EIGENVALUES OF THE ADJACENCY MATRIX OF CUBIC LATTICE GRAPHS

by

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ABSTRACT.

A cubic lattice graph is defined to be a graph G, whose vertices are the ordered triplets on n symbols, such that two vertices are adjacent if and only if they have two coordinates in common. If $n_2(x)$ denotes the number of vertices y, which are at distance 2 from x and A(G) denotes the adjacency matrix of G, then G has the following properties: P_1) the number of vertices is n^3. P_2) G is connected and regular. P_3) $n_2(x) = 3(n-1)^2$. P_4) the distinct eigenvalues of A(G) are -3, n-3, 2n-3, 3(n-1). It is shown here that if $n > 7$, any graph G (with no loops and multiple edges) having the properties P_1) - P_4) must be a cubic lattice graph. An alternative characterization of cubic lattice graphs has been given by the author (J. Comb. Theory, Vol. 3, No. 4, December 1967, 386-401).

1. Introduction.

We shall consider only finite undirected graphs without loops or multiple edges. A cubic lattice graph with characteristic n is defined to be a graph whose vertices are identified with the n^3 ordered triplets on n symbols, with two vertices adjacent if and only if their corresponding triplets have two coordinates in common. If $d(x, y)$ denotes the distance between two vertices x and y and $\Delta(x, y)$ the number of vertices adjacent to both x and y, then it has been shown by the author [6] that for $n > 7$, the following properties characterize the cubic lattice graph with characteristic n:

(b_1) The number of vertices is n^3.

(b_2) The graph is connected and regular of degree $3(n-1)$.

(b_3) If $d(x, y) = 1$, then $\Delta(x, y) = n-2$.

(b_4) If $d(x, y) = 2$, then $\Delta(x, y) = 2$.

(b_5) If $d(x, y) = 2$, there exist exactly n-1 vertices z, adjacent to x such that $d(y, z) = 3$.

Dowling [4] in a note has shown that the property (b_5) is implied by properties (b_1) - (b_4) for $n > 7$. Hence for $n > 7$, (b_1) - (b_4) characterize a cubic lattice graph with characteristic n.

The adjacency matrix A(G) of a graph G is a square (0, 1) matrix whose rows and columns correspond to the vertices of G, and $a_{ij} = 1$ if and only if i and j are adjacent. Let $n_2(x)$ denote the number of vertices y at distance 2 from x.

A cubic lattice graph G with characteristic n has the following properties:

(P_1) The number of vertices is n^3.

(P_2) G is connected and regular.

(P_3) $n_2(x) = 3(n-1)^2$ for all x in G.

(P_4) The distinct eigenvalues of A(G) are -3, n-3, 2n-3, 3(n-1).

(P_1), (P_2), (P_3) are obvious. (P_4) is proved in paragraph 2. We go on to show that (P_1), (P_2), (P_3), (P_4) characterize a cubic lattice graph with characteristic n. Similar characterization for tethrahedral graphs has been given by Bose and Laskar [1].

2. Determination of the eigenvalues of A(G).

Given v objects, a relation satisfying the following conditions is said to be an association scheme with m classes:

a) Any two objects are either 1st, 2nd, ..., or mth associates, the relation of association being symmetrical.

b) Each object α has n_i ith associates, the number n_i being independent of α.

c) If any two objects α and β are ith associates, then the number of objects which are jth associates of α, and kth associates of β, is p^i_{jk} and is independent of the pair of ith associates α and β.

The numbers v, n_i and p^i_{jk}, i, j, k = 1, 2, ..., m are the parameters of the association scheme.

The concept of an association scheme was first introduced by Bose and Shimamoto [3].

If we define

$$B_i = (b^{\beta}_{\alpha i}) = \begin{pmatrix} b^1_{1i} & b^2_{1i} & \cdots & b^v_{1i} \\ b^1_{2i} & b^2_{2i} & \cdots & b^v_{2i} \\ \cdots & \cdots & \cdots & \cdots \\ b^1_{vi} & b^2_{vi} & \cdots & b^v_{vi} \end{pmatrix},$$

i = 0, 1, 2, ..., m,

where

$$b_{\alpha i}^{\beta} = 1, \text{ if the objects } \alpha \text{ and } \beta \text{ are ith associates}$$
$$= 0, \text{ otherwise,}$$

and

$$\rho_k = (p_{ik}^j) = \begin{pmatrix} p_{0k}^0 & p_{0k}^1 & \cdots & p_{0k}^m \\ p_{1k}^0 & p_{1k}^1 & \cdots & p_{1k}^m \\ \cdots & \cdots & \cdots & \cdots \\ p_{mk}^0 & p_{mk}^1 & \cdots & p_{mk}^m \end{pmatrix}, \quad k = 0, 1, \ldots, m,$$

then it has been shown by Bose and Mesner [2], that the materices ρ_i, $i = 0, 1, \ldots, m$ are linearly independent and combine in the same way as the B's under addition as well as multiplication. It was further shown that if

$$B = \sum_{i=0}^{m} c_i B_i$$
$$\rho = \sum_{i=0}^{m} c_i \rho_i,$$

then B and ρ have the same distinct eigenvalues. If in particular we take $c_0 = 0$, $c_1 = 1$, $c_2 = c_3 = \ldots = c_m = 0$, it follows that the distinct eigenvalues of B_1 are the same as those of ρ_1.

Consider a cubic lattice graph G with characteristic n. If a relation of association on the vertices of G is defined, such that two vertices are 1st, 2nd, or 3rd associates if they are at distances 1, 2 or 3 respectively, then it can be easily checked that G yields a three-class association scheme. It may be pointed out that the matrix A(G) is the matrix B_1 and thus the distinct eigenvalues of A(G) are given by those of the matrix

$$P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ n_1 & p_{11}^1 & p_{11}^2 & p_{11}^3 \\ 0 & p_{12}^1 & p_{12}^2 & p_{12}^3 \\ 0 & p_{13}^1 & p_{13}^2 & p_{13}^3 \end{pmatrix}$$

The parameters p_{jk}^i of the association scheme corresponding to G are easily calculated. They are given by

$$\begin{aligned} n_1 = 3(n-1), \quad & p_{11}^1 = n-2, & p_{11}^2 = 2, \quad & p_{11}^3 = 0, \\ & p_{12}^1 = 2(n-1), & p_{12}^2 = 2(n-2), & p_{12}^3 = 3, \\ & p_{13}^1 = 0, & p_{13}^2 = n-1, \quad & p_{13}^3 = 3(n-2), \end{aligned}$$

Substituting these values in the matrix P_1, the eigenvalues are easily calculated. They are found to be

$$-3,\ n-3,\ 2n-3,\ 3(n-1).$$

Thus, we have the following lemma:

Lemma 2.1. If G is a cubic lattice graph with characteristic n and if A(G) is the adjacency matrix of G, then the distinct eigenvalues of A(G) are

$$(2.1) \qquad -3,\ n-3,\ 2n-3,\ 3(n-1).$$

3. Some Preliminaries on Matrices.

Before stating the next lemma, we need the concept of the polynomial of a graph introduced by Hoffman [5]. Let J be the matrix all of whose entries are unity. Then for any graph G with adjacency matrix A = A(G), there exists a polynomial P(x) such that P(A) = J if and only if G is regular and connected. The unique polynomial of least degree satisfying this equation is called the polynomial of G, and is calculated as follows: if G has v vertices, it is regular of degree d, and the other distinct eigenvalues of A(G) are $\alpha_1, \alpha_2, \ldots, \alpha_t$, then

$$(3.1) \qquad P(x) = \frac{v \prod_{i=1}^{t} (x-\alpha_i)}{\prod_{i=1}^{t} (d-\alpha_i)} .$$

Consider a regular connected graph H (with no loops and multiple edges) on $v = n^3$ vertices such that the adjacency matrix $A = A(H)$ has the distinct eigenvalues -3, $n-3$, $2n-3$, $3(n-1)$.

Lemma 3.1. The matrix A satisfies the equation

$$(3.2) \qquad A^3 - A^2(3n-9) + A(2n^2 - 18n + 27) + (6n^2 - 27n + 27)\, I = 6J,$$

where J is a $v \times v$ matrix all of whose entries are 1, and I is the $v \times v$ identity matrix.

Proof: It follows immediately by calculating the polynomial of the graph as given in (3.1).

Lemma 3.2. For any two vertices x, y in H, $d(x, y) \leq 3$.

Proof: If in (3.2) we set $A_{ij} = 0$, $A^2_{ij} = 0$, then $A^3_{ij} = 6$, but this implies that $d(i,j) \leq 3$ for all vertices i, j in H.

Lemma 3.3. Consider the matrix

$$B = \tfrac{1}{2}[A^2 - (n-2)A - 3(n-1)\, I].$$

Let $n_2(i)$ denote the number of vertices j, such that $d(i,j) = 2$, and $n_3(i)$ denote the number of vertices k, such that $d(i,k) = 3$. If $n_2(i) = 3(n-1)^2$ for all vertices i in H, then

i) B is a (0,1) matrix,

ii) $\Delta(x,y) = n-2$, for all vertices x,y in H, such that $d(x,y) = 1$,

iii) $\Delta(x,y) = 2$, for all vertices x,y in H, such that $d(x,y) = 2$.

Proof: Since H is regular and 3(n-1) is the dominant eigenvalue, it follows H is regular of degree $n_1 = 3(n-1)$.

Divide the set of vertices of H, with respect to a particular vertex i into four subsets S_0, S_1, S_2, S_3 as follows:

S_0 : i

S_1 : $j_1, j_2, \ldots, j_t \ldots, j_{n_1}$, such that $d(i,j_t) = 1$, $t=1,2,\ldots,n_1$

S_2 : $k_1,k_2,\ldots,k_s,\ldots,k_{n_2(i)}$, such that $d(i,k_s)=2, s=1,2,\ldots,n_2(i)$

S_3 : $\ell_1,\ell_2,\ldots,\ell_r,\ldots,\ell_{n_3(i)}$, such that $d(i,\ell_r)=3, r=1,2,\ldots,n_3(i)$.

Thus the vertices in S_t are tth associates of the vertex i. The following relations can be deduced easily from (3.2) by noting that AJ = JA.

(3.3) $$A^3_{ii} = \sum_{t=1}^{n_1} A^2_{ij_t} = 3(n-1)(n-2).$$

(3.4) $$A^4_{ii} = \sum_{t=1}^{n_1} A^3_{ij_t} = 3(n-1)(n^2+3n-3).$$

Also, since $A^tJ = \{3(n-1)\}^tJ$, we get

(3.5) $$\sum_{j=1}^{v} A^2_{ij} = (A^2J)_{ii} = 9(n-1)^2,$$

(3.6) $$A^2_{ii} = \sum_{t=1}^{n_1} A_{ij_t} = 3(n-1)$$

Also

$$(3.7) \qquad \sum_{r=1}^{n_3(i)} A^2_{i\ell_r} = 0.$$

Hence it follows from (3.3), (3.5), (3.6), (3.7) that

$$(3.8) \qquad \sum_{s=1}^{n_2(i)} A^2_{ik_s} = \sum_{j=1}^{v} A^2_{ij} - \sum_{t=1}^{n_1} A^2_{ij_t} - \sum_{r=1}^{n_3(i)} A^2_{i\ell_r} - A^2_{ii}$$

$$= 6(n-1)^2.$$

Consider

$$(3.9) \qquad X_i = b^2_{ii} + \sum_{t=1}^{n_1(i)} b^2_{ij_t} + \sum_{s=1}^{n_2(i)} (b_{ik_s}-1)^2 + \sum_{r=1}^{n_3(i)} b^2_{i\ell_r}$$

$$= \sum_{j=1}^{v} b^2_{ij} - 2\sum_{s=1}^{n_2(i)} b_{ik_s} + n_2(i).$$

We first show that

$$X_i = n_2(i) - 3(n-1)^2.$$

Since

$$(3.10) \qquad B = \tfrac{1}{2}[A^2-(n-2)A-3(n-1)I], \text{ we get}$$

$$(3.11) \qquad B^2_{ii} = \tfrac{1}{4}[A^4_{ii}-2(n-2)A^3_{ii}+(n^2-10n+10)A^2_{ii}+6(n^2-3n+2)A_{ii}+9(n-1)^2I_{ii}]$$

Substituting values from (3.3), (3.4), (3.6) in (3.11) we get

$$B^2_{ii} = 3(n-1)^2.$$

But

$$\sum_{j=1}^{v} b^2_{ij} = B^2_{ii}$$

Hence

(3.12) $$\sum_{j=1}^{v} b_{ij}^2 = 3(n-1)^2.$$

Also from (3.10)

$$\sum_{s=1}^{n_2(i)} b_{ik_s} = \tfrac{1}{2}\sum_{s=1}^{n_2(i)} A_{ik_s}^2$$

It follows from (3.8) that

(3.13) $$\sum_{s=1}^{n_2(i)} b_{ik_s} = 3(n-1)^2.$$

Substituting values from (3.12), (3.13) in (3.9) we get

$$X_i = n_2(i) - 3(n-1)^2.$$

Now if $n_2(i) = 3(n-1)^2$ for all i in H, then $X_i = 0$ for all i in H. Then it follows from (3.9) that B is a (0,1) matrix which proves i).

To prove ii), we note that if $A_{ij_t} = 1$, then from (3.10),(3.3) and (3.6) it follows

$$\sum_{t=1}^{n_1} b_{ij_t} = 0.$$

But since $b_{ij} = 0$ or 1, this implies $b_{ij_t} = 0$, and hence from (3.10) it follows that $A_{ij}^2 = n-2$.

To prove iii) we note that if $A_{ij} = 0$, $A_{ij}^2 \neq 0$, then $b_{ij} \neq 0$ and hence $A_{ij}^2 = 2$.

4. <u>Theorem</u>. If H is a graph satisfying the following properties:

P_1) The number of vertices is n^3.

P_2) H is connected and regular.

P_3) $n_2(x) = 3(n-1)^2$ for all x in H.

P_4) The distinct eigenvalues of A(H) are -3, n-3, 2n-3, 3(n-1). Then, for $n > 7$, H is cubic lattice.

Proof: From lemmas (3.1) - (3.3) and the hypothesis H clearly satisfies the following conditions:

(b_1) The number of vertices is n^3.

(b_2) H is connected and regular of degree 3(n-1).

(b_3) $\Delta(x,y) = n-2$ for $d(x,y) = 1$.

(b_4) $\Delta(x,y) = 2$, for $d(x,y) = 2$.

Hence if $n > 7$, H is cubic lattice [6], [4].

Note: It is conjectured that the property P_3) of the theorem is implied by other properties P_1), P_2), P_4).

It may be pointed out that the main purpose of assuming P_3) is to prove that B is a (0,1) matrix. If we replace P_3) by P_3') and P_3'') as follows:

P_3'). H is edge-regular, i.e., $\Delta(x,y) = \Delta$ for all x,y, such that $\Delta(x,y) = 1$,

P_3'') $\Delta(x,y)$ = even, for all x,y, such that $d(x,y) = 2$,

then it can be shown that B is a (0,1) matrix. The proof goes like this: From P_3') and (3.3) it follows that $\Delta = n-2$. Substituting value for Δ in (3.10) and noting P_3'') we get $b_{ij} = 0$ if $A_{ij} = 1$, and b_{ij} = an integer if $A_{ij} = 0$. Again from (3.10) and (3.12) it follows that

$$\sum_{j=1}^{v} b_{ij} = \sum_{j=1}^{v} b_{ij}^2 .$$

Thus B is a matrix whose entries are either 0 or integer such that for any row, sum of the elements is equal to the sum of the squares of the elements, but this implies that B is a (0,1) matrix.

Hence we can also state that for $n > 7$, $P_1)$, $P_2)$, $P_3')$, $P_3'')$, $P_4)$ characterize a cubic lattice graph with characteristic n.

REFERENCES

1. R. C. BOSE and R. LASKAR, Eigenvalues of the Adjacency Matrix of Tetrahedral Graphs, Institute of Statistics Mimeo Series No. 571, University of North Carolina March 1968.

2. R. C. BOSE and D. M. MESNER, On Linear Associative Algebras Corresponding to Association Schemes of Partially Balanced Designs, Ann. Math. Statist, 30 (1959), 21-36.

3. R. C. BOSE and T. SHIMAMOTO, Classification and Analysis of Partially Balanced Incomplete Block Designs with Two Associate Classes, J. Amer. Stat. Assn., Vol 47 (1952), 151-184.

4. T. A. DOWLING, Note on "A Characterization of Cubic Lattice Graphs", J. Combinatorial Theory (to appear)

5. A. J. HOFFMAN, On the Polynomial of a Graph, Am. Math. Monthly, 70 (1963), 30-36.

6. R. LASKAR, A Characterization of Cubic Lattice Graphs, J. Combinatorial Theory, Vol. 3, No 4, December 1967, 386-401.