( U_1, U_2, \ldots \) as \( U \). Then it follows by another application of Lemma 5.1 that \( \xi_k \xrightarrow{d} \) some \( \xi \) with the asserted properties.
THEOREM 5.4. For any \( V \in B_k \), \( k \in \mathbb{N} \), let \( \xi_V \) be a random vector in \( R^k_+ \). Then there exists some random measure \( \xi \) on \( S \) satisfying \( \xi V \overset{d}{=} \xi_V \), \( V \in B^k \), \( k \in \mathbb{N} \), iff

(i) the distributions \( P_{\xi_V}^{-1} \) are consistent,

(ii) \( \xi_V \{1\} + \xi_V \{2\} = \xi_V \{3\} \) a.s. whenever \( V = (B_1, B_2, B_1 \cup B_2) \in B^3 \) with disjoint \( B_1 \) and \( B_2 \),

(iii) \( \xi_{B_n} \overset{d}{=} 0 \) whenever \( B_1, B_2, \ldots \in B \) with \( B_n \to \emptyset \).

PROOF. The necessity is again obvious. Conversely, suppose that (i) - (iii) hold. Define \( U = \{ B \in B: \xi_{3B} = 0 \text{ a.s.} \} \), and verify that \( U \) is a ring. To see that \( U \) has the DC-property, let \( t \in S \) be fixed and put \( S(t,r) = \{ s \in S: \rho(s,t) < r \} \). Choose \( \varepsilon > 0 \) with \( S(t,\varepsilon) \in B \), and suppose that \( \delta > 0 \) is such that

\[
(5) \quad P(\xi_{3S(t,r)} > \delta) > \delta
\]

for infinitely many \( r \in (0, \varepsilon) \), say for \( r_1, r_2, \ldots \). By (i) and Kolmogorov's consistency theorem, there exist some random variables \( \xi_0, \xi_1, \xi_2, \ldots \) such that

\[
(\xi_0, \xi_1, \ldots, \xi_n) \overset{d}{=} \left( \xi_{S(t,\varepsilon)}, \xi_{3S(t,r_1)}, \ldots, \xi_{3S(t,r_n)} \right), \quad n \in \mathbb{N},
\]

and we get by Fatou's lemma
\[ P \left\{ \sum_{i=1}^{\infty} x_i = \infty \right\} \geq \limsup_{n \to \infty} P [ x_n \geq \delta ] \geq \limsup_{n \to \infty} P [ x_n > \delta ] \geq \delta , \]

contradicting the fact that, by (ii), \( \sum_{i=1}^{\infty} x_i \leq x_0 < \infty \) a.s. Hence (5) can hold with fixed \( \delta > 0 \) for at most finitely many \( r \in (0, \varepsilon) \), and so the set \( \{ r \in (0, \varepsilon) : S(t, r) = 0 \text{ a.s.} \} \) must be dense in \((0, \varepsilon)\). This proves that \( \mathcal{U} \) contains a base and hence has the DC-property.

Applying Theorem 5.3, it is seen that there exists some random measure \( \xi \) on \( S \) satisfying

\[ \xi_{V} \overset{d}{=} \xi_{V} , \quad V \in \mathcal{U}^k , \quad k \in \mathbb{N}. \]

To extend (6) to \( B \), let \( k \in \mathbb{Z}_+ \) and \( V \in \mathcal{U}^k \) be fixed and let \( B, B_1, B_2, \ldots \in \mathcal{B} \) with \( B_n \uparrow B \). By Theorem 4.4 in Billingsley (1968) we get

\[ \xi_{V, B, B \setminus \bigcup_{n} B_n} \overset{d}{=} \xi_{V, B, \emptyset} , \]

and so by continuity

\[ \xi_{V, B_n} \overset{d}{=} \xi_{V, B} . \]

A similar argument shows that (7) is also true when \( B_n \uparrow B \). On the other hand

\[ \xi(V, B_n) \rightarrow \xi(V, B) \text{ a.s.}, \quad B_n \rightarrow B , \]

so by the monotone class theorem in Loève (1963) we have \( \xi(V, B) \overset{d}{=} \xi_{V, B} \), \( B \in \mathcal{B} \). Proceeding inductively, we obtain the desired extension of (6).
Turning to the existence problem for point processes, let us first consider a necessary condition. Define for any set function $\phi$ on $\mathcal{B}$

$$\Lambda_A \phi(B) = \phi(A \cup B) - \phi(B) , \quad A, B \in \mathcal{B}.$$ 

Similarly, let $\Lambda_{A_1} \ldots \Lambda_{A_n} \phi(B)$ be the set function obtained by forming successive differences with respect to $\mathcal{B}$. Say that $\phi$ is completely monotone if

$$(-1)^n \Lambda_{A_1} \ldots \Lambda_{A_n} \phi(B) \geq 0 , \quad A_1, \ldots, A_n, B \in \mathcal{B} , \quad n \in \mathbb{N}.$$ 

**Lemma 5.5.** Let $\xi$ be a random measure on $\mathcal{S}$ and let $\tau \in (0, \infty]$ be fixed. Then the set function $\mathbb{E} e^{-\tau \xi_B}$, $B \in \mathcal{B}$, is completely monotone.

**Proof.** Let us first suppose that $\tau = \infty$. For the first order difference we get

$$0 \leq P(\xi A > 0, \xi B = 0) = P(\xi B = 0) - P(\xi A = 0, \xi B = 0) =$$

$$= P(\xi B = 0) - P(\xi (A \cup B) = 0) = -\Lambda_A P(\xi B = 0),$$

and continuing inductively, we obtain

$$0 \leq P(\xi A_1 > 0, \ldots, \xi A_n > 0, \xi B = 0) = (-1)^n \Lambda_{A_1} \ldots \Lambda_{A_n} P(\xi B = 0) , \quad n \in \mathbb{N}.$$ 

In the case $\tau < \infty$, let $\eta$ be a Cox process directed by $t \xi$ and note that

$$P\left\{\eta B = 0\right\} = \mathbb{E} e^{-t \xi_B} , \quad B \in \mathcal{B}.$$
Now suppose conversely that $\phi$ is a completely monotone set function defined on the DC-ring $U$ and such that $\phi(U_n) \to 1$ whenever $U, U_1, U_2, \ldots \in U$ with $U_n \to U$. Let $V \in U$ be arbitrary and let $U_{n1}, \ldots, U_{nk_n} \in U$, $n \in N$, be a null-array of nested disjoint partitions of $V$. For fixed $n \in N$, choose non-random points $s_{nj} \in U_{nj}$, $j = 1, \ldots, k_n$. Let $\xi_n$ be a simple point process on $S$ with all its mass confined to $s_{n1}, \ldots, s_{nk_n}$ and such that

\[
(8) \quad P\left\{ \xi_n(s_{nj}) = 1, j \in J; \xi_n(s_{nj}) = 0, j \notin J \right\} = \frac{P\left\{ \xi_n U_{nj} > 0, j \in J; \xi_n U_{nj} = 0 \right\}}{\phi\left\{ \left( \prod_{j \in J} \left( -A_{U_{nj}} \right) \right) \right\}} = \left\{ \prod_{j \in J} \left( -A_{U_{nj}} \right) \right\} \phi\left\{ U_{nj} \right\}
\]

for each subset $J \subset \{1, \ldots, k_n\}$. Note that $\xi_n$ exists, since the right-hand sides of (8) are non-negative, and summation over $J$ yields 1. (The latter fact is easily established by induction in the number of partitioning sets.) By (8),

\[
P\{\xi_n U = 0\} = \phi(U), \quad U \in U_n, \quad n \in N,
\]

where $U_n$ is the ring generated by $U_{n1}, \ldots, U_{nk_n}$, so (1) is satisfied with $U$ replaced by $U' = U U_n$. If $\{\xi_n\}$ is known to be tight, it follows as in the proof of Theorem 5.3 that (2) is satisfied for some point process $\xi$ on $S$. Since (2) remains true for $\xi^*$, we reach the following result.

**Theorem 5.6.** Let $\phi$ be a set function on some DC-ring $U \subset B$. Then there exists some simple point process $\xi$ on $S$ satisfying (2) and such
that $U \in B_\xi$ iff

(i) $\phi$ is completely monotone on $U$,

(ii) $\phi(U_n) \to 1$ whenever $U, U_1, U_2, \ldots \in U$ with $U_n \to U$,

(iii) for every $V \in U$ there exists some null-array $(U_{nj}) \subset U$ of nested disjoint partitions of $V$ such that the sequence of point processes $\xi_n$ defined by (8) is tight.

NOTES. Theorem 5.4 is due with an entirely different proof to Harris (1968) and (1971). For point processes, a much more general result (containing in particular the point process case of Theorem 5.3) is given by Kerstan, Matthes and Mecke (1974), page 17. See also Prohorov (1961) and Mecke (1972) for further general existence criteria. Finally, Theorem 5.6 is contained in the general results by Kurtz (1973) and Karbe (1973), (cf. Kerstan, Matthes and Mecke (1974), page 35).

Our approach via Lemmas 5.1 and 5.2 seems to be new, although the idea to prove existence by means of tightness arguments is well-established in other fields, (see e.g. Billingsley (1968)).

PROBLEMS.

5.1. Prove Lemma 5.2 and the corresponding result for diffuse random measures.

5.2. Show that, in the particular case of non-random $\xi_V$, Theorem 5.3 reduces to a version of Carathéodory's extension theorem.
5.3. Prove directly from (i) and (ii) of Theorem 5.4 that (ii) holds with any finite number of terms. By (i) and Kolmogorov's consistency theorem, \( P_{\xi V}^{-1} \) can be consistently defined even for \( V \in \mathcal{B}^\infty \). Show by means of (iii) that (ii) then extends to infinite sums.

5.4. Suppose that \( S \) is discrete, i.e. that all its points are isolated, and let \( \phi \) be a set function on \( \mathcal{U} = \mathcal{B} \). Show that (2) holds for some point process \( \xi \) on \( S \) iff \( \phi \) is completely monotone and \( \phi(\emptyset) = 1 \).

5.5. Show that Theorem 5.6 is false in general without condition (ii).
(Hint: let \( S = \mathbb{R} \) and place a fictitious atom at the point \( 0^+ \).
Let \( \mathcal{U} \) be the ring generated by all intervals \((a,b] \).)

5.6. Show that Theorem 5.6 is also false in general without condition (iii). (Hint: let \( \xi \) be a stationary "point process" on the unit circle, whose atom positions have exactly one limit point.)

5.7. Show by a direct argument that the set function \( \mathbb{E}e^{-t\xi_B} \), \( B \in \mathcal{B} \), is completely monotone for any random measure \( \xi \) and fixed \( t < \infty \).

6. NULL-ARRAYS AND INFINITE DIVISIBILITY

In this section we consider the general central limit problem for random measures, i.e. the problem of convergence in distribution of sums

\[
\sum_j \xi_{nj}, \text{ where the } \xi_{nj}, j, n \in \mathbb{N}, \text{ form a null-array of random measures in the sense that the } \xi_{nj} \text{ are independent for fixed } n \text{ and such that }
\]

\[
\lim_{n \to \infty} \sup \mathbb{P}(\xi_{nj} B > \varepsilon) = 0, \quad \varepsilon > 0, \quad B \in \mathcal{B}.
\]
Note that, in the particular case of point processes, (1) reduces to

$$\lim_{n \to \infty} \sup_{j} P(\xi_{nj} > 0) = 0, \quad B \in B.$$

Just as for distributions on $\mathbb{R}$, the class of limiting distributions of the sums $\sum_{j} \xi_{nj}$ coincides with the class of infinitely divisible distributions. A random measure $\xi$ is said to be infinitely divisible$^4$ if, for every fixed $n \in N$, we have $\xi \overset{d}{=} \xi_1 + \ldots + \xi_n$ for some independent and identically distributed random measures $\xi_1, \ldots, \xi_n$. In the point process case, we require $\xi_1, \ldots, \xi_n$ to be point processes. Note that, since a point process may be regarded as an $N$-valued random measure, we have two notions of infinite divisibility for point processes, (which are not equivalent since the measure $\mu$ may belong to $N$ even for $\mu$ in $M\setminus N$). If nothing else is stated, we mean infinite divisibility as a point process.

In our derivation of canonical representations and convergence criteria, we shall proceed in steps, beginning with the simple particular case of $\mathbb{R}_+$-valued random variables. In this case, (1) takes the form

$$\lim_{n \to \infty} \sup_{j} P(\xi_{nj} > \varepsilon) = 0, \quad \varepsilon > 0.$$

We shall use $\kappa$ to denote the function $1 - e^{-x}$. Furthermore, the projections $\pi_t$, $\pi_f$, and $\pi_V$ are defined by

$$\pi_t x = x \cdot t, \quad \pi_f \mu = \mu f, \quad \pi_V \mu = \mu V.$$

---

$^4$ For a different notion of infinite divisibility, see O. Kallenberg, Infinitely divisible processes with interchangeable increments and random measures under convolution, Tech. Report, Dept. of Mathematics, Göteborg, 1974.
Lemma 6.1. An $\mathbb{R}_+$-valued random variable $\xi$ is infinitely divisible iff

\begin{equation}
-\log \mathbb{E} e^{-\xi t} = ta + \lambda(1 - e^{-\pi t}), \quad t \geq 0,
\end{equation}

for some $a \geq 0$ and some $\lambda \in M(\mathbb{R}_+)$ with $\lambda < \infty$, and in that case $a$ and $\lambda$ are unique. Moreover, any $a$ and $\lambda$ with the stated properties may occur in (2). If $\{\xi_{nj}\}$ is a null-array of $\mathbb{R}_+$-valued random variables, then $\sum_j \xi_{nj} \overset{d}{=} \text{some } \xi$ iff

\begin{equation}
\kappa \sum_j \mathbb{P} e^{-\xi_{nj}} \overset{d}{=} \text{some } \alpha \delta_0 + \kappa \lambda,
\end{equation}

and then $\xi$, $\alpha$ and $\lambda$ satisfy (2). In the case of $\mathbb{L}_+$-valued random variables, (2) holds with $\alpha = 0$ and bounded $\lambda \in M(N)$, while (3) is equivalent to $\sum_j \mathbb{P} e^{-\xi_{nj}} \overset{d}{=} \lambda$ on $N$.

Note that (3) is equivalent to the conditions

(i) $\sum_j \mathbb{P} e^{-\xi_{nj}} \overset{d}{=} \lambda$ on $\mathbb{R}_+$ with $\lambda\{\infty\} = 0$,

(ii) $\lim_{\epsilon \to 0} \limsup_{n \to \infty} \sum_j \mathbb{E} \xi_{nj} = \alpha$,

where $\mathbb{R}_+ = (0, \infty]$, $\limsup$ stands for both $\limsup$ and $\liminf$, and $\mathbb{E}_\epsilon$ denotes the truncated expectation at the level $\epsilon > 0$.

Proof. Writing $\phi_{nj} = L_{\xi_{nj}}$, we claim that $\sum_j \xi_{nj} \overset{d}{=} \text{some } \xi$ iff

\begin{equation}
\psi_n = \sum_j (1 - \phi_{nj}) \overset{d}{=} \text{some continuous } \psi,
\end{equation}

and that in this case $\phi = L_{\xi} = e^{-\psi}$. In fact, $\sum_j \xi_{nj} \overset{d}{=} \xi$ implies...
\[ \pi \phi_{nj} \xrightarrow{\psi} \phi, \text{ whence} \]
\[ -\sum_j \log \phi_{nj} - \log \phi = \psi, \]
and here \( \psi \) is continuous since \( \phi \) is an L-transform. Since (1) is clearly equivalent to \( \max_j (1 - \phi_{nj}) \to 0 \), (4) follows from this by considering a Taylor expansion of \( \log \phi_{nj} \). Conversely, by a similar argument, (4) implies \( \pi \phi_{nj} \to \text{some continuous } \phi \), and it follows that \( \sum_j \xi \xrightarrow{\psi} \text{some } \xi \).

We next introduce the measures \( \lambda_n = \sum_j \frac{\xi_{nj}}{\phi_{nj}}, n \in N \), on \( R^+ \) and note that for \( t \geq 0 \),
\[ \psi_n(t) = \sum_j \left(1 - e^{-t\xi_{nj}}\right) = \sum_j E(1 - e^{-t\xi_{nj}}) = \sum_j \int_0^\infty (1 - e^{-tx}) \phi_{nj}^{-1}(dx) = \int_0^\infty (1 - e^{-tx}) \lambda_n(dx) = \int_0^\infty \frac{1-e^{-tx}}{1-e^{-x}} (\kappa \lambda_n)(dx). \]
Hence (3) implies (4) with \( \psi \) given by
\[ \psi(t) = ta + \int_0^\infty \frac{1-e^{-tx}}{1-e^{-x}} (\kappa \lambda)(dx) = ta + \int_0^\infty (1 - e^{-tx}) \lambda(dx), \ t \geq 0, \]
\( \psi \) being continuous by monotone convergence. Conversely, suppose that (4) holds for some continuous \( \psi \), and note that
\[ \psi_n(t+1) - \psi_n(t) = \int_0^\infty (e^{-tx} - e^{-(t+1)x}) \lambda_n(dx) = \int_0^\infty e^{-tx} (\kappa \lambda)(dx). \]
If \( \psi(1) \neq \psi(0) \), we may apply the continuity theorem for L-transforms to the probability measures \( \kappa \lambda_n / \lambda_n \kappa \), and conclude that \( \kappa \lambda_n \) converges weakly on \( R^+ \). But this is also true when \( \psi(1) = \psi(0) \), since then
\[ \lambda_n^\kappa = \psi_n(1) - \psi_n(0) + \psi(1) - \psi(0) = 0. \]

The uniqueness of \( \alpha \) and \( \lambda \) in (2) follows from the formula

\[
\psi(t + 1) - \psi(t) = \int_0^\infty e^{-t\alpha} (\delta_0 + \kappa \lambda) (dx)
\]

and the uniqueness theorem for L-transforms. To see that every choice of \( \alpha \) and \( \lambda \) is possible, use (3) with \( \lambda_n = [n^{-1}, \infty) \lambda, \ n \in \mathbb{N} \), to construct a \( \xi \) with \( \alpha = 0 \) and arbitrary \( \lambda \), and then replace \( \xi \) by \( \xi + \alpha \). In particular, the infinite divisibility of \( \xi \) in (2) follows by dividing this relation by an arbitrary \( n \in \mathbb{N} \). Conversely, any infinitely divisible L-transform \( \phi = L_{\xi} \) can be defined be written in the form \( \phi = \phi_n^j \) for every \( n \in \mathbb{N} \), so the above argument applies with \( \phi_n^{j} = \phi_n \), \( j = 1, \ldots, n \), showing that \( \xi \) has indeed the representation (2). The last assertion follows easily from (3).

We state explicitly the corresponding tightness criterion.

**LEMMA 6.2.** Let \( \{\xi_{nj}\} \) be a null-array of \( \mathbb{R}_+ \)-valued random variables. Then \( \{\sum_j \xi_{nj}\} \) is tight iff

1. \( \limsup_{n \to \infty} \sum_j E_t \xi_{nj} < \infty, \ t > 0, \)

2. \( \lim \limsup_{t \to \infty} \sum_{n \to \infty} P[\xi_{nj} > t] = 0. \)

If the \( \xi_{nj} \) are \( \mathbb{Z}_+ \)-valued, then (i) may be replaced by the condition

3. \( \limsup_{n \to \infty} \sum_j P[\xi_{nj} > 0] < \infty. \)
PROOF. The sequence \( \{ \kappa \lambda_n \} \) of the last proof is vaguely relatively compact on \( R_+ \) iff \( (\kappa \lambda_n) [0,t] \) is bounded for each fixed \( t > 0 \), i.e. iff (i) holds. The role of (ii) is to prevent mass from escaping to \( +\infty \), thus ensuring every vaguely convergent sub-sequence to be weakly convergent. For point processes, (i) and (ii) imply (i)', which in turn implies (i), so in this case, (i)' may replace (i).

We now turn to the m-dimensional version of Lemma 6.1, \( m \in \mathbb{N} \).

Let \( \overline{R}_+^m \) be a one-point compactification of \( R_+^m \), and for \( x,t \in R_+^m \), write \( xt = x \cdot t \) for the inner product of \( x \) and \( t \).

**Lemma 6.3.** An \( R_+^m \)-valued random vector \( \xi \) is infinitely divisible iff

(2) holds for some \( \alpha \in R_+^m \) and some \( \lambda \in M(R_+^m \setminus \{0\}) \) with \( (\lambda \pi_t^{-1}) \kappa \leq \infty \), \( t \in R_+^m \), and in that case \( \alpha \) and \( \lambda \) are unique. Moreover, any \( \alpha \) and \( \lambda \) with the stated properties may occur in (2). If \( \{ \xi_{nj} \} \) is a null-array of \( R_+^m \)-valued random vectors, then \( \sum_j \xi_{nj} \) is some \( \xi \) iff there exist some \( \alpha \) and \( \lambda \) as above such that

\[
\text{(i)} \quad \sum_j P \xi_{nj}^{-1} \not\subset \lambda \quad \text{on} \quad \overline{R}_+^m \setminus \{0\} ,
\]

\[
\text{(ii)} \quad \lim_{\epsilon \to 0} \limsup_{n \to \infty} \sum_j E \xi_{nj}^k = \alpha(k) \quad \text{on} \quad R_+^m \setminus \{0\} ,
\]

and then \( \xi \), \( \alpha \) and \( \lambda \) satisfy (2). In the case of \( Z_+^m \)-valued random vectors, (2) holds with \( \alpha = 0 \) and bounded \( \lambda \in M(Z_+^m \setminus \{0\}) \), and we may replace (i) and (ii) by the condition \( \sum_j P \xi_{nj}^{-1} \not\subset \lambda \quad \text{on} \quad Z_+^m \setminus \{0\} . \)
PROOF. Let \( \{\xi_{nj}\} \) be a null-array satisfying (i) and (ii) for some \( \alpha \) and \( \lambda \) with the stated properties. Then it is easily seen that

\[
\kappa \sum_j P(t\xi_{nj})^{-1} \forall \theta_0 + \kappa(\lambda t^{-1})^\pi, \quad t \in \mathbb{R}^m_+,
\]

so by Lemma 6.1,

\[
E \exp(-\sum_j t\xi_{nj}) = \exp(-t\alpha - \lambda(1 - e^{-t})) \quad t \in \mathbb{R}^m_+,
\]

and it follows by the continuity theorem for \( m \)-dimensional L-transforms that \( \sum_j \xi_{nj} \) \( \overset{d}{\to} \) some \( \xi \) satisfying (2).

Conversely, suppose that \( \sum_j \xi_{nj} \) \( \overset{d}{\to} \) some \( \xi \). By Lemma 6.2, there exist for every sequence \( N' \subseteq N \) some subsequence \( N'' \subseteq N' \) and some \( \alpha \) and \( \lambda \) the stated properties such that (i) and (ii) hold as \( n \to \infty \) through \( N'' \). By the direct assertion of the present lemma, it follows that \( \xi, \alpha \) and \( \lambda \) satisfy (2). Now at is unique for every \( t \in \mathbb{R}^m_+ \) by Lemma 6.1, and therefore \( \alpha \) must be unique. As for \( \lambda \), write

\[
1 = (1, \ldots, 1) \in \mathbb{R}^m_+ \quad \text{and note that by (2)}
\]

\[
\int e^{-tx}(1 - e^{-1x})\lambda(dx) = \log E e^{-\xi t} - \log E e^{-\xi(t+1)} - \lambda, \quad t \in \mathbb{R}^m_+.
\]

Applying the uniqueness theorem for \( m \)-dimensional L-transforms, it follows that the bounded measure \( (1 - e^{-1})\lambda \) is unique, and hence so is \( \lambda \) since \( 1 - e^{-1} > 0 \) on \( \mathbb{R}^m_+ \setminus \{0\} \). This proves that \( \alpha \) and \( \lambda \) are independent of the choice of \( N' \), and hence that (i) and (ii) remain true as \( n \to \infty \) through \( N \), (cf. Theorem 2.3 in Billingsley (1968)). The remaining assertions now follow as in the proof of Lemma 6.1.
Before proceeding to arbitrary random measures, we prove a lemma.

**Lemma 6.4.** Let \( \xi \) be a random measure on \( S \) and let \( I \in \mathcal{B} \) be a DC-semiring. Then \( \xi \) is infinitely divisible iff \( \xi^V \) is for every \( V \in I^k, k \in \mathbb{N} \). In the point process case it suffices that \( \sum_j \xi_{I_j} \) be infinitely divisible for every \( I_1, \ldots, I_k \in I, k \in \mathbb{N} \).

**Proof.** The necessity of our conditions is obvious. Conversely, suppose that \( \xi^V \) is infinitely divisible in \( \mathbb{R}^k_+ \) for every \( V \in I^k, k \in \mathbb{N} \).

Since infinite divisibility of random vectors is preserved under convergence in distribution, (as may be seen from the continuity theorem for L-transforms), we may use the monotone class theorem in Loève (1963) to extend this property to arbitrary \( V \in \mathcal{B}^k, k \in \mathbb{N} \). It follows that there exist for every \( n \in \mathbb{N} \) some random vectors \( \xi^V_n, \) satisfying

\[
(L_{\xi^V})^{-n} = L_{\xi^V_n}, \quad V \in \mathcal{B}^k, \quad k, n \in \mathbb{N}.
\]

From (5) it is easily seen that the family \( \{\xi^V_n\} \) satisfies the conditions in Theorem 5.4, and hence there exists some random measure \( \xi_n \) on \( S \) with \( \xi^V_n \overset{d}{=} \xi^V_n, \quad V \in \mathcal{B}^k, \quad k \in \mathbb{N} \). Writing (5) in the form

\[
L_{\xi^V} = (L_{\xi^V_n})^n, \quad V \in \mathcal{B}^k, \quad k, n \in \mathbb{N},
\]

and using Theorem 3.1, it is seen that \( \xi \) is infinitely divisible.

Turning to the point process case, suppose that \( \sum_j \xi_{I_j} \) is infinitely divisible (as a random variable in \( \mathbb{Z}_+ \)) for every \( I_1, \ldots, I_k \in I, k \in \mathbb{N} \). Since repetitions of \( I \)-sets may occur, this proves that, for
any fixed $V = (I_1, \ldots, I_k) \in I^k$, $k \in \mathbb{N}$, $t \xi V = \sum_j t_j \xi I_j$ is infinitely divisible in $\mathbb{Z}_+$ for every $t = (t_1, \ldots, t_k) \in \mathbb{N}^k$. Turning to rationals and using the above closure property, it follows that $t \xi V$ is indeed infinitely divisible in $R_+$ for every $t = (t_1, \ldots, t_k) \in R_+^k$. Hence, by Lemma 6.1, there exist some constants $\lambda_p \geq 0$, $p = (p_1, \ldots, p_k) \in Z_+^k \setminus \{0\}$, such that

$$- \log E e^{-ut \xi V} = \sum_p (1 - e^{-up})^\lambda_p, \quad u \in R_+.$$  

In fact, there can be no constant component $\alpha$ here since the $\xi I_j$ are infinitely divisible in $\mathbb{Z}_+$ and hence $P(\xi I_j = 0) > 0$. Moreover, making a Taylor expansion in (2), it is easily seen that the support of $\lambda$ is contained in the support of $P_{\xi}^{-1}$. (This is a simple special case of Theorem 6.9 below.) Now suppose that $\eta$ is an infinitely divisible random vector in $Z_+^k$ with spectral measure $\lambda = \sum_p \lambda_p \delta_p$, the existence of which is ensured by Lemma 6.3 since $\sum_p \lambda_p < \infty$. Then $L_t \eta$ coincides with $L_t \xi V$ above, and so $t \eta \overset{d}{=} t \xi V$. If the $t_j$ are chosen rationally independent, it follows that $\eta \overset{d}{=} \xi V$, which proves that $\xi V$ is infinitely divisible.

Since $V$ was arbitrary, we may now proceed as in the first part of the proof to conclude that $\xi$ itself is infinitely divisible.

We are now ready to extend Lemmas 6.1 and 6.3 to random measures defined on general spaces $S$.

**Theorem 6.5.** A random measure $\xi$ on $S$ is infinitely divisible iff

$$- \log E e^{-\xi f} = \alpha f + \lambda(1 - c^{-\eta f}), \quad f \in F,$$  

for any fixed $(I_1, \ldots, I_k) \in I^k$, $k \in \mathbb{N}$, $t \xi V = \sum_j t_j \xi I_j$ is infinitely divisible in $\mathbb{Z}_+$ for every $t = (t_1, \ldots, t_k) \in \mathbb{N}^k$. Turning to rationals and using the above closure property, it follows that $t \xi V$ is indeed infinitely divisible in $R_+$ for every $t = (t_1, \ldots, t_k) \in R_+^k$. Hence, by Lemma 6.1, there exist some constants $\lambda_p \geq 0$, $p = (p_1, \ldots, p_k) \in Z_+^k \setminus \{0\}$, such that

$$- \log E e^{-ut \xi V} = \sum_p (1 - e^{-up})^\lambda_p, \quad u \in R_+.$$  

In fact, there can be no constant component $\alpha$ here since the $\xi I_j$ are infinitely divisible in $\mathbb{Z}_+$ and hence $P(\xi I_j = 0) > 0$. Moreover, making a Taylor expansion in (2), it is easily seen that the support of $\lambda$ is contained in the support of $P_{\xi}^{-1}$. (This is a simple special case of Theorem 6.9 below.) Now suppose that $\eta$ is an infinitely divisible random vector in $Z_+^k$ with spectral measure $\lambda = \sum_p \lambda_p \delta_p$, the existence of which is ensured by Lemma 6.3 since $\sum_p \lambda_p < \infty$. Then $L_t \eta$ coincides with $L_t \xi V$ above, and so $t \eta \overset{d}{=} t \xi V$. If the $t_j$ are chosen rationally independent, it follows that $\eta \overset{d}{=} \xi V$, which proves that $\xi V$ is infinitely divisible.

Since $V$ was arbitrary, we may now proceed as in the first part of the proof to conclude that $\xi$ itself is infinitely divisible.

We are now ready to extend Lemmas 6.1 and 6.3 to random measures defined on general spaces $S$.

**Theorem 6.5.** A random measure $\xi$ on $S$ is infinitely divisible iff

$$- \log E e^{-\xi f} = \alpha f + \lambda(1 - c^{-\eta f}), \quad f \in F,$$
for some $\alpha \in \mathcal{M}$ and some measure $\lambda$ on $\mathcal{M}\backslash\{0\}$ satisfying $(\lambda \pi_B^{-1})_B < \infty$, $B \in \mathcal{B}$, and in this case $\alpha$ and $\lambda$ are unique. Moreover, any $\alpha$ and $\lambda$ with the stated properties may occur in (6). If $\{\xi_{nj}\}$ is a null-array of random measures on $S$, then $\sum_j \xi_{nj} \overset{d}{=} \xi$ iff there exist some $\alpha$ and $\lambda$ as above such that

$$
\kappa \sum_j P(\xi_{nj} f)^{-1} \overset{w}{=} \alpha f \delta_0 + \kappa (\lambda \pi_f^{-1}) , \quad f \in F_c ,
$$

and then $\xi$, $\alpha$ and $\lambda$ satisfy (6). If $I \in \mathcal{B}$ is a DC-semiring satisfying $\alpha I = \lambda \{\mu \in I > 0\} = 0$, $I \in I$, then (7) is equivalent to the conditions

$$
\sum_j P(\xi_{nj} V)^{-1} \overset{w}{=} \lambda \pi_V^{-1} \text{ on } R^k_+ \backslash \{0\} , \quad V \in I^k , \quad k \in \mathbb{N} ,
$$

$$
\lim_{I \in I} \lim_{n \to \infty} \sum_j E \xi_{nj} I = \alpha I , \quad I \in I .
$$

In the point process case, we have $\alpha = 0$ in (6) while $\lambda$ is confined to $\mathcal{N}\backslash\{0\}$ and satisfies $(\lambda \pi_B^{-1})_B < \infty$, $B \in \mathcal{B}$. In this case, (8) and (9) can be replaced by (8) alone, provided the vague convergence on $R^k_+ \backslash \{0\}$ is strengthened to weak convergence on $Z^k_+ \backslash \{0\}$.

**PROOF.** Suppose that $\xi$ is an infinitely divisible random measure. Then $\xi V$ is infinitely divisible for every $V = (B_1, \ldots, B_k) \in \mathcal{B}^k$, $k \in \mathbb{N}$, so by Lemma 6.4 there exist some unique $\alpha_V \in R^k_+$ and $\lambda_V \in \mathcal{M}(R^k_+ \backslash \{0\})$ such that

$$
- \log E e^{-t(\xi V)} = t\alpha_V + \lambda_V (1 - e^{-\pi t}) , \quad t \in R^k_+ .
$$
Choosing all components \( t[j] \) but one equal to 0, we obtain
\[
\alpha_V = (\alpha_{B_1}, \ldots, \alpha_{B_k}),
\]
while choosing exactly one component equal to 0, it is seen that \( \{\lambda_V\} \) is in the obvious sense a consistent family of measures. Let us now choose \( V = (A, B, A \cup B) \) in (10), where \( A \) and \( B \) are arbitrary disjoint sets in \( B \). Defining \( \lambda' \in M(\mathbb{R}_+^3 \setminus \{0\}) \) by
\[
\lambda'C \equiv \lambda_{A,B}(x,y) \colon (x,y, x+y) \in C,
\]
we get for any \( t = (u,v,w) \in \mathbb{R}_+^3 \)
\[
u \alpha_A + v \alpha_B + w \alpha_{A \cup B} + \lambda_V(1 - e^{-\pi t}) = -\log Ee^{-t(xV)} =
\]
\[
= -\log Ee^{-(u+w)\xi A-(v+w)\xi B}
\]
\[
= (u+w)\alpha_A + (v+w)\alpha_B + \int (1 - e^{-(u+w)x-(v+w)y}) \lambda_{A,B}(dx \, dy)
\]
\[
= u \alpha_A + v \alpha_B + w(\alpha_A + \alpha_B) + \lambda'(1 - e^{-\pi t}).
\]
By the uniqueness assertion in Lemma 6.4, it follows that

\[
(11) \quad \alpha_{A \cup B} = \alpha_A + \alpha_B
\]

and \( \lambda_V = \lambda' \), and in particular
\[
\lambda_V\{(x,y,z) \in \mathbb{R}_+^3 \colon x + y \neq z\} = \lambda'\{(x,y,z) \in \mathbb{R}_+^3 \colon x + y \neq z\} =
\]
\[
(12) \quad = \lambda_{A,B}\{(x,y) \in \mathbb{R}_+^2 \colon x + y \neq x + y\} = 0.
\]

Next suppose that \( B_1, B_2, \ldots \in B \) with \( B_n \rightarrow \emptyset \), and conclude from (10) that \( \alpha_{B_n} \rightarrow 0 \) while
\[(13) \quad \lambda_{B_n}^{(\varepsilon, \infty)} \to 0, \quad \varepsilon > 0.\]

Combining the former relation with (11), it is seen that the set function 
\[B \to \sigma_B, \quad B \in \mathcal{B},\]
is in fact a measure. As for the family \(\{\lambda_V\}\), let 
\[B \in \mathcal{B}\]  
be fixed and define the measures 
\[\lambda_V^k \in M(\mathbb{R}^k_+), \quad V \in \mathcal{B}_k, \quad k \in \mathbb{N},\]
by
\[\lambda_V^C \equiv \int (1 - e^{-x}) \lambda_{B,V}(dx \times C).\]

Since \(\lambda_V^R \equiv \lambda_B < \infty\) and the consistency of \(\{\lambda_B, V\}\) carries over to 
\(\{\lambda_V\}\), we may apply Theorem 5.4 to the latter family, and conclude from 
(12) and (13) that there exists some unique measure \(\Lambda_B^V\) on \(\{\mu B > 0\} \in \mathcal{M}\)
satisfying \(\Lambda_B^V \pi_V^{-1} \equiv \lambda_V^V\). Defining the measure \(\Lambda_B\) on the same set by
\[\Lambda_B(d\mu) \equiv (1 - e^{-\mu B})^{-1} \Lambda_B^V(d\mu),\]
we obtain for any measurable set \(C \in \mathbb{R}_+^k\)
\[\Lambda_B\{\mu V \in C\} = \int_{\{\mu V \in C\}} (1 - e^{-\mu B})^{-1} \Lambda_B^V(d\mu) = \int_{\mathbb{R}_+^k} (1 - e^{-x})^{-1} \lambda_{B,V}(dx \times C) =
\int_{\mathbb{R}_+^2} (1 - e^{-x})^{-1} (1 - e^{-y}) \lambda_{B,B,V}(dx dy \times C) =
\lambda_{B,B,V}(\mathbb{R}_+^2 \times C) = \lambda_{B,V}(\mathbb{R}_+^k \times C),\]

and it follows that for any \(B' \in \mathcal{B}\) with \(B' \in \mathcal{B}\)
\[\Lambda_B\{\mu V \in C, \mu B' > 0\} = \lambda_{B,B',V}(\mathbb{R}_+^2 \times C) = \lambda_{B',V}(\mathbb{R}_+^k \times C) = \Lambda_{B'}\{\mu V \in C\}.\]

This proves that the measures \(\Lambda_B\) are all restrictions of \(\lambda = \sup_{B \in \mathcal{B}} \Lambda_B\).
It is now easy to verify that this $\lambda$ is the unique measure satisfying (6). Using Theorem 5.4, it is further seen that any $\alpha$ and $\lambda$ with the stated properties define the distribution of some random measure $\xi$ satisfying (6), and it follows in particular that any such $\xi$ is infinitely divisible.

Now suppose that $\{\xi_n\}$ is a null-array of random measures satisfying (7) for some $\alpha$ and $\lambda$ with the stated properties. If $\xi$ is a random measure satisfying (6), it follows by Lemma 6.1 that $\sum_j \xi_n \xi f \stackrel{d}{=} \xi f$, $f \in F_c$, and hence by Theorem 4.5, $\sum_j \xi_n \xi \stackrel{d}{=} \xi$. Conversely, suppose that $\sum_j \xi_n \xi \stackrel{d}{=} \xi$. Then $\xi$ is infinitely divisible by Lemma 6.4, and hence it satisfies (6) for some $\alpha$ and $\lambda$. But then (7) follows by Lemma 6.1. The same argument may be used to show that (8) and (9) are necessary and sufficient, and therefore are equivalent to (7). Finally, the assertions for point processes may easily be deduced from (8).

We shall now give two useful and illuminating interpretations of the canonical representation (6). Let us write $I(\alpha, \lambda)$ for any infinitely divisible random measure $\xi$ satisfying (6). In the point process case we write $I(0, \lambda) = I(\lambda)$.

**Lemma 6.6.** Let $\alpha$ and $\lambda$ be such as in Theorem 6.5 and let $n$ be a Poisson process on $\mathbb{M}\{0\}$ with intensity $\lambda$. Then

(14) \hspace{1cm} I(\alpha, \lambda) \stackrel{d}{=} \alpha + \int \eta_n(du) \, .
PROOF. Denote the left- and right-hand sides of (14) by $\xi$ and $\tilde{\xi}$.

Since for fixed $f \in F$ the mapping $\pi_f: \mu \mapsto \mu f$ is measurable, it follows as in Section 1 that the function $\tilde{\xi} f = \alpha f + \eta \pi_f$ is a possibly infinite-valued random variable, and we obtain by (1.2) and (6)

\begin{equation}
E e^{-\tilde{\xi} f} = E e^{-\alpha f} e^{-\eta \pi_f} = e^{-\alpha f} \exp(-\lambda (1 - e^{-\pi_f})) = E e^{-\xi f}.
\end{equation}

In particular, $\tilde{\xi} B \stackrel{d}{=} \xi B < \infty$ a.s., $B \in \mathcal{B}$, and it follows as in Lemma 1.3 that $\tilde{\xi}$ is a random measure. Finally, (14) follows from (15) by Theorem 3.1.

**Lemma 6.7.** Let $\alpha$ and $\lambda$ be such as in Theorem 6.5, and let $M_1, M_2, \ldots$ be a disjoint partition of $M \setminus \{0\}$ such that $\lambda M_n < \infty$, $n \in N$. For each $n \in N$, we further assume that $\nu_n$ is a Poissonian random variable with mean $\lambda M_n$ and that $\xi_{n1}, \xi_{n2}, \ldots$ are random measures on $M_n$ with the common distribution $M_n \lambda/\lambda M_n$.

Letting all these random elements be independent, we have

\begin{equation}
I(\alpha, \lambda) \stackrel{d}{=} \alpha + \sum_{n \in N} \sum_{j \leq \nu_n} \xi_{nj}.
\end{equation}

PROOF. The right-hand side of (16) has the L-transform, evaluated at an arbitrary $f \in F$,
\[ e^{-\alpha f} \prod_{n=1}^{\infty} e^{-\lambda M_n} \sum_{k=0}^{\infty} \frac{(\frac{\lambda M_n}{\lambda M_n})^k}{k!} \left( \frac{M_n \lambda}{\lambda M_n} \right)^{e^{-\pi f}} \left( \frac{1}{\lambda M_n} \right)^k = e^{-\alpha f} \prod_{n=1}^{\infty} e^{-\lambda M_n} \exp \left\{ -(M_n \lambda)(1 - e^{-\pi f}) \right\} = e^{-\alpha f} \exp \left\{ -\sum_{n=1}^{\infty} (M_n \lambda)(1 - e^{-\pi f}) \right\} = e^{-\alpha f} \exp \left\{ -\lambda(1 - e^{-\pi f}) \right\} = \exp \left\{ -\alpha f - \lambda(1 - e^{-\pi f}) \right\}, \]

and by Theorem 6.5, this agrees with the L-transform of \( I(\alpha, \lambda) \). Thus the assertion follows by Lemma 1.3 and Theorem 3.1.

The last two lemmas may be used to establish various relationships between an infinitely divisible random measure \( \xi \) and the corresponding canonical measures \( \alpha \) and \( \lambda \). We shall give two results of this type.

**Theorem 6.8.** Let \( \xi \overset{d}{=} I(\alpha, \lambda) \) on \( S \) and let \( \alpha > 0 \). Then \( \xi^*_\alpha = 0 \) a.s. iff (i) \( \alpha^*_\alpha = 0 \), (ii) \( \lambda\{\mu^*_\alpha \neq 0\} = 0 \), and (iii) \( \lambda\{\mu(s) > 0\} = 0 \), \( s \in S \). In particular, \( \xi \in M_\lambda \) a.s. iff \( \alpha \in M_\lambda \) and \( \lambda M_\lambda = 0 \).

**Proof.** Suppose that \( \xi^*_\alpha = 0 \) a.s. By Lemma 6.7 we obtain (i), and furthermore, \( (\xi^*_{nj})^*_\alpha = 0 \) a.s. whenever \( \lambda M_n > 0 \). But then

\[ M_n \lambda\{\mu_n \neq 0\} = 0, \quad n \in \mathbb{N}, \quad \text{and summing over } n \ \text{yields } (ii). \] To prove (iii), suppose conversely that \( M_n \lambda\{\mu(s) > \epsilon\} > 0 \) for some \( n \in \mathbb{N}, \ s \in S \) and \( \epsilon > 0 \). Then we get the contradiction

\[ P\{\xi^*_\alpha > 0\} \geq P\{\xi(s) > 0\} \geq P\left\{ v_n > \left[ a/\epsilon \right], \ \xi_{nj}(s) > \epsilon, \ j \leq \left[ a/\epsilon \right] + 1 \right\} = P\{v_n > \left[ a/\epsilon \right]\} P\left\{ \xi_{nj}(s) > \epsilon \right\} \left[ a/\epsilon \right]+1 > 0, \]
so (iii) must indeed be true.

Conversely, suppose that (i) - (iii) hold. From (ii) it follows that \( (\xi_{nj})^* \equiv 0 \) a.s. whenever \( \lambda M_n > 0 \), so it suffices to prove that, with probability one, no two terms in (16) can have atom positions in common. But this follows by Fubini's theorem from the fact, implied by (iii), that \( \xi_{nj} C = 0 \) a.s. for any countable set \( C \subset S \).

To state the next result, let \( S(\lambda) \) denote the support of a measure \( \lambda \) on \( M \) w.r.t. the vague topology in \( M \). Moreover, given any \( M \subset M \), define \( G_M \) as the additive semigroup in \( M \) generated by \( M \cup \{0\} \), and write \( \alpha = \min M \) whenever \( \alpha \in M \) and \( \alpha \leq \mu, \mu \in M \).

**THEOREM 6.9.** Let \( \xi \overset{d}{=} I(\alpha, \lambda) \) on \( S \). Then

\[
\alpha = \min S(P_{\xi^{-1}}),
\]

(17)

\[
\sup(\lambda) = S(P_{\xi^{-1}}) - \alpha.
\]

(18)

**PROOF.** To avoid trivialities, we may assume that \( \lambda \neq 0 \). Let us first consider the case of bounded \( \lambda \). By Lemma 6.7, we may then assume that

\[
\xi \overset{d}{=} \alpha + \sum_{j=1}^{\nu} \xi_j,
\]

(19)

where \( \nu \) is Poissonian with mean \( \lambda M \) while the \( \xi_j \) are independent of \( \nu \) and mutually independent with distribution \( \lambda/\lambda M \). From (19) it is seen that \( \xi \geq \alpha \) a.s. and \( P(\xi = \alpha) = e^{-\lambda M} > 0 \), proving (17). Since addition and subtraction of \( \alpha \) are continuous operations on \( M \) and
M + α respectively, we further obtain $S(P_\xi^{-1}) - α = S(P(ξ - α)^{-1})$, and so it suffices to prove (18) in the case $α = 0$.

Suppose that $μ_0 \in GS(λ) \setminus \{0\}$, say $μ_0 = μ_1 + ... + μ_k$, where $k ∈ N$ and $μ_1, ..., μ_k ∈ S(λ)$. By the continuity of addition in $M$, there exists for every $ε > 0$ some $δ > 0$ such that

$$P(ρ(ξ_0, μ_0) < ε) ≥ P(ν = k, ρ(ξ_j, μ_j) < δ, j = 1, ..., k) =$$

$$= P(ν = k) \prod_{j=1}^{k} P(ρ(ξ_j, μ_j) < δ) =$$

$$= e^{-λM} \frac{(λM)^k}{k!} \prod_{j=1}^{k} λ{ρ(μ, μ_j) < δ} / λM > 0,$$

proving that $μ_0 ∈ S(P_ξ^{-1})$. Since $S(P_ξ^{-1})$ is closed, this proves that $\overline{GS(λ)} ⊂ S(P_ξ^{-1})$.

Conversely, suppose that $μ_0 ∈ S(P_ξ^{-1})$. For $n ∈ N$, let $η_n$ be obtained as $ξ$ in (19) but with $ν$ replaced by $ν ∧ n$. Then $η_n \xrightarrow{P} ξ$ a.s., so we get

$$0 < P(ρ(ξ_0, μ_0) < ε) ≤ \liminf_{n→∞} P(ρ(η_n, μ_0) < ε), \ ε > 0,$$

and hence there exist some $μ_n ∈ S(Pn^{-1})$, $n ∈ N$, with $μ_n \xrightarrow{P} μ_0$. If we can show that $μ_n ∈ GS(λ)$, $n ∈ N$, it will follow that $μ_0 ∈ \overline{GS(λ)}$, proving the relation $S(P_ξ^{-1}) ⊂ \overline{GS(λ)}$. Keeping $n ∈ N$ fixed, we thus assume that $m ∈ S(Pn^{-1})$. Then there exists for every $p ∈ N$ some $k ∈ \{0, 1, ..., n\}$ satisfying
(21) \(0 < \mathbb{P}(\rho(n, m) < p^{-1}, \nu = k) = \mathbb{P}(\rho(\xi_1 + \ldots + \xi_k, m) < p^{-1}, \nu = k)\)
\[= \mathbb{P}(\rho(\xi_1 + \ldots + \xi_k, m) < p^{-1}) \mathbb{P}(\nu = k),\]
and since the set of possible \(k\)-values is non-increasing in \(p\), we can choose \(k\) independent of \(p\). Moreover, \(k = 0\) implies \(m = 0 \in \text{GS}(\lambda)\), so we can further assume that \(k \neq 0\). Let us now define the sets
\[A_p = \{(\mu_1, \ldots, \mu_k) \in M^k: \rho(\mu_1 + \ldots + \mu_k, m) < p^{-1}\}, \quad p \in \mathbb{N}.
\]
Since \(M^k\) is separable in the product topology, each \(A_p\) can be covered by countably many sets
\[\{(\mu_1, \ldots, \mu_k) \in M^k: \rho(\mu_j, m) < p^{-1}, j = 1, \ldots, k\}\]
with \((m_1, \ldots, m_k) \in A_p\), so by (21) there must exist some vector \((m_{1p}, \ldots, m_{kp}) \in A_p\) satisfying
(22) \(0 < \mathbb{P}(\rho(\xi, m_{jp}) < p^{-1}, j = 1, \ldots, k) = \prod_{j=1}^{k} \mathbb{P}(\rho(\xi_j, m_{jp}) < p^{-1}).\)

From \((m_{1p}, \ldots, m_{kp}) \in A_p\) we get \(m_{1p} + \ldots + m_{kp} \leq m\) as \(p \to \infty\), so
\[\limsup_{p \to \infty} m_{jp} f \leq \lim_{p \to \infty} (m_{1p} + \ldots + m_{kp}) f = mf < \infty, \quad f \in F_c, \quad j = 1, \ldots, k,
\]
showing that the sequences \(\{m_{1p}\}, \ldots, \{m_{kp}\}\) are relatively compact. We may therefore choose some sequence \(N' \subset \mathbb{N}\) and some \(m_1, \ldots, m_k \in \mathcal{M}\)
such that \(m_{jp} \leq m_j\) \((p \in N')\), \(j = 1, \ldots, k\). It follows in particular that \(m_{1p} + \ldots + m_{kp} \leq m_1 + \ldots + m_k\) \((p \in N')\), so \(m = m_1 + \ldots + m_k\).

Let us now fix \(j \in \{1, \ldots, k\}\) and \(\varepsilon > 0\), and choose \(p \in N'\) so large
that \( \rho(m_j, m_j) < \epsilon/2 \) and \( p^{-1} < \epsilon/2 \). By (22), we then obtain

\[
P\{\rho(\xi_j, m_j) < \epsilon\} \geq P\{\rho(\xi_j, m_{jp}) < \epsilon/2\} \geq P\{\rho(\xi_j, m_{jp}) < p^{-1}\} > 0,
\]

so we get \( m_j \in S(p_{\xi_j}^{-1}) = S(\lambda) \). This shows that \( m \in G(\lambda) \) and hence completes the proof in the case of bounded \( \lambda \).

In the case of unbounded \( \lambda \), we still have \( \xi \geq \alpha \) a.s. by Lemma 6.7, and if (18) is true, we get in addition \( \alpha \in S(p_{\xi_j}^{-1}) \), proving (17). It thus remains to prove (18), and again we may assume that \( \alpha = 0 \). For \( M_n, \nu_n \) and \( \xi_{nj} \) as in Lemma 6.7, define

\[
\lambda_n = \sum_{k=1}^{n} M_k \lambda, \quad \xi_n = \sum_{k=1}^{n} \sum_{j=1}^{\nu_k} \xi_{kj}, \quad n \in N.
\]

Then \( \xi_n \Rightarrow \xi \) a.s., and moreover,

\[
(23) \quad \overline{G(\lambda)} \supset \overline{G(\lambda_n)} = S(p_{\xi_n}^{-1}), \quad n \in N,
\]

since the \( \lambda_n \) are bounded and form a non-decreasing sequence. Suppose that \( \mu_0 \in S(p_{\xi_n}^{-1}) \). Proceeding as in (20), it is seen that there exist some \( \mu_n \in S(p_{\xi_n}^{-1}) \), \( n \in N \), with \( \mu_n \Rightarrow \mu \), and since \( \mu_1, \mu_2, \ldots \in \overline{G(\lambda)} \) by (23), it follows that \( \mu_0 \in \overline{G(\lambda)} \).

Conversely suppose that \( \mu_0 \in G(\lambda) \), say \( \mu_0 = \mu_1 + \ldots + \mu_k \) for some \( k \in \mathbb{Z}_+ \) and \( \mu_1, \ldots, \mu_k \in S(\lambda) \). Here we may assume without loss that \( \mu_j \neq 0 \) for each \( j \in \{1, \ldots, k\} \), say that \( \mu_j f_j \neq 0 \) where \( f_j \in F_c \). As \( M_1 \) in Lemma 6.7 we can choose

\[
M_1 = \bigcup_{j=1}^{k} \{ \mu \in M: \mu f_j > \frac{1}{2} \mu_j f_j \},
\]
since by Theorem 6.5
\[ \lambda M_1 \leq \sum_{j=1}^{k} \lambda \{ f_j > \frac{1}{2} \mu_j f_j \} = \sum_{j=1}^{k} \lambda \pi_f^{-1} \left( \frac{1}{2} \mu_j f_j, \infty \right) < \infty. \]

The set \( M_1 \) being open and containing \( \mu_1, \ldots, \mu_k \), we get
\[ \mu_0 \in GS(\lambda_1) \subset GS(\lambda_2) \subset \ldots, \]
and hence by (23)
\[ \mu_0 \in S(\rho \xi_n^{-1}), \ n \in N. \]

Let \( \epsilon > 0 \) be arbitrary, and choose \( \delta \in (0, \epsilon/2) \) such that
\[ \rho(\mu, \mu_0) \lor \rho(\mu', 0) < \delta \Rightarrow \rho(\mu + \mu', \mu) < \epsilon/2, \]
which is possible since the mapping \((\mu, \mu') \mapsto \rho(\mu + \mu', \mu)\) is continuous.

Since \( \xi - \xi_n \xrightarrow{v} 0 \) a.s., we may further choose \( n \in N \) such that
\[ P\{\rho(\xi - \xi_n, 0) < \delta\} > 0. \]

By (24) - (26) and the independence of \( \xi_n \) and \( \xi - \xi_n \), we obtain
\[ P\{\rho(\xi, \mu_0) < \epsilon\} \geq P\{\rho(\xi, \xi_n) < \epsilon/2, \rho(\xi_n, \mu_0) < \epsilon/2\} \geq P\{\rho(\xi - \xi_n, 0) < \delta, \rho(\xi_n, \mu_0) < \delta\} = P\{\rho(\xi - \xi_n, 0) < \delta\} P\{\rho(\xi_n, \mu_0) < \delta\} > 0, \]
and since \( \epsilon \) was arbitrary, it follows that \( \mu_0 \in S(\rho \xi_n^{-1}) \), as desired.

This completes the proof.
NOTES. The canonical representations occurring in Lemmas 6.1 and 6.3 are classical, (see e.g. Feller (1968) and (1971)), while the one in Theorem 6.5 is due to Matthes (1963) and Lee (1967), (see also Mecke (1972)). The corresponding convergence criteria are due to Kallenberg (1973a), (though the proof of Lemma 6.1 follows that of Feller (1971) for general random variables). As for Lemma 6.4, the first part is given in Kerstan, Matthes, and Mecke (1974), page 49, while the second part is new. Lee (1967) gave a proof of Lemma 6.7 along with a heuristic argument for Lemma 6.6; the present proof was supplied for point processes by Jagers (1973). Finally, Theorem 6.8 extends a result for point processes by Kerstan, Matthes and Mecke (1974), page 95, (see also Kallenberg (1973a)), while Theorem 6.9 is new.

The reader should consult Kerstan, Matthes and Mecke (1974) for a different approach to the theory of infinitely divisible point processes, where (16) plays the role of a basis for the whole theory while (14) and (16) fit into their general theory of "showerfields".

PROBLEMS.

6.1. Prove the $Z_+\text{-version}$ of Lemma 6.1 directly by means of generating functions.

6.2. State and prove tightness criteria corresponding to the convergence assertions in Lemma 6.3 and Theorem 6.5.

6.3. Let $\xi_n \overset{d}{=} I(\alpha_n, \lambda_n)$, $n \in \mathbb{N}$, in Lemmas 6.1 and 6.3 and in Theorem 6.5. Give criteria for convergence in distribution of $\{\xi_n\}$ in terms of $\{\alpha_n\}$ and $\{\lambda_n\}$.

6.4. Assume that $S$ is countable. Show that there exists some fixed $f \in F$ such that, for any point process $\xi$ on $S$ with $\xi_S < \infty$ a.s., $\xi$ is infinitely divisible iff $\xi f$ is infinitely divisible.
vanishing constant component. (Hint: use the result in Problem 3.8.)

6.5. Let $\xi$ be a point process on $S$. Show that $\xi$ is infinitely divisible iff $\xi f$ is infinitely divisible with $P(\xi f = 0) > 0$ for every $f \in F_c$.

6.6. For $\xi = I(0, \lambda)$ on $S$ with $\lambda M < \infty$, state and prove a relationship between the set of atom positions of $P_{\xi}^{-1}$ and that of $\lambda$. Show also that $P_{\xi}^{-1}$ can have no atoms if $\lambda M = \infty$. (Hint: Let $f \in F$ be strictly positive and such that $\xi f < \infty$ a.s. Then $\lambda_{\xi f}^{-1} R^+ = \lambda(\mu f > 0) = \lambda M$, so the last assertion follows from the corresponding fact for random variables due to Blum and Rosenblatt, see Lukacs (1970), page 124.)

7. FURTHER RESULTS FOR NULL-ARRAYS.

In this section we shall see how the results of Section 6 can be improved in certain special cases. Let us first consider the case of random measures with independent increments, (sometimes called completely random measures). By this we mean random measures $\xi$ such that $\xi_{B_1}, \ldots, \xi_{B_k}$ are independent for arbitrary disjoint $B_1, \ldots, B_k \in \mathcal{B}$, $k \in \mathbb{N}$. Let us say that a measure $\mu \in M$ is degenerate if $\mu = b \delta_s$ for some $b > 0$ and $s \in S$.

**Lemma 7.1.** Let $\xi \overset{d}{=} I(\alpha, \lambda)$ on $S$. Then $\xi$ has independent increments iff $\lambda$ is confined to the class of degenerate measures.
PROOF. Suppose that \( \lambda \) has the asserted property, and consider arbitrary \( k \in \mathbb{N} \) and disjoint \( B_1, \ldots, B_k \in \mathcal{B} \). Writing \( M' = M \setminus \{0\} \), we obtain for any \( t_1, \ldots, t_k \in \mathbb{R}_+ \)

\[
- \log E \exp(- \sum_j t_j \xi_{B_j}) = \sum_j t_j \alpha_{B_j} + \int_{M'} \left\{ (1 - \exp(- \sum_j t_j \mu_{B_j})) \lambda(d\mu) \right\} = \int_{M'} \left( \sum_j (t_j \alpha_{B_j} + \int_{\mu_{B_j} > 0} (1 - \exp(- t_j \mu_{B_j})) \lambda(d\mu)) \right) = - \log \prod_j E \exp(- t_j \xi_{B_j}),
\]

proving that \( \xi_{B_1}, \ldots, \xi_{B_k} \) are independent. Conversely, suppose that \( \xi \) has independent increments, and let \( A \) and \( B \in \mathcal{B} \) be disjoint. Since \( \alpha \) is additive, we obtain as above

\[
(1) \quad \int_{M'} \left\{ (1 - e^{-\mu A}) + (1 - e^{-\mu B}) - (1 - e^{-\mu A - \mu B}) \right\} \lambda(d\mu) = 0.
\]

But the function \( f(x) = 1 - e^{-x} \) is strictly concave on \( \mathbb{R}_+ \), so \( f(x + y) < f(x) + f(y) \) for every positive \( x \) and \( y \), and hence \( (1) \) is only possible provided \( \lambda\{\mu A > 0, \mu B > 0\} = 0 \). Making a partition of \( S \) into countably many sets of diameter \( < \varepsilon \), it follows that, for \( \mu \in M' \) a.e. \( \lambda \), at most one of these sets can have \( \mu \)-measure \( > 0 \). Since \( \varepsilon \) can be chosen arbitrarily small, this proves that \( \mu \) is degenerate a.e. \( \lambda \).
THEOREM 7.2. A random measure $\xi$ on $S$ is infinitely divisible with independent increments iff
\[ -\log E e^{-\xi f} = af + \int_{R^+_* \times S} \left( 1 - e^{-x f(t)} \right) \gamma(dx dt), \quad f \in F, \]
for some $\alpha \in M$ and some $\gamma \in M(R^*_+ \times S)$ with $\gamma(\cdot \times B) \kappa < \infty$, $B \in B$, and in this case $\alpha$ and $\gamma$ are unique. Moreover, any $\alpha$ and $\gamma$ with the stated properties may occur in (2). If $\{\xi_{n_j}\}$ is a null-array of random measures on $S$ and if $I \subseteq B_\xi$ is a DC-semiring, then $\sum_j \xi_{n_j} \xi I$ iff
\[ (i) \quad \kappa \sum_j P(\xi_{n_j} I)^{-1} \omega \alpha I \delta_0 + \kappa \gamma(\cdot \times I), \quad I \in I, \text{ and} \]
\[ (ii) \quad \sum_j P(\xi_{n_j} I_1 \wedge \xi_{n_j} I_2 > \epsilon) \to 0 \quad \text{for every } \epsilon > 0 \text{ and disjoint} \]
$I_1, I_2 \in I$.

In the point process case we have $\alpha = 0$ while $\gamma$ is confined to $N \times S$. In this case it suffices to take $\epsilon = 0$ in (ii) and to replace (i) by
\[ (i)' \quad \sum_j P(\xi_{n_j} I)^{-1} \omega \gamma(\cdot \times I) \text{ on } N, \quad I \in I. \]

PROOF. Suppose that $\xi$ is infinitely divisible with independent increments.

Let $M \in M$ be the class of degenerate measures, and for $\mu \in M$, let $t_\mu$ be the unique atom position of $\mu$. From Lemma 2.1 it is seen that $M \in M$ and that $\mu \mapsto (\mu S, t_\mu)$ is a measurable mapping of $M$ into $R^*_+ \times S$. Let $\gamma$ be the measure on $R^*_+ \times S$ induced by $\lambda$ via this mapping. Then (2) is obtained by transformation of the integral in (6.6), and since
\[ \gamma(dx \times B) = \lambda\{\mu B \in dx\} \quad x > 0 \quad B \in \mathcal{B}, \]

we have \( \gamma(\cdot \times B) = (\lambda \pi_0 - 1)B < \infty \), \( B \in \mathcal{B} \). To see that \( \gamma \) is unique, note that the inverse mapping \( (x,t) \rightarrow x \delta_t \) transforms \( \gamma \) into some \( \lambda \) and (2) into (6.6). From the above mappings it is further seen that any \( \alpha \) and \( \gamma \) with the stated properties are possible in (2).

As for the convergence assertion, the necessity of (i) and (ii) follows easily from (3) and Theorem 6.5. Conversely, suppose that (i) and (ii) hold. Then (6.9) follows from (i), and it remains to prove (6.8). It is then enough to take the components \( I_i \), \( \ldots, I_k \) of \( V \) in (6.8) disjoint and to verify that

\[ \sum_{j=1}^{k} P \cap \{ \xi_j \geq t_j \} + \lambda \cap \{ \mu I_i \geq t_i \} \]

for any \( t = (t_1, \ldots, t_k) \in \mathbb{R}_+^k \setminus \{0\} \) with \( t_i = 0 \) or \( \lambda \{ \mu I_i = t_i \} = 0 \), \( i = 1, \ldots, k \). Since events corresponding to \( t_i = 0 \) can be omitted from (4), we can assume that all the \( t_i \) are > 0 . But then (4) follows from (i) for \( k = 1 \) and from (ii) and Lemma 7.1 for \( k \geq 2 \). The last assertion follows from the corresponding fact in Theorem 6.5.

The most general random measure with independent increments is obtained from the one in Theorem 7.2 by addition of at most countably many fixed atoms:

**COROLLARY 7.3.** A random measure \( \eta \) on \( S \) has independent increments iff it can be written in the form

\[ \eta = \xi + \sum_{j=1}^{k} \beta_j \delta_{t_j} \]

(5)
where \( \xi \) satisfies (2) for some \( \alpha \) and \( \gamma \) with \( \alpha(s) = \gamma(R^*_+ \times \{s\}) = 0 \), \( s \in S \), some \( k \in \mathbb{Z}_+ \) and distinct \( t_j \leq S \), \( j \leq k \), and some \( R_+ \)-valued random variables \( \beta_j \) with \( P(\beta_j > 0) > 0 \), \( j \leq k \), which are mutually independent and independent of \( \xi \). In this case, the above decomposition is unique apart from the order of terms.

PROOF. The sufficiency and uniqueness assertions are obvious. Next suppose that \( \eta \) has independent increments. Like any random measure, \( \eta \) can have at most countably many fixed atoms, i.e. \( \eta(s) = 0 \) a.s. for all but countably many \( s \in S \), (cf. the argument in Billingsley (1968), page 124). Let \( t_j \) and \( \beta_j \), \( j \leq k \), be the corresponding atom positions and sizes, and let \( \xi \) be defined by (5). Since \( \eta \) has independent increments, it is easily seen that the \( \beta_j \) are mutually independent and independent of \( \xi \), and further that \( \xi \) has independent increments. Moreover, \( \xi \) has no fixed atoms, so if \( \{B_{nj}\} \subset B \) is a null-array of partitions of some fixed set \( B \in B \), it follows by a simple compactness argument that \( \{\xi B_{nj}\} \) is a null-array of random variables. Hence, by Lemma 6.1, the common row-sum \( \xi B \) must be infinitely divisible. A similar argument shows more generally that \( (\xi B_1, \ldots, \xi B_k) \) is infinitely divisible for any \( k \in \mathbb{N} \) and disjoint \( B_1, \ldots, B_k \in B \), and so we may conclude from Lemma 6.4 that \( \xi \) is itself infinitely divisible. In view of Theorem 7.2, this completes the proof.

By comparison of (2) and (1.2), it is seen that a point process \( \xi \) on \( S \) is Poissonian iff it is infinitely divisible with independent increments and with the measure \( \gamma \) in (2) confined to the set \( \{1\} \times S \).
In this case, $\xi$ has intensity

$$E\xi B = \gamma([1] \times B), \quad B \in \mathcal{B}. $$

Specializing Theorem 7.2 to this case, we obtain the important

**COROLLARY 7.4.** Let $\xi$ be a Poisson process on $S$ with intensity $\lambda \in M$, and let $\{\xi_{nj}\}$ be a null-array of point processes on $S$. Further suppose that $I \in \mathcal{B}_\lambda$ is a DC-semiring. Then $\sum_j \xi_{nj} \xrightarrow{d} \xi$ iff

(i) $\sum_j P(\xi_{nj} I > 0) \xrightarrow{} \lambda I$, $I \in I$, and

(ii) $\sum_j P(\xi_{nj} B > 1) \xrightarrow{} 0$, $B \in \mathcal{B}$.

We are now going to consider the convergence of null-arrays towards two special types of infinitely divisible random measures which play the same role here as do the simple point processes and diffuse random measures in Sections 3 - 5. But first we define regularity for canonical measures $\lambda$ and for null-arrays $\{\xi_{nj}\}$ as follows. Given any covering class $C \subset \mathcal{B}$ and any $a > 0$, we shall say that $\lambda$ is a-regular w.r.t. $C$ if there exists for every $C \in C$ some array $\{C_{mk}\} \subset C$ of finite coverings of $C$ such that

$$\lim_{m \to \infty} \sum_{k} \lambda(\mu C_{mk} \geq a) = 0. $$

The notion of a-regularity of $\{\xi_{nj}\}$ is defined analogously but with (6) replaced by

(6) $$\lim_{m \to \infty} \sum_{k} \lambda(\mu C_{mk} \geq a) = 0. $$
\[
\lim \limsup_{m \to \infty} \sum_{n \to \infty} \sum_{j} \sum_{k} P(\xi_{nj} \land C_{mk} \geq a) = 0.
\]

As before, regularity means a-regularity for every \(a > 0\).

**Theorem 7.5.** Let \(\{\xi_{nj}\}\) be a null-array of point processes on \(S\) and let \(\xi \overset{d}{=} I(\lambda)\) with \(\lambda\{\mu = \mu^*\} = 0\). Further suppose that \(U \subset B_\xi\) is a DC-ring and \(I \subset B_\xi\) a DC-semiring. Then \(\sum_j \xi_{nj} \overset{d}{=} \xi\) iff

(i) \(\lim_{n \to \infty} \frac{1}{n} \sum_{j} \llbracket \xi_{nj} \land U > 0 \rrbracket = \lambda\{\mu \uparrow U > 0\},\) \(U \in U,\)

(ii) \(\limsup_{n \to \infty} \frac{1}{n} \sum_{j} \llbracket \xi_{nj} \land I > 1 \rrbracket \leq \lambda\{\mu \downarrow I > 1\},\) \(I \in I,\)

(iii) \(\lim_{k \to \infty} \limsup_{n \to \infty} \frac{1}{n} \sum_{j} \llbracket \xi_{nj} \land B > k \rrbracket = 0,\) \(B \in B.\)

Moreover, \(\sum_j \xi_{nj} \overset{d}{=} \xi\) follows from (i) and (ii) if \(\lambda\) is 2-regular w.r.t. \(I\) and from (i) alone if \(\{\xi_{nj}\}\) is 2-regular w.r.t. \(I\) or if

(7) \(\limsup_{n \to \infty} \frac{1}{n} \sum_{j} \llbracket \xi_{nj} \land I \leq \lambda \uparrow I < \infty \rrbracket,\) \(I \in I.\)

The proof of this theorem is similar to that of Theorem 4.8, except that we now rely on the following two lemmas, playing the roles of Theorem 3.3 and Lemma 4.7 respectively.

**Lemma 7.6.** Let \(\lambda\) be the canonical measure of an infinitely divisible point process, and suppose that \(U \subset B\) is a DC-ring and \(I \subset B\) a DC-semi-ring. Then \(\lambda\{\mu = \mu^*\} = 0\) iff \(\lambda\{\mu \uparrow I > 1\} \leq \lambda\{\mu^* \uparrow I > 1\},\) \(I \in I,\)

and in that case \(\lambda\) is uniquely determined by \(\lambda\{\mu \uparrow U > 0\},\) \(U \in U.\)
PROOF. For fixed $C \in \mathcal{U}$, let $\lambda_C$ be the measure on $N' = N \setminus \{0\}$ induced by $\lambda$ via the mapping $\mu \mapsto C\mu$, i.e.

$$\lambda_C M = \lambda\{C\mu \in M\}, \quad M \in N \cap N'.$$

Then

$$\lambda_C N' = \lambda\{C\mu \in N'\} = \lambda\{\mu > 0\} < \infty,$$

so either this quantity is zero, or $\lambda_C$ can be normalized to a probability measure on $N'$, i.e. to the distribution of some simple point process. Hence, by Theorem 3.3, $\lambda_C$ is completely determined by the quantities

$$\lambda_C\{\mu U > 0\} = \lambda\{(C\mu)U > 0\} = \lambda\{\mu (C \cap U) > 0\}, \quad U \in \mathcal{U},$$

i.e. by all $\lambda\{\mu U > 0\}, \quad U \in \mathcal{U}$. But $\lambda_C$ determines the measure $\lambda_{\pi_f}^{-1}$ for any $f \in F$ with support in $C$, so by Theorem 6.5, the family $\{\lambda_C\}$ determines $\lambda$. We may now complete the proof as in case of Theorem 3.3.

**Lemma 7.7.** Let $\{\xi_{nj}\}$ be a null-array of random measures on $S$, and suppose that $\sum_j \xi_{nj} \overset{d}{\rightarrow} \text{some } \eta$. Further suppose that $\xi \overset{d}{\equiv} I(\alpha, \lambda)$ and that $U \subset B$ is a DC-ring satisfying

$$\liminf_{n \to \infty} \sum_j \left(1 - Ee^{-t\xi_{nj}}\right) \leq t\eta U + \lambda\left(1 - e^{-t\pi U}\right), \quad U \in \mathcal{U},$$

for some $t > 0$. Then $\eta C = 0$ a.s. for every compact $C \in \mathcal{B}$ with $\xi C = 0$ a.s. In the point process case, (8) can be replaced by

$$\liminf_{n \to \infty} \sum_j P\{\xi_{nj} U > 0\} \leq \lambda\{\mu U > 0\}, \quad U \in \mathcal{U}.$$
This may be proved in the same way as Lemma 4.7.

We now turn to the analogue of Theorem 4.9.

**Theorem 7.8.** Let \( \{\xi_{nj}\} \) be a null-array of point processes (random measures) on \( S \) and let \( \xi \overset{d}{=} I(a, \lambda) \) with \( a = 0 \) and \( \lambda \{ \mu \neq \mu^* \} = 0 \) (or with \( \lambda \mu^c = 0 \) respectively). Further suppose that \( s \) and \( t \) are fixed real numbers with \( 0 < s < t \) and that \( U \in \mathcal{B}_\xi \) is a DC-ring and \( I \in \mathcal{B}_\xi \) a covering class (DC-semi-ring). Then \( \sum_j \xi_{nj} \overset{d}{=} \xi \) iff

\[
\begin{align*}
(i) \quad & \lim_{n \to \infty} \sum_j \left( 1 - E^{\xi_{nj} U} \right) = taU + \lambda \left( 1 - e^{-t\pi U} \right), \quad U \in U, \\
(ii) \quad & \limsup_{n \to \infty} \sum_j \left( 1 - E^{\xi_{nj} I} \right) \leq saI + \lambda \left( 1 - e^{-s\pi I} \right), \quad I \in I, \\
(iii) \quad & \lim_{t \to \infty} \limsup_{n \to \infty} \sum_j P(\xi_{nj} I > t) = 0, \quad I \in I.
\end{align*}
\]

Moreover, \( \sum_j \xi_{nj} \overset{d}{=} \xi \) follows from (i) alone if \( \{\xi_{nj}\} \) is 2-regular (regular) w.r.t. \( I \) or if

\[
(9) \quad \limsup_{n \to \infty} \sum_j E \xi_{nj} I \leq aC + \lambda \pi I < \infty, \quad I \in I.
\]

**Proof.** We confine our attention to the random measure case, since the point process case is similar but simpler. Suppose that \( \sum_j \xi_{nj} \overset{d}{=} \xi \) and let \( B \in U \) or \( I \). Then \( \sum_j \xi_{nj} B \overset{d}{=} \xi B \) by Lemma 4.3, so by Lemma 6.1

\[
\sum_j P(\xi_{nj} B)^{-1} \overset{w}{\to} aB^0 + \kappa \lambda^{-1} B.
\]

This implies (i) and (ii), while (iii) follows by Lemmas 4.6 and 6.2.
Conversely, suppose that (i) - (iii) hold. Since \( (1 - e^{-tx})/x \) is a decreasing function of \( x > 0 \), we have

\[
x \leq r(1 - e^{-tx})/(1 - e^{-tr}) , \quad 0 \leq x \leq r ,
\]

and so we get for any \( r > 0 \) and \( U \in \mathcal{U} \)

\[
\sum_j E \xi_{nj} U \leq r(1 - e^{-tr})^{-1} \sum_j E \left(1 - e^{-t\xi_{nj} U} \right) ,
\]

By Lemmas 4.6 and 6.2, it therefore follows from (i) and (iii) that \( \left\{ \sum_j \xi_{nj} \right\} \) is tight. But in that case every sequence \( N' \subset N \) has a subsequence \( N'' \) such that \( \sum_j \xi_{nj} \leq \eta \) \( (n \in N'') \). Now \( \eta \equiv \text{some I(}\delta, \lambda\text{)} \) by Theorem 6.5, and since \( U \subset B_\eta \) by Lemma 7.7, it follows from (i), (ii) and the necessity part of the present theorem that

\[
t \delta \{1 - e^{-t\pi U} \} = t \sigma U + \lambda \left(1 - e^{-t\pi U} \right) , \quad U \in \mathcal{U} ,
\]

\[
s \delta I + \lambda \left(1 - e^{-s\pi I} \right) \leq s \alpha I + \lambda \left(1 - e^{-s\pi I} \right) , \quad I \in I ,
\]

which by Theorem 6.5 is equivalent to

(10) \[
E e^{-t\eta U} = E e^{-t\xi U} , \quad U \in \mathcal{U} ,
\]

(11) \[
E e^{-s\eta I} \geq E e^{-s\xi I} , \quad I \in I
\]

Furthermore, it follows from Theorem 6.8 that \( \xi \) is a.s. diffuse, possibly apart from atoms of fixed size and location which arise from the atoms of \( \eta \). Let us now consider any fixed \( x \in S \). Choosing \( B_1, B_2, \ldots \) in \( \mathcal{U} \) or \( I \) respectively such that \( B_n \downarrow \{x\} \), it follows by monotone convergence
from (10) and (11) that

\begin{equation}
Ee^{-t\eta(x)} = Ee^{-t\xi(x)} = e^{-t\alpha(x)},
\end{equation}

\begin{equation}
Ee^{-s\eta(x)} \geq Ee^{-s\xi(x)} = e^{-s\alpha(x)}.
\end{equation}

If \( \eta(x) \) were non-degenerate, it would follow from Jensen's inequality for strictly convex functions that

\( e^{-t\alpha(x)} = Ee^{-t\eta(x)} > (Ee^{-s\eta(x)}t/s) \geq (e^{-s\alpha(x)}t/s) = e^{-t\alpha(x)}, \)

which is impossible. Hence \( \eta(x) \) must be degenerate, and by (12) we obtain \( \eta(x) = \alpha(x) \) a.s. Writing \( \alpha' \) for the purely atomic part of \( \alpha \), it follows that \( \eta \geq \alpha' \) a.s., and since (10) and (11) may be written in the form

\begin{equation}
Ee^{-t(\eta-\alpha')}U = Ee^{-t(\xi-\alpha')}U, \quad U \in \mathcal{U},
\end{equation}

\begin{equation}
Ee^{-s(\eta-\alpha')}I \geq Ee^{-s(\xi-\alpha')}I, \quad I \in \mathcal{I},
\end{equation}

it is seen from Theorem 3.4 that \( \xi - \alpha' \overset{d}{=} \eta - \alpha' \), i.e. that \( \xi \overset{d}{=} \eta \).

This shows that \( \sum_j \xi_{nj} \overset{d}{=} \xi \) (\( n \in N'' \)) , and since \( N' \) was arbitrary, even that \( \sum_j \xi_{nj} \overset{d}{=} \xi \) (\( n \in N \)) , proving the first assertion.

The assertion involving regularity of \( \{\xi_{nj}\} \) follows by the usual arguments, so it remains to prove the assertion involving (9). In this case we get by Fatou's lemma

\begin{equation}
E\eta I \leq \liminf_{n \to \infty} \sum_{n \in N''} E\xi_{nj} I \leq \limsup_{n \to \infty} \sum_{n \in N'} E\xi_{nj} I \leq \alpha I + \lambda \pi I = E\xi I, \quad I \in \mathcal{I},
\end{equation}
replacing (11), so (13) is now replaced by

(16) \[ \text{E} \eta(x) \leq \text{E} \xi(x) = \alpha(x). \]

If \( \eta(x) \) were non-degenerate, we would obtain from (12), (16) and Jensen's inequality

\[ e^{-ta(x)} \leq e^{-t\eta(x)} \geq e^{-t\text{E}\eta(x)} \geq e^{-ta(x)} , \]

which is impossible, so again we get \( \eta(x) = \alpha(x) \) a.s., \( x \in S \), proving that \( \eta \geq \alpha' \) a.s. Thus it follows by (14), (15) and Theorem 3.4 that \( \xi - \alpha' = \eta - \alpha' \), and the proof may be completed as before.

**NOTES.** The canonical representation in Theorem 7.2 and Corollary 7.3, which is a well-known classical result for random measures on \( \mathbb{R} \), is due in the present generality to Kingman (1967). The present approach via Lemma 7.1 as well as the convergence criterion in Theorem 7.2 were given in Kallenberg (1973a). Corollary 7.4 is a classical result, proved in particular cases by Palm (1943), Hinchin (1960) and Ososkov (1956), and in general (on \( \mathbb{R} \)) by Grigelionis (1963), (see also Goldman (1967) and Jagers (1972)). Finally, Theorem 7.5 is taken from Kallenberg (1973a) while Theorem 7.8 is new.

**PROBLEMS.**

7.1. Show in analogy with Lemma 6.6 that the random measure \( \xi \) in Theorem 7.2 is distributed like \( \alpha + \int x \delta_t \eta(dxdt) \), where \( \eta \) is a Poisson process on \( \mathbb{R}_+^* \times S \) with intensity \( \gamma \).
7.2. Let \( \{\xi_{nj}\} \) be a null-array of point processes on \( S \). Show that 
\[
\sum_j \ell_j \xi_{nj} \text{ is tight with only Poissonian limit distributions iff}
\]
\[
\sum_j \ell_j P(\xi_{nj} B > 0) \text{ is bounded and } \sum_j P(\xi_{nj} B > 1) \to 0 \text{ for every } B \in \mathcal{B}.
\]

7.3. State and prove a similar tightness criterion corresponding to the convergence assertion in Theorem 7.2.

7.4. Prove Corollary 7.4 directly from Theorem 4.8 and Lemma 6.1 in the particular case when \( \lambda \in M_d \).

7.5. Prove Theorem 7.8 in the point process case. Show that \( I \) may be assumed to be a covering class even in the random measure case, provided \( \alpha \) is known to be diffuse.

7.6. Let \( \xi \overset{d}{=} I(\alpha, \lambda) \) on \( S \), let \( C \subset \mathcal{B} \) be a covering class and let \( a > 0 \). Show that \( \lambda(\mu_a \neq 0) = 0 \) if \( \lambda \) is a-regular w.r.t. \( C \), and that the converse is also true provided \( C \) is a DC-semiring and \( E \xi \in M \).

7.7. Let \( \{\xi_{nj}\} \) be a null-array of random measures on \( S \) such that 
\[
\sum_j \ell_j \xi_{nj} \overset{d}{=} I(\alpha, \lambda).
\]
Show that \( \lambda(\mu^*_\varepsilon \neq 0) = 0 \) if \( \{\xi_{nj}\} \) is a-regular w.r.t. \( B_\varepsilon \) for every \( a > \varepsilon \), and that the converse is also true provided \( E \xi \in M \).

8. COMPOUND AND THINNED POINT PROCESSES.

The notions of compounding and thinning were introduced in Section 1, and in Corollaries 3.2 and 3.5 we proved two uniqueness results for compound point processes. The aim of the present section is to study the convergence in distribution of such random measures, and we shall
first consider a case when the limits are themselves compound point processes.

**THEOREM 8.1.** For \( n \in \mathbb{Z}_+ \), let \( \xi_n \) be a \( \beta_n \)-compound of \( \eta_n \). Then \( \beta_n \overset{d}{\to} \beta_0 \) and \( \eta_n \overset{d}{\to} \eta_0 \) imply that \( \xi_n \overset{d}{\to} \xi_0 \). Conversely, suppose that \( \xi_n \overset{d}{\to} \xi \) and that \( \{\eta_n\} \) is tight. Then \( \xi \) is a \( \beta_0 \)-compound of \( \eta_0 \) for some \( \beta_0 \) and \( \eta_0 \), and if \( \{C\eta_n\} \) is 2-regular w.r.t. \( \beta_\xi \) for some \( C \in \mathcal{B} \) with \( \xi C \overset{d}{\to} 0 \), there exist some constants \( p_0, p_1, \ldots \in (0,1] \) with \( p = \inf p_n > 0 \) such that \( \beta'_n \overset{d}{\to} \beta'_0 \) and \( \eta'_n \overset{d}{\to} \eta'_0 \), where \( \eta'_n \) is a \( p_n \)-thinning of \( \eta_n \), \( n \in \mathbb{Z}_+ \), while

\[
(1) \quad P \{ \beta'_n > x \} = p p^{-1} P \{ \beta_n > x \} , \quad x \in \mathbb{R}_+ , \quad n \in \mathbb{Z}_+ .
\]

**PROOF.** Let us first suppose that \( \beta_n \overset{d}{\to} \beta_0 \) and \( \eta_n \overset{d}{\to} \eta_0 \), and let \( \xi_n \) be a \( \beta_n \)-compound of \( \eta_n \), \( n \in \mathbb{Z}_+ \). By (1.5) and Theorem 4.5, \( \xi_n \overset{d}{\to} \xi_0 \) is then equivalent to

\[
(2) \quad E \exp(\eta_n \log \phi_n \circ f) \to E \exp(\eta_0 \log \phi_0 \circ f) , \quad f \in F_c ,
\]

where \( \phi_n = L_{\beta_n} , \quad n \in \mathbb{Z}_+ \), and by continuity and boundedness, (2) will follow if we can prove that

\[
(3) \quad \eta_n \log \phi_n \circ f \overset{d}{\to} \eta_0 \log \phi_0 \circ f , \quad f \in F_c .
\]

By Theorem 5.5 in Billingsley (1968), it is enough to prove (3) for non-random \( \eta_n \), i.e. to show that \( \nu_n \overset{v}{\to} \nu_0 \) implies

\[
(4) \quad \nu_n \log \phi_n \circ f \overset{d}{\to} \nu_0 \log \phi_0 \circ f , \quad f \in F_c .
\]
Considering a fixed \( f \in F_c \), we may assume that \( \mu_n \xrightarrow{w} \mu_0 \neq 0 \) (cf. Problem 4.5), and after a suitable normalization that the \( \mu_n \) are probability measures. But then (4) may be written in the form

\[
(5) \quad E \log \phi_n \circ f(\tau_n) + E \log \phi_0 \circ f(\tau_0)
\]

for some random elements \( \tau_0, \tau_1, \ldots \) in \( S \) with \( \tau_n \xrightarrow{d} \tau_0 \). Since \( f \) is bounded, the sequence \( \{\log \phi_n \circ f\} \) is uniformly bounded, and hence (5) is a consequence of the relation

\[
(6) \quad \phi_n \circ f(\tau_n) \xrightarrow{d} \phi_0 \circ f(\tau_0).
\]

Applying Theorem 5.5 in Billingsley (1968) once more, it is seen that the quantities \( \tau_n = s_n \) may be considered as non-random with \( s_n \xrightarrow{d} s_0 \), and since in that case \( x_n = f(s_n) \xrightarrow{d} f(s) = X \), it remains to prove that \( \phi_n(x_n) \xrightarrow{d} \phi_0(x_0) \) whenever \( x_n \xrightarrow{d} x_0 \). But this follows from the fact that the convergence \( \phi_n \xrightarrow{d} \phi_0 \) is uniform on bounded intervals.

Let us now suppose conversely that \( \xi_n \xrightarrow{d} \xi \) and that \( \{n_n\} \) is tight. Since \( 0 \in M \) is a 0-compound of \( 0 \in N \), we may assume that \( \xi \xrightarrow{d} 0 \). Let the sequence \( N' \subset N \) be such that \( n_n \xrightarrow{d} \) some \( n \) (\( n \in N' \)). If \( n \xrightarrow{d} 0 \), we obtain for any fixed \( B \in \mathcal{B} \)

\[
1 - \exp(\eta_n B \log \phi_n(1)) \leq n_n B \xrightarrow{d} 0,
\]

so by (1.5) and bounded convergence

\[
E e^{-\xi_n B} = E \exp(\eta_n B \log \phi_n(1)) \to 1,
\]
and hence $\xiB = 0$, $B \in B$, yielding the contradiction $\xi = 0$ a.s.

Thus $\eta \not\xrightarrow{d} 0$.

We shall use this fact to show that $\{\beta_n\}$ is tight or, what amounts to the same, that

$$\lim_{t \to 0} \limsup_{n \to \infty} \left(1 - \phi_n(t)\right) = 0.$$

Suppose on the contrary that the limit in (7) is greater than some $\epsilon > 0$.

Then there exist arbitrarily small numbers $t > 0$ such that, for some sequence $N'' \subset N'$,

$$1 - \phi_n(t) > \epsilon, \quad n \in N''.$$

Choosing $B \in B_\xi \cap B_\eta$ such that $\etaB \not\xrightarrow{d} 0$ and noting that

$$1 - x \leq -\log x \leq 2(1 - x), \quad x \in \left[\frac{1}{2}, 1\right],$$

it follows by (1.5) and (8) that

$$Ee^{-t\xi_nB} = E e^{\log(\etaB \log \phi_n(t))} \leq E e^{\log(-\etaB(1 - \phi_n(t)))} \leq E e^{-\epsilon nB},$$

and since $\xi_nB \not\xrightarrow{d} \xiB$ and $\eta_nB \not\xrightarrow{d} \etaB$, we obtain $Ee^{-t\xiB} \leq E e^{-\epsilon nB}$. Letting $t \to 0$, we reach the contradiction $1 \leq E e^{-\epsilon nB} < 1$, proving that (7) is indeed true. Thus the sequences $\{\beta_n\}$ and $\{\eta_n\}$ are both tight, and in particular we may consider convergent subsequences and use the direct part of the theorem to conclude that $\xi$ has the asserted form.

Let us now suppose that $\{C\eta_n\}$ is 2-regular w.r.t. $B_\xi$ for some $C \in B_\xi$ with $\xiC \not\xrightarrow{d} 0$, and assume that $\beta_n \xrightarrow{d} \beta'$ and $\eta_n \xrightarrow{d} \eta'$ as $n \to \infty$.
through \( N' \subset N \). Since \( \xi \) is then a \( \beta' \)-compound of \( \eta' \), and moreover \( \beta' \not\overset{d}{=} 0 \), it follows from (1.5) that \( B_{\eta'} = B_{\xi} \), and so \( C_{\eta'} \) must be simple (cf. Problem 4.9). Put \( p = P[\beta' > 0] \), let \( \eta_0 \) be a \( p \)-thinning of \( \eta' \) and let \( \beta_0 \) be an \( R_+ \)-valued random variable satisfying

\[
P[\beta_0 > x] = p^{-1}P[\beta' > x] , \quad x \in R_+ .
\]

Then \( \phi_0 = L_{\beta_0} \) and \( \phi' = L_{\beta'} \) satisfy the relation \( \phi' = p\phi_0 + (1 - p) \), so letting \( \xi_0 \) be a \( \beta_0 \)-compound of \( \eta_0 \) and using (1.5) and (1.6), we obtain for any \( f \in F \)

\[
L_{\xi_0}(f) = L_{\eta_0}(-\log \phi_0 \circ f) = L_{\eta_1}(-\log[1 - p(1 - \phi_0 \circ f)]) =
\]

\[
= L_{\eta_1}(-\log[1 - (1 - \phi' \circ f)]) = L_{\eta_1}(-\log \phi' \circ f) = L_{\xi}(f) ,
\]

and hence \( \xi_0 \overset{d}{=} \xi \) by Theorem 3.1. Now \( P[\beta_0 > 0] = 1 \), and moreover \( C_{\eta_0} \) is simple by Problem 2.4, so it follows from Corollary 3.5 that \( P_{\beta_0}^{-1} \) and \( P_{\eta_0}^{-1} \) are uniquely determined by \( P_{\xi}^{-1} \).

Choose any fixed \( y > 0 \) satisfying \( P[\beta_0 > y] > 0 \) and \( P[\beta_0 = y] = 0 \), and define \( p_n = P[\beta_n > y] , \quad n \in Z_+ . \) Then \( p_0 > 0 \), and moreover \( \liminf_n p_n > 0 \) since if \( p_n \to 0 \) as \( n \to \infty \) through some sequence \( N' \subset N \), we could choose some further subsequence \( N'' \subset N' \) such that \( \beta_n \overset{d}{=} \) some \( \beta' \), and then (10) would hold with \( p = P[\beta' > 0] > 0 \), and we would get the contradiction

\[
p_n = P[\beta_n > y] + P[\beta' > y] = P[\beta' > 0]P[\beta_0 > y] > 0 \quad (n \in N'') .
\]
To obtain \( p = \inf p_n > 0 \), we may replace the finitely many zeros in the sequence \( \{p_n\} \) by arbitrary positive numbers. We can now let \( n' \)
be a \( p_n \)-thinning of \( n_n \), \( n \in \mathbb{Z}_+ \), and define random variables \( b_n' \)
satisfying (1).

Given any sequence \( N' \subset N \), we may choose a subsequence \( N'' \subset N' \)
such that \( b_n \approx_{d} b', \eta_n' \approx_{d} \eta' \) and \( p_n + p' = p\{b' > y\} (n \in N') \).
In this case the constant \( p \) in (10) equals \( p'p_0^{-1} \), and in particular
\( p\{b' > 0\} = p'p_0^{-1} \). Using the direct part of the theorem, it is seen
that, on \( N'' \), \( n_n' \approx_{d} n_0'' \), a \( p' \)-thinning of \( n' \). On the other hand, it
was shown above that \( n \) is a \( (p'p_0^{-1}) \)-thinning of \( n' \), and so \( n_0'' \) is
like \( n_0' \) a \( p_0 \)-thinning of \( n \), and we get \( n_n' \approx_{d} n_0' \) \( (n \in N'') \). Furthermore,
it is seen from (1) that \( b_n' \approx_{d} b_0' \) \( (n \in N'') \), where
\[
P\{b_0'' > x\} = pp_0^{-1}p\{b' > x\} = pp_0^{-1}p'p_0^{-1}p\{b_0 > x\} = pp_0^{-1}p\{b_0 > x\} = p\{b_0' > x\} , \quad x \in \mathbb{R}_+ ,
\]
so \( b_0'' \approx_{d} b_0'' \) and it follows that \( b_n' \approx_{d} b_0' \) \( (n \in N'') \). In view of Theorem
2.3 in Billingsley (1968), this completes the proof.

We are now going to consider the entirely different case when the
compounding variables \( b_n \) satisfy \( b_n \approx_{d} 0 \). The class of possible limits
may then be described as follows.

Consider arbitrary \( \alpha \in \mathbb{R}_+ \) and \( \lambda \in \mathcal{M}(\mathbb{R}_+) \) with \( \lambda \alpha < \infty \), and define

\[
(11) \quad \psi(t) = \alpha t + \lambda (1 - e^{-\eta t}) , \quad t \in \mathbb{R}_+ .
\]
By Theorem 7.2, there exists for every \( \mu \in \mathcal{M} \) some random measure \( \xi \) with homogeneous (w.r.t. \( \mu \)) independent increments satisfying, for any \( f \in F \),

\[
- \log \mathbb{E} e^{-\xi f} = \alpha \mu f + \int_{\mathbb{R}^+ \times \mathcal{S}} (1 - e^{-xf(t)}) (\lambda \times \mu) d dx dt =
\]

\[
= \int_{\mathcal{S}} \{\alpha f(t) + \int_{\mathbb{R}^+} (1 - e^{-xf(t)}) \lambda dx \} \mu (dt) = \mu (\psi \circ f).
\]

By Lemma 1.4, we may consider \( \mu = \eta \) as a random measure and mix w.r.t. its distribution, thus obtaining

\[
(12) \quad L_\xi (f) = \mathbb{E} e^{-\xi f} = \mathbb{E} e^{-\eta (\psi \circ f)} = L_\eta (\psi \circ f), \quad f \in F.
\]

Note in particular that, if \( \alpha = 0 \) and \( \lambda = \delta_1 \), then \( \xi \) is a Cox process directed by \( \eta \). We shall need the following uniqueness result which is similar to Corollary 3.5.

**LEMMA 8.2.** Let \( \xi \) be a random measure on \( \mathcal{S} \) satisfying (11) and (12) for some \( \eta \), \( \alpha \) and \( \lambda \), and suppose that \( \alpha + \lambda \kappa = 1 \) while \( C_\eta \) is diffuse for some \( C \in \mathcal{B} \) with \( \eta C \overset{d}{=} 0 \). Then \( P_{\eta}^{-1} \), \( \alpha \) and \( \lambda \) are uniquely determined by \( P_{\xi}^{-1} \).

**PROOF.** By (12),

\[
\mathbb{E} e^{-\xi B} = \mathbb{E} e^{-\eta (\psi(1))} = \mathbb{E} e^{-\eta B}, \quad B \in \mathcal{B},
\]

and replacing \( B \) by \( B \cap C \), it follows by Theorem 3.4 that \( P(C_\eta)^{-1} \) is uniquely determined by \( P_{\xi}^{-1} \). But then it follows as in the proof
of Corollary 3.5 that even $\psi$ is uniquely determined, and by Lemma 6.1, $\psi$ determines $\alpha$ and $\lambda$. The uniqueness of $P_n^{-1}$ now follows as in case of Corollary 3.5.

To avoid trivial complications, let us assume that any compounding variable $\beta$ is such that $\beta \not\preceq 0$, and further that the $\alpha$ and $\lambda$ in (11) are normalized in such a way that $\alpha + \lambda\kappa = 1$.

**Theorem 8.3.** For $n \in N$, let $\xi_n$ be a $\beta_n$ compound of $\eta_n$ and put $c_n = E\kappa(\beta_n)$. Suppose that $\beta_n \not\preceq 0$. Then $\xi_n \not\preceq 0$ iff $c_n\eta_n \not\preceq 0$.

Furthermore, the conditions

(i) $c_n\eta_n \not\preceq \eta_n$,

(ii) $c_n^{-1}\kappa(P_n^{-1}) \not\preceq \alpha\delta_0 + \kappa\lambda$

imply that $\xi_n \not\preceq$ some $\xi$ satisfying (11) and (12). Conversely, $\xi_n \not\preceq \xi$ implies that $\xi$ satisfies (11) and (12) for some $\eta$, $\alpha$ and $\lambda$, and if moreover \( \{c_n\eta_n\} \) is regular w.r.t. $B_\xi$ for some $C \in B_\xi$ with $\xi \not\preceq C \not\preceq 0$, then $P_n^{-1}$, $\alpha$ and $\lambda$ are unique and satisfy (i) and (ii).

**Proof.** Writing $\phi_n = L_{\beta_n}$, $n \in N$, we get as before

(13) \[ E\exp(-t\xi_n B) = E \exp(\eta_n B \log \phi_n(t)), \quad t \in \mathbb{R}_+, \quad B \in B, \quad n \in N. \]

Now $\beta_n \not\preceq 0$ implies that $\phi_n \to 1$, so by (9) we get for sufficiently large $n \in N$
\[ c_n = 1 - \phi_n(1) \leq -\log \phi_n(1) \leq 2\{1 - \phi_n(1)\} = 2c_n, \]

and it follows from (13) that

\[ E \exp(-2c_n n_B) \leq E e^{-c_n n_B} \leq E \exp(-c_n n_B) , \quad B \in \mathcal{B}, \]

showing that \( \xi_n B \overset{d}{\rightarrow} 0 \) iff \( c_n n_B \overset{d}{\rightarrow} 0 \), \( B \in \mathcal{B} \). This proves the first assertion.

Next suppose that (i) and (ii) are satisfied. Proceeding as in the proof of Theorem 8.1, we obtain \( \xi_n \overset{d}{\rightarrow} \xi \) with \( P_\xi^{-1} \) determined by (11) and (12), provided we can show that

\[ -c_n^{-1} \log \phi_n(x_n) \rightarrow \psi(x) \]

whenever \( x, x_1, x_2, \ldots \in R_+ \) with \( x_n \rightarrow x \). But this follows from the fact, essentially being proved in Lemma 6.1, that \( -c_n^{-1} \log \phi_n \) tends to \( \psi \) uniformly on bounded intervals.

Suppose conversely that \( \xi_n \overset{d}{\rightarrow} \xi \), and let \( B \in \mathcal{B}_\xi \) be arbitrary. Then \( \xi_n B \overset{d}{\rightarrow} \xi B \) by Lemma 4.3, so by (13)

\[ E \exp(\eta_n B \log \phi_n(t)) \rightarrow E e^{-t\xi B} , \quad t \in R_+, \]

and since this limit tends to one as \( t \rightarrow 0 \), there exists for each \( \varepsilon > 0 \) some \( t \in (0,1) \) satisfying

\[ E \exp(\eta_n B \log \phi_n(t)) \geq 1 - \varepsilon/2 , \quad n \in N. \]

Now it follows from (9) and the elementary inequality
that

\[- \log \phi_n(t) \geq 1 - \phi_n(t) = E\left(1 - e^{-n t}\right) \geq tE\left(1 - e^{-n}\right) = tc_n, \quad n \in \mathbb{N},\]

and so by (14) and Chebyshev's inequality

\[P\{c_n^{-1}nB > t^{-1}\log 2\} = P\{tc_n^{-1}nB > \log 2\} \leq P\{-nB \log \phi_n(t) > \log 2\} =
\]

\[= P\{1 - \exp\left(nB \log \phi_n(t)\right) > \frac{1}{2}\} \leq
\]

\[\leq 2E\{1 - \exp\left(nB \log \phi_n(t)\right)\} \leq \varepsilon,
\]

proving tightness of \(\{c_n^{-1}nB\}\). Since \(B \in \mathcal{B}_\xi\) was arbitrary, it follows by Lemma 4.6 that \(\{c_n^{-1}n\}\) is tight.

Let us now assume in addition that \(\xi \not\equiv 0\). Proceeding as in the proof of Theorem 8.1, we can then prove that

\[
\lim_{t \to 0} \limsup_{n \to \infty} c_n^{-1}(1 - \phi_n(t)) = 0.
\]

By Chebyshev's inequality, we further obtain for any \(r, t > 0\)

\[
c_n^{-1}P\{\beta_n > r\} \leq \frac{c_n^{-1}E\left(1 - e^{-n t}\right)}{1 - e^{-r t}} = \frac{c_n^{-1}(1 - \phi_n(t))}{1 - e^{-r t}},
\]

and letting in order \(n \to \infty\), \(r \to \infty\) and \(t \to 0\), we get by (15)

\[
\lim_{r \to \infty} \limsup_{n \to \infty} c_n^{-1}P\{\beta_n > r\} = 0,
\]

which shows that the sequence \(\{c_n^{-1}\kappa(P_n^{-1})\}\) of probability measures on \(\mathbb{R}_+\) is tight. In particular there exist some sequence \(N' \subseteq N\) and some
\( \eta, \alpha \) and \( \lambda \) such that (i) and (ii) hold on \( N' \), and we may conclude from the sufficiency part of the theorem that \( \xi \) has the asserted form. (In the case \( \xi \overset{d}{=} 0 \) we may take \( \eta = 0 \) and choose arbitrary \( \alpha \) and \( \lambda \).)

Let us finally assume that \( \xi_n \overset{d}{=} \xi \) and that \( \{c_n C_n\} \) is regular w.r.t. \( B_\xi \) for some \( C \in B_\xi \) with \( \xi C \overset{d}{=} 0 \). Since the sequences \( \{c_n \eta_n\} \) and \( \{c_n^{-1} \kappa(\beta_n^{-1})\} \) are both tight, any sequence \( N' \subset N \) must contain a subsequence \( N'' \subset N' \) such that (i) and (ii) hold on \( N'' \) for some \( \eta, \alpha \) and \( \lambda \). But then \( \xi \) satisfies (11) and (12), and in particular \( B_\xi = B_\eta \), so it follows by Problem 4.9 that \( C_\eta \) is a.s. diffuse. We may thus conclude from Lemma 8.2 that \( P_{\eta^{-1}} \), \( \alpha \) and \( \lambda \) are uniquely determined by \( P_{\xi^{-1}} \), and hence, by Theorem 2.3 in Billingsley (1968), that (i) and (ii) remain true for the original sequence. This completes the proof.

The above convergence criterion takes a particularly simple form in the case of thinnings:

**Corollary 8.4.** For \( n \in N \), let \( p_n \in (0,1] \) and let \( \xi_n \) be a \( p_n \)-thinning of some point process \( \eta_n \). Suppose that \( p_n \to 0 \). Then \( \xi_n \overset{d}{=} \xi \) iff \( p_n \eta_n \overset{d}{=} \eta \), and in this case \( \xi \) is a Cox process directed by \( \eta \).

**Proof.** The preceding proof applies with \( c_n = p_n \), \( \alpha = 0 \) and \( \lambda = \delta_1 \), (and it may even be simplified in the present case, since (ii) is automatically satisfied). No regularity assumption is required on account of Corollary 3.2.
The last result leads to an interesting characterization of the class of Cox processes:

**Corollary 8.5.** Let \( \xi \) be a point process on \( S \). Then \( \xi \) is a Cox process directed by some random measure \( \eta \) iff \( \xi \) is for every \( p \in (0,1) \) a \( p \)-thinning of some point process \( \eta_p \). In this case \( \eta_p \) is a Cox process directed by \( p^{-1}\eta \).

**Proof.** Let \( p \in (0,1) \), let \( \xi \) and \( \eta_p \) be Cox processes directed by \( \eta \) and \( p^{-1}\eta \) respectively, and let \( \xi_p \) be a \( p \)-thinning of \( \eta_p \). By (1.4) and (1.6) we then obtain for any \( f \in F \)

\[
L_{\xi_p}(f) = L_{\eta_p}(-\log[1 - p(1 - e^{-f})]) = L_{p^{-1}\eta}(1 - [1 - p(1 - e^{-f})]) = L_{p^{-1}\eta}(p(1 - e^{-f})) = L_{\eta}(1 - e^{-f}) = L_{\xi}(f),
\]

so \( \xi \equiv \xi_p \). Conversely, suppose that \( \xi \) is for every \( p \in (0,1) \) a \( p \)-thinning of some \( \eta_p \). Applying Corollary 8.4 with \( p_n = n^{-1} \) and \( \xi_n = \xi \), \( n \in \mathbb{N} \), it is seen that \( \xi \) must be a Cox process.

We conclude this section with a useful criterion for infinite divisibility.

**Lemma 8.6.** Let \( \xi \) be a random measure on \( S \) and, for any \( c > 0 \), let \( \eta_c \) be a Cox process directed by \( c\xi \). Then \( \xi \) is infinitely divisible iff \( \eta_c \) is infinitely divisible for every \( c > 0 \).

**Proof.** The necessity part is obvious. Suppose conversely that \( \eta_c \) is infinitely divisible for every \( c > 0 \). Put \( P_{c^{-1}} = D \) and introduce
in the spaces of distributions on \( N \) and \( M \) the operators \( \Omega_p, \mathcal{D}_p \) and \( J \) corresponding to division by \( p \), \( p \)-thinning and formation of Cox processes respectively, \( p \in (0,1] \). From (1.4) and (1.6) it is seen that \( \mathcal{D}_p \) and \( J \) commute with convolution. Using this fact and the last assertion in Corollary 8.5, we obtain

\[
(\mathcal{D}_p (J \Omega_p D)^{1/n})^n = \mathcal{D}_p J \Omega_p D = JD, \quad p \in (0,1], \quad n \in N.
\]

Thus \((JD)^{1/n}\) is a \( p \)-thinning for every \( p \in (0,1] \) and \( n \in N \), so by Corollary 8.5 there exists for every \( n \in N \) some distribution \( D_n \) on \( M \) such that \((JD)^{1/n} = JD_n\). But then

\[
J(D_n)^n = (JD_n)^n = JD, \quad n \in N,
\]

and it follows by Corollary 3.2 that \( D = (D_n)^n, \quad n \in N \), proving that \( D \) is infinitely divisible.

**NOTES.** Theorem 8.1 is new while Theorem 8.3 and Corollary 8.4 were given in Kallenberg (1975). The latter result extends some classical work of Rényi (1956), Nawrotzki (1962), Belyaev (1963) and Goldman (1967), who all consider \( p \)-thinnings of some fixed point process and change the "time" scale by the factor \( p^{-1} \) to be able to obtain convergence. Finally, Corollary 8.5 is due with a direct proof to Mecke (1968) and (1972), (cf. Kerstan, Matthes and Mecke (1974), page 318), while Lemma 8.6 is due to Kummer and Matthes (1970).
PROBLEMS.

8.1. State and prove the thinning case version of Theorem 8.1. Note that no regularity condition is required in this case.

8.2. Let $\xi$ be a $\beta$-compound of some simple point process $\eta$ and put $p = P(\beta > 0)$. Show that $\xi_{0+}^*$ is then a $p$-thinning of $\eta$. Use this fact to give a new proof of Corollary 3.5.

8.3. For $n \in \mathbb{N}$, let $\xi_n$ be a $\beta_n$-compound of $\eta_n$, and suppose that $\beta_n \overset{d}{=} \beta, \beta \overset{d}{=} 0$. Show that $\xi_n \overset{d}{=} \xi$ iff $\eta_n \overset{d}{=} \eta$, and that $\xi$ is then a $\beta$-compound of $\eta$. State and prove the corresponding version of Theorem 8.3.

8.4. Let $\xi$ be a $\beta$-compound of $\eta$ and suppose that $\beta$ is infinitely divisible. Show that $\xi$ has then infinitely many essentially different representations as a $\beta'$-compound of $\eta'$ with $\beta' > 0$ a.s. (Hint: Replace $\eta$ by $n\eta$, $n \in \mathbb{N}$. Proceed as in the proof of Theorem 8.1 to reduce to the case $\beta > 0$ a.s. Finally prove that the representations thus obtained are really different.)

8.5. Let $\xi$ be a $\beta$-compound of $\eta$ and suppose that $\eta$ is infinitely divisible. Show that $\xi$ is then also infinitely divisible. Prove the corresponding result for the random measures defined by (11) and (12).

8.6. Let $\xi$ be a Cox process directed by $\eta$ or a $p$-thinning of $\eta$. Show that it is possible that $\xi$ is infinitely divisible while $\eta$ is not. (Hint: It is enough to consider $\mathbb{Z}_+$-valued random variables. To construct our $\eta$, we may start with the familiar Raikov-Lévy example reproduced in Lukacs (1970), page 251, and add a suitable infinitely divisible variable to make the thinning or Cox variable $\xi$ infinitely divisible.)
9. SYMMETRICALLY DISTRIBUTED RANDOM MEASURES.

Let \( \omega \in M_d \) be fixed. We shall say that a random measure \( \xi \) is *symmetrically distributed with respect to* \( \omega \), if the distribution of \((\xi B_1, \ldots, \xi B_k)\) for disjoint \( B_1, \ldots, B_k \in \mathcal{B}, \ k \in \mathbb{N} \), only depends on \((\omega B_1, \ldots, \omega B_k)\). If \( \omega S \in R^+_1 \) and if \( \tau \) is a random element in \( S \) with distribution \( \omega / \omega S \), then this property is obvious for the point process \( \delta_\tau \), and since the property is persistent under addition of independent random measures and under mixing, it is shared by all mixed Poisson and sample processes which are defined w.r.t. \( \omega \). In particular, any such process must satisfy

\[
(1) \quad P(\xi B = 0) = \phi(\omega B), \quad B \in \mathcal{B},
\]

for some non-increasing function \( \phi \). If \( \xi \) is a mixed Poisson process w.r.t. \( \omega \) with mixing random variable \( \alpha \), then

\[
(2) \quad P(\xi B = 0) = E e^{-\alpha \omega B} = \phi(\omega B), \quad B \in \mathcal{B},
\]

so in this case \( \phi = \mathcal{L}_\alpha \). On the other hand, if \( \omega S \in R^+_1 \) and if \( \xi \) is a mixed sample process w.r.t. \( \omega / \omega S \) with mixing variable \( \nu \), then

\[
(3) \quad P(\xi B = 0) = E (1 - \frac{\omega B}{\omega S})^{\nu} = \psi(1 - \frac{\omega B}{\omega S}) = \phi(\omega B), \quad B \in \mathcal{B},
\]

where \( \psi \) is the generating function of \( \nu \), so in this case

\[
(4) \quad \phi(t) = \psi(1 - \frac{t}{\omega S}), \quad t \in [0, \omega S);
\]

\[
\psi(t) = \phi(\omega S(1 - t)), \quad t \in (0,1].
\]
If $\xi$ is a mixed Poisson or sample process, then $P_{\xi^{-1}}$ is easily seen to be determined by $\omega$ and $\phi$. We shall use the symbol $M(\omega, \phi)$ to denote such a process $\xi$. (Note, however, that the pair $(\omega, \phi)$ is not uniquely determined by $P_{\xi^{-1}}$ since it can be replaced by $(a\omega, \phi(\cdot/a))$ for any $a > 0$.)

It turns out that every symmetrically distributed simple point process is a mixed Poisson or sample process. We shall even be able to prove the following stronger result.

**THEOREM 9.1.** Let $\xi$ be a simple point process on $S$ and let $\omega \in M_d$. Further suppose that $U \subset B$ is a DC-ring and that $t \in (0, \omega]$ is fixed. Then $\xi \overset{d}{=} M(\omega, \phi)$ iff there exists some non-increasing function $\phi_t$ satisfying

$$Ee^{-t\xi U} = \phi_t(\omega U), \ U \subset U.$$

In that case

$$\phi_t(x) = \phi(x(1 - e^{-t})), \ x \in [0, \omega S].$$

**PROOF.** Let $t \in R^+_d$ and $B \in B$, and suppose that $\xi$ is a mixed Poisson process with mixing variable $\alpha$. Then

$$Ee^{-t\xi B} = E \exp(-\alpha_t \omega B(1 - e^{-t})) = \phi(\omega B(1 - e^{-t})).$$

On the other hand, we get in the mixed sample case, on account of (4)
\[ E e^{-t \xi} = E \left( \frac{\omega^B}{\omega^S} e^{-t} + \frac{\omega^C}{\omega^S} \right) = \psi \left( 1 - \frac{\omega^B}{\omega^S} (1 - e^{-t}) \right) = \phi(\omega^B(1 - e^{-t})) \, . \]

Thus (5) holds in both cases with \( \phi_t \) defined by (5'). For \( t = \infty \), this statement was proved in (2) and (3).

Suppose conversely that (5) holds, and let us first assume that \( t = \infty \). Since \( \omega = 0 \) clearly implies \( \xi = 0 \) a.s., we may assume that \( \omega \neq 0 \). Our first aim is to extend (5) to \( B \), i.e. to prove (1). For this purpose, let \( s_1, s_2, \ldots \in S \) be a bounded sequence of distinct points belonging to the support of \( \omega \), and consider for fixed \( s = s_n \) some sequence of sets \( U_k \in \mathcal{U} \) with \( s \in U_k \), \( k \in \mathbb{N} \), satisfying \( U_k \uparrow \{ s \} \). Then \( 0 < \omega U_k \downarrow \omega \{ s \} = 0 \) while \( \xi U_k \downarrow \xi \{ s \} \) a.s., so we get

\[ 1 - \phi(\lambda U_k) = P\{ \xi U_k > 0 \} + P\{ \xi \{ s \} > 0 \} = 1 - \phi(0^+) \, . \]

Hence by Fatou's lemma

\[ 0 = P \limsup_{n \to \infty} \left\{ \xi(s_n) > 0 \right\} \geq \limsup_{n \to \infty} P\{ \xi \{ s_n \} > 0 \} = 1 - \phi(0^+) \geq 0 \, , \]

so \( \phi(0^+) = 1 \). To extend this continuity at 0 to (uniform) continuity on \( [0, \omega S] \) (or even on \( [0, \omega S] \) if \( \omega \) has bounded support), let \( \epsilon > 0 \) be fixed and choose \( \delta > 0 \) such that \( 1 - \phi(3\delta) < \epsilon \). For any \( s, t \in [0, \omega S] \) with \( s < t < s + \delta \), we can choose sets \( U, V \in \mathcal{U} \) with \( U \subset V \) and such that

\[ s - \delta < \omega U \leq s < t \leq \omega V < t + \delta \, . \]

In fact, let \( C \in \mathcal{U} \) be such that \( \omega C \geq t \) and let \( \{ C_n \} \subset \mathcal{U} \) be a null-array
of partitions of $C$. Then $\max_j \omega_{C_{nj}} < \delta$ since $\omega \in M_d$, so it suffices for construction of $U$ and $V$ to consider an $n \in N$ with $\max_j \omega_{C_{nj}} < \delta$. From (5), (6) and the monotonicity of $\phi$ we get

$$\phi(s) - \phi(t) \leq \phi(\omega U) - \phi(\omega V) = P(\xi U = 0) - P(\xi V = 0) = P(\xi U = 0, \xi V > 0) \leq P(\xi (V \setminus U) > 0) = 1 - \phi(\omega (V \setminus U)) =$$

$$= 1 - \phi(\omega V - \omega U) \leq 1 - \phi(3\delta) < \varepsilon.$$  

We may now use the monotone class theorem to extend (5) to $B$.

For arbitrary $n \in N$ and $h < \omega S/n$, it is possible to choose disjoint sets $B_1, \ldots, B_n \in B$ with $\omega B_1 = \ldots = \omega B_n = h$. In fact, let $C \in B$ with $\omega C > nh$ and let $\{C_n\}$ be a null-array of nested partitions of $C$. For fixed $k > 0$ we can choose $m$ so large that $\max_j \omega_{C_{nj}} < k^{-1}$, and then there exist disjoint unions $B_{k_1}, \ldots, B_{k_n}$ of the partitioning sets $C_{m_1}, C_{m_2}, \ldots$ such that $h - k^{-1} < \omega B_{k_j} \leq h$, $j = 1, \ldots, n$.

The sets $B_{k_j}$ may clearly be chosen non-decreasing in $k$ for each $j$, and in that case the sets $B_j = \bigcup_k B_{k_j}$, $j = 1, \ldots, n$, have the desired property.

With this choice of $B_1, \ldots, B_n$, it is seen from (1) and Lemma 5.7 that

$$(-1)^n \Delta_{\omega}^{\Delta} \phi(x) \bigg|_{x=0} = (-1)^n \Delta_{B_1}^{\cdots} \Delta_{B_n}^{\omega B} \bigg|_{B=\emptyset} \geq 0,$$

which means that the function $\phi$ is completely monotone (cf. Feller (1971), pp. 223, 439). If $\omega S = \infty$, it follows that $\phi$ is the L-transform of some

\[\text{together with the corresponding inequality for arbitrary } x\]
$R_+^*$-valued random variable, and we get $\xi \overset{d}{=} M(\omega, \phi)$ by (2) and Theorem 3.3. On the other hand, if $\omega S < \infty$, then the function $\psi(t) = \phi(\omega S(1 - t))$, $t \in (0,1)$, being absolutely monotone, is the probability generating function of some $Z_+^*$-valued random variable, and again $\xi \overset{d}{=} M(\omega, \phi)$, now by (3) and Theorem 3.3. This completes the proof in the case $t = \infty$.

In the case of finite $t$, let $\xi$ be a $p$-thinning of $\xi$, where $p = 1 - e^{-t}$. As in the proof of Theorem 3.4, we obtain

$$P(\eta U = 0) = \phi_t(\omega U), \quad U \in \mathcal{U},$$

and it follows as above that $\eta \overset{d}{=} M(\omega, \phi_t)$. But then $\eta$ is symmetrically distributed w.r.t. $\omega$, and from (1.6) it is easily seen by analytic continuation that this is also true for $\xi$. In particular, (1) must hold for some $\phi$, and it follows as above that $\xi \overset{d}{=} M(\omega, \phi)$. The proof is now complete.

**COROLLARY 9.2.** Let $\xi$ be a point process on $S$ and let $\omega \in \mathcal{M}$. Then $\xi \overset{d}{=} \text{some } M(\omega, \phi)$ iff $B\xi \overset{d}{=} \text{some } M(\omega_B, \phi_B)$ for every $B \in \mathcal{B}$. We may then take $\omega_B = B\omega$, in which case $\phi_B$ becomes the restriction of $\phi$ to $[0, \omega B]$. Furthermore, $P\xi^{-1}$ is then uniquely determined by $P(\xi B)^{-1}$ for any fixed $B \in \mathcal{B}$ with $\omega B > 0$.

Note that the only if part of the first assertion contains a well-known classical property of the Poisson process.

**PROOF.** Let us first suppose that $\omega \in M_d$. Then the first assertion is an immediate consequence of Theorem 9.2, while the second one follows
by comparison of (2) and (3). As for the last assertion, note that $\xi_B$ has probability generating function $\phi(\omega B(1 - t))$, $t \in [0,1]$. Thus $P(\xi_B)^{-1}$ determines $\phi$ on $[0, \omega B]$ and hence, by the uniqueness of analytic continuations, on $[0, \omega S]$.

The proof for general $\omega \in M$ requires randomization. Write $\xi = \sum_j \delta_{\tau_j}$ and choose independently of $\{\tau_j\}$ a sequence $\{\eta_j\}$ of independent random variables which are uniformly distributed on $[0,1]$. It is then easily seen that $\hat{\xi} = \sum_j \delta_{(\tau_j, \eta_j)}$ is a point process on $S \times [0,1]$ and that $\hat{\xi} \overset{d}{=} M(\omega \times \lambda, \phi)$ iff $\xi \overset{d}{=} M(\omega, \phi)$, where $\lambda$ denotes Lebesgue measure on $[0,1]$. Since $\omega \times \lambda$ is diffuse, the assertion is true for $\hat{\xi}$, and so it holds for $\xi$ as well.

We next indicate by an example how Theorem 9.1 can be used to improve Theorems 3.3 and 3.4 in the particular case of Poisson and sample processes.

**Corollary 9.3.** Let $\omega \in M_d$ and, for $\omega S = \infty$ or $< \infty$, let $\xi$ be a Poisson or sample process respectively with intensity $\omega$. Further suppose that $\eta$ is a simple point process on $S$ satisfying

$$P(\eta_U = 0) = \phi(\omega U), \quad U \in U,$$

for some DC-ring $U \subset B$ and some non-increasing function $\phi$. Then $\xi \overset{d}{=} \eta$ iff there exist two sets $B_1, B_2 \in B$ with $0 < \omega B_1 < \omega B_2$ satisfying

$$P(\xi B_j = 0) = P(\eta B_j = 0), \quad j = 1,2.$$
PROOF. By Theorem 9.1 we have \( \eta \equiv M(\omega, \phi) \). In the case \( \omega S = \infty \) we get by (2) and (7)

\[
E e^{-\alpha s} = e^{-s}, \quad E e^{-\alpha t} = e^{-t},
\]

where \( s = \omega B_1 \) and \( t = \omega B_2 \) while \( \alpha \) is a random variable with \( L \)-transform \( \phi \). If \( \alpha \) were non-degenerate, it would follow by Jensen's inequality for strictly convex functions that

\[
e^{-t} = (e^{-s})^{t/s} = (E e^{-\alpha s})^{t/s} < E(e^{-\alpha s})^{t/s} = E e^{-\alpha t} = e^{-t},
\]

which is impossible and proves that \( \alpha \) is indeed degenerate. From (8) we get \( \alpha = 1 \) a.s., and so \( \phi(x) \equiv e^{-x} \) which means that \( \eta \) is a Poisson process with intensity \( \omega \). A similar proof applies to the case \( \omega S < \infty \).

We are now going to characterize the class of arbitrary symmetrically distributed random measures. Consider an infinitely divisible random measure \( \xi \) with homogeneous w.r.t. \( \omega \) independent increments such that

\[
(9) \quad \log E e^{-xf} = \alpha f + \int_{S \times \mathbb{R}_+} (1 - e^{-xf(s)}) (\omega \times \lambda) (dsdx), \quad f \in F,
\]

where \( \alpha \in \mathbb{R}_+ \) and \( \lambda \in M(\mathbb{R}_+) \) with \( \lambda x < \infty \). By Fubini's theorem, the integral \( \int_S (1 - e^{-xf(s)}) \omega(ds) \) is a measurable function of \( x \), and therefore the right-hand side of (9) is jointly measurable in \( \alpha \) and \( \lambda \). Hence, by Lemma 1.4, we may perform mixing with respect to \( \alpha \) and \( \lambda \), considered as random elements. This gives the most general distribution in case \( \omega S = \infty \). The case \( \omega S < \infty \) is entirely different:
THEOREM 9.4. Let $\omega \in M_d\setminus\{0\}$ and let $\xi$ be a random measure on $S$.

Then $\xi$ is symmetrically distributed w.r.t. $\omega$ iff

(i) in the case $\omega S = \infty$, there exist some $R_+^*$-valued random variable $\alpha$ and some random measure $\lambda$ on $R_+^*$ with $\lambda < \infty$ a.s. such that, given $\alpha$ and $\lambda$, the conditional L-transform of $\xi$ is given by (9);

(ii) in the case $\omega S < \infty$, there exist some $R_+^*$-valued random variable $\alpha$ and some point process $\beta = \sum_j \delta_{\tau_j}$ on $R_+^*$ with $\sum_j \beta_j < \infty$ a.s. such that, given $\alpha$ and $\beta$, $\xi$ is conditionally distributed as $\omega' + \sum_j \beta_j \delta_{\tau_j}$, where $\omega' = \omega/\omega S$ while $\tau_1, \tau_2, \ldots$ are independent random elements in $S$ with distribution $\omega'$.

The random elements $(\alpha, \lambda)$ and $(\alpha, \beta)$ respectively are a.s. unique measurable functions of $\xi$.

We shall call $(\alpha, \lambda)$ or $(\alpha, \beta)$ respectively the canonical random elements of $\xi$.

PROOF. The sufficiency of (i) and (ii) is obvious. Conversely, suppose that $\xi$ is symmetrically distributed w.r.t. $\omega$, and let us first assume that $\omega S < \infty$, say that $\omega S = 1$. The symmetry assumption then extends by monotone convergence to $\overline{B}$, and in particular it it seen that $\xi S < \infty$ a.s. Hence by Lemma 2.1, $\xi$ has the representation

$$\xi = \xi_d + \sum_j \beta_j \delta_{\tau_j},$$

(10)
for some random elements $\xi_d \in M_d$, $\beta_1 \geq \beta_2 \geq \ldots \geq 0$ and $\tau_1, \tau_2, \ldots \in S$, the latter being a.s. distinct. Whenever two or more $\beta_j$ coincide, we may assume that the corresponding $\tau_j$ are taken in random order.

We shall first consider the case when $S$ is the unit interval $[0,1]$ while $\omega$ is Lebesgue measure. Using Theorem 3.1, it is easily seen that $P_{\xi^{-1}}$ is symmetric in the sense that $\xi_{T^{-1}} \overset{d}{=} \xi$ for every $\omega$-preserving bi-measurable 1-1 mapping $T$ of $[0,1]$ onto itself. Since $\alpha \equiv \xi_d S$ and $\beta_1, \beta_2, \ldots$ are invariant under such transformations, the conditional distributions of $\xi$, given $(\alpha, \beta_1, \beta_2, \ldots)$, are again symmetric, so we may consider $\alpha, \beta_1, \beta_2, \ldots$ as constants. Since $\xi_d$ is by Lemma 2.1 a measurable function of $\xi$, we have for any $T$ as above $\xi_d T^{-1} = (\xi T^{-1})_d \overset{d}{=} \xi_d$, so even $P_{\xi^{-1}}^{d}$ is symmetric. Moreover, the process $\xi_d[0,t)$ - at, $t \in [0,1]$ is clearly a.s. uniformly continuous and of bounded variation so, writing $I_{nj} = [(j-1)/n, j/n)$ and $\eta_{nj} = \xi_d I_{nj} - \alpha/n$, $j = 1, \ldots, n$, $n \in N$, we get

$$\sum_{j} \frac{\eta_{nj}^2}{n_{nj}} \leq \left(\sum_{j} |\eta_{nj}|\right) \max_{j} |\eta_{nj}| \to 0 \text{ a.s., } n \to \infty.$$  

From the symmetry and the obvious relations $\sum_{j} \eta_{nj} = 0$ and $\sum_{j} \eta_{nj}^2 \leq 4\alpha^2$ we hence obtain by bounded convergence

$$-n(n-1)E_n\eta_{n1}\eta_{n2} = nE_n^2\eta_{n1}^2 = E\sum_{j} \eta_{nj}^2 \to 0,$$

and so

$$\max_{k} \left\{ \frac{1}{k} \sum_{j=1}^{K} \eta_{nj} \right\}^2 = \max_{k} \left\{ kE_n^2\eta_{n1}^2 + k(k - 1)E_n\eta_{n1}\eta_{n2} \right\} \leq$$

$$\leq nE_n^2\eta_{n1}^2 + n(n-1) |E_n\eta_{n1}\eta_{n2}| \to 0.$$
But this implies \( \xi_d[0,t) = \alpha t \) a.s. for every rational \( t \in [0,1] \),
and therefore \( \xi_d = \alpha \lambda \) a.s.

Next suppose that \( \beta_1 > 0 \), and let \( \nu = \nu(\xi) \) be defined for fixed \( n \in N \) by \( \tau_1 \in I_{n \nu} \). For \( k,m \in Z \) (mod \( n \)), let \( T_{km} \) be the transformation of \( [0,1) \) which maps \( I_{nk} \) onto \( I_{n,k+m} \) and \( I_{n,k+m} \) onto \( I_{nk} \), and put \( \xi_{nm} = \xi_{T_{km}}^{-1} \). By symmetry we get for any \( A \in M \)

\[
P(\xi \in A) = \sum_{k=1}^{n} P(\xi \in A, \nu = k + m) = \sum_{k=1}^{n} P(\xi_{T_{km}}^{-1} \in A, \nu(\xi_{T_{km}}^{-1}) = k + m) = \sum_{k=1}^{n} P(\xi_{T_{km}}^{-1} \in A, \nu(\xi) = k) = \sum_{k=1}^{n} P(\xi_{T_{vm}}^{-1} \in A, \nu(\xi) = k) = P(\xi_{nm}^{-1} \in A) = P(\xi_{nm} \subseteq A).
\]

(Note that this relation is even true when \( \beta_1 = \beta_2 \), although \( \nu \) is not uniquely determined by \( \xi \) in this case.) This proves that \( \xi_{nm} \equiv \xi \) for any \( n \) and \( m \).

For fixed \( s \in (0,1] \) we now define \( \xi_s \) as the random measure obtained from \( \xi \) by replacing \( \tau_1 \) by \( \tau_1 + s \) (mod \( 1 \)). Let \( m,n \to \infty \) in such a way that \( m/n \to s \), and suppose that \( \tau_1 + s \equiv 0 \) (mod \( 1 \)). Then \( \xi_{nm} I \to \xi_s I \) for any interval \( I \) with \( s \notin \overline{I} \), and hence also when \( s \in \overline{I} \) since \( \xi_{nm}[0,1) = \xi[0,1) = \xi_s[0,1) \). Therefore \( \xi_{nm} \Rightarrow \xi_s \) a.s., and we get \( \xi \equiv \xi_{nm} \subseteq \xi_s \), i.e. \( \xi_s \Rightarrow \xi \). By Fubini's theorem, this implies for any measurable function \( f : R^+ \to R^+ \), \( k = \sup \{ j : \beta_j > 0 \} \),
\[ Ef(\tau_1, \tau_2, \ldots) = Ef(\tau_1 + s, \tau_2, \ldots) = \int_0^1 Ef(\tau_1 + s, \tau_2, \ldots) ds = \]
\[ = E \int_0^1 f(\tau_1 + s, \tau_2, \ldots) ds = E \int_0^1 f(s, \tau_2, \ldots) ds = \]
\[ = Ef(\tau_1', \tau_2, \ldots) , \]

where \( \tau_1' \) is a random variable in \([0,1]\) which is independent of \( (\tau_2, \tau_3, \ldots) \) with distribution \( \omega \). Hence \( \tau_1 \) has the same properties, and continuing inductively, it is seen that \( \tau_1', \tau_2, \ldots \) are independent with distribution \( \omega \). This completes the proof of (ii) when \( S = [0,1] \).

Turning to the case of general \( S \) and \( \omega \) with \( \omega S = 1 \), note first that the symmetry property of \( \xi \) extends by monotone convergence from \( B \) to \( \overline{B} \). For convenience, we may compactify \( S \) or rather assume from the beginning that \( S \) is compact. Let \( S = \{ S_{nj} \} \subset B \) be a null-array of nested partitions of \( S \), and note that \( \max \omega S_{nj} \to 0 \) by compactness of \( S \) and diffuseness of \( \omega \). To each set \( S_{nj} \) we can make correspond a subinterval \( I_{nj} \) of \([0,1]\) satisfying \( |I_{nj}| = \omega S_{nj} \) in such a way that \( I = \{ I_{nj} \} \) forms a null-array of nested partitions of \([0,1]\). For each \( s \in S \) there is a unique decreasing sequence of sets \( S_{nj} \subset S_{n} \) containing \( s \), and hence we can define a function \( f: S \to [0,1] \) by \( f(s) = \cap_n I_{nj}^- \). Each \( f^{-1}(0,t) \), \( t \in [0,1] \), may be written as a countable union of \( S \)-sets, so \( f \) is measurable, and it follows in particular that every measure \( \mu \in M \) induces a measure \( \mu f^{-1} \) on \([0,1]\). Moreover, the mapping \( \mu \to \mu f^{-1} \) is measurable by definition of \( M \), so \( \xi \) induces a random measure \( \xi f^{-1} \) on \([0,1]\) .
Writing $D$ for the set of $I$-interval endpoints, we get by construction of $f$

$$\omega f^{-1}[0,t] \leq t \leq \omega f^{-1}[0,1], \quad t \in D,$$

and hence for any $t \in D \setminus \{1\}$

$$t < \omega f^{-1}[0,t] = \inf(\omega f^{-1}[0,s], s \in D, s > t) \leq \inf(s \in D: s > t) = t,$$

which yields $\omega f^{-1}[0,t] = t$. Since $D$ is dense in $[0,1]$, this together with the fact that $\omega f^{-1}[0,1] = \omega S = 1$ implies that $\omega f^{-1}$ equals Lebesgue measure. Moreover, $\xi f^{-1}$ is symmetrically distributed with respect to $\omega f^{-1}$ since $\xi$ is symmetrically distributed with respect to $\omega$.

Using the completeness of $S$, it is easily seen that $f$ has a unique inverse $f^{-1}$ on $f(S) \setminus D$, and since $\omega f^{-1}D = 0$ and therefore $\xi f^{-1}D = 0$ a.s., there is a.s. a unique correspondence between the atoms of $\xi$ and $\xi f^{-1}$, and (10) is equivalent to

$$\xi f^{-1} = \xi_d f^{-1} + \sum_j \beta_j \delta f(\tau_j),$$

with a.s. diffuse $\xi_d f^{-1}$ and distinct $f(\tau_1), f(\tau_2), \ldots$. Now the statement of the theorem is known to be true for $\xi f^{-1}$, and so $\xi_d f^{-1} = a \omega f^{-1}$ a.s. with $a = \xi_d S$, while $f(\tau_1), f(\tau_2), \ldots$ are mutually independent and independent of $(a, \beta_1, \beta_2, \ldots)$ with common distribution $\omega f^{-1}$. For every $S_{nj} \in S$ we obtain a.s.

$$\xi_d S_{nj} = \xi_d (S_{nj} \setminus f^{-1}D) = \xi_d f^{-1}(I_{nj} \setminus D) = a \omega f^{-1}(I_{nj} \setminus D) = a \omega f^{-1} I_{nj} =$$

$$= a |I_{nj}| = a \omega S_{nj},$$
and $S$ being a DC-semi-ring, it follows by Dynkin's theorem that $\xi_d = \alpha \omega$ a.s. Similarly, we have a.s. for any $n, j$ and $k$

$$\{\tau_k \in S_{nj} \} = \{\tau_k \in S_{nj} \setminus f^{-1}D\} = \{f(\tau_k) \in I_{nj} \setminus \overline{D}\} = \{f(\tau_k) \in I_{nj}\} ,$$

so we get

$$P\left(\{(\alpha, \beta_1, \beta_2, \ldots) \in A\} \cap \bigcap_{j=1}^{k} \{\tau_j \in B_j\} \right) = P(\{(\alpha, \beta_1, \beta_2, \ldots) \in A\} \cap \bigcap_{j=1}^{k} \omega B_j$$

for any measurable set $A$ and for arbitrary $k \in \mathbb{N}$ and $B_1, \ldots, B_k \in S$, and by Dynkin's theorem, this extends to general $B_1, \ldots, B_k \in \overline{S}$. The proof of (ii) is now complete. The measurability of $(\alpha, \beta)$ is obvious.

We now consider the case when $\omega S = \infty$. Choose $C_1, C_2, \ldots \in \mathbb{B}$ with $C_n^c + S$ and $\omega C_1 > 0$. Since $C_n \xi$ is symmetrically distributed with respect to $C_n \omega$ for each $n \in \mathbb{N}$, we know that

$$C_n \xi = \frac{\alpha}{n} (C_n \omega) + \sum_{j} \beta_j \delta_{\tau_{nj}} , \quad n \in \mathbb{N} ,$$

for some $\alpha_n, \beta_{n1}, \beta_{n2}, \ldots$ and some $\tau_{n1}, \tau_{n2}, \ldots$, the latter being independent of $(\alpha_n, \beta_{n1}, \beta_{n2}, \ldots)$ and mutually independent with common distribution $C_n \omega / \omega C_n$. Now $C_m \xi = C_m (C_n \xi)$ for $m \leq n$, so writing

$$\alpha = \alpha_1, \quad \beta = \sum_{j} \beta_{nj} , \quad p_{mn} = \omega C_m / \omega C_n , \quad m \leq n \in \mathbb{N} ,$$

it is seen by identification that $\alpha_1 = \alpha_2 = \ldots = \alpha$ a.s. while $\beta_m$ is a $p_{mn}$-thinning of $\beta_n$ for $m \leq n \in \mathbb{N}$. Since $p_{mn} \to 0$ as $n \to \infty$ for each fixed $m \in \mathbb{N}$, it follows by Corollary 8.5 that $\beta_m$ is distributed as a Cox process on $\mathbb{R}_+$ directed by some $\lambda_m$, and that $\lambda = \lambda / \omega C_1$. Putting $\lambda = \lambda_1 / \omega C_1$, it is seen that we may take $\lambda_n = (\omega C_n) \lambda$ for each $n$. Let $f \in F_c$ be arbitrary, and
choose $n \in \mathbb{N}$ so large that the support of $f$ is contained in $\mathbb{C}_n$.

Then

$$E e^{-\xi} f = E \exp\left\{- \sum_j \beta_{n_j} \tau_{n_j} \right\} = E \exp\left\{- \sum_j \beta_{n_j} f(\tau_{n_j}) \right\} | \left\{ \beta_{n_j} \right\} =$$

$$= E \Pi E \exp\left\{- \beta_{n_j} \tau_{n_j} \right\} | \beta_{n_j} = E \Pi \int_S \exp\left\{- \beta_{n_j} f(s) \right\} \omega(ds)/\omega C_n ,$$

and if we define

$$h(x) = - \log \int_S e^{-xf(s)} \omega(ds)/\omega C_n , \quad x \in \mathbb{R}_+ ,$$

this expression may be written

$$E \Pi \exp\left\{-h(\beta_{n_j})\right\} = \exp\left\{- \sum_j h(\beta_{n_j})\right\} = \exp\left\{- \beta_n h\right\} = E \exp\left\{- \beta_n h\right\} | \lambda =$$

$$= \exp\left\{- \lambda_n (1 - e^{-h})\right\} =$$

$$= \exp\left\{- \omega C_n \int_{\mathbb{R}_+} \left\{1 - \int_S e^{-xf(s)} \omega(ds)/\omega C_n \right\} \lambda(dx)\right\} =$$

$$= \exp\left\{- \int_{\mathbb{R}_+ < S} (1 - e^{-xf(s)}) (\lambda \times \omega)(dxds)\right\} .$$

Putting $f(s) = t 1_{\mathbb{C}_1}(s)$, $t \in \mathbb{R}_+$, it follows by Fubini's theorem and monotone convergence that $\lim_{t \to 0} \Omega(x)(t) = 1$ can only be true if $\lambda < \infty$ a.s. This proves that $\xi$ is distributed as the random measure described in (i). But for the $\xi$ in (i) we easily obtain $\beta_n/\omega C_n \xrightarrow{\text{a.s.}} \lambda$ by the strong law of large numbers, so $\lambda$ is a measurable function of $\xi$. For the $\xi$ under consideration we may therefore redefine $\lambda$ according to
this mapping, and doing so, (i) will automatically be satisfied. This completes the proof of the theorem.

Of particular interest is the class of random measures which are at the same time symmetrically distributed and infinitely divisible. Here is a simple characterization of this class:

**Theorem 9.5.** Let the random measure \( \xi \) be symmetrically distributed with respect to some \( \omega \in M_d \) with canonical random elements \((\alpha, \lambda)\) or \((\alpha, \beta)\). Then \( \xi \) is infinitely divisible iff this is true for \((\alpha, \lambda)\) or \((\alpha, \beta)\) respectively.

**Proof.** First suppose that \( \omega S < \infty \), and let \((\alpha, \beta)\) be infinitely divisible. For each \( n \in \mathbb{N} \) we may choose independent random elements 
\[(\alpha_1, \beta_1) \overset{d}{=} \ldots \overset{d}{=} (\alpha_n, \beta_n) \text{ such that } (\alpha_1 + \ldots + \alpha_n, \beta_1 + \ldots + \beta_n) \overset{d}{=} (\alpha, \beta).\]

By randomization we may then construct independent symmetrically distributed random measures \( \xi_1, \ldots, \xi_n \) with canonical random elements \((\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\). From Theorem 9.4 it is seen that \( \xi_1 + \ldots + \xi_n \) is symmetrically distributed with canonical random element \((\alpha_1 + \ldots + \alpha_n, \beta_1 + \ldots + \beta_n) \overset{d}{=} (\alpha, \beta)\), and so \( \xi_1 + \ldots + \xi_n \overset{d}{=} \xi \). Since \( n \) was arbitrary, this proves that \( \xi \) is infinitely divisible.

Conversely, suppose that \( \xi \) is infinitely divisible and choose for fixed \( n \in \mathbb{N} \) some independent random measures \( \xi_1 \overset{d}{=} \ldots \overset{d}{=} \xi_n \) such that \( \xi_1 + \ldots + \xi_n \overset{d}{=} \xi \). By considering the corresponding L-transforms, it is easily seen that \( \xi_1, \ldots, \xi_n \) are symmetrically distributed, say with
canonical random elements \((a_1, \beta_1), \ldots, (a_n, \beta_n)\). Again we may conclude from Theorem 9.4 that \(\xi_1 + \ldots + \xi_n \overset{d}{=} \xi\) has the canonical random element 
\((a_1 + \ldots + a_n, \beta_1 + \ldots + \beta_n)\), so we must have
\((a_1 + \ldots + a_n, \beta_1 + \ldots + \beta_n) \overset{d}{=} (a, \beta)\). Furthermore, the random elements 
\((a_1, \beta_1), \ldots, (a_n, \beta_n)\), being measurable functions of the independent
random measures \(\xi_1, \ldots, \xi_n\) respectively, are themselves independent. Since \(n\) was arbitrary, this proves that \((a, \beta)\) is infinitely divisible.

In the case \(\omega S = \infty\), let us first show that, if \(\xi_1\) and \(\xi_2\) are
independent and symmetrically distributed with canonical random elements 
\((a_1, \lambda_1)\) and \((a_2, \lambda_2)\) respectively, then \(\xi_1 + \xi_2\) has canonical random
elements \((a_1 + a_2, \lambda_1 + \lambda_2)\). To see this, note that \((a_1, \lambda_1)\) and 
\((a_2, \lambda_2)\) are measurable functions of \(\xi_1\) and \(\xi_2\) respectively, and hence that \(\xi_1\) and \(\xi_2\) are conditionally independent with canonical
elements \((a_1, \lambda_1)\) and \((a_2, \lambda_2)\), given \((a_1, a_2, \lambda_1, \lambda_2)\). Using (9),
it follows that \(\xi_1 + \xi_2\) has conditionally the canonical elements 
\((a_1 + a_2, \lambda_1 + \lambda_2)\), given \((a_1, a_2, \lambda_1, \lambda_2)\), and the assertion follows
by taking conditional expectations with respect to \((a_1 + a_2, \lambda_1 + \lambda_2)\).
Once this fact is established, we may proceed exactly as in the case
\(\omega S < \infty\). The proof is thus complete.

According to the last theorem, a symmetrically distributed random
measure \(\xi\) is infinitely divisible iff this is true for the corresponding
pair of canonical random elements. In view of Theorem 6.5, the distribution
of \(\xi\) may thus be described in two different ways by means of spectral
representations of the form (6.6), and we may ask for the relationship
between the corresponding canonical measures. More precisely, it follows
from Theorem 6.5 that, if \( \xi, (\alpha, \lambda) \) or \( (\alpha, \beta) \) is infinitely divisible,
then

\[
- \log E e^{-\xi f} = \Gamma f + \int (1 - e^{-\mu f}) \lambda(du), \quad f \in F(S),
\]

\[
- \log E e^{-at-\lambda f} = a_0 t + \lambda_0 f + \int (1 - e^{-xt-\mu f}) \gamma(dxdu), \quad t \in \mathbb{R}_+, \quad f \in F(\mathbb{R}_+),
\]

or

\[
- \log E e^{-at-\beta f} = a_0 t + \int (1 - e^{-xt-\mu f}) \gamma(dxdu), \quad t \in \mathbb{R}_+, \quad f \in F(\mathbb{R}_+),
\]

respectively, where in (11) \( \Gamma \in \mathcal{M} \) while \( \Lambda \) is a suitable measure on
\( \mathcal{M}\setminus\{0\} \), in (12) \( a_0 \in \mathbb{R}_+, \lambda_0 \in \mathcal{M}(\mathbb{R}_+) \) while \( \gamma \) is a measure on
\( \mathcal{M}(\mathbb{R}_+)\setminus\{0\} \), and in (13) \( a_0 \in \mathbb{R}_+ \) while \( \gamma \) is a measure on \( \mathcal{M}(\mathbb{R}_+)\setminus\{0\} \).

and we may ask for the relationship between these quantities. For a simple
description of \( \Gamma \) and \( \Lambda \) in terms of \( (a_0, \lambda_0, \gamma) \) or \( (a_0, \gamma) \), let
\( P_{x,\mu} \) and \( Q_{x,\mu} \) be the distributions of symmetrically distributed random
measures on \( S \) with non-random canonical elements \((x,\mu)\) in the cases
\( \omega S = \infty \) and \( \omega S = 1 \) respectively. In the former case we further denote
by \( R_x \) the measure on \( \mathcal{M}\setminus\{0\} \) induced by \( \omega \) via the mapping \( s \mapsto x\delta_S \),
s \in S, where \( x > 0 \) is arbitrary but fixed.

**Theorem 9.6.** Let \( \xi \) be symmetrically distributed with respect to \( \omega \in \mathcal{M}_d \)
with canonical random elements \((\alpha, \lambda)\) or \((\alpha, \beta)\). Further suppose that
\( \xi \) is infinitely divisible, and let the quantities \((\Gamma, \Lambda)\) and 
\((\alpha_0, \lambda_0, \gamma)\) or \((\alpha_0, \gamma)\) respectively be defined by (11) - (13). Then we have in the case \( \omega S = \infty \)

\[
\Gamma = \alpha_0 \omega \quad , \quad \Lambda = \int P_{x, \mu} \gamma(dx \mu) + \int R_x \lambda_0(dx) ,
\]
while in the case \( \omega S = 1 \)

\[
\Gamma = \alpha_0 \omega \quad , \quad \Lambda = \int Q_{x, \mu} \gamma(dx \mu) .
\]

By comparison with Theorem 9.4, it is seen that the integrals 
\[
\int P_{x, \mu} dy \quad \text{and} \quad \int Q_{x, \mu} dy
\]
are formed in exactly the same way as the probability distributions of general symmetrically distributed random measures, except that the "mixing" measures \( \gamma \) need no longer be normalized and can in fact be infinite. In addition we have in (14) a term \( \int R_x d\lambda_0 \)
which can not occur in the corresponding representation of distributions, since the measure \( R_x \) is infinite and can not be normalized.

**PROOF.** Let us first consider the case \( \omega S = \infty \). By Theorem 9.4 we get for any \( f \in F \)

\[
- \log E[e^{-\xi f} | \alpha, \lambda] = \alpha \omega f + \int (1 - e^{-xf(s)}) (\omega \times \lambda)(dsdx) .
\]

Writing

\[
g(x) = \int (1 - e^{-xf(s)}) \omega(ds) \quad , \quad x \in R_+ ,
\]

we thus obtain
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\[- \log E e^{-\xi f} = - \log E e^{-\alpha \omega f - \lambda g}, \]

and hence by (12)

\[- \log E e^{-\xi f} = \alpha_0 \omega f + \lambda_0 g + \int (1 - e^{-x\omega f - \mu g}) d\gamma(x, \mu). \]

Now

\[
\lambda_0 g = \int \lambda_0(dx) \int (1 - e^{-xf(s)}) \omega(ds) = \int \lambda_0(dx) \int (1 - e^{-x\delta f(s)}) \omega(ds) =
\]

\[
= \int \lambda_0(dx) \int (1 - e^{-\mu f}) R_x(du) = \int (1 - e^{-\mu f}) R_x(du) \lambda_0(dx),
\]
giving rise to the second \(\Lambda\)-term in (14). Further observe that, by (9),

\[e^{-x\omega f - \mu g} = \int e^{-mf} p_{x,\mu}(dm), \quad x \in \mathbb{R}_+, \quad \mu \in \mathcal{M}(\mathbb{R}_+),\]

and so

\[
\int (1 - e^{-x\omega f - \mu g}) d\gamma(x, \mu) = \int d\gamma(x, \mu) \int (1 - e^{-mf}) p_{x,\mu}(dm) =
\]

\[
= \int (1 - e^{-mf}) \int p_{x,\mu}(dm) d\gamma(x, \mu),
\]
giving rise to the first \(\Lambda\)-term in (14). Finally, the term \(\alpha_0 \omega f\) is of the form \(\Gamma f\) with \(\Gamma = \alpha_0 \omega\). This completes the proof of (14).

Next suppose that \(\omega S = 1\). We then get by Theorem 9.4 for any \(f \in F\)
\[ E e^{-\xi f} = E \exp(-\alpha \omega f - \sum_j \beta_j f(\tau_j)) = EE[\exp(-\alpha \omega f - \sum_j \beta_j f(\tau_j) \mid \alpha, \beta] = \]

\[ = E e^{-\alpha \omega f} \prod_j E[e^{-\beta_j f(\tau_j)} \mid \beta_j] = \]

\[ = E e^{-\alpha \omega f} \exp \left( \sum_j \log E[e^{-\beta_j f(\tau_j)} \mid \beta_j] \right) = \]

\[ = E e^{-\alpha \omega f} \exp \left( - \sum_j g(\beta_j) \right) = E e^{-\alpha \omega f - \beta g}, \]

where

\[ g(x) = - \log \int e^{-xf(s)} \omega(ds), \quad x \in \mathbb{R}_+. \]

Using (13), we thus obtain

\[ - \log E e^{-\xi f} = \alpha_0 \omega f + \int (1 - e^{-x \omega f - \mu g}) d\gamma(x, \mu). \]

But from the above calculation it is seen that

\[ e^{-x \omega f - \mu g} = \int e^{-mf} Q_{x, \mu}(dm), \quad x \in \mathbb{R}_+, \quad \mu \in \mathcal{N}(\mathbb{R}_+^*), \]

and hence

\[ \int (1 - e^{-x \omega f - \mu g}) d\gamma(x, \mu) = \int d\gamma(x, \mu) \int (1 - e^{-mf}) Q_{x, \mu}(dm) = \]

\[ = \int (1 - e^{-mf}) Q_{x, \mu}(dm) d\gamma(x, \mu). \]

Hence (15) holds, and the proof is complete.
We are now able to characterize the class of measures $\Gamma$ and $\Lambda$ which can occur in the representation (11) of infinitely divisible symmetrically distributed random measure. Write for brevity $\kappa'(x) \equiv x$.

**Theorem 9.7.** Let $\omega \in M_d$ and $\Gamma \in M$, and let $\Lambda$ be a measure on $M \setminus \{0\}$. Then there exists some infinitely divisible random measure $\xi$ which is symmetrically distributed with respect to $\omega$ and satisfies (11) iff

(i) in the case $\omega S = \infty$, (14) holds for some $\alpha_0 \geq 0$, some $\lambda_0 \in M_{R'}$ and some measure $\gamma$ on $(R_+ \times M_{R'}) \setminus \{0\}$ satisfying

$$\lambda_0 \kappa' < \infty, \quad \gamma\{\mu \kappa = \infty\} = 0, \quad \int (1 - e^{-x - \mu \kappa}) \gamma(dx d\mu) < \infty,$$

(ii) in the case $\omega S = 1$, (15) holds for some $\alpha_0 \geq 0$ and some measure $\gamma$ on $(R_+ \times (R_+')) \setminus \{0\}$ satisfying

$$\gamma\{\mu \kappa' = \infty\} = 0, \quad \int (1 - e^{-x - \mu \kappa'}) \gamma(dx d\mu) < \infty.$$

The quantities $(\alpha_0, \lambda_0, \gamma)$ or $(\alpha_0, \gamma)$ are then unique.

**Proof.** Suppose that $\xi$ is an infinitely divisible symmetrically distributed random measure satisfying (11). Then (11) must be the canonical representation occurring in Theorem 6.5, and therefore $\Gamma$ and $\Lambda$ are necessarily unique. But then (14) or (15) holds by Theorem 9.6 for some $(\alpha_0, \lambda_0, \gamma)$ or $(\alpha_0, \gamma)$ satisfying (12) or (13) respectively. In the case $\omega S = \infty$ we get in particular
\begin{align*}
(18) \quad & \log E e^{-t(\alpha + \lambda \kappa)} = a_0 t + t\lambda_0 \kappa + \int (1 - e^{-t(x + \mu \kappa)}) \gamma(dx \mu), \quad t \geq 0,
\end{align*}

and since $\lambda \kappa < \infty$ a.s. by Theorem 9.4, the last two terms on the right-hand side of (18) must be finite for all $t$. Furthermore, it follows from (18) by monotone convergence as $t \to 0^+$ that $0 = \gamma(\mu \kappa = \infty)$, which completes the proof of (16). In the case $\omega S = 1$, note that $\beta \kappa < \infty$ a.s., and use a similar argument to prove (17). The uniqueness of $(a_0, \lambda_0, \gamma)$ or $(a_0, \gamma)$ follows by the uniqueness assertions in Theorems 6.5 and 9.4.

Conversely, let $(a_0, \lambda_0, \gamma)$ be such as described in (i), and define $\Gamma$ and $\Lambda$ by (14). Note that $\Lambda$ exists according to the second relation in (16). By Theorem 6.5 and the last relation in (16), there exist some random variable $\alpha \geq 0$ and some random measure $\lambda$ on $\mathbb{R}_+^\ast$ such that $(\alpha, \lambda)$ is infinitely divisible and satisfies (12). Using monotone convergence as above, it follows from (12) that $\lambda \kappa < \infty$ a.s., and hence by Theorem 9.4, $(\alpha, \lambda)$ can serve as the canonical random elements of some symmetrically distributed random measure $\xi$ which by Theorem 9.5 has to be infinitely divisible. Hence by Theorem 9.6, $\Gamma$ and $\Lambda$ are related to $\xi$ by (11), and therefore have the asserted property. A similar proof works in the case $\omega S = 1$.

NOTES. Theorem 9.1 is known for $t = \infty$, in which case it was proved, independently, by Davidson (1974) (in a special case) and Kallenberg (1973a). The latter paper is also the source for Corollary 9.2, which extends a result by Nawrotzki (1962)\(^6\). Furthermore, Corollary 9.3 is a typical result from Kallenberg (1974a), while Theorem 9.4 was proved by Böhm (1960) and

by Kallenberg (1973b) and (1974b), (see also Kallenberg (1973c)). Finally, Theorem 9.5 extends two results for point processes given by Kerstan, Matthes and Mecke (1974), pages 66-67, while Theorems 9.6 and 9.7 are new.

PROBLEMS.

9.1. Let $S = \{1, \ldots , n\}$, let $\omega$ be counting measure on $S$ and let $\xi$ be counting measure on a subset of $k$ elements in $S$ chosen at random. Show that $\xi$ is (in the obvious sense) symmetrically distributed w.r.t. $\omega$ and still not a mixed sample process (unless $k = 0$ or $n$). Thus the assumption $\omega \in \mathcal{M}_d$ is essential in Theorem 9.1.

9.2. Let $\xi$ be a sample process on $[0,1]$ with intensity $n$ times Lebesgue measure. Show that there does not exist any symmetrically distributed (w.r.t. Lebesgue measure) point process $\eta$ on some interval $[0,c]$, $c > 1$, such that the restriction of $\eta$ to $[0,1]$ is distributed like $\xi$. (Hint: $\eta[0,1]$ cannot be non-random unless $\eta = 0$ a.s.)

9.3. Extend Corollary 9.2 to general symmetrically distributed random measures. (Cf. Westcott (1973).)

9.4. Let $\omega \in \mathcal{M}_d$ and suppose that $\xi$ is a $\beta$-compound of $\eta$, where $\beta \neq 0$. Show that $\xi$ and $\eta$ are simultaneously symmetrically distributed w.r.t. $\omega$. Prove the corresponding result for the random measures defined by (8.11) and (8.12). (Hint: use analytic continuation.)

9.5. Let $\omega \in \mathcal{M}_d$ be fixed and let $\xi$ be symmetrically distributed w.r.t. $\omega$. Show that $P_\xi^{-1}$ is uniquely determined by $P(C\xi)^{-1}$ for any

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7 Extensions of Theorems 9.5 - 9.6 are proved in Kallenberg: "Infinitely divisible processes with interchangeable increments and random measures under convolution", *Tech. Report*, Dept. of Mathematics, Göteborg University.
fixed \( C \in \mathcal{B} \) with \( \omega C > 0 \). (Hint: use the connection with thinnings exploited in the proof of Theorem 9.4. Cf. Kallenberg (1973c).)


10. PALM DISTRIBUTIONS.

Let \( \xi \) be a random measure on \( S \), and define the corresponding \textit{Campbell measure} \( (P_{\xi}^{-1})^1 \) on \( S \times M \) by

\[
(1) \quad (P_{\xi}^{-1})^1(B \times M) = E[\xi B; \xi \in M], \quad B \in \mathcal{B}, \quad M \in M.
\]

If \( E\xi \) is \( \sigma \)-finite, which holds e.g. when \( E\xi \in M \), we can define the corresponding Radon-Nikodym derivatives \( P_{S}M \) by

\[
P_{S}M = \frac{(P_{\xi}^{-1})^1(ds\times M)}{(P_{\xi}^{-1})^1(ds\times M)} = \frac{E[\xi(ds); \xi \in M]}{E\xi(ds)}, \quad s \in S \text{ a.e. } E\xi, \quad M \in M.
\]

More generally, suppose that \( f \in F \) is such that \( Ef \xi \) is \( \sigma \)-finite.

We can then define

\[
P_{S}M = \frac{Ef\xi(ds); \xi \in M]}{Ef\xi(ds)}, \quad s \in S \text{ a.e. } Ef\xi, \quad M \in M,
\]

without ambiguity, since the two expressions for \( P_{S}M \) clearly coincide a.e. \( E\xi \) on the support of \( f \). We shall call \( P_{S}M \) the \textit{Palm probability} of \( M \) \( w.r.t. \) \( s \). Note that, by definition, \( P_{S}M \) is \( \mathcal{B} \)-measurable in \( s \) for fixed \( M \in M \) and that \( P_{S}M \) is almost a probability measure in \( M \) for fixed \( s \in S \), in the sense that every probability defining property holds a.s. \( E\xi \). Just as in the case of conditional probabilities, we
may choose a version which is actually a probability measure in $M$ (to be called the Palm distribution) for every fixed $s \in S$. This is possible by the usual arguments for construction of conditional distributions (cf. Bauer (1972)), on account of the following fact.

**Lemma 10.1.** $M$ and $N$ are Polish spaces in their vague topologies.

**Proof.** By Lemma 4.4, it is enough to consider $M$. Let $G$ be a countable base in $S$, and assume without loss that $G$ is a semi-ring.

Approximate each indicator $1_G$, $G \in G$, from below by a sequence of functions in $F_c$. Let $f_1, f_2, \ldots$ be an enumeration of all these functions for all $G \in G$, and note that, by Dynkin's theorem, each $\mu \in M$ is uniquely determined by $\mu f_j$, $j \in N$. Furthermore, we have $\mu_n \to \mu$ iff $\mu_n f_j \to \mu f_j$ for all $j \in N$. In fact, the latter condition implies compactness of $\{\mu_n\}$, so any sequence $N' \subset N$ must contain a subsequence $N''$ such that $\mu_n \to \mu'$ ($n \in N''$). But then $\mu_n f_j \to \mu' f_j$ ($n \in N''$), $j \in N$, so we get $\mu f_j = \mu' f_j$ and hence $\mu = \mu'$, proving that indeed $\mu_n \to \mu$. It is now easily seen that the function $\rho: M^2 \to \mathbb{R}_+$ defined by

$$\rho(\mu, \mu') = \sum_{j=1}^{\infty} 2^{-j} \left| \frac{\mu f_j - \mu' f_j}{1 + |\mu f_j - \mu' f_j|} \right|, \quad \mu, \mu' \in M,$$

is a complete metric on $M$ generating the vague topology. To see that $M$ is separable, let $A$ and $B$ be countable dense subsets of $S$ and $\mathbb{R}_+$ respectively. Then the class of all finite sums of measures $b \delta_a$, $a \in A$, $b \in B$, is clearly countable and dense in $M$. 
Henceforth, we shall always assume the set functions $P_S$ to be probabilities on $M$ (and even on $N$ if $\xi$ is a point process), and for convenience of writing, we introduce for every $s \in S$ a random measure $\xi_s$, defined on the same probability space as $\xi$ and with distribution $P_S$.

Just as in the case of conditional distributions, the expectations corresponding to Palm distributions can be computed in two different ways. More precisely, we have the following result, known as Campbell's theorem.

**Lemma 10.2.** Let $\xi$ be a random measure on $S$ with $E\xi$ $\sigma$-finite, and let $f: S \times M \to \mathbb{R}_+$ be measurable. Then

$$ Ef(s, \xi_s) = \int_\Omega f(s, \xi_s(\omega)) P(d\omega) = \int_M f(s, \mu) P_S(d\mu) = $$

$$ = \frac{Ef(s, \xi) \xi(ds)}{E\xi(ds)} , \ s \in S \ a.e. \ E\xi \ . $$

**Proof.** We have to prove that $Ef(s, \xi_s)$ is $\overline{B}$-measurable and satisfies

(2) $$ \int_B Ef(s, \xi_s) E\xi(ds) = E \int_B f(s, \xi) \xi(ds) , \ B \subset \overline{B} \ . $$

Now this is true by definition of $\xi_s$ for indicators $f$ of product sets $A \times B$, $A \subset B$, $M \subset M$, and it easily extends by Dynkin's theorem to indicators of arbitrary sets in $B \times M$. But then it holds for simple functions, and finally, by monotone convergence, for arbitrary $f$. 
In the particular case of simple point processes, $P_{\xi}^{-1}$ should be interpreted as the conditional distribution of $\xi$, given that $\xi(s) = 1$. (Note, however, that a definition along these lines only makes sense when $P\{\xi(s) > 0\} > 0$.) We could then expect $\xi_s$ to have an atom at $s$, and this can in fact be achieved by changing the definition on an $E_\xi$-null set:

**Lemma 10.3.** Let $\xi$ be a point process on $S$ with $E_\xi$ $\sigma$-finite. Then there exist versions of $\xi_s$ such that $\xi_s - \delta_s$ is a point process for each $s \in S$. For any measurable function $f: S \times N \to R_+$ we then have

$$Ef(s, \xi_s - \delta_s) = \frac{Ef(s, \xi_s - \delta_s)\xi(ds)}{E_\xi(ds)}, \quad s \in S \text{ a.e. } E_\xi,$$

where $f(s, \mu)$ may be arbitrarily defined for $\mu \notin N$.

**Proof.** Clearly

$$(4) \quad \{(s, \mu) \in S \times M: \mu - \delta_s \in N\} = (S \times N) \cap \{(s, \mu) \in S \times N: \mu(s) \geq 1\}.$$ 

To see that the second set on the right of (4) belongs to $B \times N$, let $B \in B$ be arbitrary and let $\{B_{nj}\} \in B$ be a null-array of partitions of $B$. Then

$$\{(s, \mu) \in B \times N: \mu(s) \geq 1\} = \bigcap\{B_{nj} \times \{\mu \in N: \mu B_{nj} \geq 1\}\} \in B \times N,$$

and the measurability in (4) follows from the fact that $S$ is $\sigma$-compact.

By (2) and (4) we now obtain for any $B \in B$
\begin{align*}
\int_B P(\xi_s - \delta_s \in \mathbb{N})E\xi(ds) &= E\int_B 1_{\{\xi-\delta_s \in \mathbb{N}\}}(s,\xi)\xi(ds) = \\
&= E\int_B 1_{\{\xi(s) \geq 1\}}(s,\xi)\xi(ds) = E\xi B ,
\end{align*}
so
\begin{align*}
\int_B (1 - P(\xi_s - \delta_s \in \mathbb{N}))E\xi(ds) &= 0 ,
\end{align*}
and hence
\begin{align*}
P(\xi_s - \delta_s \in \mathbb{N}) &= 1 \ a.e. \ E\xi .
\end{align*}

Redefining \( \xi_s \equiv \delta_s \) on the exceptional \( s \)-set in (5), we can make (5) hold for all \( s \in S \). For each \( s \in S \) we then redefine \( \xi_s = \delta_s \) on the exceptional \( \omega \)-set where \( \xi_s - \delta_s \not\in N \) to make \( \xi_s - \delta_s \in N \) hold for all \( s \) and \( \omega \). Since \( \xi_s B - \delta_s B \) remains \( A \)-measurable for each \( s \in S \) and \( B \in B \), it is now a point process. But then (5) is an immediate consequence of Lemma 10.2.

Besides the random measures \( \xi_s \) we shall also consider, for arbitrary \( f \in F \) with \( E\xi f \in \mathbb{R}^+ \), the mixture \( \xi_f \) with distribution defined by
\begin{align*}
P(\xi_f \in M) &= \frac{1}{E\xi f} \int S P(\xi_s \in M)f(s)E\xi(ds) = \frac{E[\xi f; \xi \in M]}{E\xi f} , \ M \in M .
\end{align*}
When \( \xi \) is a point process, we also consider the point processes \( \hat{\xi}_f \) with distributions defined by
\begin{align*}
P(\hat{\xi}_f \in M) &= \frac{1}{E\xi f} \int S P(\xi_s - \delta_s \in M)f(s)E\xi(ds) = \\
&= \frac{1}{E\xi f} \int S 1_{\{\xi - \delta_s \in M\}}(s,\xi)f(s)\xi(ds) , \ M \in M .
\end{align*}
It is easily verified that Campbell's theorem remains valid for \( \xi_f \) and \( \hat{\xi}_f \), in the sense that

\[
\text{Eh}(\xi_f) = \frac{1}{E\xi_f} \int_S \text{Eh}(\xi_s)f(s)E\xi(ds) = \frac{\text{Eh}(\xi)\xi_f}{E\xi_f},
\]

(6)

\[
\text{Eh}(\hat{\xi}_f) = \frac{1}{E\hat{\xi}_f} \int_S \text{Eh}(\xi_s - \delta_s)f(s)E\xi(ds) = \frac{1}{E\hat{\xi}_f} E \int_S h(\xi_s - \delta_s)f(s)\xi(ds).
\]

(7)

Here \( h \) is any measurable function from \( M \) or \( N \) respectively to \( R_+ \).

From the definitions of Palm distributions of a given random measure \( \xi \) it is obvious that they determine, together with the intensity \( E\xi \), the Campbell measure \( (P_{\xi^{-1}})_1 \) and hence also the distribution \( P_{\xi^{-1}} \).

Indeed, a much stronger result is true, indicating a close relationship between the Palm distributions for different \( s \in S \):

**Lemma 10.4.** Let \( \xi \) be a random measure on \( S \) and let \( f \in F \) with \( E\xi f \in R_+ \). Then \( E\xi f \) and \( P_{\xi^{-1}} \) determine the restriction of \( P_{\xi^{-1}} \) to the set \( \{ \mu : \mu f > 0 \} \). They further determine \( P_{\xi^{-1}} \) for all \( s \in S \) a.e. \( E(f\xi) \).

Note that \( f \) may always be chosen such that \( f(s) > 0 \) for all \( s \in S \). Then \( \mu f = 0 \) implies \( \mu = 0 \), so in this case \( E\xi f \) and \( P_{\xi^{-1}} \) alone determine the whole distribution of \( \xi \).

**Proof.** The first assertion follows from (6) by choosing for fixed \( M \in M \)

\[
h(\mu) = \begin{cases} (\mu f)^{-1}1_M(\mu), & \mu f > 0, \\ 0, & \mu f = 0, \end{cases}
\]
since for this \( h \), (6) takes the form

\[
(8) \quad P\{\xi \in M, \xi f > 0\} = E_{\xi f} Eh(\xi f).
\]

Furthermore, the probabilities in (8) determine the measure

\[
E[(f\xi)B; \xi \in M] = E[(f\xi)B; \xi \in M, \xi f > 0], \quad B \in B, \quad M \in M,
\]

and so the second assertion follows from the fact that

\[
P\{\xi_s \in M\} = \frac{E[\xi(ds); \xi \in M]}{E\xi(ds)} = \frac{E[(f\xi)(ds); \xi \in M]}{E(f\xi)(ds)}, \quad s \in S \text{ a.e. } E(f\xi).
\]

For \( f \in F_c \), the above uniqueness assertion can essentially be strengthened to continuity:

**THEOREM 10.5.** Let \( \xi, \xi_1, \xi_2, \ldots \) be random measures on \( S \) and let \( f \in F_c \) be such that \( E\xi f \in R_+^* \). Under the assumption \( E\xi_n f \to E\xi f \), the condition \( \xi_n \overset{d}{\to} \xi \) then implies \( \xi_n f \overset{d}{\to} \xi f \), while conversely \( \xi_n \overset{d}{\to} \xi_f \) implies \( f\xi_n \overset{wd}{\to} f\xi \). Furthermore, \( \xi_n \overset{d}{\to} \xi \) and \( \xi_n \overset{d}{\to} \xi f \) imply that

\[
E\xi_n f \to E\xi f.
\]

**PROOF.** First assume that \( \xi_n \overset{d}{\to} \xi \) and \( E\xi_n f \to E\xi f \), and let \( h: M \to R_+ \) be bounded and continuous. Then the mapping \( \mu \to h(\mu)f \) is continuous, so we get

\[
(9) \quad h(\xi_n)\xi_n f \overset{d}{\to} h(\xi)\xi f
\]

by Theorem 5.1 in Billingsley (1968). Furthermore, it follows from the relations \( \xi_n f \overset{d}{\to} \xi f \) and \( E\xi_n f \to E\xi f \) that \( \{\xi_n f\} \) is uniformly integrable.
(cf. Theorem 5.4 in Billingsley (1968)), and hence so is \( \{h(\xi_n)\xi_n f\} \).

By (9) we thus obtain

\[
(10) \quad Eh(\xi_n)\xi_n f \rightarrow Eh(\xi)\xi f ,
\]

so by (6)

\[
(11) \quad Eh(\xi_n f) \rightarrow Eh(\xi f) .
\]

Since \( h \) was arbitrary, this proves that \( \xi_n f \xrightarrow{d} \xi f \).

Suppose conversely that \( \xi_n f \xrightarrow{d} \xi f \) and \( E\xi_n f \rightarrow E\xi f \). If \( h: M \rightarrow R_+ \) is bounded and measurable with \( \xi f \notin D_f \) a.s., it follows by Theorem 5.2 in Billingsley (1968) that (11) holds, and by (6) this implies (10).

For any fixed bounded and continuous function \( g: S \rightarrow R_+ \) we may choose

\[
h(\mu) = \begin{cases} 
\mu(fg)/\mu f , & \mu f > 0 , \\
0 , & \mu f = 0 .
\end{cases}
\]

In fact, \( h \) is bounded and measurable, and by (6) the set \( D_h = \{\mu f = 0\} \) satisfies

\[
P(\xi f \in D_h) = P(\xi f = 0) = \frac{E[\xi f; \xi f = 0]}{E\xi f} = 0 .
\]

For this particular \( h \), (10) takes the form

\[
E(f\xi_n)g = E\xi_n (fg) \rightarrow E\xi (fg) = E(f\xi)g ,
\]

and since \( g \) was arbitrary, it follows by Theorem 4.10 that \( f\xi_n \xrightarrow{wd} f\xi \), as asserted.
Finally suppose that $\xi_n \overset{d}{\to} \xi$ and $\xi_{nf} \overset{d}{\to} \xi_f$, and choose $h(\mu) = (1 + \mu f)^{-1}$. Then $h(\mu)$ and $h(\mu)f$ are both bounded and continuous, so (10) and (11) are satisfied. Since moreover $E h(\xi_f) > 0$, it follows from (6) that $E \xi_n f \to E \xi f$.

In case of point processes, we may also prove a continuity theorem for $\hat{\xi}_f$, $f \in F_c$.

**Theorem 10.6.** Let $\xi, \xi_1, \xi_2, \ldots$ be point processes on $S$, and suppose that $E \xi \in M \backslash \{0\}$. Then any two of the following statements imply the third one.

(i) $E \xi_n \overset{\mathcal{V}}{\to} E \xi$,

(ii) $\xi_n \overset{d}{\to} \xi$,

(iii) $\hat{\xi}_{nf} \overset{d}{\to} \hat{\xi}_f$ for all $f \in F_c$ with $E \xi f > 0$.

**Proof.** Let $f \in F_c$ and let $h: \mathbb{N} \to \mathbb{R}_+$ be bounded and continuous. Then the function

$$g(\mu) = \int_S h(\mu - \delta_s) f(s) \mu(ds), \quad \mu \in \mathbb{N}, \quad (12)$$

is vaguely continuous. In fact, suppose that $\mu(s) \geq 1$ and $\mu_n(s_n) \geq 1$, $n \in \mathbb{N}$, and that $\mu_n \overset{\mathcal{V}}{\to} \mu$ in $\mathbb{N}$ and $s_n \to s$ in $S$. Then

$$\mu_n - \delta_{s_n} \overset{\mathcal{V}}{\to} \mu - \delta_s$$

holds by definition of vague convergence, and hence by continuity.
h(\mu_n - \delta_n) f(s_n) \rightarrow h(\mu - \delta_s) f(s).

Since \( h \) and \( f \) has bounded support, it follows by Theorem 5.5 in Billingsley (1968) that \( g(\mu_n) \rightarrow g(\mu) \). (Note that, in Billingsley's notation, we need only assume that \( h_n(x_n) \rightarrow h(x) \) for \( x_n \rightarrow x \) such that \( x \in E \) and \( x_n \in E_n \), \( n \in \mathbb{N} \), where \( PE = P_1 E_1 = P_2 E_2 = \ldots = 1 \).

Now suppose that (i) and (ii) hold, and let \( h \) and \( g \) be as above. Then \( E g(\xi_n) \overset{d}{\rightarrow} E g(\xi) \) follows by uniform integrability as in the preceding proof, and it follows by (i) and (7) that \( E h(\xi_n f) \) \( \overset{d}{\rightarrow} E h(\xi_f) \) provided \( E \xi f > 0 \). Since \( h \) was arbitrary, this implies (iii).

Next suppose that (ii) and (iii) hold, and define \( g \) by (12) with

\[
h(\mu) = (1 + \mu f + ||f||)^{-1}, \quad \mu \in \mathbb{N}.
\]

Then \( h \) is bounded and continuous, so by (iii)

\[
(13) \quad E h(\xi_n f) \rightarrow E h(\xi_f) > 0.
\]

Moreover, \( g \) is continuous, and it is even bounded since

\[
g(\mu) = \int_S \frac{f(s)\mu(ds)}{1 + (\mu - \delta_s)f^+ ||f||} = \int_S \frac{f(s)\mu(ds)}{1 + \mu f - f(s) + ||f||} \leq \int_S \frac{f(s)\mu(ds)}{1 + \mu f} = \frac{\mu f}{1 + \mu f} < 1.
\]

Hence by (ii)

\[
(14) \quad E g(\xi_n) \rightarrow E g(\xi).
\]

Using (7), (13) and (14), we obtain \( E \xi_n f \rightarrow E \xi f \). If \( E \xi f = 0 \), we may
choose \( f_1, f_2 \in F_c \) with \( f = f_1 - f_2 \) and \( E\xi f_1 = E\xi f_2 > 0 \), and conclude that

\[
E\xi_n f = E\xi_n f_1 - E\xi_n f_2 \to E\xi f_1 - E\xi f_2 = E\xi f.
\]

Thus \( E\xi_n f \to E\xi f \) is valid for all \( f \in F_c \), and (i) follows.

Finally suppose that (i) and (iii) hold. By (7) we get for any \( f \in F_c \) and any bounded continuous \( h: \mathbb{N} \to \mathbb{R}_+ \)

\[
E \int_S h(\xi_n - \delta) f(s) \xi_n (ds) \to E \int_S h(\xi - \delta) f(s) \xi (ds).
\]

For any fixed \( f \in F_c \) and \( t \in \mathbb{R}_+ \), we may replace the \( f \) in (15) by \( fe^{-tf} \) and put \( h(\mu) = e^{-t\mu f} \), thus obtaining

\[
E \int_S \exp(-t\xi_n f + tf(s)) f(s) e^{-tf(s)} \xi_n (ds) \to E \int_S \exp(-t\xi f + tf(s)) f(s) e^{-tf(s)} \xi (ds),
\]

or equivalently

\[
E\xi_n fe^{-t\xi_n f} \to E\xi fe^{-t\xi f}.
\]

By Fubini's theorem and dominated convergence, we get

\[
Ee^{-\xi_n f} = 1 - E \int_0^1 e^{-t\xi_n f} dt = 1 - \int_0^1 E\xi_n f e^{-t\xi_n f} dt \to 1 - \int_0^1 E\xi f e^{-t\xi f} dt = 1 - E \int_0^1 \xi e^{-t\xi f} dt = E e^{-\xi f},
\]

and since \( f \) was arbitrary, it follows by Theorem 4.5 that \( f_n \xrightarrow{d} f \).
This completes the proof.

Palm distributions are often easily calculated by means of L-transforms. In fact, we get by dominated convergence for any \( f \in F \) with \( \mathbb{E} f \in R^+_1 \)

\[
(16) \quad \mathbb{E} \xi(ds)e^{-\xi f} = -\frac{\partial}{\partial t} L_{\xi}(f + t1_ds) \bigg|_{t = 0},
\]

and \( L_{\xi} \) is obtained from this derivative by normalization. As a simple example, suppose that \( \xi \) is a Poisson process with intensity \( \lambda \in M\setminus\{0\} \).

By dominated convergence we get

\[
-\frac{\partial}{\partial t} L_{\xi}(f + t1_ds) = -\frac{\partial}{\partial t} \exp\left\{ -\lambda \left[ 1 - e^{-f-t1_ds} \right] \right\} = \exp\left\{ -\lambda \left[ 1 - e^{-f-t1_ds} \right] \right\} \lambda (1_ds) e^{-f-t1_ds},
\]

so putting \( t = 0 \),

\[
\mathbb{E} \xi(ds)e^{-\xi f} = L_{\xi}(f)e^{-f(s)}\lambda(ds),
\]

and hence by normalization

\[
L_{\xi_s}(f) = L_{\xi}(f)e^{-f(s)} = L_{\xi}(f)L_{\delta_s}(f) = L_{\xi_s\delta_s}(f), \quad s \in S \text{ a.e. } \mathbb{E} \xi.
\]

By Theorem 3.1 we thus obtain \( \xi_s \overset{d}{=} \xi + \delta_s \) or equivalently \( \xi_s - \delta_s \overset{d}{=} \xi \) a.e. \( \mathbb{E} \xi \), which yields \( \xi_f \overset{d}{=} \xi \) for every \( f \in F \) with \( \mathbb{E} f \in R^+_1 \). We shall use this fact to prove the following complement to Corollary 7.4.

**Corollary 10.7.** Let \( \xi \) be a Poisson process on \( S \) with intensity \( \lambda \in M\setminus\{0\} \), let \( I : R_\lambda \) be a DC-semi-ring and let \( \{\xi_{nj}\} \) be a null-array of point processes on \( S \). Put \( \xi_n = \sum_j \xi_{nj} \). Then \( \xi_n \overset{d}{=} \xi \) and \( \xi_n \overset{d}{=} \xi \).
for all $f \in F$ with $Ef \in R'_+$ if and only if

(i) $\sum_j P(\xi_{nj} I > 0) \to \lambda I$ , $I \in I$,

(ii) $\sum_j E[\xi_{nj} B; \xi_{nj} B > 1] \to 0$ , $B \in B$.

**Proof.** Suppose that (i) and (ii) hold. Then $\xi_n \overset{d}{\to} \xi$ follows by Corollary 7.4 since

(17) $\sum_j P(\xi_{nj} B > 1) \leq \frac{1}{2} \sum_j E[\xi_{nj} B; \xi_{nj} B > 1] \to 0$ , $B \in B$.

Furthermore, we get by (i), (ii) and (17) for any $I \in I$

$\sum_j E\xi_{nj} I = \sum_j P(\xi_{nj} I > 0) - \sum_j P(\xi_{nj} I > 1) + \sum_j E[\xi_{nj} I; \xi_{nj} I > 1] \to \lambda I$,

proving that

$E\xi_n = \sum_j E\xi_{nj} \overset{d}{\to} \lambda \xi = E\xi$.

Hence it follows by Theorem 10.6 that $\hat{\xi}_n \overset{d}{\to} \hat{\xi}_f$ for any $f \in F$ with $Ef \in R'_+$.

Suppose conversely that $\xi_n \overset{d}{\to} \xi$ and that $\hat{\xi}_n \overset{d}{\to} \hat{\xi}_f$ for any $f \in F$ with $Ef \in R'_+$. By Corollary 7.4 and Theorem 10.6, we get for any $B \in B$,

$\sum_j P(\xi_{nj} B > 0) \to \lambda B$ , $\sum_j P(\xi_{nj} B > 1) \to 0$ , $\sum_j E\xi_{nj} B \to \lambda B$.

Here the first relation contains (i), and moreover
\[ \sum_{j} E[\xi_{nj}^B; \xi_{nj}^B > 1] = \sum_{j} E\xi_{nj}^B - \sum_{j} P(\xi_{nj}^B > 0) + \sum_{j} P(\xi_{nj}^B > 1) + 0, \]

proving (ii).

**NOTES.** Palm probabilities were introduced for stationary point processes on \( \mathbb{R} \) by Palm himself (1943), whose studies were continued by several authors, including U. H. Uhlenbeck and S. Glasser (1960) and I. S. Gradshteyn (1962). The present approach is essentially due to Ryll-Nardzewski (1961), (see also Jagers (1973), Papangelou (1974), and Kerstan, Matthes and Mecke (1974), the latter authors working directly with the Campbell measures). Campbell's theorem, in the formulation of Lemma 10.2, was first stated by Kummer and Matthes (1979), while Lemma 10.3 is implicit in Kallenberg (1973a). The uniqueness Lemma 10.4 is new. As for the corresponding continuity problem, a discussion for stationary point processes is essentially given by Kerstan and Matthes (1965), and for general point processes by Kallenberg (1973a), from which Theorem 10.6 and Corollary 10.7 are taken. Theorem 10.5 extends a result in the same paper.

**PROBLEMS.**

10.1. Prove the version of Theorem 5.5 in Billingsley (1968) which is used in the proof of Theorem 10.6.

10.2. Let \( \xi \) be a random measure on \( S \) with \( E\xi \) \( \sigma \)-finite, and let \( M \in \mathcal{M} \). Show that \( P(\xi \in M) = 1 \iff P(\xi_{S} \in M) = 1 \), \( s \in S \) a.e. \( E\xi \).

10.3. Let \( \xi \) be a random measure on \( S \), let \( B \in \mathcal{B} \) with \( E\xi B < \infty \), and let \( x > 0 \) be fixed. Show that \( P(\xi B = x) = x^{-1} \int_{B} P(\xi_{S} B = x)E\xi(ds) \), (cf. Jagers (1973)).
10.4. Let $\xi$ and $\eta$ be independent random measures on $S$ with $E\xi$ and $E\eta$ in $M$. Calculate $P(\xi + \eta)^{-1}$ in terms of $P\xi^{-1}$ and $P\eta^{-1}$.

10.5. Let $S = \mathbb{R}$ and define the translation operators $T_s$, $s \in S$, on $M$ by $(T_s \mu)(B) = \mu(B - s)$, $B \in \mathcal{B}$. Say that a random measure $\xi$ on $\mathbb{R}$ is stationary if $P(T_s \xi)^{-1}$ is independent of $s \in \mathbb{R}$. Show that in this case $E\xi$ equals some constant $\lambda$ times Lebesgue measure, and that if $\lambda \in \mathbb{R}_+^*$, then $P(T_s \xi)^{-1}$ is independent of $s$ a.e. $E\xi$. The converse is also true. (Cf. Mecke (1967) and Jagers (1973).)

10.6. Let $\xi$ be a random measure on $\mathbb{R}$ and let $f \in \mathcal{F}(\mathbb{R})$ with $E\xi f \in \mathbb{R}_+^*$. Show that it is possible to define a random measure $\bar{\xi}_f$ on $\mathbb{R}$ such that

$$P(\bar{\xi}_f \in M) = \int P(T_s \xi \in M)f(s)E\xi(ds)/E\xi f =$$

$$= E \int 1_{\{T_s \xi \in M\}}f(s)(ds)/E\xi f, \quad M \in M.$$

State and prove the corresponding version of Campbell's theorem.

10.7. State and prove a version of Theorem 10.6 for the random measures $\bar{\xi}_f$ defined in Problem 10.6. Specialize to the case of stationary random measures. (Cf. Kerstan and Matthes (1965) as well as Kallenberg (1973a).)

10.8. Let $\xi$ be a random measure on $S$ and let $f \in \mathcal{F}$ be such that $E\xi f \in \mathbb{R}_+^*$. Define the distribution of a random element $(\eta_f, \xi_f)$ on $\mathbb{R}_+ \times M$ by

$$P\{(\eta_f, \xi_f) \in A\} = \frac{1}{E\xi f} \int_S \left\{ P\left( (\xi_S(s), \xi_S - \xi_S(s)\delta_s) \in A \right) f(s)E\xi(ds) \right\}$$
for any measurable set \( A \) in \( \mathbb{R}_+ \times M \), and show that this probability equals

\[
\frac{1}{E \xi f} E \int_S 1_{\{((\xi(s), \xi(s)\delta_s) \in A\}} f(s)\xi(ds).
\]

State and prove the corresponding version of Campbell's theorem.

11. CHARACTERIZATIONS INVOLVING PALM DISTRIBUTIONS.

In the preceding section we saw that any Poisson process \( \xi \) on \( S \) satisfies the relation \( \xi_s - \delta_s \overset{d}{=} \xi \), \( s \in S \) a.e. \( E\xi \). Actually, this relation characterizes the class of Poisson processes among arbitrary random measures:

**Proposition 11.1.** Let \( \xi \) be a random measure on \( S \) with \( E\xi \in M \).

Then \( \xi \) is a Poisson process iff \( \xi_s \overset{d}{=} \xi + \delta_s \), \( s \in S \) a.e. \( E\xi \).

**Proof.** Suppose that \( \xi_s \overset{d}{=} \xi + \delta_s \) a.e. Then we get for any \( B \in \mathcal{B} \) and \( n \in \mathbb{N} \)

\[
P(\xi_B = n) = \frac{1}{n} E(\xi_B; \xi_B = n) = \frac{1}{n} \int_B E[\xi(ds); \xi_B = n] = \frac{1}{n}\int_B P(\xi_B = n)E\xi(ds)
\]

\[
= \frac{1}{n} \int_B P(\xi_B = n - 1)E\xi(ds) = \frac{1}{n} P(\xi_B = n - 1)E\xi_B,
\]

and hence by induction

\[
P(\xi_B = n) = P(\xi_B = 0) \cdot \frac{(E\xi_B)^n}{n!}.
\]

In the same way we get more generally for any \( k \in \mathbb{N} \), \( n_1, \ldots, n_k \in \mathbb{Z}_+ \).
and disjoint $B_1, \ldots, B_k \in \mathcal{B}$

$$
\prod_{j=1}^{k} \left( \mathbb{P}(\xi B_j = n_j) = \prod_{j=1}^{k} \frac{P(\xi \cup V_j = 0)}{(n_j)!} \right).
$$

A similar argument also shows that $\mathbb{P}(\xi B \not\in \mathcal{Z}_+) = 0$, $B \in \mathcal{B}$. Hence $\xi$ must be a Poisson process, and the proof is complete.

The aim of the present section is to prove extensions in various directions of the above proposition. We shall first consider the case of infinitely divisible random measures $\xi$ with $E\xi \in \mathcal{M}$. If $\alpha$ and $\lambda$ are the corresponding canonical measures defined in Theorem 6.4, we can define the corresponding Campbell measure $\Lambda$ on $S \times \mathcal{M}$ by the relation

$$
\Lambda(B \times \mathcal{M}) = \alpha \delta_0 \mathcal{M} + \int_{\mathcal{M}} \mu \lambda(d\mu), \quad B \in \mathcal{B}, \quad \mathcal{M} \in \mathcal{M}.
$$

Formally, we may write $\Lambda = \alpha \delta_0 + \lambda^1$, where $\lambda^1 = \int \mu \lambda(d\mu)$, cf. (10.1).

Since $\int \mu \lambda(d\mu) = E\xi < \infty$, we can proceed as in the definition of Palm distributions and define the distributions $\Lambda_s$, $s \in S$ a.e. $E\xi$ by

$$
\Lambda_s = \frac{\Lambda(ds \times \mathcal{M})}{\Lambda(ds \times \mathcal{M})} = \frac{\alpha(ds) \delta_0 \mathcal{M} + \int_{\mathcal{M}} \mu(ds) \lambda(d\mu)}{\alpha(ds) + \int_{\mathcal{M}} \mu(ds) \lambda(d\mu)}.
$$

Campbell's theorem remains valid for the $\Lambda_s$, i.e. we have for any non-negative function $f$ on $S$

$$
\Lambda_s f = \frac{\alpha(ds) f(s,0) + \int f(s,\mu) \mu(ds) \lambda(d\mu)}{\alpha(ds) + \int \mu(ds) \lambda(d\mu)} \quad a.e. \quad E\xi.
$$
Let us now use the method of the preceding section to calculate $P^*_{\xi}$. For any $f \in F$, $t \in \mathbb{R}_+$, and $B \in \mathcal{B}$ we obtain

$$-\frac{\partial}{\partial t} L^\xi(f + t1_B) = -\frac{\partial}{\partial t} \exp\{\alpha f - t\alpha B - \int (1 - e^{-\mu f + t\mu B}) \lambda(d\mu)\}$$

$$= L^\xi(f + t1_B)\{\alpha B + \int \mu Be^{-\mu f + t\mu B} \lambda(d\mu)\}$$

by dominated convergence. Hence

$$-\frac{\partial}{\partial t} L^\xi(f + t1_B) \bigg|_{t=0} = L^\xi(f)\{\alpha B + \int \mu Be^{-\mu f} \lambda(d\mu)\}$$

$$= L^\xi(f) \int e^{-\mu f} \Lambda(B \times d\mu),$$

and so, by Campbell's theorem,

$$L^\xi_{\xi S}(f) = L^\xi(f) \int \frac{e^{-\mu f} \Lambda(ds \times d\mu)}{\Lambda(ds \times d\mu)} = L^\xi(f) \int e^{-\mu f} \Lambda_S(ds) \Lambda(d\mu).$$

Since the last expression is the $L$-transform of $P^*_{\xi}1^*_{A_S}$, it follows by Corollary 3.2 that $P^*_{\xi S} = P^*_{\xi}1^*_{A_S}$ a.e., and in particular

$P^*_{\xi} \mid P^*_{\xi S}^{-1}$ ( $P^*_{\xi}^{-1}$ is a convolution factor of $P^*_{\xi S}^{-1}$). This factorization property characterizes the class of infinitely divisible random measures:

**Theorem 11.2.** Let $\xi$ be a random measure on $S$ with $E\xi \in \mathcal{M}$. Then $\xi$ is infinitely divisible if $P^*_{\xi} \mid P^*_{\xi S}^{-1}$, $s \in S$ a.e. Further, in this case $P^*_{\xi S}^{-1} = P^*_{\xi}1^*_{A_S}$ a.e.

A much stronger result, to be proved first, holds in the point process case:
THEOREM 11.3. Let $\xi$ be a point process on $S$ with $\mathbb{E} \xi \in M$. Then $\xi$ is infinitely divisible iff $P(\xi f = 0) > 0$ and $P(\xi f)^{-1} \mid P(\xi f)^{-1}$ for every $f \in F_c$ with $\xi f \not\equiv 0$.

PROOF. If $\xi$ is infinitely divisible and $f \in F_c$ with $\xi f \not\equiv 0$, then $P(\xi f)^{-1} \mid P(\xi f)^{-1}$ follows by the above calculations. Furthermore, Lemma 6.1 yields $P(\xi B = 0) > 0$, $B \in \mathcal{B}$, which implies $P(\xi f = 0) > 0$, $f \in F_c$.

Conversely, let $f \in F_c$ be fixed with $\xi f \not\equiv 0$ and suppose that $P(\xi f)^{-1} \mid P(\xi f)^{-1}$. Writing $\phi = L_{\xi f}$, it follows easily that

$$- \frac{\phi'(t)}{\mathbb{E} \xi f} = \phi(t) \mathbb{E} e^{-nt}, \quad t > 0,$$

for some random variable $\eta \in \mathbb{Z}_+$, so if we divide by $\phi(t)$ and integrate from 0 to $t$, we obtain

$$- \log \phi(t) = \left\{ tP(\eta = 0) + E[\eta^{-1}(1 - e^{-nt}); \eta > 0] \right\} \mathbb{E} \xi f = at + \lambda(1 - e^{-nt}),$$

where

$$a = P(\eta = 0) \mathbb{E} \xi f; \lambda(dx) = x^{-1}P_{\eta^{-1}}(dx) \mathbb{E} \xi f, \quad x > 0.$$  

By Lemma 6.1, this proves that $\xi f$ is infinitely divisible. The sufficiency assertion now follows by Problem 6.5.

Proof of Theorem 11.2. Suppose that $P_{\xi}^{-1} \mid P_{\xi}^{-1}$, $s \in S$ a.e. $\mathbb{E} \xi$.

To show that $\xi$ is infinitely divisible, it suffices by Lemma 6.4 to prove that $(\xi B_1, \ldots, \xi B_k)$ is infinitely divisible for every $k \in \mathbb{N}$ and every disjoint $B_1, \ldots, B_k \in \mathcal{B}$. Since clearly $P_{\xi}^{-1} \mid P_{\xi}^{-1}$ for
any \( B \in \mathcal{B} \) with \( \xi_B \not\equiv 0 \), it is enough to prove the theorem for finite \( S \), say for \( S = \{1, \ldots, k\} \). For fixed \( c > 0 \) and \( s \in S \) with \( \xi(s) \not\equiv 0 \), let \( \eta \) and \( \eta' \) be Cox processes directed by \( c(\xi(1), \ldots, \xi(k)) \) and \( c(\xi_S(1), \ldots, \xi_S(k)) \) respectively. From (1.4) it is seen that \( P_{n^{-1}} \mid P_{n'^{-1}} \) and hence also \( P_n^{-1} \mid P(n' + \delta_S)^{-1} \). Writing \( \phi = L_{c\xi_S} \), we further obtain from (1.4)

\[
L_{\eta' + \delta_S}(t_1, \ldots, t_k) = \frac{\phi'_S(1-e^{-t_1}, \ldots, 1-e^{-t_k})}{\phi'_S(0, \ldots, 0)} e^{-t_S} = L_{\eta_S}(t_1, \ldots, t_k),
\]

\( t_1, \ldots, t_k \in \mathbb{R}_+ \),

and so \( \eta' + \delta_S \not\equiv \eta_S \), proving that \( P_n^{-1} \mid P_{\eta_S}^{-1} \). Since \( s \in S \) was arbitrary with \( \xi(s) \not\equiv 0 \), we may conclude that \( P(nf)^{-1} \mid P(nf)^{-1} \) for every \( f \in F_c \), and it follows by Theorem 11.3 that \( \eta \) is infinitely divisible. The factor \( c > 0 \) being arbitrary, this implies by Lemma 8.6 that \( \xi \) itself is infinitely divisible, and the proof is complete.

We are now going to show that it is enough in Theorem 11.2 to consider one particular mixture of Palm distributions, provided we add an assumption about the constant part of \( \xi \). Given any \( f \in F \) with \( \xi f < \infty \) a.s., we define the non-random set function \( f\xi \) on \( B \) by \( f\xi B = \text{essinf} (f\xi)_B = \inf\{x > 0: P((f\xi)_B < x) > 0\} \). Since \( f\xi \) is automatically finitely superadditive and continuous at \( \emptyset \), only finite subadditivity is needed to ensure \( f\xi \) to be a measure. This holds in particular if \( \text{essinf} \xi f = 0 \).
THEOREM 11.4. Let \( \xi \) be a random measure on \( S \) and let \( f \in F \) be strictly positive with \( 0 < E[f] < \infty \). Then \( \xi \) is infinitely divisible iff \( f\xi \) is finitely subadditive and \( p_{\xi}^{-1} | p_{f\xi}^{-1} \).

PROOF. Suppose that \( \xi \overset{d}{=} I(\alpha,\lambda) \). Then \( f\xi = f\alpha \in M \), (this is a simple special case of Theorem 6.9), and it follows in particular that \( f\xi \) is subadditive.

Suppose conversely that \( f\xi \) is subadditive (and hence a measure) and that \( p_{\xi}^{-1} | p_{f\xi}^{-1} \). Since \( f \) is strictly positive by assumption, it is easily seen that \( \xi \) and \( f\xi \) are simultaneously infinitely divisible, and so it is enough to assume that \( f = 1 = 1_S \). By Lemma 6.4, it suffices to prove that \( (\xi B_1, \ldots, \xi B_k) \) is infinitely divisible for every \( k \in \mathbb{N} \) and \( B_1, \ldots, B_k \in \mathcal{B} \). From the assumed factorization property it is seen that

\[
E[\xi S \exp(-\sum_j t_j \xi B_j)] \quad \frac{E[S \exp(-\sum_j t_j \xi B_j)]}{E[S]} = E \exp(-\sum_j t_j \xi B_j) \quad E \exp(-\sum_j t_j \eta_j)
\]

for some \( R \)-valued random variables \( \eta_1, \ldots, \eta_k \), and writing \( \tilde{\xi} = \xi - \xi \)
we get

\[
E[\tilde{\xi} S \exp(-\sum_j t_j \tilde{\xi} B_j)] \quad \frac{E[S \exp(-\sum_j t_j \tilde{\xi} B_j)]}{E[S]} = E \exp(-\sum_j t_j \tilde{\xi} B_j) \quad E \exp(-\sum_j t_j \eta_j) - \xi S \quad \frac{E[S]}{E[S]}
\]

Thus the numerator on the right is positive, and letting \( t_1, \ldots, t_k \to \infty \), we obtain \( E[S \xi P(\eta_1 = \ldots = \eta_k = 0)] \geq E[S] \). But then the second factor on the right of (1) must be an L-transform, and (1) shows that the assumed factorization property of \( \xi \) remains true for \( \tilde{\xi} \). We may therefore assume from now on that \( \xi = 0 \).
Let us now suppose that \( \eta = (\eta_0, \eta_1, \ldots, \eta_k) \) is a Cox process on \( \{0,1, \ldots, k\} \) directed by \( (\xi_S, \xi_{B_1}, \ldots, \xi_{B_k}) \). Proceeding as in the proof of Theorem 11.2, it is seen that \( P_{\eta^{-1}} \mid P_{\eta'^{-1}} \), where \( P_{\eta^{-1}} \) denotes the Palm distribution of \( \eta \) w.r.t. the point \( 0 \), and if \( \psi \) is the generating function of \( \eta \), we obtain

\[
\frac{t \psi'(t, t_1, \ldots, t_k)}{E_\xi S} = \psi(t, t_1, \ldots, t_k) E_t^{\zeta_1} t_1^{\zeta_1} \ldots t_k^{\zeta_k}, \ t, t_1, \ldots, t_k \in [0,1],
\]

for some \( \mathbb{Z}_+ \)-valued random variables \( \zeta, \zeta_1, \ldots, \zeta_k \), where the prime stands for differentiation with respect to \( t \). Now

\[
\psi(0, t_1, \ldots, t_k) = E[t_1^{\eta_1} \ldots t_k^{\eta_k}; \eta_0 = 0] > 0,
\]

so it follows from (2) that \( t \) is a factor of the expectation on the right, i.e. that \( P\{\zeta = 0\} = 0 \). Rewriting (2) in the form

\[
\frac{\psi'(t, t_1, \ldots, t_k)}{\psi(t, t_1, \ldots, t_k)} = E_\xi S E_t^{\zeta-1} t_1^{\zeta_1} \ldots t_k^{\zeta_k}
\]

and integrating w.r.t. \( t \) from 0 to 1, we thus obtain

\[
- \log \psi(1, t_1, \ldots, t_k) = - \log \psi(0, t_1, \ldots, t_k) - E_\xi S E_t^{\zeta-1} t_1^{\zeta_1} \ldots t_k^{\zeta_k}.
\]

In particular

\[
0 = - \log \psi(1, \ldots, 1) = - \log \psi(0, 1, \ldots, 1) - E_\xi S E_t^{\zeta-1},
\]

and hence by subtraction

\[
- \log \psi(1, t_1, \ldots, t_k) =
\]

\[
= \log \psi(0, 1, \ldots, 1) - \log \psi(0, t_1, \ldots, t_k) + E_\xi S E_t^{\zeta-1} (1 - t_1^{\zeta_1} \ldots t_k^{\zeta_k}).
\]
Since $\xi^{-1} < \infty$, it is seen from Lemma 6.3 that the function $\theta$ on $[0,1]^k$ defined by

$$-\log \theta(t_1, \ldots, t_k) = E\xi\xi^{-1}(1 - \frac{\xi}{t_1} \ldots \frac{\xi}{t_k}) , t_1, \ldots, t_k \in [0,1] ,$$

is the generating function of some infinitely divisible random vector on $\mathbb{R}^k$.

Letting $\eta_p$ be a Cox process directed by $p^{-1}(\xi, B_1, \ldots, B_k)$ and writing $\psi_p$ for its generating function, we obtain more generally for any $p \in (0,1]$

$$(3) \quad -\log \psi_p(1, t_1, \ldots, t_k) =$$

$$= \log \psi_p(0,1, \ldots, 1) - \log \psi_p(0, t_1, \ldots, t_k) - \log \theta_p(t_1, \ldots, t_k) ,$$

where $\theta_p$ is an infinitely divisible generating function. But $\eta$ is by Corollary 8.5 a $p$-thinning of $\eta_p$, and so by (1.6) and (3)

$$-\log \psi(1, t_1, \ldots, t_k) = -\log \psi_p(1, q+pt_1, \ldots, q+pt_k) =$$

$$= \log \psi_p(0,1, \ldots, 1) - \log \psi_p(0, q+pt_1, \ldots, q+pt_k) -$$

$$- \log \theta_p(q+pt_1, \ldots, q+pt_k) ,$$

where $q = 1 - p$. Here the last term corresponds by Lemma 8.6 to an infinitely divisible distribution, so if we can show that

$$(4) \quad \log \psi_p(0,1, \ldots, 1) - \log \psi_p(0, q+pt_1, \ldots, q+pt_k) \to 0 \quad \text{as} \quad p \to 0 ,$$
it will follow that \( \psi(1, t_1, \ldots, t_k) \) is the limit of a sequence of
infinitely divisible generating functions, and hence itself infinitely
divisible.

To prove (4), put \( \phi = L_{\xi S, \xi B_1, \ldots, \xi B_k} \), and note that by (1.4)
\[
\psi_p(t, t_1, \ldots, t_k) = E \exp\left\{-p^{-1}\left[\xi S(1 - t) + \sum_j \xi B_j (1 - t_j)\right]\right\} = \\
= \phi(p^{-1}(1 - t), p^{-1}(1 - t_1), \ldots, p^{-1}(1 - t_k)) .
\]

Hence
\[
\psi_p(0, 1, \ldots, 1) = \phi(p^{-1}, 0, \ldots, 0) = E \exp(-p^{-1} \xi S) ,
\]
while
\[
\psi_p(0, q + pt_1, \ldots, q + pt_k) \geq \psi_p(0, q, \ldots, q) = \phi(p^{-1}, 1, \ldots, 1) = \\
= E \exp(-p^{-1} \xi S - \sum_j \xi B_j) \geq E \exp(-p^{-1} + k) \xi S .
\]

Now it may be seen as in the proof of Theorem 11.3 that \( \xi S \overset{d}{=} \text{some } I(\alpha, \lambda) \),
and since \( \xi = 0 \) we have \( \alpha = 0 \). Thus by Lemma 6.1, the difference in (4)
is at most equal to
\[
\log E \exp(-p^{-1} \xi S) - \log E \exp(-(\alpha p^{-1} + k) \xi S) = \\
= \int \left\{\exp(-p^{-1}x) - \exp(-(\alpha p^{-1} + k)x)\right\} \lambda(dx) = \int e^{-x/p}(1 - e^{-kx}) \lambda(dx) ,
\]
which tends to zero as \( p \to 0 \) by dominated convergence. This proves
that the Cox process \( \{n_1, \ldots, n_k\} \) directed by \( (\xi B_1, \ldots, \xi B_k) \) is
infinitely divisible, and exactly the same argument shows more generally
that Cox processes directed by \( c(\xi B_1, \ldots, \xi B_k) \) are infinitely divisible for any \( c > 0 \). By Lemma 8.6, \((\xi B_1, \ldots, \xi B_k)\) is then itself infinitely divisible, and the proof is complete.

We shall now prove a related but quite different criterion for infinite divisibility which applies to the class of purely atomic random measures with diffuse intensity. Given any measure \( \mu \in \mathcal{M} \) and any \( \epsilon > 0 \), we define the measure \( \mu_\epsilon \in \mathcal{M} \) by

\[
\mu_\epsilon B = \sum_{s \in B} \mu(s)1_{\{\mu(s) > \epsilon\}}(u), \quad B \in \mathcal{B},
\]

and note that \( \mu_\epsilon \) is discrete in the sense that it is purely atomic with finitely many atoms in each compact set. Note also that the mapping \( \mu \mapsto \mu_\epsilon \) is measurable by Lemma 2.1. Given any random measure \( \xi \) on \( S \), and any \( B \in \mathcal{B} \) and \( \epsilon > 0 \) with \( P(\xi_\epsilon B = 0) > 0 \), we further define the random measure \( \xi_{B,\epsilon} \) on \( S \) as one with distribution

\[
P(\xi_{B,\epsilon} \in M) = P(\xi \in M \mid \xi_\epsilon B = 0), \quad M \in \mathcal{M}.
\]

**Theorem 11.5.** Let \( \xi \) be a purely atomic random measure on \( S \) with \( E \xi \in M_d \). Then \( \xi \) is infinitely divisible iff \( P_{B,\epsilon}^{-1} \mid P_{\xi,\epsilon}^{-1} \) for every \( \epsilon > 0 \) and \( B \in \mathcal{B} \) with \( P(\xi_\epsilon B = 0) > 0 \). In the case when \( \xi \) is a.s. discrete, this statement remains true with \( \epsilon = 0 \).

**Proof.** The necessity is an immediate consequence of Lemma 6.6 and the fact that Poisson processes have independent increments. To prove the
converse, let us first assume that \( \xi \) is a.s. discrete and that \( P_{\xi^{-1}} \mid P_{\xi^{-1}} \) for every \( B \in \mathcal{B} \) satisfying \( P\{\xi B = 0\} > 0 \). For any such \( B \) we may assume that \( \xi \overset{d}{=} \xi_B + \eta_B \) where \( \xi_B \overset{d}{=} \xi_{B,0} \) is independent of \( \eta_B \). For \( s \in B \) a.e. \( E\xi \) and any \( f \in F_c \) we obtain

\[
L_{\xi_B}(f) = \frac{E_{\xi_B}(ds) e^{-\xi_B f}}{E_{\xi}(ds)} = \frac{E_{\eta_B}(ds) e^{-f}}{E_{\eta_B}(ds)}
\]

(5)

\[
= \frac{E e^{-\xi_B f}}{E e^{-\eta_B f}} = \frac{E e^{-f}}{E e^{-\eta_B f}} = L_{\xi_B}(f) L_{\eta_B}(f),
\]

and this holds simultaneously for all \( B \) belonging to some countable base \( G \) and all \( s \in B \) except for points \( s \in S \) in some fixed \( E\xi \)-null set \( C \). For every fixed \( s \in C \), consider a sequence \( B_1, B_2, \ldots \in G \) such that \( B_n \uparrow \{s\} \). Since \( \xi \) is a.s. discrete and has no fixed atoms, we have \( P\{\xi B_n = 0\} \rightarrow 1 \), and so \( \xi_B \overset{d}{=} \xi \). Hence by (5) and Lemma 5.1, \( \eta_B \overset{n}{\rightarrow} \eta \), and we get \( P_{\xi^{-1}} \mid P_{\xi^{-1}} \). We may thus conclude from Theorem 11.2 that \( \xi \) is infinitely divisible.

We now consider the general case and hence assume that \( P_{\xi^{-1}} \mid P_{\xi^{-1}} \) whenever \( P\{\xi B = 0\} > 0 \). Writing \( \xi \overset{d}{=} \xi_{B_e} + \eta_{B_e} \) with independent terms, we get in place of (5)

\[
L_{\xi}(f) = \frac{E_{\xi}(ds)}{E_{\xi}(ds)} L_{\xi_{B_e}}(f) L_{\eta_{B_e}}(f) + \frac{E_{\eta}(ds)}{E_{\eta}(ds)} L_{\xi_{B_e}}(f) L_{\eta_{B_e}}(f),
\]

and so
\[ |L_{\xi} (f) - L_{\xi}(f)L_{\eta_{B \epsilon}} (f)| \]
\[ \leq |L_{\xi} (f) - L_{\xi_{B \epsilon}} (f)L_{\eta_{B \epsilon}} (f)| + |L_{\xi} (f) - L_{\xi}(f)|l_{\eta_{B \epsilon}} (f) \]
\[ \leq 2 \frac{E_{B \epsilon} (ds)}{E_{\xi}(ds)} + |L_{\xi} (f) - L_{\xi}(f)| \]
\[ = 2 \frac{E[\xi (ds); \xi_{B \epsilon} = 0]}{E_{\xi}(ds)P\{\xi_{B \epsilon} = 0\}} + \left| \frac{E[e^{-\xi f}; \xi_{B \epsilon} = 0]}{P\{\xi_{B \epsilon} = 0\}} - Ee^{-\xi f} \right| \]
\[ = 2 \frac{E(\xi - \xi_{\epsilon}) (ds)}{E_{\xi}(ds)} \frac{1}{P\{\xi_{B \epsilon} = 0\}} + 2 \frac{1-P\{\xi_{B \epsilon} = 0\}}{P\{\xi_{B \epsilon} = 0\}} . \]

As before, this holds for all \( s \) outside some \( E_{\epsilon}\)-null set \( C \), simultaneously for all \( \epsilon \) in some sequence \( E \) tending to 0 and all \( B \) with \( P\{\xi_{B \epsilon} = 0\} > 0 \) belonging to some countable base \( G \). Now \( \xi - \xi_{\epsilon} \triangledown 0 \) a.s. as \( \epsilon \to 0 \), so \( E(\xi - \xi_{\epsilon}) \triangledown 0 \) by dominated convergence. Choosing \( g_{\epsilon}(s) \equiv E(\xi - \xi_{\epsilon})(ds)/E_{\xi}(ds) \) non-decreasing in \( \epsilon \in E \), it follows from this by dominated convergence that \( g_{\epsilon}(s) \to 0 \) as \( \epsilon \to 0 \) for \( s \in S \) a.e. \( E_{\xi} \), say for \( s \) outside some null-set \( C' \). Furthermore, every \( \xi_{\epsilon} \) is clearly a.s. discrete and has no fixed atoms, so for fixed \( \epsilon \) we have \( P\{\xi_{B \epsilon} = 0\} \to 0 \) as \( B \uparrow \{s\} \).

Given arbitrary \( n \in N \) and \( s \notin C \cup C' \), we may therefore choose \( \epsilon \in E \) so small that \( g_{\epsilon}(s) < (8n)^{-1} \) and then \( B \in G \) so small that \( P\{\xi_{B \epsilon} > 0\} < (8n)^{-1} \). Writing \( \eta_{n} \) for the corresponding random measure \( \eta_{B \epsilon s} \), we then obtain

\[ |L_{\xi} (f) - L_{\xi}(f)L_{\eta_{n}} (f)| < n^{-1} , \ n \in N , \ f \in F_{c} , \]

proving that \( L_{\xi_{n}} \eta_{n} \to L_{\xi} \). By Lemma 5.1 it follows that \( \eta_{n} \overset{d}{\rightarrow} \) some \( \eta \).
satisfying $L_{\xi} L_{\eta} = L_{\xi S}$, and again $P_{\xi}^{-1} \mid P_{\xi S}^{-1}$. As before, this implies infinite divisibility of $\xi$, and the proof is complete.

We are now going to consider extensions of Proposition 11.1 in another direction.

**THEOREM 11.6.** Let $\xi$ be a point process on $S$ with $E\xi \in M$. Then $P(\xi_S - \delta_S)^{-1}$ is independent of $s \in S$ a.e. iff $\xi \overset{d}{=} M(E\xi, \phi)$ for some $\phi$. In this case $\phi'(0) = -1$, and $\xi_S - \delta_S \overset{d}{=} M(E\xi, -\phi')$ a.e.

It is interesting to see how the specialization of Proposition 11.1 to point processes follows from this result simply by solving the differential equation $\phi = -\phi'$ subject to the initial condition $\phi(0) = 1$.

The proof of Theorem 11.6 is based on the following lemma, giving an interpretation of the notion of Palm distributions in terms of ordinary conditional distributions.

**LEMMA 11.7.** Let $\xi$ be a point process on $S$ with $E\xi \in M$, and let $n \in n$, $C \in B$ and $M \in N$. For $\xi C > 0$, we further assume that $\tau$ is one of the atoms of $\xi$ in $C$, chosen at random. Then

$$P(\xi \in M \mid \xi C = n, \tau = s) = P(\xi_S \in M \mid \xi_S C = n), \quad s \in C,$$

a.e. with respect to the measure

$$E[\xi(ds); \xi C = n] = P(\xi_S C = n)E\xi(ds), \quad s \in S.$$

**PROOF.** The $n$ atoms in $C$ being symmetric, we get for $s \in C$
\[ P(\xi \in M, \xi \in C = n, \tau \in ds) = E_\tau [\delta_{\xi}(ds); \xi \in M, \xi \in C = n] = n^{-1}E[\xi(ds); \xi \in M, \xi \in C = n] = n^{-1}P(\xi_S \in M, \xi_S \in C = n)E\xi(ds), \]

and in particular

\[ P(\xi \in C = n, \tau \in ds) = n^{-1}P(\xi_S \in C = n)E\xi(ds). \]

By the chain rule for Radon-Nikodym derivatives, we thus obtain a.e.

\[
P(\xi \in M \mid \xi \in C = n, \tau = s) = \frac{P(\xi \in M, \xi \in C = n, \tau \in ds)}{P(\xi \in C = n, \tau \in ds)} = \frac{P(\xi_S \in M, \xi_S \in C = n)}{P(\xi_S \in C = n)}
\]

\[= P(\xi_S \in M \mid \xi_S \in C = n), \]

as desired.

**Proof of Theorem 11.6.** Suppose that \( \xi \overset{d}{=} M(\xi, \phi) \). As shown in Section 9, \( \xi_B \) has then the generating function \( \phi(E\xi_B(1 - t)) \), \( t \in [0, 1] \), and we get in particular

\[
P(\xi_B = 1) = \left. \frac{d}{dt} \phi(E\xi_B(1 - t)) \right|_{t = 0} = -E\xi_B \phi'(E\xi_B), \ B \in B.
\]

Using this fact and Corollary 9.2, we obtain for any \( B, C \in B \) with \( B \subseteq C \)

\[
\int_CP((\xi_S - \delta_S)B = 0)E\xi(ds) = \int_CP(\xi_S B = 1)E\xi(ds) = E[\xi; \xi_B = 1]
\]

\[= P(\xi_B = 1)E[\xi; \xi_B = 1] = -E\xi_B \phi'(E\xi_B) \frac{\partial E\xi}{\partial E\xi_B}
\]

\[= -\phi'(E\xi_B)E\xi_C.
\]

On the other hand we get by Corollary 9.2 for any disjoint \( B, C \in B \)
\[
\int_C \mathbb{P}\{\xi_s - \delta_s \in B = 0\} \mathbb{E}_\xi(ds) = \int_C \mathbb{P}\{\xi_s = 0\} \mathbb{E}_\xi(ds) = \mathbb{E}[\xi_C; \xi_B = 0] \\
= \sum_{n=1}^{\infty} n \mathbb{P}\{\xi_C = n, \xi_B = 0\} \\
= \sum_{n=1}^{\infty} n \mathbb{P}\{\xi(B \cup C) = n\} \mathbb{P}\{\xi_C = n \mid \xi(B \cup C) = n\} \\
= \sum_{n=1}^{\infty} n \mathbb{P}\{\xi(B \cup C) = n\} \left(\frac{\mathbb{E}_C}{\mathbb{E}(B \cup C)}\right)^n \\
= \frac{\mathbb{E}_C}{\mathbb{E}(B \cup C)} \frac{d}{dt} \sum_{n=0}^{\infty} t^n \mathbb{P}\{\xi(B \cup C) = n\} \bigg|_{t = \mathbb{E}_C/\mathbb{E}(B \cup C)} \\
= \frac{\mathbb{E}_C}{\mathbb{E}(B \cup C)} \mathbb{E}(B \cup C) \phi'(\mathbb{E}(B \cup C) - \mathbb{E}_C) \\
= -\phi'(\mathbb{E}_B)\mathbb{E}_C .
\]

These calculations show that for any \( B \in \mathcal{B} \)

\[ \mathbb{P}\{\xi_s - \delta_s \in B = 0\} = -\phi'(\mathbb{E}_B), \quad s \in S \text{ a.e. } \mathbb{E}_\xi . \]

If \( \mathbb{E}_\xi \in \mathbb{M}_d \), then \( \xi \) and hence also \( \xi_s - \delta_s \) is simple, and it follows by Theorem 3.3 that \( \xi_s - \delta_s \overset{d}{=} \mathbb{M}(\mathbb{E}_\xi, -\phi') \). In the general case we have to evaluate

\[ \mathbb{P}\{\xi_s - \delta_s \in B_1 = n_1, \ldots, (\xi_s - \delta_s)B_k = n_k\}, \quad k \in \mathbb{N}, \quad n_1, \ldots, n_k \in \mathbb{Z}_+, \quad B_1, \ldots, B_k \in \mathcal{B} . \]

But these probabilities are determined by the expectations
E[ξ(ds); ξB_1 = n_1, ..., ξB_k = n_k], so they are uniquely determined by the finite-dimensional distributions of ξ which in turn depend on Eξ and ϕ only. Thus it makes no difference whether Eξ ∈ M_d or not, and the above conclusion for diffuse Eξ must be generally valid.

Conversely, suppose that ξS_δ_s d some n a.e. Eξ. Let C ∈ B and n ∈ N be fixed with P{ξC = n} > 0, let B_1, ..., B_m ∈ B be an arbitrary disjoint partition of C, and let k_1, ..., k_m ∈ Z_+. Defining τ as in Lemma 11.7, we get for s ∈ B_1 a.e. E[ξ(ds); ξC = n]

P{(ξ - δ_τ)B_1 = k_1, ..., (ξ - δ_τ)B_m = k_m | ξC = n, τ = s}

= P{ξB_1 = k_1 + 1, ξB_2 = k_2, ..., ξB_m = k_m | ξC = n, τ = s}

= P{ξS_1 = k_1 + 1, ξS_2 = k_2, ..., ξS_m = k_m | ξS_C = n}

= P{(ξS - δ_s)B_1 = k_1, ..., (ξS - δ_s)B_m = k_m | (ξS - δ_s)C = n - 1}

= P{nB_1 = k_1, ..., nB_m = k_m | nC = n - 1},

and by the symmetry in B_1, ..., B_m, this remains true for s ∈ C a.e. E[ξ(ds); ξC = n]. By Dynkin's theorem, we can extend this result to

P{ξ - δ_τ ∈ M | ξC = n, τ = s} = P{n ∈ M | nC = n - 1}, M ∈ N,

proving that τ and ξ - δ_τ are conditionally independent, given that ξC = n. If τ_1, ..., τ_n is a random enumeration of the atom positions in C, we may easily conclude that τ_j is conditionally independent of \{τ_i, i ≠ j\} for every j ∈ \{1, ..., n\}. But then τ_1, ..., τ_n must be conditionally independent, and since they are also equally distributed,
\( \xi \) is conditionally a sample process. Furthermore, we have for any \( B \in \mathcal{B} \) with \( B \subset C \)

\[
P(\tau_1 < B \mid \xi_C = n) = \frac{\mathbb{E}\{\delta_1 B \mid \xi_C = n\}}{n} = \frac{\mathbb{E}\{\xi_B \mid \xi_C = n\}}{n} = \frac{\mathbb{E}\{\xi_B ; \xi_C = n\}}{n} = \frac{\mathbb{E}\{\xi_C \mid \xi_C = n\}}{n}
\]

\[
= \frac{\mathbb{P}\{\xi_C = n\} \mathbb{E}\{\xi(\mathcal{S}) \mid \xi_C = n\}}{n \mathbb{P}\{\xi_C = n\}} = \frac{\mathbb{P}\{\xi_C = n\} \mathbb{E}\{\xi(\mathcal{S}) \mid \xi_C = n\}}{n \mathbb{E}\{\xi_C \mid \xi_C = n\}}
\]

\[
= \mathbb{E}\{\xi_B \mathbb{E}\{\xi_C \mid \xi_C = n\}\} = \frac{\mathbb{E}\{\xi_B \mathbb{E}\{\xi_C \mid \xi_C = n\}\}}{n \mathbb{E}\{\xi_C \mid \xi_C = n\}} = \frac{\mathbb{E}\{\xi_B \mathbb{E}\{\xi_C \mid \xi_C = n\}\}}{n \mathbb{E}\{\xi_C \mid \xi_C = n\}}
\]

showing that the common conditional distribution of the \( \tau_j \) is independent of \( n \). It follows that \( \xi \) is unconditionally a mixed sample process, and an application of Corollary 9.2 completes the proof.

We shall finally consider an extension of Theorem 11.6 to general random measures. Recall that a measure is degenerate if its mass is confined to one single point. If \( n \) is a random variable while \( \xi \) is a random measure on \( \mathcal{S} \), then \( (n, \xi) \) is said to be symmetrically distributed w.r.t. some \( \omega \in M \), if the distribution of \( (n, \xi_{B_1}, \ldots, \xi_{B_k}) \) for arbitrary \( k \in \mathbb{N} \) and disjoint \( B_1, \ldots, B_k \in \mathcal{B} \) only depends on \( \omega_{B_1}, \ldots, \omega_{B_k} \). For convenience, the random measures described in parts (i) and (ii) of Theorem 9.4 are said to be of Type A and B respectively.

We shall allow the measure \( \omega \) there to be finite in both cases. Note that the two classes of random measures will then no longer be mutually exclusive. The canonical random elements \( (\alpha, \lambda) \) and \( (\alpha, \beta) \) are defined as in Theorem 9.4, and in case of Type A random measures, we further
define the canonical random measure \( \Lambda \) on \( \mathbb{R}_+ \) by \( \Lambda(dx) = a\delta_0(dx) + x\lambda(dx) \), \( x \geq 0 \).

**Theorem 11.8.** Let \( \xi \) be a random measure on \( S \) with \( E\xi \in M \). Then

\[(\xi_s\{s\}, \xi_s - \xi_s\{s\}\delta_s) \sim \text{some } (n, \zeta) \text{ independently of } s \text{ a.e. } E\xi \text{ iff}

\( E\xi \in M_d \) (except possibly for a.s. degenerate \( \xi \)) and \( \xi \) is symmetrically distributed w.r.t. \( E\xi \). In this case, \( (n, \zeta) \) is also symmetrically distributed w.r.t. \( E\xi \), and furthermore, \( n \) is independent of \( \xi \) iff either

(i) \( \xi \) is of Type A with \( \Lambda = \rho M \) a.s. for some random variable \( \rho \geq 0 \) and some non-random \( M \in M(\mathbb{R}_+) \), or

(ii) \( \xi \) is of Type B with \( \alpha = 0 \) and \( \beta \) a mixed sample process on \( \mathbb{R}_+ \).

Note by comparison with (i) that the point process \( \beta \) in (ii) is also allowed to be a mixed Poisson process. In this case we can even take \( \alpha \) non-zero, proportional to the mixing variable.

To save space, we omit the rather complicated proof of this theorem and refer the reader to Kallenberg (1974b).

**Notes.** Proposition 11.1 was proved for point processes by Kerstan and Matthes (1964) and Mecke (1967), but the present proof is adapted from Jagers (1973). Theorem 11.2 is due to Kerstan and Matthes (1964), Mecke (1967), Kummer and Matthes (1970) and others (cf. Kerstan, Matthes and Mecke (1974), page 116), while the extensions given in Theorems 11.3 and
11.4 are new. Theorem 11.5 extends a result for point processes, proved by Matthes (1969) with entirely different methods, (cf. Kerstan, Matthes and Mecke (1974), page 97). Finally, Theorem 11.6 is due to Slivnyak (1962), Papangelou (1974) and Kallenberg (1973a), while Theorem 11.8 was given in Kallenberg (1974b).

PROBLEMS.

11.1. Let \( \xi \) be a point process on \( S \) with \( E\xi \in M \) and let \( I \subset B \) be a \( DC \)-semiring. Show that \( \xi \) is infinitely divisible iff
\[
P(\xi f)^{-1} \mid P(\xi_f f)^{-1}
\]
for every function \( f = \sum_{j=1}^{k} I_j \), \( k \in \mathbb{N} \), \( I_1, \ldots, I_k \subset I \), such that \( \xi f \notin 0 \).

11.2. Suppose that \( S \) is countable. Show that there exists some fixed function \( f \in F \) such that, given any point process \( \xi \) on \( S \) with \( \xi S < \infty \) a.s., \( \xi \) is infinitely divisible iff \( P(\xi = 0) > 0 \) and \( P(\xi f)^{-1} \mid P(\xi_f f)^{-1} \). (Hint: use the result in Problem 6.4.)

11.3. Let \( \xi \) be a point process on \( S \) and let \( B \in B \) be such that \( P(\xi \neq 0, \xi B = 0) = 0 \). Show that \( \xi \) is infinitely divisible iff \( P(\xi = 0) > 0 \) and \( P\xi^{-1} \mid P\xi^{-1} \), \( f = 1_B \). (Hint: argue with \( \xi \) as with the Cox processes in the proof of Theorem 11.4, and note that the counterpart of (4) holds trivially in the present case.)

11.4. Show that the condition \( E\xi \in M_\mu \) in Theorem 11.5 can be replaced by the assumption that \( E\xi \) be \( \sigma \)-finite and diffuse.
11.5. Let $\xi$ be a random measure on $S$ with $\xi S < \infty$ a.s., and let $\tau$ be a random element in $S$ which for given $\xi \neq 0$ has the conditional distribution $\xi/\xi S$. Show that, for $u > 0$, $s \in S$ and measurable $f: M \times S \rightarrow \mathbb{R}_+$,

$$E[f(\xi, \tau) \mid \xi S = u, \tau = s] = E[f(\xi_s, s) \mid \xi S = u]$$

holds a.e. $E[\xi(\text{ds}); \xi S < du]$, (cf. Kallenberg (1974b)).

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