ON THE DISTRIBUTION OF THE LAST OCCURRENCE TIME IN AN INTERVAL FOR A REGENERATIVE PHENOMENON.

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1. Introduction and Summary

Let \((\Omega, \mathcal{G}, \text{Pr})\) be a probability space on which a regenerative phenomenon \(\mathcal{E}\) is defined. That is, \(\mathcal{E}\) is a family \(\{E(t), t > 0\}\) of subsets of \(\Omega\), each belonging to \(\mathcal{G}\), and having the property that whenever

\[0 < t_1 < t_2 < \ldots < t_k,
\]

then

\[
\text{Pr}\{E(t_1), E(t_2), \ldots, E(t_k)\} = \text{Pr}\{E(t_1)\} \cdot \text{Pr}\{E(t_2 - t_1), \ldots, E(t_k - t_1)\}.
\]

Let \(\mathcal{E}\) be such that

\[
p(t) = \text{Pr}\{E(t)\} \to 1
\]
as \(t \to \infty\). That is, in the terminology of KINGMAN [14], \(\mathcal{E}\) is standard. We write

\(Z(t, \omega)\) for the indicator process of \(\mathcal{E}\) defined by

\[
Z(t, \omega) = \begin{cases} 1 & \text{if } \omega \in E(t) \\ 0 & \text{if } \omega \notin E(t), \end{cases}
\]

and we shall suppose, as we may do without essential loss of generality, that
this process is separable ([14], Section 13). Take \(Z(0, \omega) = 1\) for convenience. Then, associated with \(\mathcal{E}\) there is a stochastic process \(T_t\) defined by

\[T_t = \sup\{u : 0 \leq u \leq t; Z(u, \omega) = 1\}.
\]

That is, \(T_t\) is the time of last occurrence of \(\mathcal{E}\) in the interval \([0, t]\).

This paper is concerned with a study of the process \(T_t\). We shall obtain various representation results for \(T_t\) and examine aspects of its limit behaviour as \(t \to \infty\).

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2. Representation Results

Theorem 1. For \( \theta > 0 \), define
\[
r(\theta) = \int_0^\infty e^{-\theta t} p(t) dt.
\]
Then, for \( s > 0, z > 0 \),
\[
z \int_0^\infty \left( e^{-st} \right) e^{-zt} dt.
\]

\[
= \exp \left\{ -s \int_0^\infty t^2 E_t e^{-zt} \left[ (1 + tz) - (1 + (z+s) \left( e^{-st} \right) \right] dt \right\}
\]
\[
= r(z + s) \left[ r(z) \right]^{-1}.
\]

Proof. For integer valued \( n \geq 1 \), consider the family \( \{ E(2^{-m} n), n = 0, 1, 2, \ldots \} \) of subsets of \( \Omega \). This discrete skeleton forms a recurrent event in the sense of FELLER ([1], I Chapter 13). Now, define the sequences \( \{ u_n, n \geq 0 \} \) and \( \{ f_n, n \geq 1 \} \) by
\[
u_n = \Pr\{ E(2^{-m} n) \}, \quad n \geq 0,
\]
\[
f_n = \Pr\{ E(2^{-m}), E(2^{-(m-1)}), \ldots, E(2^{-m}(n-1)), E(2^{-m} n) \}, \quad n \geq 1,
\]
where \( E(2^{-m}n) \) denotes the complement of \( E(2^{-m} n) \). These sequences are related by the well-known power series identity
\[
U(t) = \left[ 1 - F(t) \right]^{-1}, \quad 0 \leq t < 1,
\]
where
\[
U(t) = \sum_{n=0}^{\infty} u_n t^n, \quad F(t) = \sum_{n=1}^{\infty} f_n t^n, \quad 0 \leq t < 1.
\]

Let \( 2^{-n} T_n^{(m)} \) denote the time of last occurrence of \( \epsilon \) in the set \( \{ 0, 2^{-m}, 2^{-(m-1)}, \ldots, 2^{-m} n \} \). Then, the representations
\[
u_n = \Pr\{ T_n^{(m)} = n \}, \quad q_n = \sum_{r=n+1}^{\infty} f_r = \Pr\{ T_n^{(m)} = o \}, \quad n \geq 0,
\]
are obvious, together with the extremal factorization property
\[
\Pr\{ T_n^{(m)} = k \} = \Pr\{ T_k^{(m)} = k \} \Pr\{ T_n^{(m)} = o \}
\]
\[
= u_k q_n k^n k^k,
\]
o \leq k \leq n. Then, upon taking generating functions we readily find that for \( s > 0, z > 0 \),
(2) 
\[ (1 - e^{-2^{-m}z}) \sum_{n=0}^{m} E(e^{-2^{-m}sT_n^{(m)}}) e^{-2^{-m}zn} = U(e^{-2^{-m}(z + s)} \left[ U(e^{-2^{-m}z}) \right]^{-1}. \]

The next step is to take the limit as \( m \to \infty \) in (2) and in order to proceed with this we need the following lemma.

**Lemma:** Write 
\[ \phi_{m}^{(m)}(\lambda, t) = E(e^{-2^{-m}\lambda t \left[ m \right]}), \quad \phi(\lambda, t) = E(e^{-\lambda t}), \]
where \( \left[ 2^{-m} \right] \) denotes the integer part of \( 2^{-m}t \). Then, 
\[ \lim_{m \to \infty} \phi_{m}^{(m)}(\lambda, t) = \phi(\lambda, t) \]
uniformly in \( t \) in any finite interval.

**Proof:** Firstly, we note that \( 2^{-m}\lambda t \left[ m \right] \) is monotone non-decreasing in \( m \) and hence that \( \phi_{m}^{(m)}(\lambda, t) \) is monotone non-increasing in \( m \). Consequently, it will be sufficient to show that 
\[ \lim_{m \to \infty} \phi_{m}^{(m)}(\lambda, t) = \phi(\lambda, t) \]
for each fixed \( t \) and that \( \phi(\lambda, t) \) is a continuous function of \( t \).

Now, the indicator process of \( \mathcal{E} \) is separable so that for any \( x \geq 0 \) and any countably dense subset \( \{ t_1, t_2, \ldots \} \) of \([0, t] \),
\[ \Pr(T_t \leq x) = \Pr(Z(u, w) = 0 \text{ for } x < u \leq t) = \Pr(T_t^* \leq x) \]
where \( T_t^* = \sup \{ t_j : Z(t_j, w) = 1 \} \). It follows then, using the Helly-Bray Theorem, that
\[ \lim_{m \to \infty} \phi_{m}^{(m)}(\lambda, t) = \phi(\lambda, t) \]
for fixed \( t \). Also, if 
\[ \tau(t) = \tau(t, w) = \int_{0}^{t} Z(u, w)du, \]
then 
\[ \Pr(T_t + \Delta t > T_t) \leq \Pr(\tau(t + \Delta t) > \tau(t)) \to 0 \]
as \( \Delta t \to 0 \) so that \( T_t \) is continuous in probability. Finally, since \( T_t \) is monotone non-decreasing in \( t \),
\[ \phi(\lambda, t + \Delta t) \leq \phi(\lambda, t), \]
while, letting \( \delta \) be arbitrarily small and positive and using integration by parts,
\[ \phi(\lambda, t + \Delta t) = 1 - \int_0^\infty e^{-\lambda x} \Pr(T_t + \Delta t \geq x)dx \]
\[ = 1 - \int_0^\infty e^{-\lambda x} \Pr(T_t + \Delta t - T_t + T_t \geq x)dx \]
\[ \geq 1 - \int_0^\infty e^{-\lambda x} \left[ \Pr(T_t + \Delta t - T_t \geq \delta) + \Pr(T_t \geq x - \delta) \right]dx \]
\[ = 1 - \Pr(T_t + \Delta t - T_t \geq \delta) - \lambda e^{-\lambda \delta} \int_0^\infty e^{-\lambda y} \Pr(T_t \geq y)dy \]
\[ = 1 - \Pr(T_t + \Delta t - T_t \geq \delta) - e^{-\lambda \delta} \left[ 1 - \phi(\lambda, t) \right] \]
\[ - \lambda e^{-\lambda \delta} \int_0^\infty e^{-\lambda y} \Pr(T_t \geq y)dy, \]

so that, from (3) and (4) and since \( T_t \) is continuous in probability,
\[ \lim_{\Delta t \to 0} \phi(\lambda, t + \Delta t) = \phi(\lambda, t). \]

This completes the proof of the lemma and we resume the proof of the theorem.

We can write,
\[ z \int_0^\infty E(e^{-sT_t})e^{-zt}dt = \lim_{m \to \infty} (1 - e^{-2^{-m}z}) \sum_{n=0}^\infty \phi(s, 2^{-m}n)e^{-2^{-m}nz}, \]
and we shall show that it is possible to replace \( \phi(s, 2^{-m}n) \) by \( \phi_m(s, 2^{-m}n) \) on the right hand side of (5). To show this, let \( \epsilon > 0 \) be arbitrarily small and let \( N \) be a positive integer so large that for fixed \( z \), \( e^{-2^Nz} < (\frac{1}{4})\epsilon \). Then, from the lemma, we have for sufficiently large \( m \),
\[ |\phi_m(s, 2^{-m}k) - \phi(s, 2^{-m}n)| \leq (1/2)\epsilon, \quad k = 0, 1, 2, \ldots, 2^m + N. \]

Therefore, for sufficiently large \( m \),
\[ |(1 - e^{-2^{-m}z})\sum_{n=0}^\infty \phi_m(s, 2^{-m}n) - \phi(s, 2^{-m}n)|e^{-2^{-m}nz}| \]
\[ \leq (1 - e^{-2^{-m}z})\sum_{n=0}^\infty |\phi_m(s, 2^{-m}n) - \phi(s, 2^{-m}n)|e^{-2^{-m}nz} \]
\[ + 2(1 - e^{-2^{-m}z})\sum_{n=2^m+N}^\infty e^{-2^{-m}nz} \]
\[ \leq \frac{\epsilon}{2} + 2 \cdot \frac{\epsilon}{4} = \epsilon, \]
and from (5) and (6) we obtain

$$
\int_0^\infty E(e^{-st})e^{-zt}dt = \lim_{m \to \infty} (1 - e^{-2^m z}) \sum_{n=0}^\infty \Pr(T_n = n)e^{-2^m n z},
$$

which deals with the left hand side of (2).

For the right hand side of (2), we firstly make use of a result of DWASS (see PORT [7]) which allows us to express $U(t)$ in the form

$$
U(t) = \exp\left\{ \sum_{k=1}^\infty t^k \Delta_{km} k^{-1} \right\}, \quad 0 \leq t < 1,
$$

where

$$
\Delta_{km} = 2^{-m} E\left[ T_k^{(m)} - T_{k-1}^{(m)} \right], \quad k \geq 1.
$$

Then,

$$
U(e^{-2^m z + s}) \left[ U(e^{-2^m z}) \right]^{-1} = \exp\left\{ -\sum_{k=1}^\infty k^{-1} e^{-2^m k z} (1 - e^{-2^m k s}) \Delta_{km} \right\}.
$$

(7)

Now, using summation by parts,

$$
\sum_{k=1}^\infty k^{-1} e^{-2^m k z} (1 - e^{-2^m k s}) \Delta_{km}
$$

(8)

$$
= \sum_{k=1}^\infty 2^{-m} E_k^{(m)} \left\{ \frac{1}{k} e^{-2^m k s} - \frac{1}{(k+1)} e^{-2^m (k+1) z} (1 - e^{-2^m (k+1) s}) \right\},
$$

while, again making use of the lemma, we obtain without difficulty that

$$
\lim_{m \to \infty} \sum_{k=1}^\infty 2^{-m} E_k^{(m)} \left\{ \frac{1}{k} e^{-2^m k z} (1 - e^{-2^m k s}) - \frac{1}{(k+1)} e^{-2^m (k+1) z} (1 - e^{-2^m (k+1) s}) \right\}
$$

$$
= \int_0^\infty t \left\{ \int_0^t \left[ (1 + tz) - (1 + t(z + s) e^{-st}) \right] dt \right\} dt.
$$

(9)

The second part of (1) follows immediately from (7), (8) and (9).

Finally, for $\theta > 0$,

$$
2^{-m} U(e^{-2^m \theta}) = 2^{-m} \sum_{n=0}^\infty \Pr(T_n^{(m)} = n) e^{-2^m \theta}
$$
\[ r(\theta) = \int_0^\infty e^{-\theta t} p(t) dt \]

as \( m \to \infty \) and, consequently, the right hand side of (2) also converges to \( r(z+1)[r(z)]^{-1} \) as \( m \to \infty \). This provides the third part of (1) and thus completes the proof of the theorem.

**Corollary 1.** For \( z > 0 \),

\[ (10) \quad zr(z) = \exp\left\{-\int_0^\infty t^{-1} e^{tz} d_t(t - ET_t)\right\}. \]

**Proof.** Using Theorem 1, we have with the aid of easy calculations that

\[ \frac{zr(z)}{(z+1)r(z+1)} = \exp\left\{\int_0^\infty t^{-1} e^{-zt} [1+tz] - (1 + t(z+1)e^{-zt}] dt - \log(1 + z^{-1}s)\right\} \]

\[ = \exp\left\{\int_0^\infty t^{-1} e^{-zt} (1 + tz) - (1 + t(z+1)e^{-zt}] dt\right\} \]

\[ = \exp\left\{\int_0^\infty t^{-1} e^{-zt} d_t(t - ET_t)\right\} \]

\[ = \exp\left\{\int_0^\infty t^{-1} e^{-zt} (1 - e^{-st}) d_t(t - ET_t)\right\}. \]

(11)

Furthermore, in Theorem 3 of [4] it is shown that there exists a unique positive measure \( \mu \) on \((0, \infty)\) with

\[ \int_{(0, \infty)} (1 - e^{-x}) \mu(dx) < \infty, \]

such that for \( \theta > 0 \),

\[ r(\theta) = \left[ \theta + \int_{(0, \infty)} (1 - e^{-\theta x}) \mu(dx) \right]^{-1}, \]

and from this representation it follows immediately that \( \Theta r(\theta) \to 1 \) as \( \theta \to \infty \). The result (10) is then obtained by letting \( s \to \infty \) in (11).

**Theorem 2.** Let \( \mu \) be the canonical measure of the regenerative phenomenon \( \xi \).

That is, \( \mu \) is a positive measure on \((0, \infty)\) for which

\[ \int_{(0, \infty)} (1 - e^{-x}) \mu(dx) < \infty, \]

(12)
and such that for $\theta > 0$,

$$r(\theta) = \int_0^\infty e^{-\theta t} p(t)dt = \left[ \theta + \int_{(0, \infty)} (1 - e^{-\theta x}) \mu(dx) \right]^{-1}.$$  

Then, the distribution of $T_t$ is given by

$$\Pr(T_t < u) = \int_0^u \mu(t - v, \infty) p(v)dv, \quad 0 \leq u < t,$$

$$\Pr(T_t = t) = p(t).$$

Proof. In view of (12), $\mu(x, \infty)$ is bounded in $(a, \infty)$ and integrable over $(0, a)$ for each $a > 0$. Furthermore, its Laplace transform $\int_0^\infty e^{-\theta x} \mu(x, \infty) dx$ exists for $\theta > 0$ and

$$\int_0^\infty e^{-\theta x} \mu(x, \infty) dx = \theta^{-1} \int_{(0, \infty)} (1 - e^{-\theta x}) \mu(dx).$$

Consequently, from (13),

$$\left[ \theta r(\theta) \right]^{-1} = 1 + \int_0^\infty e^{-\theta x} \mu(x, \infty) dx.$$

It then follows from Theorem 1 that

$$\int_0^\infty E(e^{-ST_t})e^{-zt}dt = r(z + s) \left[ 1 + \int_0^\infty e^{-z x} \mu(x, \infty) dx \right]$$

$$= \int_0^\infty e^{-z t} p(t)dt + \int_0^\infty e^{-z t} \int_0^t e^{-s v} p(v) \mu(t - v, \infty) dv dt,$$

so that

$$E(e^{-ST_t}) = e^{-st} p(t) + \int_0^t e^{-s v} p(v) \mu(t - v, \infty) dv,$$

and a further inversion yields (14). We note that

$$\Pr(T_t \leq t) = 1 = p(t) + \int_0^t p(t - v) \mu(v, \infty) dv,$$

which is the Volterra integral equation obtained in Proposition 7 of [4].

Theorem 2 shows us clearly that the function $p(t)$ uniquely determines the distribution of $T_t$ for all $t > 0$ and vice-versa. We note also the following result which is obtained by differentiating in (14),

$$\mu(t - u, \infty) = \frac{d}{du} \Pr(T_t < u), \quad 0 < u < t.$$
3. Limit Behavior

In [4], it is shown that there are three possibilities for the ergodic behavior of a standard regenerative phenomenon $\mathcal{E}$:

(I) $\mu(\infty) > 0$ (transient),

(II) $\mu(\infty) = 0$, $\int_{(0, \infty)} x \mu(dx) = \infty$ (null),

(III) $\mu(\infty) = 0$, $\int_{(0, \infty)} x \mu(dx) < \infty$ (positive).

Theorem 3. If $\mathcal{E}$ is transient, then

(15) $\lim_{t \to \infty} \Pr(T_t < u) = \left[\mu(\infty)\right]^{-1} \int_0^u p(x) dx$.

If $\mathcal{E}$ is positive, then

$\lim_{t \to \infty} \Pr(t - T_t < u) = \frac{1 + \int_0^u \mu(x, \infty) dx}{1 + \int_0^\infty \mu(x, \infty) dx}$.

Proof. Suppose firstly that $\mathcal{E}$ is transient. Then, from Proposition 8 of [4],

$\left[\mu(\infty)\right]^{-1} = \int_0^\infty p(t) dt < \infty$.

The result (15) follows immediately upon proceeding to the limit in (14).

Next, suppose that $\mathcal{E}$ is positive. From Theorem 6 of [4],

$p(t) \to \frac{1}{1 + \int_0^\infty \mu(x, \infty) dx}$

as $t \to \infty$. Then, from Theorem 2,

$\Pr(t - T_t < u) = 1 - \int_0^{t-u} \mu(t - v, \infty)p(v) dv$

$= p(t) + \int_t^{t-u} \mu(t - v, \infty)p(v) dv$

$= p(t) + \int_0^u \mu(x, \infty)p(t - x) dx$

$\to \frac{1 + \int_0^u \mu(x, \infty) dx}{1 + \int_0^\infty \mu(x, \infty) dx}$

as $t \to \infty$. This completes the proof of the theorem.
Definition. A regenerative phenomenon \( \mathcal{E} \) will be called \( \beta \)-regular if
\[
\lim_{t \to \infty} t^{-1} \mathbb{E}^\mathcal{E}_t = \beta \text{ (obviously } 0 \leq \beta \leq 1)\).
\]
The concept of \( \beta \)-regularity is important by virtue of the following theorem which is the regenerative phenomenon analogue of Theorem 3.2 of LAMPERTI [6] for the recurrent event context.

**Theorem 4.** The limiting distribution
\[
(17) \quad \lim_{t \to \infty} \Pr(t^{-1} \mathbb{E}^\mathcal{E}_t < x) = F(x)
\]
exists if and only if \( \mathcal{E} \) is \( \beta \)-regular and then \( F(x) \) is related to \( \beta \) by
\[
F(x) = F_{\beta}(x) = \frac{\sin \pi \beta}{\pi} \int_0^x (1 - \beta)(1 - v)^{-\beta} dv, \quad 0 < \beta < 1, \quad 0 \leq x \leq 1,
\]
\[
(18) \quad F_0(x) = 0 \text{ if } x < 0, \quad 1 \text{ if } x \geq 0,
\]
\[
F_1(x) = 0 \text{ if } x < 1, \quad 1 \text{ if } x \geq 1.
\]

**Proof.** We shall first establish that the condition of \( \beta \)-regularity is sufficient for the existence of the limiting distribution (17). In order to do this, we show firstly that under the condition of \( \beta \)-regularity and when \( 0 < \lambda < 1 \),
\[
(19) \quad \lim_{z \to 0} \int_0^\infty t^{-1} \mathbb{E}^\mathcal{E}_t e^{-zt} \left[ (1 + tz) - (1 + tz(1+\lambda)) e^{-\lambda zt} \right] dt = \beta \log(1 + \lambda).
\]
Write
\[
C(z, t) = t^{-1} e^{-zt} \left[ (1 + tz) - (1 + tz(1+\lambda)) e^{-\lambda zt} \right]
\]
and note that \( C(z, t) \geq 0 \) and \( \lim_{z \to 0} C(z, t) = 0 \). Thus, for \( z > 0 \),
\[
\int_0^\infty t^{-1} \mathbb{E}^\mathcal{E}_t C(z, t) dt = \int_0^\infty \left[ t^{-1} \mathbb{E}^\mathcal{E}_t - \beta \right] C(z, t) dt + \beta \int_0^\infty C(z, t) dt
\]
\[
= \int_0^\infty \left[ t^{-1} \mathbb{E}^\mathcal{E}_t - \beta \right] C(z, t) dt + \beta \log(1 + \lambda),
\]
upon performing a simple integration. Now, in view of the \( \beta \)-regularity condition we can, given \( \epsilon > 0 \) arbitrarily small, choose \( T \) so large that
\[
|t^{-1} \mathbb{E}^\mathcal{E}_t - \beta| < \epsilon \text{ for } t \geq T \text{ and then}
\]
\[ \left| \int_0^\infty \left[ t^{-1}\mathcal{E}_t - \beta \right] C(z, t) \, dt \right| \leq \int_0^T \left| t^{-1}\mathcal{E}_t - \beta \right| C(z, t) \, dt + \epsilon \log(1 + \lambda) \rightarrow \epsilon \log(1 + \lambda) \]

as \( z \to 0 \) since \( \lim_{z \to 0} C(z, t) = 0 \). The result (19) follows immediately. Then, putting \( s = \lambda z \) where \( 0 < \lambda < 1 \) in the result of Theorem 1 and making use of (19), we obtain

\[ \lim_{z \to 0} \int_0^\infty e^{-zt}E(e^{-\lambda zT_t}) \, dt = (1 + \lambda)^{-\beta}. \]

Now,

\[ \int_0^\infty e^{-zt}E(e^{-\lambda zT_t}) \, dt = \int_0^\infty e^{-zt} \sum_{k=0}^{\infty} \frac{(-\lambda z)^k T_t^k}{k!} \, dt \]

\[ = \sum_{k=0}^{\infty} \lambda^k A_k(z), \]

where

\[ A_k(z) = \frac{(-z)^k + 1}{k!} \int_0^\infty e^{-zt} T_t^k \, dt, \]

so that from (20),

\[ \lim_{z \to 0} \sum_{k=0}^{\infty} \lambda^k A_k(z) = (1 + \lambda)^{-\beta} = \sum_{k=0}^{\infty} \lambda^k \binom{-\beta}{k}, \]

and consequently,

\[ \lim_{z \to 0} A_k(z) = \binom{-\beta}{k}, \quad k \geq 0. \]

But, \( T_t^k \) is monotone in \( t \) so, using Theorem 4, 423, Vol. II of [1], it follows from (21) that as \( t \to \infty \),

\[ E(t^{-1}T_t)^k \to (-1)^k \binom{-\beta}{k}, \quad k \geq 0. \]

Furthermore, it is easy to verify that

\[ (-1)^k \binom{-\beta}{k} = \int_0^1 x^k F_\beta(dx), \]

where \( F_\beta(x) \) is given by (18) and, since the moment problem in this case has a unique solution, the proof of the sufficiency part of the theorem is complete.

Finally, suppose that \( t^{-1}T_t \) has a proper limiting distribution. Then, necessarily, \( t^{-1}\mathcal{E}_t \to \beta \) for some \( 0 \leq \beta \leq 1 \) so that \( \epsilon \) is \( \beta \)-regular. This completes the proof of the theorem.
In view of the importance of the $\beta$-regularity concept, we shall next give some equivalent forms which may provide more useful criteria under certain circumstances.

Theorem 5. A regenerative phenomenon $\mathcal{E}$ is $\beta$-regular if and only if

\begin{equation}
(23) \quad r(z) \sim z^{-\beta} L(z^{-1})
\end{equation}

as $z \to 0$ or equivalently,

\begin{equation}
(24) \quad \int_0^t p(u) du \sim \frac{1}{\Gamma(1 + \beta)} t^\beta L(t)
\end{equation}

as $t \to \infty$, $L(x)$ being a slowly varying function as $x \to \infty$. In the particular case where $p(t)$ is ultimately monotone and $\beta > 0$, the conditions (23) and (24) are also equivalent to

\begin{equation}
(25) \quad p(t) \sim \frac{1}{\Gamma(\beta)} t^{-(1 - \beta)} L(t)
\end{equation}

as $t \to \infty$.

Proof. We shall deal firstly with the condition (23). Suppose that $\mathcal{E}$ is $\beta$-regular. Then, making use of Theorem 1 and equation (20),

\begin{equation}
\lim_{z \to 0} \frac{r((1 + \lambda)z)}{r(z)} = (1 + \lambda)^{-\beta}
\end{equation}

for $0 < \lambda < 1$ and (23) follows by use of the Theorem, 270 Vol. II of [1].

Conversely, suppose that the condition (23) holds. Then, again making use of Theorem 1, we have for $0 < \lambda < 1$,

\begin{equation}
\lim_{z \to 0} \int_0^\infty e^{-zt} E(e^{-sT_t}) dt = \lim_{z \to 0} \frac{r((1 + \lambda)z)}{r(z)} = (1 + \lambda)^{-\beta},
\end{equation}

which is equation (20). Following the proof of Theorem 4, we then deduce from (22) the required $\beta$-regularity condition. This completes the proof that (23) is a necessary and sufficient condition for $\beta$-regularity. The remainder of the proof is then immediately completed by appeal to Theorem 2, 421 Vol. II of [1] for condition (24) and Theorem 4, 423, Vol. II of [1] for condition (25).

For $s > 0$, $t > 0$, write

\begin{equation}
P(s, t) = \Pr\left(\sup_{s \leq u \leq s + t} Z(u, w) = 1\right).
\end{equation}

That is, $P(s, t)$ is the probability that $\mathcal{E}$ will occur in the time interval $[s, s + t]$. 
Theorem 6. Suppose $\mathcal{E}$ is $\beta$-regular. Then, for any $\alpha > 0$,

\[
\lim_{t \to \infty} P(t, \alpha t) = \begin{cases} 
\frac{\sin \pi \beta}{\pi} \int_0^1 x^{-(1 - \beta)} (1 - x)^{-\beta} \, dx, & 0 < \beta < 1 \\
0, & \beta = 0, \\
1, & \beta = 1.
\end{cases}
\]

If $\mathcal{E}$ is not $\beta$-regular for some $\beta$, $0 \leq \beta < 1$, then the limit of $P(t, \alpha t)$ as $t \to \infty$ does not exist. In the particular case $\beta = 1/2$, (26) yields

\[
\lim_{t \to \infty} P(t, \alpha t) = 1 - 2^n^{-1} \arcsin \left[ (1 + \alpha)^{-\frac{1}{2}} \right],
\]

so that

\[
\lim_{t \to \infty} \left[ P(t, \alpha t) + P(\alpha t, t) \right] = 1.
\]

Proof. We have

\[P(s, t) = \text{Pr}(T_s + t \geq s),\]

so that

\[
\lim_{t \to \infty} P(t, \alpha t) = \lim_{t \to \infty} \text{Pr}(T_t (1 + \alpha) \geq t) = \lim_{t \to \infty} \text{Pr}(t^{-1} T_t \geq (1 + \alpha)^{-1}),
\]

and the result (26) follows from Theorem 4. It also follows from Theorem 4 that the limit as $t \to \infty$ of $P(t, \alpha t)$ only exists in the case of $\beta$-regularity. In the case $\beta = 1/2$, we have

\[
\lim_{t \to \infty} P(t, \alpha t) = \pi^{-1} \int_0^1 x^{-\frac{1}{2}} (1 - x)^{-\frac{1}{2}} \, dx
\]

\[= 1 - \pi^{-1} \int_0^{(1+\alpha)^{-1}} x^{-\frac{1}{2}} (1 - x)^{-\frac{1}{2}} \, dx = 1 - 2^n^{-1} \arcsin \left[ (1 + \alpha)^{-\frac{1}{2}} \right] \]

by a simple transformation. The result (27) follows as

\[
\arcsin \left[ (1 + \alpha)^{-\frac{1}{2}} \right] + \arcsin \left[ \frac{1}{2} (1 - \alpha)^{-\frac{1}{2}} \right] = \frac{1}{2} \pi.
\]

This completes the proof of the theorem.

Theorem 6 is the counterpart of Theorem 2 of HEYDE [2] for recurrent events. Unlike the recurrent event case, however, it cannot happen that

\[
P(s, t) + P(t, s) = 1
\]

for all $s > 0$, $t > 0$. In order to see this, we note that (28) is precisely
the condition that $T_t$ and $t - T_t$ should have the same distribution for each $t > 0$. This is clearly impossible by Theorem 2.

4. Examples

Let $\xi(t), t \geq 0, \xi(0) = 0$ be a separable stochastic process with stationary independent increments whose sample functions are continuous on the left. Write $\bar{\xi}(t) = \sup_{0 \leq s \leq t} \xi(s)$ and $T_t = \min[u : \xi(u) = \bar{\xi}(t)]$.

We shall show that $\xi$, where $E(t)$ is the event $\{T_t = t\}$, is a proper regenerative phenomenon if and only if

$$\int_0^1 t^{-1} \Pr(\xi(t) < o) dt < \infty.$$

This is in contrast with the corresponding discrete case where $\xi$ is always a recurrent event. A discussion of the condition

$$\int_0^1 t^{-1} \Pr(\xi(t) < o) dt < \infty$$

can be found in ROGOZIN [8]. It is satisfied, for example, if $\xi(t)$ is a process with negative jumps and positive drift.

Write $\eta(t) = -\xi(t)$ and $\bar{\eta}(t) = \sup_{0 \leq s \leq t} \eta(s)$. Then, it can readily be verified that

$$p(t) = \Pr\{E(t)\} = \Pr(\bar{\eta}(t) = o).$$

Furthermore, from equation (1) of ROGOZIN [8], we have for $\text{Re}\lambda \leq o$,

$$u \int_0^\infty e^{-ut} \int_0^\infty e^{\lambda x} \Pr(\bar{\eta}(t) < x) dx dt = \exp \left\{ -\int_0^\infty (e^{\lambda x} - 1) dx \int_0^t t^{-1} \Pr(\eta(t) > x) e^{-ut} dt \right\},$$

and upon letting $\lambda \to -\infty$ we obtain

$$ur(u) = u \int_0^\infty e^{-ut} \Pr(\bar{\eta}(t) = o) dt = \exp \left\{ -\int_0^\infty t^{-1} \Pr(\eta(t) > o) e^{-ut} dt \right\}$$

when the integral

$$\int_0^1 t^{-1} \Pr(\xi(t) < o) dt = \int_0^1 t^{-1} \Pr(\eta(t) > o) dt$$

converges and

$$u \int_0^\infty e^{-ut} \Pr(\bar{\eta}(t) = o) dt = o.$$
when (30) diverges. The equation (31) of course implies that $p(t) = \circ$ for $t > o$ and the regenerative phenomenon definition breaks down. In the case where (30) converges we let $u \rightarrow \infty$ in (29) and obtain $\lim_{t \rightarrow o} p(t) = 1$. It is easily checked in this case that whenever

$$0 < t_1 < t_2 < \ldots < t_k,$$

then

$$\Pr\{E(t_1), E(t_2), \ldots, E(t_k)\} = \Pr\{E(t_1)\} \Pr\{E(t_2 - t_1), \ldots, E(t_k - t_1)\}.$$

Equation (29) can, of course, be indentified with equation (10) (Corollary 1) so we see that

$$ET_t = \int_0^t \Pr(\xi(u) \geq o)du,$$

and therefore $E$ is $\beta$-regular if and only if

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \Pr(\xi(u) \geq o)du = \beta.$$

The use of Theorem 4 in this context provides a result which is a special case of that of HEYDE [3].

As a final example, we mention a regenerative phenomenon which occurs in connection with the queueing system $M|G|1$. Customers arrive at a single server in a Poisson process of rate $\lambda$ and the service time distribution is of arbitrary type. Then, if the server is initially idle, the event of the server being idle forms a regenerative phenomenon $E$ (KINGMAN [4], [5]). We have

$$p(t) = \Pr\{\text{queue empty at time } t\},$$

and the random variable $t - T_t$ represents the time for which the current busy period has been in progress. It is not very difficult to show, making use of the results of Section 3.5 of [5] and of Theorem 6 and Section 16 of [4], that $E$ is $1$-regular if and only if the mean service time is less than $\lambda^{-1}$, $o$-regular if and only if the mean service time is greater than $\lambda^{-1}$, and $\beta$-regular, $\frac{1}{2} \leq \beta < 1$, if and only if the mean service time is equal to $\lambda^{-1}$ and its distribution belongs to the domain of attraction of a stable law of index $\beta^{-1}$. $E$ cannot be $\beta$-regular for $o < \beta < \frac{1}{2}$. 
References


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13. ABSTRACT
    Let \( \{E(t), t > 0\} \) be a family of regenerative events, as originally introduced by KINGMAN, and let \( Z(t, \omega) = 1 \) if \( \omega \in E(t) \) and be zero otherwise. Define

    \[
    T_t = \sup\{u : 0 \leq u \leq t; Z(u, \omega) = 1\}
    \]

    This paper obtains various theorems about \( T_t \) and studies its limiting behavior as \( t \to \infty \).
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REGENERATIVE EVENTS

OCCURRENCE TIMES